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Boundedness of the Hilbert Transform on Besov Spaces.

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Dedicate

The simplest words are the strongest,

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I am very grateful to your prayers and passionate encouragements,

you have followed me everywhere to give me a lot of power.

To the spirit of my father immaculate.

To the best person in my life,

to the person who taught me the meaning

of patience and respect, to my husband thank you for your help and understanding.

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and my heart always remembers them,

to the most precious to my heart, to my sons and daughters.

To my sisters and brothers, to all my family.

I dedicate this modest work.

Aldjia MAATOUG.

Contents

Notations

- $\mathbb{Z}$ is the set of all integer numbers.
- $\mathbb{N}$ is the set of all positive integer numbers including 0.
- $\mathbb{R}^n$ is the n-dimensional real Euclidean space.
- All considered spaces are defined on the Euclidean space $\mathbb{R}^n$.
- $(e_1, \ldots, e_n)$ is the canonical basis of $\mathbb{R}^n$.
- $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$ is the scalar product of $x = (x_1, \ldots, x_n)$ and $\xi = (\xi_1, \ldots, \xi_n)$ in $\mathbb{R}^n$.
- For multi-indices α and β we define the following operations
  - i) Addition and subtraction: $\alpha \pm \beta = (\alpha_1 \pm \beta_1, \alpha_2 \pm \beta_2, \cdots, \alpha_n \pm \beta_n)$
  - ii) Partial order: $\alpha \leq \beta$ means $\alpha_j \leq \beta_j, \quad \forall j \in \{1, \ldots, n\}$.
  - iii) Factorial: $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$.
  - iv) Power: $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$.
- For each $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\partial^\alpha f$ denotes the partial derivative $\partial^\alpha f = f^{(\alpha)} = \dfrac{\partial^{|\alpha|} f}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}}$ with $|\alpha| = \alpha_1 + \ldots + \alpha_n$.
- For a function $f : \mathbb{R}^n \to \mathbb{C}$, the support of f is $suppf = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}$.
- The spectrum of a temperate distribution f is the support of $\widehat{f}$.
- $L^1_{loc}(\mathbb{R}^n)$ is the collection of all locally integrable functions on $\mathbb{R}^n$.
- $\mathcal{D}(\mathbb{R}^n)$ is the space of all infinitely differentiable and compactly supported functions on $\mathbb{R}^n$.
- $\mathcal{D}'(\mathbb{R}^n)$ is the topological dual of $\mathcal{D}(\mathbb{R}^n)$ called the space of distributions on $\mathbb{R}^n$.

- $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space of all complex-valued, infinitely differentiable, and rapidly decreasing functions on $\mathbb{R}^n$ equipped with the semi-norm

$$\mathcal{N}_k(f) = \sup_{x\in\mathbb{R}^n} (1+|x|)^k \sum_{|\alpha|\leq k} |\partial^\alpha f(x)|,\ k = 1, 2, \ldots,$$

where $f \in \mathcal{S}(\mathbb{R}^n)$.

- $\mathcal{S}'(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}(\mathbb{R}^n)$ called the space of tempered distributions.

- For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $f \in \mathcal{S}'(\mathbb{R}^n)$, the Fourier transform defined on both $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ is given by $(\mathcal{F}f)(\varphi) = f(\mathcal{F}\varphi)$ where

$$(\mathcal{F}\varphi)(\xi) = \widehat{\varphi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)dx, \quad \xi \in \mathbb{R}^n,$$

- The mapping $\mathcal{F}$ is a bijection (in both cases) and its inverse is given by

$$(\mathcal{F}^{-1}\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(x)dx, \quad \xi \in \mathbb{R}^n.$$

- The convolution $\varphi * \psi$ of two integrable functions φ, ψ is defined via the integral

$$(\varphi * \psi)(x) = \int_{\mathbb{R}^n} \varphi(x-y)\psi(y)dy.$$

- For $1 \leq p \leq \infty$ we denote by p' the conjugate exponent of p such that $\dfrac{1}{p} + \dfrac{1}{p'} = 1.$

- The space $L^p(\mathbb{R}^n)$ denotes the set of all Lebesgue-measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{\frac{1}{p}} < \infty,\ 0 < p < \infty,$$

$$\|f\|_{L^\infty(\mathbb{R}^n)} = ess \sup_{x\in\mathbb{R}^n} |f(x)| < \infty.$$

With $\|\cdot\|_p$ we denote the L_p-norm.

- For $0 < q \leq \infty$, ℓ^q denotes the set of all (complex) sequences $\{a_k\}_{k\in\mathbb{N}}$ such that

$$\|\{a_k\}_{k\in\mathbb{N}}\|_{\ell^q} = \left(\sum_{k=0}^{\infty} |a_k|^q\right)^{1/q} < \infty,$$

(with the usual modification if $q = \infty$).

- For p and $q, 0 < p \leq \infty, 0 < q \leq \infty$; we put

$$\|\{f_k\}_k\|_{\ell^q(L^p)} = \left(\sum_{k=0}^{\infty} \|f_k(x)\|_p^q\right)^{\frac{1}{q}} < \infty,$$

$$\|\{f_k\}_k\|_{L^p(\ell^q)} = \| \left(\sum_{k=0}^{\infty} |f_k(x)|^q\right)^{\frac{1}{q}} \|_p < \infty.$$

$\ell^q(L^p)$ and $L^p(\ell^q)$ are quasi-Banach spaces (Banach spaces if $p \geq 1$ *and* $q \geq 1$).

- $L^p(\mathbb{R}^n, X)$ denotes the Lebesgue-Bôchner space of vector-valued functions.

- $(E)_{\ell^r}$ denotes the localized space in the ℓ^r norm, $1 \leq r \leq \infty$.

- For $m \in \mathbb{N}$, $W_p^m(\mathbb{R}^n)$ is the Sobolev space of all functions $f \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\|f\|_{W_p^m(\mathbb{R}^n)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p < \infty.$$

- $\Delta f = \sum_{i=1}^n \frac{\partial^2 f}{\partial {x_i}^2}$ is the Laplacian of f.

- For $s \in \mathbb{R}$, $p \in (1, \infty)$, $H_p^s(\mathbb{R}^n)$ is the Bessel-potential space (or Sobolev space of fractional order) of all functions $f \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\|f\|_{H_p^s(\mathbb{R}^n)} = \| \ (I - \Delta)^{s/2} f\|_p < \infty.$$

where I is the identity operator.

- $\mathcal{L}(A, B)$ denotes the set of all bounded linear operators from A into B.

- For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $\mathbb{B}(a, r)$ the ball in $\mathbb{R}^n$ with center x and radius r.

- "i.e." stands simply for "in other words".

- "a.e." stands simply for "almost everywhere".

- $C, c, c_1, c_2, \ldots$ are strictly positive constants, their values may depend on certain parameters, and change from one line to another.

- For $k \in \mathbb{Z}^n$, τ_k is the translation operator defined by $\tau_k f(\cdot) = f(\cdot - k)$.

- Difference operator:

$$
\begin{aligned}
\Delta_h^1 f(x) =& f(x+h) - f(x), \quad x, h \in \mathbb{R}^n \\
\Delta_h^m f(x) =& \Delta_h^1(\Delta_h^{m-1} f(x)), \quad m = 2, 3, \cdots \\
\Delta_h^m f(x) =& \sum_{j=0}^{m} \begin{pmatrix} j \\ m \end{pmatrix} (-1)^j f(x + (m-j)h), \quad m = 1, 2, \cdots
\end{aligned}
$$

Introduction

The Hilbert transform along curves is of a great importance in harmonic analysis, and also in applications to many fields of applied mathematics. Several authors have studied the boundedness of a such transform on different function spaces.

Let $\Gamma : \mathbb{R} \to \mathbb{R}^n, n \geq 2$, be a continuous curve passing through the origin, i.e. $\Gamma(0) = 0$. We define the Hilbert transform along Γ by the principal-valued integral

$$\mathcal{H}f(x) = p.v. \int_{-\infty}^{\infty} f(x - \Gamma(t))\frac{dt}{t}, \quad \forall f \in \mathcal{D}(\mathbb{R}^n). \tag{1}$$

It is known that its boundedness on $L^p(\mathbb{R}^n)$ has been extensively studied by various authors in different contexts, the authors gave positive results for some or all $p, 1 < p < \infty$. Many results are found in [9, 12, 13, 16, 29, 30, 31, 32, 33]. Recently, these results have been extended to the $L^p(\mathbb{R}^n; X)$, the spaces of vector-valued functions which have been taken up by several authors, where X is some suitable Banach spaces, but the Banach space of most interest to us will be ℓ^q for $1 < q < \infty$, see [3, 22, 35, 47]. Let us state some results which establish in this situation, the first is the work done in 1986 by Rubio de Francia et al. [35]. Now, we restrict our attention to $\Gamma(t)$ that are given as polynomial functions, because it is a model problem in this situation, they dealt with well-curved curves, see [20, 21].

At first, note that a simple calculation shows that

$$\widehat{\mathcal{H}f}(\xi) = m(\xi) \cdot \widehat{f},$$

where the "Fourier multiplier" m is the function

$$m(\xi) = p.v. \int_{-\infty}^{\infty} e^{-i\xi\cdot\Gamma(t)}\frac{dt}{t}, \quad \xi \in \mathbb{R}^n.$$

Next, it is known that for proving the boundedness on $L^2(\mathbb{R}^n)$, it suffices to show that $m(\xi)$ is a bounded function on $\mathbb{R}^n$ and to use the Van der Corput lemma and Plancherel's

theorem, (see[48, p.197] and [33]). Littlewood-Paley theory provides alternative methods for studying singular integrals. The Hilbert transform along curves, the classical example of a singular integral operator, led to the extensive modern theory of Calderón-Zygmund operators, mostly studied on different functional spaces such as Lebesgue spaces $L^p(\mathbb{R}^n)$, $L^p(\mathbb{R}^n; X)$ the space of vector-valued functions as we have seen above.
In the first part of this thesis, we will use the Littlewood-Paley theory to prove that the boundedness of the Hilbert transform along curve Γ on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ can be obtained by its $L^p(\mathbb{R}^n)$-boundedness, where $s \in \mathbb{R}, p, q \in (1, \infty)$, and $\Gamma(t)$ is an appropriate curve in $\mathbb{R}^n$, also, it is known that the Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ are embedded into $L^p(\mathbb{R}^n)$ spaces for $s > 0$ (*i.e.* $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n), s > 0$). Thus, our first result may be viewed as an extension to the besov spaces $B^s_{p,q}(\mathbb{R}^n)$ for general values of s in $\mathbb{R}$.

When Γ is of finite type, i.e. the set $\{\Gamma^{(k)}(0) : k \geq 1\}$ spans $\mathbb{R}^n$, we must consider the local version $\mathcal{H}_{loc}$ of the operator $\mathcal{H}$, where the integral defining $\mathcal{H}$ is restricted to $[-1, 1]$. In [38], it is shown that, in this case $\mathcal{H}_{loc}$ is bounded on $L^p(\mathbb{R}^n)$ for every $p, 1 < p < \infty$. Thus, what can happen in the case when Γ is not of finite type? This brings us to the simplest case $\Gamma(t) = (t, \gamma(t))$. And we restrict our attention to curves γ satisfying

$$\gamma \in \mathcal{C}^2((0, \infty)),\ \textit{convex on } [0, \infty) \textit{ and } \gamma(0) = \gamma'(0) = 0; \tag{2}$$

and γ is either even or odd. The convexity hypothesis means that

$$[\gamma(c) - \gamma(b)]/(c - b) \geq [\gamma(b) - \gamma(a)]/(b - a) \ \textit{ for } \ 0 \leq a < b < c.$$

The following notions naturally arise

Definition 0.1. *(i) A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ belongs to $\mathcal{C}^1$ if there exists $\lambda, 1 < \lambda < \infty$, such that for each $t > 0$ the inequality $f(\lambda t) \geq 2f(t)$ holds. Such a function f is said to be doubling.*

(ii) A differentiable function $f : \mathbb{R} \longrightarrow \mathbb{R}$ belongs to $\mathcal{C}^2$ if there exists $\epsilon_0 > 0$ such that for $t > 0$ the inequality $f'(t) \geq \epsilon_0 f(t)/t$ holds. Such a function f is said to be infinitesimally doubling.

If f is nondecreasing on $(0, \infty)$, then $f \in \mathcal{C}^2$ implies $f \in \mathcal{C}^1$.

We will also use the function h defined for $t>0$ by $h(t)=t\gamma'(t)-\gamma(t)$. Because of γ is convex and $\gamma(0)=0$ we get the following important inequality

$$t\gamma'(t) \geq \gamma(t) \;\; \textit{for all} \;\; t>0.$$

In [10], it was proved that if γ is even and satisfies (2) for $p \in (1,\infty)$, then $\mathcal{H}$ is $L^p(\mathbb{R}^n)$-bounded if and only if $\gamma' \in \mathcal{C}^1$. This is the case when γ is convex and even. In the odd case, the current situation is less satisfactory. In [30], it is shown that if γ is odd and satisfies (2), then $\mathcal{H}$ is L^2-bounded if and only if $h \in \mathcal{C}^1$. This means that for each $p \in (1,\infty)$ a necessary condition for $\mathcal{H}$ to be L^p-bounded is $h \in \mathcal{C}^1$. However, it was demonstrated in [8] that this condition is far from sufficient. It is shown that when γ is odd satisfies (2) and if $h \in \mathcal{C}^2$, then $\mathcal{H}$ is L^p-bounded for all $p \in (1,\infty)$. In the same case, we have the L^p result for any $p \in (1,\infty)$ if $\gamma' \in \mathcal{C}^1$ (see [10]).

For the case of polynomial curve Γ in $\mathbb{R}^n$ some of related known results are Theorems 2.7 and 2.8 (see below and [5, 37]). Indeed, the subject of bounds on Hilbert transforms and singular integrals has a rich history and has been studied by many authors on different spaces such as Lebesgue, Sobolev spaces, which are special cases of Besov spaces. For different states of the Hilbert transform we refer the reader to [5, 18, 23, 39] and many references therein. In particular, and in connection with our work, we mention the work of U. Luther and M.G. Russo [23]. The authors have studied the Hilbert transform on new weighted Besov spaces which touched our topic but did not approach exactly. Our method is also different from [23]. Besov spaces are the natural spaces in which many operators related to functional equations, many papers appeared on Besov spaces and some possible related applications, for example Lagrange interpolation in Besov spaces and Cauchy singular integral equations in Sobolev spaces (see [19, 25, 26]).

On some conditions on the curve Γ, we confirm that the Hilbert transform preserves the boundedness property on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, for all $s \in \mathbb{R}$ and some or all $p, q \in (1,\infty)$. In this thesis we will affirm this.

In the second part of this thesis, we will use the boundedness of the Hilbert transform on $L^p(\mathbb{R}^n;\ell^q)$ the spaces of vector-valued functions to prove the main results of Chapter 3 and

Chapter 4, we prove the boundedness on Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$, and more than on localized Lizorkin-Triebel spaces $(F^s_{p,q}(\mathbb{R}^n))_{\ell^r}$ with $r = p$.

The aim of this thesis is to study the boundedness of the Hilbert transform on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$, as well as on Localized Lizorkin-Triebel spaces $(F^s_{p,q}(\mathbb{R}^n))_{\ell^r}$ with $r = p$. And the main results of this thesis are related to the following chapters:

We start with a reminder on some classical inequalities of harmonic analysis which will be used in the following. And before giving the definitions of Besov and Lizorkin-Triebel spaces, we start by recalling the Littlewood-Palley decomposition by using a suitable resolution of the unity.

In Chapter 2, we present the boundedness of the Hilbert transform on $L^p(\mathbb{R}^n)$ and we use it to prove the boundedness on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$.

In Chapter 3, we give the ℓ^q-valued inequlity

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{H} f_j|^q \right)^{1/q} \right\|_{L^p} \leqslant c_{p,q} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p}$$

which is held for some or all p, q with $1 < p, q < \infty$, and we use it to prove the boundedness properties of the Hilbert transform on Lizorkin-Triebel spaces.

In Chapter 4, we present the boundedness properties of the Hilbert transform on localized Lizorkin-Triebel spaces $(F^s_{p,q}(\mathbb{R}^n))_{\ell^r}$ with $r = p$. Before that, we give the notion of Pointwise multipliers for Besov and Lizorkin-Triebel spaces, and the characterization of the localized Lizorkin-Triebel spaces, we need to introduce a new resolution of unity which is used to define this characterization. Let $\psi \in \mathcal{D}(\mathbb{R}^n)$, supported in the ball $\mathbb{B}(0, R)$ with $R > \sqrt{n}$ and satisfying

$$\sum_{k\in\mathbb{Z}^n} \psi(x - k) = 1 \quad (\forall x \in \mathbb{R}^n),$$

and we give some properties of localized Lizorkin-Triebel spaces. Finally to prove the boundedness of the Hilbert transform on localized Lizorkin-Triebel spaces $(F^s_{p,q}(\mathbb{R}^n))_{\ell^r}$ for $r = p$, we just give the result of [17] which proved that the Lizorkin-Triebel spaces are localizable in the ℓ^p norm.

Finally, we finish with a general conclusion and some open future works.

The main results of this thesis are:

In Chapter 2, we will prove the following theorem:

Theorem 0.1. *Let $s \in \mathbb{R}$, $1 < p, q < \infty$. If $\mathcal{H}$ is bounded on $L^p(\mathbb{R}^n)$, then $\mathcal{H}$ is bounded on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$.*

In Chapter 3, we will prove the following theorem:

Theorem 0.2. *Let $\gamma(t)$ be a continuous odd function, twice continuously differentiable, increasing and convex for $t \geq 0$. Suppose that γ'' is monotone for $t > 0$ and that there exists $C > 0$ so that $\gamma'(t) \leq Ct\gamma''(t)$ for $t > 0$. Then $\mathcal{H}$ is bounded on $F^s_{p,q}(\mathbb{R}^2)$ for all $s \in \mathbb{R}$, and all p, q with $5/3 < p, q < 5/2$.*

In the context of Theorem 0.2 we have the following theorems:

Theorem 0.3. *Let $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be a well-curved curve in $\mathbb{R}^n$ with $\Gamma(0) = 0$. Then*

$$\|\mathcal{H}f\|_{F^s_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{F^s_{p,q}(\mathbb{R}^n)},$$

for all $s \in \mathbb{R}$, and all $1 < p, q < \infty$, with $n \geq 2$.

Theorem 0.4. *(1) Let $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be a polynomial function. Then*

$$\|\mathcal{H}f\|_{F^s_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{F^s_{p,q}(\mathbb{R}^n)},$$

for all $s \in \mathbb{R}$, and all $1 < p, q < \infty$, with $n \geqslant 2$.

(2) Let the image of $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be of finite type at 0. Then

$$\|\mathcal{H}f\|_{F^s_{p,q}(\mathbb{R}^n)} \leq C\|f\|_{F^s_{p,q}(\mathbb{R}^n)},$$

for all $s \in \mathbb{R}$, all $1 < p, q < \infty$, with $n \geqslant 2$.

In Chapter 4, we will prove the following theorem:

Theorem 0.5. *Suppose that $\gamma(t)$ is a continuous odd function, twice continuously differentiable, increasing and convex for $t \geq 0$. Suppose also that γ'' is monotone for $t > 0$ and that there exists $C > 0$ so that $\gamma'(t) \leq Ct\gamma''(t)$ for $t > 0$. Then $\mathcal{H}$ is bounded on localized Lizorkin-Triebel $(F^s_{p,q}\mathbb{R}^n)_{\ell^p}$ spaces for all $s \in \mathbb{R}$, and all $5/3 < p, q < 5/2$.*

In the context of Theorem 0.5 we have the following theorems:

Theorem 0.6. *Let* $\Gamma : \mathbb{R} \to \mathbb{R}^n$ *be a well-curved curve in* $\mathbb{R}^n$ *with* $\Gamma(0) = 0$*. Then*

$$\|\mathcal{H}f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}} \leq C\|f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}}, \tag{3}$$

for all $s \in \mathbb{R}$*, and all* $1 < p, q < \infty$ *with* $n \geq 2$*.*

Theorem 0.7. *(1) Let* $\Gamma : \mathbb{R} \to \mathbb{R}^n$ *be a polynomial function. Then*

$$\|\mathcal{H}f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}} \leq C\|f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}}, \tag{4}$$

for all $s \in \mathbb{R}$*, and all* $1 < p, q < \infty$*, with* $n \geqslant 2$*.*

(2) Let the image of $\Gamma : \mathbb{R} \to \mathbb{R}^n$ *be of finite type at* 0*. Then*

$$\|\mathcal{H}f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}} \leq C\|f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}}, \tag{5}$$

for all $s \in \mathbb{R}$*, and all* $1 < p, q < \infty$*, with* $n \geqslant 2$*.*

The Chapter 2 is a paper published in Journal of Carpathian Mathematical publication "Boundedness of The Hilbert Transform on Besov Spaces". (2020), **12** (2), 443-450, see [24], while the Chapter 3 and the Chapter 4 are papers in preparation.

CHAPTER 1

Besov and Lizorkin-Triebel Spaces

The theory of Besov and Lizorkin-Triebel spaces has a long history and plays a central part in harmonic analysis and partial diferential equations, it has been an active area of research in the last few decades because of its important role in the study of approximation of functions and regularity of solutions to partial differential equations.
In this chapter, we present some results, which remain fixed throughout this thesis. We will recall the essential concepts that we will use subsequently, we then define the Besov and the Lizorkin-Triebel spaces and give some of their basic properties, we start with a reminder on some classical inequalities of harmonic analysis.

1.1 Classical inequalities

In the following paragraph, we recall some classical inequalities of harmonic analysis in $L^p(\mathbb{R}^n)$ spaces which are necessary for the continuation of this thesis. We will apply several times the Hölder and the Young inequalities. Sometimes we need also the use of, both, the Minkowski and the Bernstein inequalities.

Proposition. 1.1. ***(Hölder's inequality).*** *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with $1 \leq p, q, r \leq \infty$ and $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then $f \cdot g \in L^r(\mathbb{R}^n)$ and*

$$\|f \cdot g\|_r \leq \|f\|_p \|g\|_q.$$

Proposition. 1.2. ***(Young's inequality).*** *Let $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ with p, q, $r \in [1, \infty]$ where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then $f * g \in L^r(\mathbb{R}^n)$ and*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

Proof. We fix $g \in L^q(\mathbb{R}^n)$ and we consider the operator $Tf = f * g$. So we have

$$|Tf(x)| \leq \int_{\mathbb{R}^n} |f(y)|^{\frac{1}{q}} |g(x-y)||f(y)|^{\frac{1}{q'}} dy.$$

Using the Hölder's inequality, we get on the one hand

$$\|Tf(x)\|_q \leq \|f\|_1 \|g\|_{q'},$$

on the other hand, Hölder's inequality gives

$$|Tf(x)| \leq \|g\|_q \|f\|_{q'}.$$

Then applying the Riesz-Thorin interpolation theorem, we have

$$\begin{aligned} T : L^1(\mathbb{R}^n) &\longrightarrow L^q(\mathbb{R}^n) \\ L^q(\mathbb{R}^n) &\longrightarrow L^\infty(\mathbb{R}^n), \end{aligned}$$

so we have

$$T : L^p(\mathbb{R}^n) \longrightarrow L^r(\mathbb{R}^n) \quad \textit{with} \quad \frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}.$$

□

Proposition. 1.3. *(Minkowski's inequality).* *For all $1 \leq p \leq q \leq \infty$, and X an element in $\ell^p(\ell^q)$, then we have*

$$\|X\|_{\ell^q(\ell^p)} \leq \|X\|_{\ell^p(\ell^q)}.$$

Proposition. 1.4. *(Bernstein's inequality).* *[40, Remark 1, p.18].*
Let $R > 0$, $\alpha \in \mathbb{N}_0^n$ and $1 \leq p \leq q \leq \infty$. Then there exists a constant $c = c(\alpha, p, q, n) > 0$ such that

$$\|f^{(\alpha)}\|_q \leq cR^{|\alpha| + n(\frac{1}{p} - \frac{1}{q})} \|f\|_p$$

holds for all $f \in L^p(\mathbb{R}^n)$ with supp$\widehat{f} \subset \{\xi \in \mathbb{R}^n, \quad |\xi| \ \leq R\}$.

Proof. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ such that $\varphi(\xi) = 1$ if $|\xi| \leq 1$.
We pose $\varphi_R(\xi) = \varphi(\xi/R)$, such that $\varphi_R(\xi) = 1$ if $|\xi| \leq R$, then we have

$$\widehat{f} = \varphi_R \widehat{f} \quad \textit{and} \quad f^{(\alpha)} = (\mathcal{F}^{-1}\varphi_R)^{(\alpha)} * f.$$

From Young's inequality, we obtain

$$\|f^{(\alpha)}\|_q \leq\| (\mathcal{F}^{-1}\varphi_R)^{(\alpha)} \|_r \|f\|_p \quad \textit{with } 1+\frac{1}{q}=\frac{1}{p}+\frac{1}{r}.$$

As for all $x\in\mathbb{R}^n$, we have

$$(\mathcal{F}^{-1}\varphi_R)^{(\alpha)}(x)=R^n(\mathcal{F}^{-1}\varphi)^{(\alpha)}(Rx),$$

it becomes

$$\| (\mathcal{F}^{-1}\varphi_R)^{(\alpha)} \|_r= R^{n+|\alpha|-\frac{n}{r}} \| (\mathcal{F}^{-1}\varphi)^{(\alpha)} \|_r,$$

so we have

$$\|f^{(\alpha)}\|_q \leq cR^{|\alpha|+n(\frac{1}{p}-\frac{1}{q})}\|f\|_p \quad \textit{with } c=\|(\mathcal{F}^{-1}\varphi)^{(\alpha)}\|_r.$$

□

Proposition. 1.5. *Let $0<a<1$ and $0<q\leq\infty$. For any real sequence with positive term $\{\epsilon_j\}$ in $\ell^q(\mathbb{Z})$, the sequences $\eta_k=a^k\sum_{j=0}^k a^{-j}\epsilon_j$ and $\mu_k=a^{-k}\sum_{j\geq k}a^j\epsilon_j$ belong to $\ell^q(\mathbb{Z})$. Furthermore, there is $c=c(a,q)$ such that*

$$\|\eta_k\|_{\ell^q(\mathbb{Z})}+\|\mu_k\|_{\ell^q(\mathbb{Z})}\leq c\|\{\epsilon_j\}\|_{\ell^q(\mathbb{Z})}.$$

Proof. For $1<q<\infty$, we can write $\{\eta_k\}$ in the form

$$\eta_k=\sum_{j=0}^k a^{\frac{(k-j)}{q}}a^{\frac{(k-j)}{q'}}\epsilon_j,\ \textit{where } \frac{1}{q}+\frac{1}{q'}=1.$$

By Hölder's inequality, we find

$$\eta_k^q\leq\left(\sum_{j=0}^k a^{(k-j)}\epsilon_j^q\right)\left(\sum_{j=0}^k a^{(k-j)}\right)^{\frac{q}{q'}},$$

we deduce that the sum of η_k^q for $k=0,1,2,\cdots$ is increased by $\left(\sum_{j\geq0}a^i\right)^q\left(\sum_{j\geq0}\epsilon_j^q\right)$, which gives

$$\|\{\eta_k\}\|_{\ell^q(\mathbb{Z})}\leq\frac{1}{1-a}\|\{\epsilon_j\}\|_{\ell^q(\mathbb{Z})}.$$

The same for $\{\mu_k\}$.

For $0<q\leq1$, we have

$$\eta_k^q\leq\sum_{j=0}^k a^{(k-j)q}\epsilon_j^q,$$

which makes it possible to increase the sum of η_k^q for $k = 0, 1, 2, \cdots$ by $\left(\sum_{i\geq 0} a^{iq}\right)\left(\sum_{j\geq 0}\epsilon_j^q\right)$, i.e.

$$\|\{\eta_k\}\|_{\ell^q(\mathbb{Z})} \leq \left(\frac{1}{1-a^q}\right)^{\frac{1}{q}}\|\{\epsilon_j\}\|_{\ell^q(\mathbb{Z})}.$$

The same for $\{\mu_k\}$. By the same reasoning, we can show the case $q = \infty$. □

1.2 Definitions and basic properties of Besov and Lizorkin-Triebel spaces

In this section, we give the definition of Besov and Lizorkin-Triebel spaces using the Littlewood-Paley series which are based on the decomposition of the unity, to this decomposition, we associate a decomposition of any function f in the space of tempered distributions, we will recall the definition of finite differences which will allow us to build equivalent norms for these spaces, as well as their properties which will be useful later.

For details and further properties of Besov and Lizorkin-Triebel spaces, such equivalent norms, embeddings, etc., we refer to Bergh and Löfström [4], Peetre [34], Runst and Sickel [36], Yuan, Sickel and Yang [46] and Triebel [40, 41].

First, we recall the definition of Besov and Lizorkin-Triebel spaces using the Littlewood-Paley decomposition.

1.2.1 Littlewood-Paley decomposition

The Littlewood-Paley series play an important role in the definition of functional spaces in general, in particular Besov and Lizorkin-Triebel spaces. They were introduced by different authors, Peetre, Triebel and others, we will recall the definition of the Littlewood-Paley decomposition of tempered distributions, we first need the concept of a smooth dyadic resolution of unity.

Let ϱ be a smooth function in $\mathcal{S}(\mathbb{R}^n)$, which satisfies the conditions:

(i) $supp\, \varrho \subset \{\xi \in \mathbb{R}^n : 1 \leq |\xi| \leq 3\}$,

(ii) $\varrho(\xi) > 0$, *for* $1 \leq |\xi| \leq 3$,

(iii) $\sum_{j\in\mathbb{Z}} \varrho(2^{-j}\xi) = 1,\ \textit{for}\ \xi \in \mathbb{R}^n \setminus \{0\}$.

The construction of ϱ does not pose any difficulty, see for example [4].

We put $\Psi(\xi) = 1 - \sum_{k=1}^{\infty} \varrho(2^{-k}\xi)$, then it follows that the function $\Psi \in \mathcal{C}^\infty(\mathbb{R}^n)$, such that

$$\textit{supp}\,\Psi \subset \{\xi \in \mathbb{R}^n : |\xi| \leq 3\},$$

then for all $\xi \in \mathbb{R}^n$, we obtain a smooth dyadic decomposition of unity in $\mathbb{R}^n$, namely,

$$\Psi(\xi) + \sum_{k=1}^{\infty} \varrho(2^{-k}\xi) = 1. \tag{1.1}$$

And putting

$$\Phi_0 = \mathcal{F}^{-1}\Psi \quad \textit{and} \quad \Phi_k = \mathcal{F}^{-1}\varrho(2^{-k}\cdot), \quad k = 1, 2, \cdots . \tag{1.2}$$

Then we rewrite (1.1) as follows

$$\sum_{k=0}^{\infty} \widehat{\Phi}_k(\xi) = 1, \quad \forall \xi \in \mathbb{R}^n.$$

To this partition, we associate a sequence of convolution operators

$$\Delta_k : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \quad \textit{and} \quad Q_j : \mathcal{S}'(\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n),$$

defined by

$$\Delta_k f = \mathcal{F}^{-1}(\varrho(2^{-k}\cdot)) * f, \quad \textit{for}\ k = 1, 2, \cdots , \tag{1.3}$$

$$Q_j f = \mathcal{F}^{-1}(\Psi(2^{-j}\cdot)) * f, \quad \textit{for}\ j = 0, 1, 2, \cdots , \tag{1.4}$$

hence

$$\Delta_k f(\xi) = \Phi_k(\xi) * f(\xi), \quad \textit{for}\ k = 1, 2, \cdots , \tag{1.5}$$

with the notation $\Delta_0 = Q_0$, we write the relation (1.1) at the point $2^{-j}\xi$, then we have

$$\Psi(2^{-j}\xi) + \sum_{k=j+1}^{\infty} \varrho(2^{-k}\xi) = 1,$$

by multiplying by $\widehat{f}$, we obtain

$$\Psi(2^{-j}\xi)\widehat{f} + \sum_{k=j+1}^{\infty} \varrho(2^{-k}\xi)\widehat{f} = \widehat{f}, \tag{1.6}$$

by applying the application $\mathcal{F}^{-1}$ on (1.6) , then we obtain

$$Q_j f + \sum_{k=j+1}^{\infty} \Delta_k f = f, \quad (\forall j \in \mathbb{N}), \tag{1.7}$$

for $j = 0$, we obtain

$$Q_0 f + \sum_{k=1}^{\infty} \Delta_k f = f,$$

so, the Littlewood-Paley decomposition of f is given by

$$f = \sum_{k=0}^{\infty} \Delta_k f, \ (\textit{ convergence in } \mathcal{S}'(\mathbb{R}^n)). \tag{1.8}$$

In other words, we have

$$f = \sum_{k=0}^{\infty} \Phi_k * f, \ (\textit{ convergence in } \mathcal{S}'(\mathbb{R}^n)). \tag{1.9}$$

By replacing f of (1.8) in (1.7), we find

$$Q_j f + \sum_{k=j+1}^{\infty} \Delta_k f = \sum_{k=0}^{j} \Delta_k f + \sum_{k=j+1}^{\infty} \Delta_k f,$$

then we have

$$Q_j f = \sum_{k=0}^{j} \Delta_k f.$$

Remark 1.1. *By the Young inequality, we easily show that the sequences of operators $(\Delta_j)_{j\geq 0}$ and $(Q_j)_{j\geq 0}$ are uniformly bounded in $\mathcal{L}(L^p)$ for $1 \leq p \leq \infty$.*

Definition 1.1. *[40] Let $f \in \mathcal{S}'(\mathbb{R}^n)$ and $a > 0$. we define the maximal operators associated to the Δ_j and Q_j by*

$$\Delta_j^{*,a} f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\Delta_j f(x-y)|}{1 + (2^j|y|)^a}, \quad x \in \mathbb{R}^n, j = 0, 1, 2 \cdots ,$$

$$Q_j^{*,a} f(x) = \sup_{y\in\mathbb{R}^n} \frac{|Q_j f(x-y)|}{1 + (2^k|y|)^a}, \quad x \in \mathbb{R}^n, j = 0, 1, 2 \cdots ,$$

Definition 1.2. *Let f, g in $\mathcal{S}'(\mathbb{R}^n)$. The product $f \cdot g$ is defined by*

$$f \cdot g = \lim_{j\to\infty} (Q_j f) \cdot (Q_j g) \tag{1.10}$$

if the limit of the right hand side of (1.10) exists in $\mathcal{S}'(\mathbb{R}^n)$(see [36, 4.2]).

1.2.2 Besov and Lizorkin-Triebel spaces via convolution

Now, we are ready for the definition of the Besov and the Lizorkin-Triebel spaces via convolution. See for instance, Runst and Sickel [36] and Triebel [40].

Definition 1.3. *[36] Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. The Besov space $B^s_{p,q}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left(\sum_{j\geq 0}(2^{sj}\|\Delta_j f\|_p)^q\right)^{\frac{1}{q}} < \infty, \text{ for } q \neq \infty. \\ \sup_{j\geq 0}(2^{sj}\|\Delta_j f\|_p) < \infty, \text{ for } q = \infty. \end{cases} \tag{1.11}$$

Definition 1.4. *[36] Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < q \leq \infty$. The Lizorkin-Triebel space $F^s_{p,q}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ satisfying*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} = \begin{cases} \left\|\left(\sum_{j\geq 0}(2^{sj}|\Delta_j f|)^q\right)^{\frac{1}{q}}\right\|_p < \infty, \text{ for } q \neq \infty, \\ \left\|\sup_{j\geq 0}(2^{sj}|\Delta_j f|)\right\|_p < \infty, \text{ for } q = \infty. \end{cases} \tag{1.12}$$

$B^s_{p,q}(\mathbb{R}^n)$ (resp.$F^s_{p,q}(\mathbb{R}^n)$) endowed with the norm (1.11) (resp.(1.12)) is a quasi-Banach space (Banach space if $p \geq 1$, $q \geq 1$), and can be defined by replacing the discrete chosen system $\{\varrho(2^{-j}\xi)\}_j$ by a continuous one.

Each definition in (1.11) and (1.12) above is just one of the many possibilities for introducing Besov and Lizorkin-Triebel spaces.

Remark 1.2. *In the formula (1.11) (resp.(1.12)), we can replace Δ_j by $\Delta_j^{*,a}$ with $a > \frac{n}{p}$ (resp. $a > \frac{n}{\min(p,q)}$), and we obtain an equivalent norm in $B^s_{p,q}(\mathbb{R}^n)$ (resp.$F^s_{p,q}(\mathbb{R}^n)$). For more details, see Peetre [34] and Triebel [40].*

Let us redefine the Besov and the Lizorkin-Triebel spaces by another way that we will use in Chapter 2, see Yuan, Sickel and Yang [46] and Triebel [40, 2.3.1].

Definition 1.5. *[46] Let $s \in \mathbb{R}$ and $0 < p, q \leqslant \infty$. The Besov space $B^s_{p,q}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \left(\sum_{j=0}^{\infty} 2^{sjq}\|\Phi_j * f\|_p^q\right)^{1/q} < \infty, \tag{1.13}$$

with the usual modification if $q = \infty$.

Definition 1.6. *[46] Let $s \in \mathbb{R}, 0 < p < \infty$ and $0 < q \leqslant \infty$. Then The Lizorkin-Triebel space $F^s_{p,q}(\mathbb{R}^n)$ is the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} = \left\| \left(\sum_{j=0}^{\infty} 2^{jsq} |\left(\Phi_j * f\right)(x)|^q \right)^{\frac{1}{q}} \right\|_{Lp(\mathbb{R}^n)} < \infty, \tag{1.14}$$

with the usual modification if $q = \infty$.

Remark 1.3. *[40, 2.3.9] Beside the trivial equality $B^s_{p,p}(\mathbb{R}^n) = F^s_{p,p}(\mathbb{R}^n)$ we have always diversity. More exactly, it holds*

(i) $B^{s_1}_{p_1,q_1}(\mathbb{R}^n) = B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *implies* $s_1 = s_2, p_1 = p_2$ *and* $q_1 = q_2$,

(ii) $F^{s_1}_{p_1,q_1}(\mathbb{R}^n) = F^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *implies* $s_1 = s_2, p_1 = p_2$ *and* $q_1 = q_2$,

(iii) $F^{s_1}_{p_1,q_1}(\mathbb{R}^n) = B^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ *implies* $s_1 = s_2$ *and* $, p_1 = p_2 = q_1 = q_2$.

Proposition. 1.6. *[40, 2.3.5] The following properties are verified*

(i) $L^p(\mathbb{R}^n) = F^0_{p,2}(\mathbb{R}^n)$ *if* $1 < p < \infty$, *(the Lebesgue spaces),*

(ii) $W^m_p(\mathbb{R}^n) = F^m_{p,2}(\mathbb{R}^n)$ *if* $1 < p < \infty$, $m = 1, 2, \cdots$, *(the Sobolev spaces),*

(iii) $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$ *if* $1 < p < \infty, s \in \mathbb{R}$, *(the Bessel potential spaces),*

(iv) $F^s_{p,p}(\mathbb{R}^n) = B^s_{p,p}(\mathbb{R}^n)$ *if* $1 \leq p < \infty$, $s \in \mathbb{R}$, *(the Slobodeckij spaces),*

Proposition. 1.7. *Let $s > 0$, then a tempered distribution f belongs to $B^s_{p,q}(\mathbb{R}^n)$ if and only if $f \in L_p(\mathbb{R}^n)$ and $\partial_j f \in B^{s-1}_{p,q}(\mathbb{R}^n)$ for all $j = 1, \cdots, n$. Moreover the expression*

$$\|f\|_p + \sum_{j=1}^{n} \|\partial_j f\|_{B^{s-1}_{p,q}(\mathbb{R}^n)}$$

is an equivalent norm in $B^s_{p,q}(\mathbb{R}^n)$.

Proposition. 1.8. *Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty, \alpha \in \mathbb{N}^n$, then the partial derivative ∂^α takes continuously the space $B^s_{p,q}(\mathbb{R}^n)$ to $B^{s-|\alpha|}_{p,q}(\mathbb{R}^n)$.*
In an other words, if $f \in B^s_{p,q}(\mathbb{R}^n)$ then $\partial^\alpha f \in B^{s-|\alpha|}_{p,q}(\mathbb{R}^n)$.
Specifically, if $f \in B^s_{p,q}(\mathbb{R}^n)$ then $\partial f \in B^{s-1}_{p,q}(\mathbb{R}^n)$.

Proof. We have $\|\Delta_j(\partial^\alpha f)\|_p = \|\partial^\alpha(\Delta_j f)\|_p$.

From Bernstein's inequality (for $p = q$), we have

$$\|\Delta_j(\partial^\alpha f)\|_p \leq c 2^{j|\alpha|}\|\Delta_j f\|_p,$$

hence

$$\|2^{j(s-|\alpha|)}\|\Delta_j(\partial^\alpha f)\|_p\|_{\ell^q} \leq c\|2^{js}\|\Delta_j f\|_p\|_{\ell^q},$$

i.e.

$$\|\partial^\alpha f\|_{B^{s-|\alpha|}_{p,q}(\mathbb{R}^n)} \leq c\|f\|_{B^s_{p,q}(\mathbb{R}^n)}.$$

□

Proposition. 1.9. *(i) Let $m = 1, 2, \cdots$, for any f in $\mathcal{S}'(\mathbb{R}^n)$, the following expressions*

$$\sum_{|\alpha|\leq m} \left\|f^{(\alpha)}\right\|_{B^{s-m}_{p,q}(\mathbb{R}^n)}, \tag{1.15}$$

$$\|f\|_{B^{s-m}_{p,q}(\mathbb{R}^n)} + \sum_{j=1}^{n} \left\|\frac{\partial^m f}{\partial x_j^m}\right\|_{B^{s-m}_{p,q}(\mathbb{R}^n)}, \tag{1.16}$$

and

$$\|\Delta_0 f\|_p + \sum_{j=1}^{n} \left\|\frac{\partial^m f}{\partial x_j^m}\right\|_{B^{s-m}_{p,q}(\mathbb{R}^n)}, \tag{1.17}$$

are equivalent norms in Besov spaces $B^s_{p,q}(\mathbb{R}^n)$.

(ii) Let $m = 1, 2, \cdots$, for any f in $\mathcal{S}'(\mathbb{R}^n)$, the following expressions

$$\sum_{|\alpha|\leq m} \left\|f^{(\alpha)}\right\|_{F^{s-m}_{p,q}(\mathbb{R}^n)}, \tag{1.18}$$

$$\|f\|_{F^{s-m}_{p,q}(\mathbb{R}^n)} + \sum_{j=1}^{n} \left\|\frac{\partial^m f}{\partial x_j^m}\right\|_{F^{s-m}_{p,q}(\mathbb{R}^n)}, \tag{1.19}$$

and

$$\|\Delta_0 f\|_p + \sum_{j=1}^{n} \left\|\frac{\partial^m f}{\partial x_j^m}\right\|_{F^{s-m}_{p,q}(\mathbb{R}^n)}, \tag{1.20}$$

are equivalent norms in Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$.

Proof. See [36, P19]. □

1.2.3 Characterizations of Besov and Lizorkin-Triebel spaces via differences

To define equivalent norms of a function f of certain functional spaces, specially in Besov and Lizorkin-Triebel spaces, we need to define the finite difference operator Δ_h^m.
For any distribution f on $\mathbb{R}^n$, and all $h \in \mathbb{R}^n$, we pose

$$\Delta_h^1 = \Delta_h = \tau_{-h} f - f, \quad \textit{where} \quad \tau_h f(x) = f(x-h).$$

Recall that Δ_h^m can also be defined iteratively via

$$\Delta_h^{m+1} f(x) = \Delta_h \left(\Delta_h^m f(x) \right), \quad \forall m \in \mathbb{N}^*.$$

We easily verify the following formula

$$\Delta_h^m f(x) = \sum_{j=0}^{m} (-1)^{m-j} \begin{pmatrix} j \\ m \end{pmatrix} f(x+jh),\ m \in \mathbb{N}^*,\ h \in \mathbb{R}^n,\ x \in \mathbb{R}^n,$$

where $\begin{pmatrix} j \\ m \end{pmatrix}$ are binomial coefficients.

Lemma 1.10. *Let* $1 \leq p \leq \infty,\ k \in \mathbb{N}, f \in \mathcal{S}(\mathbb{R}^n),$ *then*

(i) $\|\tau_{-h} f - f\|_p \leq |h| \|f'\|_p,$

(ii) $\|\Delta_h^k f\|_p \leq |h|^k \|f^{(k)}\|_p,\ (k \geq 2).$

Proof. (i) By using the following inequality

$$|f(x) - f(y)| \leq |x-y|^{1-1/p} \|f'\|_p,$$

and since $p \geq 1$, given $0 \leq 1 - 1/p \leq 1$, and taking $h = x - y$, we obtain

$$\begin{aligned} \|\tau_{-h} f - f\|_p &\leq |h|^{1-1/p} \|f'\|_p \\ &\leq |h| \|f'\|_p. \end{aligned}$$

(ii) We obtain by induction

$$\begin{aligned}
\|\Delta_h^{k+1} f\|_p &\leq \|\Delta_h(\Delta_h^k f)\|_p \\
&\leq |h| \|(\Delta_h^k f)'\|_p \\
&= |h| \|\Delta_h^k f'\|_p \\
&\leq |h| |h|^k \|(f')^{(k)}\|_p \\
&\leq |h|^{k+1} \|f^{(k+1)}\|_p.
\end{aligned}$$

□

Definition 1.7. *Let $h \in \mathbb{R}^n$, $m \in \mathbb{N}^*$, $t > 0$, $1 \leq p \leq \infty$, and $f \in L^p(\mathbb{R}^n)$, we define the modulus of continuity of order m of f in $L^p(\mathbb{R}^n)$ as*

$$w_p^m(f,t) := \sup_{|h| \leq t, h \in \mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\Delta_h^m f(x)|^p dx \right)^{1/p}. \tag{1.21}$$

This modulus of continuity is used to find equivalent norms.

Remark 1.4. *(i) The continuity of a function f in x is defined by*

$$\lim_{h \longrightarrow 0} |\Delta_h^1 f(x)| = \lim_{h \longrightarrow 0} |f(x+h) - f(x)| = 0.$$

(ii) The differentiability of a function f in x is described by

$$\lim_{h \longrightarrow 0} \frac{|\Delta_h^1 f(x)|}{|h|} \leq c < \infty, \ (c > 0).$$

(iii) The modulus of continuity defined in (1.21) converges for the L^p-norm, and it is monotonic for the sup operator, and m-differentiable (regularity of order m).

(iv) $w_p^m(f,t) = 0$ if and only if f is a polynomial of degree less or equal $m-1$.

The following propositions present equivalent norms in $B_{p,q}^s(\mathbb{R}^n)$ and $F_{p,q}^s(\mathbb{R}^n)$.

Proposition. 1.11. *Let $m \in \mathbb{N}^*$, $1 \leq p, q \leq \infty$ and $0 < s < m$. Then the expression*

$$\|f\|_p + \left(\int_0^1 \left(\frac{w_p^m(f,t)}{t^s} \right)^q \frac{dt}{t} \right)^{1/q}$$

is an equivalent norm in $B_{p,q}^s(\mathbb{R}^n)$.

Proposition. 1.12. *([36, Theorem 3.1.1 p41]) Let m be a natural number, $0 < s < m$, $1 \leq p, q \leq \infty$. Then the space $B^s_{p,q}(\mathbb{R}^n)$ is the set of all temperate distributions f satisfying*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^n)} = \|f\|_p + \left(\int_{\mathbb{R}^n} |h|^{-sq} \left(\int_{\mathbb{R}^n} |\Delta^m_h f(x)|^p dx \right)^{q/p} \frac{dh}{|h|^n} \right)^{1/q} < \infty, \tag{1.22}$$

For $1 \leq p < \infty$. Then the space $F^s_{p,q}(\mathbb{R}^n)$ is the set of all temperate distributions f satisfying

$$\|f\|_{F^s_{p,q}(\mathbb{R}^n)} = \|f\|_p + \left\| \left(\int_0^1 \left(t^{-s-n} \int_{|h|<t} |\Delta^m_h f(\cdot)| dh \right)^q \frac{dt}{t} \right)^{1/q} \right\|_{Lp(\mathbb{R}^n)} < \infty. \tag{1.23}$$

In (1.22), the term $(\int_{\mathbb{R}^n} \cdots dh)$ may be replaced by $(\int_{h<\epsilon} \cdots)$ for any $\epsilon > 0$, and sometimes, in (1.23), the ball-means may be replaced by $(\frac{1}{t^n} \int_{|h|\leq t} \cdots)$.

1.3 Embeddings in $B^s_{p,q}(\mathbb{R}^n)$ and $F^s_{p,q}(\mathbb{R}^n)$

In this paragraph, we recall some embeddings and equalities in the sense of norms between Besov and Lizorkin-Triebel spaces.

Definition 1.8. *Let $(X, \|.\|_X), (Y, \|.\|_Y)$ two quasi-Banach spaces. We say that X injects into Y and we write $X \hookrightarrow Y$ if $X \subseteq Y$ and, the identity application defined from X into Y is continuous, i.e. there exists a constant c such that for any function $f \in X$ we have*

$$\|f\|_Y \leq c\|f\|_X.$$

Example 1.1. *The space $\ell_p(\mathbb{N})$ injects into $\ell_q(\mathbb{N})$ for $0 < p \leq q \leq \infty$ i.e.*

$$\ell_p(\mathbb{N}) \hookrightarrow \ell_q(\mathbb{N}), \quad 0 < p \leq q \leq \infty. \tag{1.24}$$

Proposition. 1.13. *[40, 2.3.3] Let $s \in \mathbb{R}$, and $1 \leq p, q \leq \infty$, then*

(i) We have

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n).$$

Furthermore, $\mathcal{S}(\mathbb{R}^n)$ is dense in $B^s_{p,q}(\mathbb{R}^n)$ if $\max(p, q) < \infty$.

(ii) We have

$$\mathcal{S}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n),$$

if $p < \infty$. Furthermore, $\mathcal{S}(\mathbb{R}^n)$ is dense in $F^s_{p,q}(\mathbb{R}^n)$ if $q < \infty$.

Proposition. 1.14. *(i) Let $-\infty < \sigma < s < \infty$ and $1 \leq p, r, t \leq \infty$, Then*

$$B^s_{p,r}(\mathbb{R}^n) \hookrightarrow B^\sigma_{p,t}(\mathbb{R}^n),$$

(ii) Let $s \in \mathbb{R}$, $1 \leq r \leq t \leq \infty$ and $1 \leq p \leq \infty$, then

$$B^s_{p,r}(\mathbb{R}^n) \hookrightarrow B^s_{p,t}(\mathbb{R}^n),$$

(iii) Let $1 \leq p_0 < p \leq \infty$, $1 \leq q \leq \infty$ and $s - \frac{n}{p} \geq s_0 - \frac{n}{p_0}$, then

$$B^{s_0}_{p_0,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,q}(\mathbb{R}^n),$$

(iv) Let $s \geq \frac{n}{p} - \frac{n}{r}$, $1 \leq q \leq \infty$ and $1 \leq p < r < \infty$, then

$$F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n),$$

(v) Let $s > \frac{n}{p} - \frac{n}{r}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$ or $s = \frac{n}{p} - \frac{n}{r}$ and $q \leq r$, then

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_r(\mathbb{R}^n).$$

Proof. See [36, Coro 2,p.37] □

Proposition. 1.15. *Let $s \in \mathbb{R}$, $1 \leq p < \infty$, and $1 \leq q \leq \infty$, (for the Besov spaces $0 < p \leq \infty$).*

i) For $q \leq r \leq \infty$, we have

$$F^s_{p,q}(\mathbb{R}^n) \hookrightarrow F^s_{p,r}(\mathbb{R}^n). \tag{1.25}$$

ii) For $1 \leq r \leq \infty$ and $\epsilon > 0$, we have

$$F^{s+\epsilon}_{p,q}(\mathbb{R}^n) \hookrightarrow F^s_{p,r}(\mathbb{R}^n). \tag{1.26}$$

iii)

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n). \tag{1.27}$$

Proof. i) For $1 \leq q \leq r \leq \infty$ and according to (1.24) the proof is obvious.

ii) For arbitrary $1 \leq r \leq \infty$ and $\epsilon > 0$, we can affirm that

$$\begin{aligned}
\|f\|_{F^s_{p,r}} &= \left(\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{jsq}|\Delta_j f|^q\right)^{\frac{r}{q}} dx\right)^{\frac{1}{r}} \\
&\leq \int_{\mathbb{R}^n} \sup 2^{j(s+\epsilon)}|\Delta_j f| dx \left(\sum_{j=0}^{\infty} 2^{-j\epsilon r}\right)^{\frac{1}{r}} \\
&\leq C\|f\|_{F^{s+\epsilon}_{p,\infty}}.
\end{aligned}$$

with $i)$ we finish the proof. Modification if $r = \infty$.

iii) We will study two cases:

If $1 \leq q \leq p < \infty$, we have

$$\begin{aligned}
\|f_k\|_{\ell_p(L_p(\mathbb{R}^n))} &\leq \|f_k\|_{L_p(\mathbb{R}^n,\ell_q)} \\
&= \left\|\sum_{k=0}^{\infty} |f_k|^q\right\|^{\frac{1}{q}}_{L_{\frac{p}{q}}(\mathbb{R}^n)} \\
&\leq \left(\sum_{k=0}^{\infty} |f_k|_{L_{\frac{p}{q}}(\mathbb{R}^n)}\right)^{\frac{1}{q}} \\
&= \|f_k\|_{\ell_q(L_p(\mathbb{R}^n))}.
\end{aligned}$$

At first, we consider $q < p$; then (1.27) is reduced to $B^s_{p,q} \hookrightarrow F^s_{p,q}$, since the right side of the inclusion already follows from $F^s_{p,p} = B^s_{p,p}$.

Let $f \in B^s_{p,q}$, putting $u = \frac{p}{q}$ (so $u > 1$) then

$$\begin{aligned}
\|f\|_{F^s_{p,q}} &= \left(\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{jsq}|\Delta_j f|^q\right)^{\frac{p}{q}} dx\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{jsq}|\Delta_j f|^q\right)^{u} dx\right)^{\frac{1}{uq}} \\
&\leq \left(\sum_{j=0}^{\infty} \left(\int_{\mathbb{R}^n} 2^{jsqu}|\Delta_j f|^{qu} dx\right)^{\frac{1}{u}}\right)^{\frac{1}{q}} \\
&= \left(\sum_{j=0}^{\infty} 2^{jsq}\|\Delta_j f\|^q_p\right)^{\frac{1}{p}} \\
&= \|f\|_{B^s_{p,q}}.
\end{aligned}$$

If $1 \le p \le q \le \infty$, we have

$$\begin{aligned}
\|f_k\|_{\ell_q(L_p(\mathbb{R}^n))} &\le \|f_k\|_{L_p(\mathbb{R}^n,\ell_q)} \\
&= \left\| \int_{\mathbb{R}^n} |f_k|^p dx \right\|^{\frac{1}{p}}_{L_{\frac{q}{p}}(\mathbb{R}^n)} \\
&\le \left(\int_{\mathbb{R}^n} \| |f_k|^p \|_{L_{\frac{q}{p}}(\mathbb{R}^n)} dx \right)^{\frac{1}{p}} \\
&\le \|f_k\|_{\ell_p(L_p(\mathbb{R}^n))}.
\end{aligned}$$

when $p < q$, putting $v = \frac{q}{p}$, then

$$\begin{aligned}
\|f\|_{B^s_{p,q}} &= \left(\sum_{j=0}^{\infty} 2^{jsq} \int_{\mathbb{R}^n} (|\Delta_j f|^p dx)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&= \left(\sum_{j=0}^{\infty} \left(\int_{\mathbb{R}^n} 2^{jsp} |\Delta_j f|^p dx \right)^v \right)^{\frac{1}{vp}} \\
&\le \left(\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{jspv} |\Delta_j f|^{pv} dx \right)^{\frac{1}{v}} \right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}^n} \left(\sum_{j=0}^{\infty} 2^{jsq} |\Delta_j f|^q \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\
&= \|f\|_{F^s_{p,q}}.
\end{aligned}$$

□

Proposition. 1.16. *[41, p.97] Let $0 < q \le \infty$, then*

$$B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_p \quad \textit{if} \quad 1 \le p \le \infty \quad \textit{and} \quad s > 0. \tag{1.28}$$

Definition 1.9. *[36] Let either $E^s_{p,q}(\mathbb{R}^n) = B^s_{p,q}(\mathbb{R}^n)$ or $E^s_{p,q}(\mathbb{R}^n) = F^s_{p,q}(\mathbb{R}^n)$. Then $E^s_{p,q}(\mathbb{R}^n)$ is called an algebra if*

$$E^s_{p,q}(\mathbb{R}^n) \cdot E^s_{p,q}(\mathbb{R}^n) \hookrightarrow E^s_{p,q}(\mathbb{R}^n).$$

Furthermore, for all $f \in E^s_{p,q}(\mathbb{R}^n)$ and all $g \in E^s_{p,q}(\mathbb{R}^n)$ there exists $C > 0$ such that

$$\|f \cdot g\|_{E^s_{p,q}(\mathbb{R}^n)} \le C \|f\|_{E^s_{p,q}(\mathbb{R}^n)} \|g\|_{E^s_{p,q}(\mathbb{R}^n)}. \tag{1.29}$$

Proposition. 1.17. *[36] Let $s > 0$ and $0 < p, q \le \infty$. The following statements are equivalent:*

(i) $B^s_{p,q}(\mathbb{R}^n)$ *is an algebra,*

(ii) $B^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty$,

(iii) either $s > n/p$ *or* $s = n/p$ *and* $0 < q \leq 1$.

Proposition. 1.18. *[36] Let* $s > 0$, $0 < p < \infty$ *and* $0 < q \leq \infty$. *The following statements are equivalent:*

(i) $F^s_{p,q}(\mathbb{R}^n)$ *is an algebra,*

(ii) $F^s_{p,q}(\mathbb{R}^n) \hookrightarrow L_\infty$,

(iii) either $s > n/p$ *or* $s = n/p$ *and* $0 < p \leq 1$.

1.4 Interpolation

During the last fifty years, the interpolation theory becames an important tool in the theory of linear and nonlinear partial differential equations, it is a powerful tool to study the continuity of a linear operator. In what follows, we recall the known results on the interpolation of Besov and Lizorkin-Triebel spaces. For details we refer to [4, 34, 40, 41].
Let A_0, A_1 two Banach spaces, such that the algebraic operations (addition of elements and multiplication by complex numbers) are the same in A_0 and A_1. Let $A_0 + A_1$ be the set of all elements a which can be represented as $a = a_0 + a_1$ with $a_0 \in A_0$ and $a_1 \in A_1$.
If $0 < t < \infty$ and $a \in A_0 + A_1$, puting

$$K(t,a) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}). \tag{1.30}$$

Definition 1.10. *(Real interpolation)[40] Let* $0 < \theta < 1$.
If $0 < q < \infty$, *then*

$$(A_0, A_1)_{\theta,q} = \left\{ a/a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,q}} = \left(\int_0^\infty (t^{-\theta} K(t,a))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \right\}. \tag{1.31}$$

If $q = \infty$, *then*

$$(A_0, A_1)_{\theta,\infty} = \left\{ a/a \in A_0 + A_1 : \|a\|_{(A_0,A_1)_{\theta,\infty}} = \sup_{0<t<\infty} t^{-\theta} K(t,a) < \infty \right\}. \tag{1.32}$$

Remark 1.5. *[40] By* $(A_0, A_1)_{\theta,q}$ *we denote the result of the real interpolation space of the exponent* θ*, applied to* A_0 *and* A_1*.*

Theorem 1.1. *[40] If* A_0 *and* A_1 *are Banach spaces, then* $(A_0, A_1)_{\theta,q}$ *with* $0 < \theta < 1$ *and* $1 \leq q \leq \infty$*, is also a Banach space.*

Theorem 1.2. *[36, p.82] Let* (A_0, A_1)*,* (B_0, B_1) *be two interpolation couples of compatibles Banach spaces and* T *be a linear operator takes* A_i *into* B_i*,* $(i = 0, 1)$ *such that,*

$$\|Tf\|_{B_0} \leq M_0 \|f\|_{A_0} \text{ and } \|Tf\|_{B_1} \leq M_1 \|f\|_{A_1},$$

then for all $0 < \theta < 1$ *and* $1 \leq q \leq \infty$*, the operator* T *takes* $(A_0, A_1)_{\theta,q}$ *into* $(B_0, B_1)_{\theta,q}$ *such that*

$$\|T\|_{(B_0,B_1)_{\theta,q}} \leq {M_0}^{1-\theta} {M_1}^{\theta} \|f\|_{(A_0,A_1)_{\theta,q}}. \tag{1.33}$$

Remark 1.6. *Let* A_0, A_1, B_0 *and* B_1 *be Banach spaces,* $0 < \theta < 1$ *and* $1 \leq q \leq \infty$*.*
If $A_i \hookrightarrow B_i$*,* $(i = 0, 1)$ *then* $(A_0, A_1)_{\theta,q} \hookrightarrow (B_0, B_1)_{\theta,q}$ *.*

Theorem 1.3. ***(Riesz-Thorin interpolation)****.Assume that* $p_0, p_1, q_0, q_1 \in [1, \infty]$ *with* $p_0 \neq p_1, q_0 \neq q_1$*. Let* $(X, \mu), (Y, \nu)$ *two measured spaces and* T *be a linear operator*

$$T : L^{p_0}(X, \mu) \longrightarrow L^{q_0}(Y, \nu),$$

$$T : L^{p_1}(X, \mu) \longrightarrow L^{q_1}(Y, \nu)$$

such that, for any simple function f *we have*

$$\|Tf\|_{q_i} \leqslant C_i \|f\|_{p_i} \quad (i = 0, 1).$$

Then T *takes* $(L^{p_0}, L^{p_1})_\theta = L^p$ *into* $(L^{q_0}, L^{q_1})_\theta = L^q$ *such that*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \ (0 < \theta < 1). \tag{1.34}$$

Furthermore

$$\|Tf\|_q \leqslant C_0^{1-\theta} C_1^{\theta} \|f\|_p. \tag{1.35}$$

Proof. See [4, p.2] □

Proposition. 1.19. *Let* $0 < \theta < 1,\ 1 \leq q_0 \leq \infty,\ 1 \leq q_1 \leq \infty$ *and* $1 \leq q \leq \infty$. *Furthermore, let* $s_0 \in \mathbb{R},\ s_1 \in \mathbb{R},\ s_0 \neq s_1$ *and* $s = (1-\theta)s_0 + \theta s_1$.

(i) If $1 \leq p \leq \infty$, *then*

$$\left(B^{s_0}_{p,q_0}(\mathbb{R}^n), B^{s_1}_{p,q_1}(\mathbb{R}^n)\right)_{\theta,q} = B^s_{p,q}(\mathbb{R}^n). \tag{1.36}$$

(ii) If $1 \leq p < \infty$, *then*

$$\left(F^{s_0}_{p,q_0}(\mathbb{R}^n), F^{s_1}_{p,q_1}(\mathbb{R}^n)\right)_{\theta,q} = B^s_{p,q}(\mathbb{R}^n). \tag{1.37}$$

Proof. See [40, 2.4.2]. □

Proposition. 1.20. *[40] Let* $s_0 \in \mathbb{R},\ s_1 \in \mathbb{R},\ 0 < p_0 < \infty$ *and* $0 < p_1 < \infty$. *If* $0 < \theta < 1$, *and*

$$s = (1-\theta)s_0 + \theta s_1, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},$$

then

$$\left(B^{s_0}_{p_0,p_0}(\mathbb{R}^n), B^{s_1}_{p_1,p_1}(\mathbb{R}^n)\right)_{\theta,p} = B^s_{p,p}(\mathbb{R}^n). \tag{1.38}$$

1.5 Examples of functions in Besov space

In this section, we will give examples of functions in Besov space.

Example 1.2. *[36] Let* $f = \delta \in \mathcal{S}'(\mathbb{R}^n)$ *(the Dirac delta function), we have*

$$\begin{aligned} \Delta_j f(x) &= 2^j \int_{\mathbb{R}^n} \mathcal{F}^{-1}\varphi(2^j(x-y))f(y)dy \\ &= < f, 2^j\mathcal{F}^{-1}\varphi(2^j(x-\cdot)) >, \end{aligned}$$

so we have

$$\begin{aligned} \Delta_j \delta(x) &= < \delta, 2^j\mathcal{F}^{-1}\varphi(2^j(x-\cdot)) > \\ &= 2^j\mathcal{F}^{-1}\varphi(2^j x), \end{aligned}$$

then

$$\|\Delta_j \delta\|_p = c2^{nj(1-\frac{1}{p})}.$$

Which implies

$$2^{js}\|\Delta_j \delta\|_p = c2^{nj(1-\frac{1}{p})+js},$$

for $1 \leq q \leq \infty$, *we have*

$$\left(\sum_{j\geq 0} 2^{jsq}\|\Delta_j\delta\|_p^q\right)^{\frac{1}{q}} = c\left(\sum_{j\geq 0} 2^{j(n-\frac{n}{p}+s)q}\right)^{\frac{1}{q}}.$$

So the series $\left(\sum_{j\geq 0} 2^{j(n-\frac{n}{p}+s)q}\right)^{\frac{1}{q}}$ *converges in the following two cases:*

- *If* $n - \frac{n}{p} + s < 0$ *i.e.* $s < \frac{n}{p} - n$, *then* $\delta \in B^s_{p,q}(\mathbb{R}^n)$.
- *If* $n - \frac{n}{p} + s = 0$ *i.e.* $s = \frac{n}{p} - n$, *then*

$$2^{sj}\|\Delta_j\delta\|_p = c \;\; for \;\; j \in \mathbb{N} \Longrightarrow \sup_{j\in\mathbb{N}} 2^{sj}\|\Delta_j\delta\|_p = c = \|\delta\|_{B^s_{p,\infty}(\mathbb{R}^n)}.$$

Then we have

$$\begin{cases} \delta \in B^s_{p,q}(\mathbb{R}^n) & if \quad s < \frac{n}{p} - n, 1 \leq p, q \leq \infty. \\ \delta \in B^s_{p,\infty}(\mathbb{R}^n) & if \quad s = \frac{n}{p} - n, 1 \leq p \leq \infty, q = \infty. \end{cases}$$

Example 1.3. *[36] Let* $f(x) = v.p(\frac{1}{x})$ *(the principal valued of* $\frac{1}{x}$*). We have*

$$\widehat{f}(\xi) = -i\pi sgn\xi$$

and

$$supp\widehat{\Delta_j f} \subset \{\xi \in \mathbb{R} : |\xi| \leq 2^{j+1}\}.$$

According to Bernstein's inequality, we obtain

$$\|\Delta_j f\|_p \leq c_1 2^{j(\frac{1}{2}-\frac{1}{p})}\|\Delta_j f\|_2, \; (p \geq 2). \tag{1.39}$$

On the other hand, we have

$$\begin{aligned} \|\Delta_j f\|_2 &= (2\pi)^{-\frac{1}{2}}\|\widehat{\Delta_j f}\|_2 \quad (Plancherel) \\ &= (2\pi)^{-\frac{1}{2}}\|\varphi(2^{-j}\cdot)\widehat{f}\|_2 \\ &= c_2 2^{\frac{j}{2}}, \end{aligned}$$

because $\varphi \in \mathcal{D}(\mathbb{R})$.

Then the equation (1.39) *becomes*

$$\|\Delta_j f\|_p \leq c2^{j(1-\frac{1}{p})}, \quad c = c_1c_2,$$

from where

$$2^{sj}\|\Delta_j f\|_p \leq c2^{j(s+\frac{1}{p'})},$$

the series $\sum_{j\geq 0} 2^{j(s+\frac{1}{p'})q}, 1 \leq q \leq \infty$ *converges if* $s < -\frac{1}{p'}$, *which gives* $f(x) = v.p(\frac{1}{x}) \in B^s_{p,q}(\mathbb{R})$ *in the following two cases*

$$\begin{cases} s = -\frac{1}{p'},\ 2 \leq p \leq \infty, \quad and \quad q = \infty. \\ s < -\frac{1}{p'},\ 2 \leq p \leq \infty, \quad and \quad 1 \leq q \leq \infty. \end{cases}$$

Chapter 2

Boundedness of the Hilbert Transform on Besov Spaces

In this chapter, we will study the boundedness of the Hilbert transform on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, the principal theorem of this chapter is Theorem 2.9 where we will use the Littlewood-Paley decomposition to prove that the boundedness of the Hilbert transform on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ can be obtained by its $L^p(\mathbb{R}^n)$-boundedness for $s \in \mathbb{R}$, and $p, q \in (1, \infty)$.

The chapter is arranged as follows. In Section 2.1, we will give definitions and basic properties of the Hilbert transform. In the second Section, we will give the boundedness of such a transform on Lebesgue spaces $L^p(\mathbb{R}^n)$. Finally, in Section 2.3, we will give the main result of this chapter where we will prove the boundedness on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$.

2.1 Hilbert transform

Let $\Gamma : \mathbb{R} \to \mathbb{R}^n, n \geq 2$, be a continuous curve passing through the origin, i.e. $\Gamma(0) = 0$. We define the Hilbert transform along Γ by the principal-valued integral

$$\mathcal{H}f(x) = p.v. \int_{-\infty}^{\infty} f(x - \Gamma(t)) \frac{dt}{t}, \quad \forall f \in \mathcal{D}(\mathbb{R}^n). \tag{2.1}$$

And the truncated form $\mathcal{H}_{\epsilon,N}$ is given by

$$\mathcal{H}_{\epsilon,N} f(x) = \int_{\epsilon \leq |t| \leq N} f(x - \Gamma(t)) \frac{dt}{t}. \tag{2.2}$$

It is known that for proving the boundedness of the Hilbert transform on $L^2(\mathbb{R}^n)$, it suffices to show that the "Fourier multiplier" m namely

$$m(\xi) = p.v. \int_{-\infty}^{\infty} e^{-i\xi \cdot \Gamma(t)} \frac{dt}{t}, \quad \xi \in \mathbb{R}^n,$$

is a bounded function on $\mathbb{R}^n$ and to use the Van der Corput lemma and Plancherel's theorem, (see[48, p.197] and [33]).

Lemma 2.1. *[38] Let $f(x)$ be a locally integrable function, and $\Gamma(t)$ be a continuous curve. Then $f(x-\Gamma(t))$ is a measurable function of x and t. Also for almost every x, $f(x-\Gamma(t))$ is a measurable and locally integrable function of t.*

Remark 2.1. *[38]*

(i) Note that the fact that $f(x-\Gamma(t))$ is locally integrable in t for almost every x follows from Fubini's Theorem.

(ii) Lemma 2.1 implies that for locally integrable f, then $\mathcal{H}_{\epsilon,N}f(x)$ is a well-defined measurable function of x for each fixed positive ϵ and N. Moreover for almost every x, $\mathcal{H}_{\epsilon,N}f(x)$ is continuous in ϵ and N.

Now we ask, does

$$\lim_{h\to 0}\frac{1}{h}\int_0^h f(x-\Gamma(t))dt = f(x)\ a.e.? \tag{2.3}$$

To obtain positive answers to this question, it is necessary to restrict attention to a subclass of curves. Again curvature is crucial, we obtain positive results if Γ has an appropriate amount of curvature.

Definition 2.1. *[38] We say that a C^∞ curve $\Gamma(t)$ in $\mathbb{R}^n$ is well-curved if $\Gamma(0)=0$ and for some $\epsilon>0$, $\Gamma([-\epsilon,\epsilon])$ lies in the subspace of $\mathbb{R}^n$ spanned by*

$$\frac{d^j}{dt^j}\Gamma(t)\,|_{t=0},\quad j=1,2,3,\ldots$$

We should mention that any real-analytic curve in $\mathbb{R}^n$ is well-curved.
An important feature of well-curved curves appears when we consider the simplest two dimensional examples, namely

$$\Gamma(t)=(t^j,t^k),\ t\geq 0,$$

where j and k are positive integers.
The following result holds:

Theorem 2.1. *[38] Let f be locally in $L^p(\mathbb{R}^n), p > 1$. Then*

$$\lim_{h\to 0} \frac{1}{h} \int_0^h f(x - \Gamma(t))dt = f(x) \ a.e. \tag{2.4}$$

provided $\Gamma(t)$ is well-curved.

Let us recall the definition of the maximal operator along Γ, which plays a very important role in harmonic analysis.

Definition 2.2. *Suppose that f is a locally integrable on $\mathbb{R}^n$, i.e. $f \in L^1_{loc}(\mathbb{R}^n)$. The maximal operator $\mathcal{M}$ is defined on $L^1_{loc}(\mathbb{R}^n)$ by*

$$\mathcal{M}f(x) = \sup_{h>0} \frac{1}{h} \left| \int_0^h f(x - \Gamma(t))dt \right|. \tag{2.5}$$

We should mention that the maximal operator $\mathcal{M}$ is well defined for any locally integrable f on $\mathbb{R}^n$, in general, it is not a bounded operator from $L^1(\mathbb{R}^n)$ to itself, and of course $\mathcal{M}$ is bounded on $L^\infty(\mathbb{R}^n)$, i.e.

$$\|\mathcal{M}f\|_\infty \leq \|f\|_\infty.$$

We consider the plane curve $(t, \gamma(t))$ in $\mathbb{R}^2$, and we have the following theorem for $P = 2$.

Theorem 2.2. *If $\gamma(0) = \gamma'(0) = 0$, and $\gamma(t)$, $\gamma'(t)$, and $\gamma(t)''$ are increasing, and positive when $t > 0$, then*

$$\|\mathcal{M}f\|_2 \leq c\|f\|_2.$$

Proof. See [38]. □

We set, for the following theorem:

$$\mathcal{M}_1 f(x) = \sup_{0<h\leq 1} \frac{1}{h} \int_0^h |f(x - \Gamma(t))| \, dt. \tag{2.6}$$

We should mention that $\mathcal{M}_1$ is well defined for any locally integrable f.

Theorem 2.3. *[38] If $\Gamma(t)$ is well-curved, then*

$$\|\mathcal{M}_1 f(x)\|_p \leq C_p \|f\|_p, \quad \text{for all } \ 1 < p < \infty. \tag{2.7}$$

Theorem 2.3 implies Theorem 2.1.

For a survey of this problem's history through 1977 see [38].

Now, we give the Van der Corput lemma in the simple case which has been basic in the study of the Hilbert transform.

Lemma 2.2. *(Van der Corput lemma). Suppose $f : [a, b] \to \mathbb{R}$ is in $C^1[a, b]$, f' is monotone, and there is a $\lambda > 0$ with $|f'(t)| \geq \lambda$ for $a \leq t \leq b$. Then*

$$\left| \int_a^b e^{if(t)} dt \right| \leq c/\lambda,$$

for a constant c independent of a, b, f and λ.

Proof. See [48, p.197]. □

2.2 Boundedness of the Hilbert transform on $L^p(\mathbb{R}^n)$

2.2.1 Boundedness of the Hilbert transform on $L^2(\mathbb{R}^n)$

In this paragraph, we are interested in the Boundedness of the operator $\mathcal{H}_R$ on $L^2(\mathbb{R}^n)$, we define it in its truncated form :

$$\mathcal{H}_{\varepsilon,R}(f)(x) = \int_{\varepsilon < |t| \leq R} f(x - \Gamma(t)) \frac{dt}{t}, \quad (\forall f \in \mathcal{D}(\mathbb{R}^n)), \tag{2.8}$$

where $\Gamma(t) = (\gamma_1(t), \gamma_2(t), \cdots, \gamma_n(t))$.

Theorem 2.4. *[2, 33] Let $\Gamma : [-R, R] \to \mathbb{R}^n$ be a highly monotone and odd curve with $\gamma_1(t) = t$, then*

(i) $\mathcal{H}_{\varepsilon,R}$ is bounded on $L^2(\mathbb{R}^n)$ uniformly on ε and R, i.e.

$$\exists c = c(n) > 0, \quad \|\mathcal{H}_{\varepsilon,R} f\|_2 \leq c\|f\|_2, \quad (\forall f \in L^2(\mathbb{R}^n)); \tag{2.9}$$

(ii) $\lim_{\varepsilon \to 0} \mathcal{H}_{\varepsilon,R} = \mathcal{H}_R$ exists in the $L^2(\mathbb{R}^n)$ norm and we have

$$\|\mathcal{H}_R f\|_2 \leq c\|f\|_2, \quad (\forall f \in L^2(\mathbb{R}^n)).$$

To prove this theorem, we need the following lemmas:

Lemma 2.3. *For all function f in $S(\mathbb{R}^n)$ we have*

$$\mathcal{F}(\mathcal{H}_{\varepsilon,R}f)(\xi) = m_{\varepsilon,R}(\xi)\widehat{f}(\xi),$$

where

$$m_{\varepsilon,R}(\xi) = \int_{\varepsilon<|t|\leq R} e^{-i\xi\cdot\Gamma(t)}\frac{dt}{t},$$

such that $\xi\cdot\Gamma(t) = \xi_1 t + \xi_2\gamma_2(t) + \cdots + \xi_n\gamma_n(t)$.

Lemma 2.4. *(Plancheral-Parseval formula) For all $g \in L^2(\mathbb{R}^n)$ we have*

$$\|\mathcal{F}g\|_2 = (2\pi)^{\frac{n}{2}}\|g\|_2.$$

Lemma 2.5. *(W.C.Nestlerode) There exists $c = c(n) > 0$, such that*

$$\left|\int_{\varepsilon\leq|t|\leq R} e^{-ig(t,\xi)}\frac{dt}{t}\right| \leq C. \quad \textit{pour tout } \xi \in \mathbb{R}^n,$$

where

$$g(t,\xi) = \xi_1 t + \xi_2\gamma_2(t) + \cdots + \xi_n\gamma_n(t), \quad \textit{and} \ \xi = (\xi_1, \xi_2, \cdots, \xi_n) \in \mathbb{R}^n.$$

Proof of Theorem 2.4

(i) By applying to (2.8) successively Lemma 2.3, Lemma 2.4 and Lemma 2.5 , we obtain

$$\|\mathcal{F}(\mathcal{H}_{\varepsilon,R}f)\|_2 = (2\pi)^{\frac{n}{2}}|m_{\varepsilon,R}|\|f\|_2$$

with

$$m_{\varepsilon,R}(\xi) = \int_{\varepsilon<|t|\leq R} e^{-i\xi\cdot\Gamma(t)}\frac{dt}{t},$$

then

$$\|\mathcal{F}(\mathcal{H}_{\varepsilon,R}f)\|_2 \leq (2\pi)^{\frac{n}{2}}c\,\|f\|_2,$$

that is to say

$$\|\mathcal{F}(\mathcal{H}_{\varepsilon,R}f)\|_2 \leq c_0\,\|f\|_2,$$

with

$$c_0 = (2\pi)^{\frac{n}{2}}\,c.$$

So, it suffices to prove that there is a constant $c > 0$ which depends only on n, i.e. $c = c(n)$, such that

$$|m_{\varepsilon,R}(\xi)| \leq c, \quad (\forall\xi \in \mathbb{R}^n).$$

We can prove it by induction on n (see [33]).

(ii) To show that $\lim_{\varepsilon\to 0}\mathcal{H}_{\varepsilon,R}f = \mathcal{H}_R f$ exists in the $L^2(\mathbb{R}^n)$ norm, it suffices to prove that $(\mathcal{H}_{\varepsilon,R}f)_{\varepsilon>0}$ is a sequence of Cauchy in $L^2(\mathbb{R}^n)$.

Existence.

By applying Lemma 2.3 and Lemma 2.4 to the sequence of functions

$$(\mathcal{H}_{\varepsilon,R}f - \mathcal{H}_{\varepsilon',R}f)_{\varepsilon,\varepsilon'}, \quad (\forall f \in L^2(\mathbb{R}^n)),$$

we obtain for $\varepsilon > \varepsilon'$,

$$\mathcal{F}(\mathcal{H}_{\varepsilon,R}f - \mathcal{H}_{\varepsilon',R}f)(\xi) = m_{\varepsilon,\varepsilon'}(\xi)\mathcal{F}(f)(\xi),$$

where

$$m_{\varepsilon,\varepsilon'} = -2i\int_{\varepsilon'}^{\varepsilon}\{\sin(\xi\cdot\Gamma(t))\}\frac{dt}{t}.$$

$\forall\varepsilon' : 0 < \varepsilon' < 1$, we have

$$\begin{aligned} |m_{\varepsilon,\varepsilon'}(\xi)| &\leq 2\int_{1-\varepsilon'}^{1+\varepsilon}\frac{dt}{t} \\ &= 2\ln\frac{1+\varepsilon}{1-\varepsilon'}, \end{aligned}$$

from where

$$\|\mathcal{H}_{\varepsilon,R}f - \mathcal{H}_{\varepsilon',R}f\|_2 \leq 2\|f\|_2\ln\frac{1+\varepsilon}{1-\varepsilon'}$$

which gives when $\varepsilon,\varepsilon' \to 0$

$$\|\mathcal{H}_{\varepsilon,R}f - \mathcal{H}_{\varepsilon',R}f\|_2 \to 0.$$

Then $(\mathcal{H}_{\varepsilon,R}f)_{\varepsilon>0}$ converges to a limit called $L_R f$, on $L^2(\mathbb{R}^n)$.

Uniqueness.

From the definition of the operateur $\mathcal{H}_R$, we have : $\lim_{\varepsilon\to 0}\mathcal{H}_{\varepsilon,R}f(x) = \mathcal{H}_R f(x)$ exists on $\mathbb{R}$, then

$$\mathcal{H}_R f(x) = L_R f(x), \quad (\forall x \in \mathbb{R}^n)$$

that is to say

$$\mathcal{H}_R f = L_R f.$$

It remains to prove

$$\|\mathcal{H}_R f\|_2 \leq c_0\|f\|_2, \quad (\forall f \in L^2(\mathbb{R}^n)).$$

By applying (2.9) and the fact that $\lim_{\varepsilon\to 0}\mathcal{H}_{\varepsilon,R} = \mathcal{H}_R f$ exists in the $L^2(\mathbb{R}^n)$ norm, we obtain, for all $\varepsilon : 0 < \varepsilon < R$

$$\begin{aligned} \|\mathcal{H}_R f\|_2 &= \|\mathcal{H}_R f - \mathcal{H}_{\varepsilon,R} f + \mathcal{H}_{\varepsilon,R} f\|_2 \\ &\leq \|\mathcal{H}_{\varepsilon,R} f - \mathcal{H}_R f\|_2 + c_0\|f\|_2, \quad (\forall f \in L^2(\mathbb{R}^n)). \end{aligned}$$

Passing to the limit when $\varepsilon \to 0$, then we have the result. And the proof of Theorem 2.4 is completed.

□

Let us give a counterexample, determine the class of curves γ for which there are L^p results for $\mathcal{H}$ corresponding to γ, this problem will be clarified by several examples presented in [38]. The simplest is as follows.

Remark 2.2. *[38] Let γ be an odd function such that $\gamma(t) = t$ for $0 \leq t \leq 1$, and $\gamma(t) = at+b$ for $t > 1$, with $b \neq 0$. The Hilbert transform corresponding to $(t, \gamma(t))$ is unbounded on $L^2(\mathbb{R}^2)$.*

2.2.2 Boundedness of the Hilbert transform on $L^p(\mathbb{R}^n)$

In this paragraph, we present some known results in the $L^p(\mathbb{R}^n)$-boundedness.

Theorem 2.5. *[38]*

(i) If $\Gamma(t)$ is two-sided homogeneous, and f is in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Then

$$\mathcal{H}f(x) = \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \mathcal{H}_{\epsilon,N} f(x) \tag{2.10}$$

exists almost everywhere and in $L^p(\mathbb{R}^n)$. Moreover,

$$\|\mathcal{H}f\|_p \leq A_p\|f\|_p,\ 1 < p < \infty. \tag{2.11}$$

(ii) Let $\Gamma(t)$ be in $C^\infty(\mathbb{R}^n)$. Assume that for small t, $\Gamma(t)$ lies in the subspace spanned by $\Gamma^{(j)}(0), j = 1, 2, \cdots$. Then

$$\mathcal{H}_1 f(x) = \lim_{\epsilon\to 0} \mathcal{H}_{\epsilon,1} f(x) \tag{2.12}$$

exists almost everywhere and in $L^p(\mathbb{R}^n)$. Moreover,

$$\|\mathcal{H}_1 f\|_p \leq A_p\|f\|_p,\ 1 < p < \infty. \tag{2.13}$$

Theorem 2.6. *(i) Let $\Gamma(t)$ be two-sided homogeneous curve, then for f in $\mathcal{S}(\mathbb{R}^n)$,*

$$\|\mathcal{H}_{\epsilon,N}f(x)\|_p \leq c\|f\|_p, \ 1 < p < \infty. \tag{2.14}$$

Moreover $\mathcal{H}_{\epsilon,N}f$ converges in $L^p(\mathbb{R}^n)$ to $\mathcal{H}f$, as $\epsilon \longrightarrow 0, \ N \longrightarrow \infty$, and

$$\|\mathcal{H}f\|_p \leq c\|f\|_p. \tag{2.15}$$

(ii) Let $\Gamma(t)$ be a C^∞ curve on $[-1,1]$. Assume $\Gamma(t)$ lies in the subspace spanned by $\left\{\Gamma^{(j)}(0)\right\}_{j=1}^{\infty}$. Then

$$\|\mathcal{H}_{\epsilon,1}f(x)\|_p \leq c\|f\|_p, \ 1 < p < \infty. \tag{2.16}$$

Moreover $\mathcal{H}_{\epsilon,1}f$ converges in $L^p(\mathbb{R}^n)$ to $\mathcal{H}_1 f$, as $\epsilon \longrightarrow 0$, and

$$\|\mathcal{H}_1 f\|_p \leq c\|f\|_p. \tag{2.17}$$

Proof. See [38]. □

Let us present some results which are useful for us, we give the results of [37] and [5] which guarantee the $L^p(\mathbb{R}^n)$-boundedness of the Hilbert transform, for all $p, 1 < p < \infty$, as follows:

Theorem 2.7. *[37] Let $\Gamma(t) = (P_1(t), \cdots, P_n(t))$, where $P_1, \cdots, P_n$ are real polynomials on $\mathbb{R}$. Then $\mathcal{H}$ is bounded on $L^p(\mathbb{R}^n)$ for all $p, 1 < p < \infty$, with bound independent of the coefficients of $P_1, \cdots, P_n$.*

Theorem 2.8. *Suppose that P is a real polynomial and γ is convex on $[0, \infty)$ twice differentiable, either even or odd, $\gamma(0) = 0$, and $\gamma'(0) \geq 0$.*
Let $\Gamma(t) = (t, P(\gamma(t)))$, $p \in (1, \infty)$, and either (1) $P'(0)$ is zero, or (2) $P'(0)$ is nonzero and $\gamma' \in \mathcal{C}^1$, then

$$\|\mathcal{H}f\|_p \leq c\|f\|_p.$$

Moreover the constant c depends only on p, γ and the degree of P.

Proof. See [5]. □

Remark 2.3. *(1) In Theorem 2.8, by taking $\gamma(t) = t$, we recover a form of Theorem 2.7, it is shown in [37] that for all $p \in (1, \infty)$, $L^p(\mathbb{R}^n)$-boundedness of $\mathcal{H}$ is obtained. Also taking $P(s) = s$, it is shown in [10] that if γ is odd, satisfies*

$$\gamma \in \mathcal{C}^2((0, \infty)), \ \textit{convex on } [0, \infty) \textit{ and } \gamma(0) = \gamma'(0) = 0, \tag{2.18}$$

and $\gamma' \in \mathcal{C}^1$, then the $L^p(\mathbb{R}^n)$-boundedness of $\mathcal{H}$, for all $p \in (1, \infty)$ is also obtained.

(2) Some examples of nonconvex curves were studied in [44], and later these were generalized somewhat through a technical theorem in [43]. Although the class of these curves obtained from theorem in [5].

2.3 Main results

The following theorem concerning the boundedness of the Hilbert transform on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$.

Theorem 2.9. *[24] Let $s \in \mathbb{R}$, and $1 < p, q < \infty$. If $\mathcal{H}$ is bounded on $L^p(\mathbb{R}^n)$, then $\mathcal{H}$ is bounded on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$.*

The proof of Theorem 2.9 needs the following lemma.

Lemma 2.6. *[12] For all function f in $S(\mathbb{R}^n)$ we have*

$$\widehat{\mathcal{H}f}(\xi) = m(\xi) \cdot \widehat{f}, \tag{2.19}$$

where the "Fourier multiplier" m is the function

$$m(\xi) = p.v. \int_{-\infty}^{\infty} e^{-i\xi\cdot\Gamma(t)} \frac{dt}{t}, \quad \xi \in \mathbb{R}^n.$$

Proof. By the Fubini theorem we have

$$\widehat{\mathcal{H}f}(\xi) = p.v. \int_{-\infty}^{\infty} \frac{1}{t} \left\{ \int_{\mathbb{R}^n} e^{(-i\xi\cdot x)} f(x - \Gamma(t)) dx \right\} dt.$$

By changing the variable $u = x - \Gamma(t)$, we obtain

$$\int_{\mathbb{R}^n} f(u) e^{(-i\xi\cdot(u+\Gamma(t)))} du = \widehat{f}(\xi) e^{(-i\xi\cdot\Gamma(t))}.$$

Hence the result. □

2.3.1 Proof of Theorem 2.9

With Lemma 2.6 in hand, and by applying the inverse of the Fourier transform for (2.19), (see for instance [42, p.40]), thus we can define the Hilbert transform as a convolution operator by the formula

$$\mathcal{H}f(x) = K * f(x), \tag{2.20}$$

where

$$K(\xi) = \mathcal{F}^{-1}m(\xi).$$

(See also [16] and [37, p.25]). Now by Theorem 2.7 or Theorem 2.8 we have, for any p, q such that $1 < p, q < \infty$ and any $\{f_j\}_j$ in $\ell^q(L^p)(\mathbb{R}^n)$,

$$\left(\sum_{j=0}^{\infty} \|\mathcal{H}f_j\|_p^q\right)^{1/q} \le c \left(\sum_{j=0}^{\infty} \|f_j\|_p^q\right)^{1/q}, \tag{2.21}$$

where c is independent of $\{f_j\}_j$.

It follows from formula (2.20) that

$$\begin{aligned}
\|\mathcal{H}f\|_{B^s_{p,q}(\mathbb{R}^n)} &:= \left(\sum_{j=0}^{\infty} \left(2^{sj}\|\Phi_j * \mathcal{H}f\|_p\right)^q\right)^{1/q} \\
&= \left(\sum_{j=0}^{\infty} 2^{jsq}\|\Phi_j * \mathcal{H}f\|_p^q\right)^{1/q} \\
&= \left(\sum_{j=0}^{\infty} 2^{jsq}\|\Phi_j * K * f\|_p^q\right)^{1/q}.
\end{aligned}$$

By the convolution properties and formula (2.20) we hold

$$\begin{aligned}
\|\mathcal{H}f\|_{B^s_{p,q}(\mathbb{R}^n)} &= \left(\sum_{j=0}^{\infty} \|K * (2^{js}\Phi_j * f)\|_p^q\right)^{1/q} \\
&= \left(\sum_{j=0}^{\infty} \|\mathcal{H}(2^{js}\Phi_j * f)\|_p^q\right)^{1/q}.
\end{aligned}$$

Then, by formula (2.21), there exists a constant c such that

$$\|\mathcal{H}f\|_{B^s_{p,q}(\mathbb{R}^n)} \le c \left(\sum_{j=0}^{\infty} \|2^{js}\Phi_j * f\|_p^q\right)^{1/q}.$$

Simple caculations show that

$$\begin{aligned}\|\mathcal{H}f\|_{B^s_{p,q}(\mathbb{R}^n)} &\leq c\left(\sum_{j=0}^{\infty}\big(2^{sj}\|\Phi_j * f\|_p\big)^q\right)^{1/q}\\ &= c\left(\sum_{j=0}^{\infty}2^{jsq}\|\Phi_j * f\|_p^q\right)^{1/q}\\ &= c\|f\|_{B^s_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

To achieve the proof of Theorem 2.9.

□

Chapter 3

Boundedness of the Hilbert Transform on Lizorkin-Triebel Spaces

In this chapter, we will study the Boundedness of the Hilbert transform on Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$, the principal theorem of this chapter is Theorem 3.4, where we use the boundedness of a such transform on the ℓ^q-valued function spaces $L^p(\mathbb{R}^n;\ell^q)$ to prove the main results of this chapter. It should be noted that the boundedness was extensively studied on $L^p(\mathbb{R}^n)$ and a little on $H^s_p(\mathbb{R}^n)$, $B^s_{p,q}(\mathbb{R}^n)$, while that on Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$, we have a little work or almost non-existent.

3.1 Hilbert transform along convex curves for ℓ^q-valued functions

This section concerns the boundedness properties of the Hilbert transform on the ℓ^q-valued function spaces $L^p(\mathbb{R}^n;\ell^q)$. It is shown that the boundedness of a such transform along a large class of convex curves is obtained on $L^p(\mathbb{R}^2;\ell^q)$, where $5/3 < p,q < 5/2$. In the following, we are interested in the work of Liu [22], We saw in the previous chapter that there was a great interest to determine for which curves Γ, and which indices p, one has

$$\|\mathcal{H}f\|_{L^p} \leqslant c\|f\|_{L^p}, \tag{3.1}$$

for a constant c depending only on Γ and p. The question of whether these results could be extended to the spaces $L^p(\mathbb{R}^n;\ell^q)$ of vector-valued functions was taken up by several authors recently for $1 < q < \infty$. Let us state some previous theorems which are useful for us in this chapter. The first is the work of Rubio de Francia et al. in 1986:

Theorem 3.1. *[35] Let $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be a well-curved curve in $\mathbb{R}^n$ with $\Gamma(0) = 0$. Then the ℓ^q-valued inequality*

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{H} f_j|^q \right)^{1/q} \right\|_{L^p} \leqslant c_{p,q} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p} \tag{3.2}$$

holds for all p, q with $1 < p, q < \infty$, and all $f_j \in L^p(\mathbb{R}^n)$, $n \geqslant 2$.

Recently, The authors pay attention to curves of finite type which means that the image of the mapping $t \to \Gamma(t)$ is of finite type at 0. At first, we turn our attention to $\Gamma(t)$ which is given as a polynomial functions, because it is a model problem and we have the following results.

Theorem 3.2. *[20, 21]*

(1) Let $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be a polynomial function. Then the ℓ^q-valued inequality

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{H} f_j|^q \right)^{1/q} \right\|_{L^p} \leqslant c_{p,q} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p} \tag{3.3}$$

holds for all p, q with $1 < p, q < \infty$, and all $f_j \in L^p(\mathbb{R}^n)$, $n \geqslant 2$.

(2) Let the image of $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be of finite type at 0. Then the ℓ^q-valued inequality

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{H} f_j|^q \right)^{1/q} \right\|_{L^p} \leqslant c_{p,q} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p} \tag{3.4}$$

holds for all p, q with $1 < p, q < \infty$, and all $f_j \in L^p(\mathbb{R}^n)$, $n \geqslant 2$.

In this chapter, we give the work of Liu [22], and we use it to prove our main results. We deal with plane curves of the form $\Gamma(t) = (t, \gamma(t))$, where $\gamma(t)$ is a convex function for $t \geq 0$. The result of [22] was given as follows.

Theorem 3.3. *[22] Suppose that $\gamma(t)$ is a continuous odd function, twice continuously differentiable, increasing and convex for $t \geq 0$. Suppose also that γ'' is monotone for $t > 0$ and that there exists $C > 0$ so that $\gamma'(t) \leq Ct\gamma''(t)$ for $t > 0$. Then $\mathcal{H}$ is bounded on $L^p(\mathbb{R}^2, \ell^q)$ for all p, q with $5/3 < p, q < 5/2$.*

The proof of Theorem 3.3 was given in several steps, which are similar to that in [32]. Given an analytic family of convolution operators $\mathcal{H}_z$ so that $\mathcal{H}_0 = \mathcal{H}$. To prove the $L^2(\mathbb{R}^2, \ell^2)$ result it suffices to obtain the boundedness of the multiplier associated to the operator $\mathcal{H}_z$ with $Re(z) < a$, where a is some positive constant. And by using the Van der Corput lemma and integration by parts. The vector-valued Fourier multiplier theorem is used to show that $\mathcal{H}_z$ is bounded on $L^p(\mathbb{R}^2, \ell^q)$ for p, q with $1 < p, q < \infty$, and $Re(z) < b$, where b is some negative constant, (see Lemma 3.2). Finally, the theorem follows by using generalized analytic interpolation of operators.

Remark 3.1. *[22] Theorem 3.3 covers a large class of functions $\gamma(t)$ such as*

$$\gamma(t) = sgn(t)|t|^{\alpha}, \quad (\alpha \geq 2),$$

$$\gamma(t) = te^{-1/|t|}.$$

The first one is homogeneous, while the other one is without homogeneity. In [22], the author extended the class of curves for which the $L^p(\mathbb{R}^2, \ell^q)$-result is obtained for $5/3 < p, q < 5/2$.

3.1.1 Proof of Theorem 3.3.

In the following, we will give the proof of Theorem 3.3, for more details see [22].
For $z \in \mathbb{C}$, An analytic family of operators $\mathcal{H}_z$ is defined by

$$\widehat{\mathcal{H}_z f}(\xi, \eta) = m_z(\xi, \eta)\widehat{f}(\xi, \eta), \tag{3.5}$$

where m_z is given by

$$m_z(\xi, \eta) = p.v. \int_{\mathbb{R}} e^{-2\pi i[\xi t + \eta\gamma(t)]}[1 + \eta^2\gamma^2(t)]^z \frac{dt}{t}. \tag{3.6}$$

Notice that $\mathcal{H}_0 = \mathcal{H}$.

Boundedness of $\mathcal{H}_z$ on $L^2(\mathbb{R}^2, \ell^2)$.

In this subsection, we are interested in the proof of the following estimate

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{H}_z f_j|^2 \right)^{1/2} \right\|_{L^2} \leqslant C_\delta[1 + |Im(z)|] \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{L^2}, \tag{3.7}$$

where $Re(z) = (1/4) - \delta$ for some $\delta > 0$.

It is known that the boundedness of $\mathcal{H}_z$ on $L^2(\mathbb{R}^2, \ell^2)$ is equivalent to the uniform boundedness of the multiplier $m_z(\xi, \eta)$. Thus, one can show that

$$|m_z(\xi, \eta)| \leq C_\delta[1 + |Im(z)|], \tag{3.8}$$

where the constant C_δ is independent of $Im(z)$. The proof of (3.8) is based on the Van der Corput lemma which plays an important role in estimating related multipliers. This lemma is given in several books or papers as follows

Lemma 3.1. *[37, p.332] Suppose φ is real-valued and smooth in (a, b), and that $|\varphi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$. Then*

$$\left| \int_a^b e^{i\lambda\varphi(x)} dx \right| \leq C_k \lambda^{-1/k} \tag{3.9}$$

holds when $k \geq 2$; the bound C_k is independent of φ and λ.

For more details on the proof of (3.8) see [22].

Boundedness of $\mathcal{H}_z$ on $L^p(\mathbb{R}^2, \ell^q)$.

In this subsection, one can show that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{H}_z f_j|^q \right)^{1/q} \right\|_{L^p} \leqslant C[1 + |Im(z)|]^2 \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p}, \tag{3.10}$$

where $Re(z) < -1, 1 < p, q < \infty$, the constant C depends on $Re(z)$, and is independent of $Im(z)$.

Before proving (3.10), we need a vector-valued Fourier multiplier theorem. Firstly, let us recall some definitions:

Definition 3.1. *[22] Let X be a Banach space, X is said to be an UMD-space, if the Hilbert transform is bounded on $L^p(\mathbb{R}, X)$ for some (and then all) $p \in (1, \infty)$.*

Definition 3.2. *[22] A basis $\{e_j\}_j$ in a Banach space X is called unconditional, if there is a constant C such that for every $x = \sum_j \epsilon_j t_j e_j \in X$ one has*

$$\left\| \sum_j \epsilon_j t_j e_j \right\| \leq C\|x\|,$$

for all $\epsilon_j = \pm 1$ for $j \in \mathbb{N}$.

Definition 3.3. *[22] A Banach X is said to have local unconditional structure if there exists a constant C such that for any finitedimensional subspace X_0 of X there exists a Banach space F with an unconditional basis and operators*

$$u_0 : X_0 \to F \quad \text{and} \quad w_0 : F \to X$$

such that the natural embedding

$$j : X_0 \to X$$

admits a factorization $j = w_0 u_0$ and

$$\|u_0\| \|w_0\| \leq C.$$

Remark 3.2. *[22] Well-known examples of UMD-spaces are $L^p(\mathbb{R}^n)$, ℓ^q and $L^{p,q}(\mathbb{R}^n)$, $1 < p, q < \infty$, three examples are also Banach lattices for their usual norm and the pointwise order. Further, all Banach lattices have local unconditional structure.*

Let $m : \mathbb{R}^n \to \mathbb{C}$ be a bounded function, we associate operators T_m defined on the test functions $f \in \mathcal{S}(\mathbb{R}^n, X)$ by

$$T_m f(x) = (m\widehat{f})^{\vee}(x). \tag{3.11}$$

As a vector-valued Fourier multiplier theorem, we state the following vector-valued theorem which was proved in [47].

Lemma 3.2. *[47] Let X be an UMD-space with local unconditional structure. Then for any $p \in (1, \infty)$ there is a constant $C < \infty$ such that*

$$\|T_m\|_{L^p(\mathbb{R}^n, X)} \leq C \sup\{|x^\alpha \partial^\alpha m(x)| : x \in \mathbb{R}^n \setminus \{0\}, \alpha \leq (1, 1, \cdots, 1)\}. \tag{3.12}$$

For $n = 2$, in view of Lemma 3.2 and Remark 3.2, it suffices to show that the following functions

$$m_z(\xi, \eta), \quad \xi \frac{\partial m_z}{\partial \xi}(\xi, \eta), \quad \eta \frac{\partial m_z}{\partial \eta}(\xi, \eta), \quad \xi\eta \frac{\partial^2 m_z}{\partial \xi \partial \eta}(\xi, \eta) \tag{3.13}$$

are uniformly bounded on $\mathbb{R}^2$ for $Re(z) < -1$.

The uniform boundedness of $m_z(\xi, \eta)$ is trivial.

The Boundedness of $\xi(\partial m_z/\partial\xi)(\xi,\eta)$.

Integration by parts implies that

$$\begin{aligned}
\xi\frac{\partial m_z}{\partial\xi}(\xi,\eta) &= -2\pi i\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\xi\left[1+\eta^2\gamma^2(t)\right]^z dt \\
&= \int_{\mathbb{R}}\frac{d}{dt}(e^{-2\pi i\xi t})e^{-2\pi i\eta\gamma(t))}\left[1+\eta^2\gamma^2(t)\right]^z dt \\
&= e^{-2\pi i[\xi t+\eta\gamma(t))}\left[1+\eta^2\gamma^2(t)\right]^z\Big|_{-\infty}^{\infty} \\
&+ 2\pi i\eta\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\gamma'(t)\left[1+\eta^2\gamma^2(t)\right]^z dt \\
&- 2z\eta^2\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\left[1+\eta^2\gamma^2(t)\right]^{z-1}\gamma(t)\gamma'(t)dt.
\end{aligned}$$

Note that $Re(z)<-1$, for $t\in\mathbb{R}$, we have

$$\left|[1+\eta^2\gamma^2(t)]^z\right| = [1+\eta^2\gamma^2(t)]^{Re(z)} \leq 1$$

The boundary terms is bounded by $1/\pi$.

For $Re(z)<-1$, making the change of variables $u=|\eta|\gamma(t)$, we can get the following estimates

$$\begin{aligned}
\left|\eta\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\gamma'(t)\left[1+\eta^2\gamma^2(t)\right]^z dt\right| &\leq \int_{\mathbb{R}}\gamma'(t)|\eta|\left[1+\eta^2\gamma^2(t)\right]^{Re(z)} dt \\
&\leq \int_{\mathbb{R}}(1+u^2)^{Re(z)}du \leq \pi.
\end{aligned}$$

In the similar way, the second integrated term can be estimated by

$$\left|z\eta^2\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\left[1+\eta^2\gamma^2(t)\right]^{z-1}\gamma(t)\gamma'(t)dt\right| \leq 1+|Im(z)|. \tag{3.14}$$

Therefore, for $Re(z)<-1$,

$$\left|\xi\frac{\partial m_z}{\partial\xi}(\xi,\eta)\right| \leq C[1+|Im(z)|]. \tag{3.15}$$

The Boundedness of $\eta(\partial m_z/\partial\eta)(\xi,\eta)$.

Integrating by parts, we obtain

$$\begin{aligned}
\eta\frac{\partial m_z}{\partial\eta}(\xi,\eta) &= -2\pi i\ p.v.\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\eta\gamma(t)\left[1+\eta^2\gamma^2(t)\right]^z\frac{dt}{t} \\
&+ 2z\ p.v.\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\eta^2\gamma^2(t)\left[1+\eta^2\gamma^2(t)\right]^{z-1}\frac{dt}{t}.
\end{aligned}$$

For the first integral, for any $\epsilon > 0$, it suffices to bound the following two parts

$$\int_{\epsilon<|t|<t_0} |\eta||\gamma(t)| \left[1+\eta^2\gamma^2(t)\right]^{Re(z)} \frac{dt}{|t|}, \tag{3.16}$$

$$\int_{|t|\geq t_0} |\eta||\gamma(t)| \left[1+\eta^2\gamma^2(t)\right]^{Re(z)} \frac{dt}{|t|}. \tag{3.17}$$

Recall that $t_0 > 0$ was chosen so that $|\eta|\gamma(t_0) = 1$, and $\gamma(t) \leq t\gamma'(t)$ because of the convexity. Thus, we can prove that

$$\begin{aligned} \int_{\epsilon<|t|<t_0} |\eta||\gamma(t)| \left[1+\eta^2\gamma^2(t)\right]^{Re(z)} \frac{dt}{|t|} &\leq 2|\eta| \int_0^{t_0} \gamma'(t)dt \\ &\leq 2. \end{aligned} \tag{3.18}$$

For $Re(z) < -1$, an elementary calculation shows that

$$\begin{aligned} \int_{|t|\geq t_0} |\eta||\gamma(t)| \left[1+\eta^2\gamma^2(t)\right]^{Re(z)} \frac{dt}{|t|} &\leq \frac{2}{|2Re(z)+1|} \\ &\leq 2. \end{aligned}$$

Similarly, the second integral can be estimated as

$$\left| z\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\eta^2\gamma^2(t) \left[1+\eta^2\gamma^2(t)\right]^{z-1} \frac{dt}{t}\right| \leq 2|Re(z)| \left[1+|Im(z)|\right]. \tag{3.19}$$

Therefore, for $Re(z) < -1$,

$$\left|\eta\frac{\partial m_z}{\partial\eta}(\xi,\eta)\right| \leq C\left[1+|Im(z)|\right]. \tag{3.20}$$

The Boundedness of $\xi\eta(\partial^2 m_z/\partial\xi\partial\eta)(\xi,\eta)$.

We note that $\xi\eta\frac{\partial^2 m_z}{\partial\xi\partial\eta}(\xi,\eta)$ can be written as

$$\begin{aligned} \xi\eta\frac{\partial^2 m_z}{\partial\xi\partial\eta}(\xi,\eta) &= -4\pi^2\xi\eta\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\gamma(t)\left[1+\eta^2\gamma^2(t)\right]^z dt \\ &- 4\pi i z\xi\eta\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\left[1+\eta^2\gamma^2(t)\right]^{z-1}\eta\gamma^2(t)dt. \end{aligned}$$

For the first term, integrating by parts, we obtain

$$\begin{aligned}
4\pi^2\xi\eta\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\gamma(t)\left[1+\eta^2\gamma^2(t)\right]^z dt &= 2\pi i\int_{\mathbb{R}}\frac{d}{dt}e^{(-2\pi i\xi t)}e^{-2\pi i\eta\gamma(t)}[\eta\gamma(t)]\left[1+\eta^2\gamma^2(t)\right]^z dt\\
&= 2\pi i e^{-2\pi i[\xi t+\eta\gamma(t)]}[\eta\gamma(t)]\left[1+\eta^2\gamma^2(t)\right]^z\Big|_{-\infty}^{\infty}\\
&- 4\pi^2\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\eta\gamma'(t)[\eta\gamma(t)]\left[1+\eta^2\gamma^2(t)\right]^z dt\\
&- 2\pi i\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\eta\gamma'(t)\left[1+\eta^2\gamma^2(t)\right]^z dt\\
&- 4\pi i z\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\left[1+\eta^2\gamma^2(t)\right]^{z-1}\eta^3\gamma^2\gamma'(t)dt.
\end{aligned}$$

Obviously, for $Re(z)<-1$, $t\in\mathbb{R}$,

$$\begin{aligned}
\left|2\pi i e^{-2\pi i[\xi t+\eta\gamma(t)]}[\eta\gamma(t)]\left[1+\eta^2\gamma^2(t)\right]^z\right| &\le 2\pi|\eta||\gamma(t)|\left[1+\eta^2\gamma^2(t)\right]^{Re(z)}\\
&\le 2\pi. \qquad (3.21)
\end{aligned}$$

So, the boundary terms is bounded by 2π.

By making a suitable changements of variable, one has the following estimates

$$\left|\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\eta\gamma'(t)[\eta\gamma(t)]\left[1+\eta^2\gamma^2(t)\right]^z dt\right| \le \frac{1}{|Re(z)+1|}, \qquad (3.22)$$

$$\left|\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\eta\gamma'(t)\left[1+\eta^2\gamma^2(t)\right]^z dt\right| \le \pi, \qquad (3.23)$$

and

$$\left|z\int_{\mathbb{R}} e^{-2\pi i[\xi t+\eta\gamma(t)]}\left[1+\eta^2\gamma^2(t)\right]^{z-1}\eta^3\gamma^2(t)\gamma'(t)dt\right| \le \pi|z|. \qquad (3.24)$$

The second term can be handled similarly. Finally, we can obtain

$$\left|\xi\eta\frac{\partial^2 m_z}{\partial\xi\partial\eta}(\xi,\eta)\right| \le C[1+Im(z)]^2. \qquad (3.25)$$

We omit the details.

Remark 3.3. *[22] To complete the proof we need an ℓ^q-valued analytic interpolation theorem, which is a consequence of Theorem 1 in [14], a concrete proof can be found in [11].*

Let S denote the closed strip $\{a\le Re(z)\le b\}\subset C$, $\{U_z\}_{z\in S}$ is a family of uniformly bounded linear operators on $L^2(\mathbb{R}^n)$, that is, there is an $M>0$ such that

$$\|U_z\|_{L^2\longrightarrow L^2}\le M, \quad \forall z\in S. \qquad (3.26)$$

Moreover, the function

$$z \longmapsto \int_{\mathbb{R}^n} U_z f(x) g(x) dx \tag{3.27}$$

is continuous on S and analytic in the interior of S, whenever $f, g \in \ell^2(\mathbb{R}^n)$.

Lemma 3.3. *[22] With the above assumptions, if for $1 < p_i, q_i < \infty, (i = 0, 1)$, $\{U_z\}$ satisfies the following conditions:*

$$(i) \quad \left\| \left(\sum_{j\in\mathbb{Z}} |U_z f_j|^{q_0} \right)^{1/q_0} \right\|_{L^{p_0}} \leq M_0 \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^{q_0} \right)^{1/q_0} \right\|_{L_{p_0}}, \quad \text{when } Re(z) = a;$$

$$(ii) \quad \left\| \left(\sum_{j\in\mathbb{Z}} |U_z f_j|^{q_1} \right)^{1/q_1} \right\|_{L^{p_1}} \leq M_1 \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^{q_1} \right)^{1/q_1} \right\|_{L_{p_1}}, \quad \text{when } Re(z) = b.$$

Then we have

$$\left\| \left(\sum_{j\in\mathbb{Z}} |U_{a(1-\theta)+\theta b} f_j|^{q} \right)^{1/q} \right\|_{L^{p}} \leq M_0^{1-\theta} M_1^{\theta} \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^{q} \right)^{1/q} \right\|_{L^{p}},$$

where $0 < \theta < 1, 1/q = ((1-\theta)/q_0) + \theta/q_1$ and $1/p = ((1-\theta)/q_0) + \theta/p_1$.

Let $U_z f(x) = e^{z^2} \mathcal{H}_z f(x)$. Note that

$$|e^{z^2}| = e^{Re(z)^2 - Im(z)^2} \tag{3.28}$$

then there exists a constant M_0 which is independent of $Im(z)$ such that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |U_z f_j|^{2} \right)^{1/2} \right\|_{L^{2}} \leq C_\delta e^{-Im(z)^2} [1 + |Im(z)|] \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^{2} \right)^{1/2} \right\|_{L^{2}} \tag{3.29}$$

$$\leq M_0 \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^{2} \right)^{1/2} \right\|_{L^{2}} \quad \text{when } Re(z) = 1/4 - \delta. \tag{3.30}$$

Also, for $1 < p_1, q_1 < \infty$, there exists a constant M_1 which is independent of $Im(z)$ such that

$$\left\| \left(\sum_{j\in\mathbb{Z}} |U_z f_j|^{q_1} \right)^{1/q_1} \right\|_{L^{p_1}} \leq M_1 \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^{q_1} \right)^{1/q_1} \right\|_{L^{p_1}} \quad \text{when } Re(z) < -1. \tag{3.31}$$

For $2 \leq p, q < 5/2$, there exists $1 < p_1, q_1 < \infty$ so that

$$\frac{1}{p} = \frac{\theta}{2} + \frac{1-\theta}{p_1}, \qquad \frac{1}{q} = \frac{\theta}{2} + \frac{1-\theta}{q_1} \tag{3.32}$$

where $0 < \theta < 1$ and satisfies the equation

$$\left(\frac{1}{4} - \delta\right)\theta + (-1-\epsilon)(1-\theta) = 0 \tag{3.33}$$

for some $\epsilon > 0$ and $0 < \delta < 1/4$.

Therefore, by Lemma 3.3, we obtain

$$\begin{aligned} rl\left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{H} f_j|^q \right)^{1/q} \right\|_{L^p} &= \left\| \left(\sum_{j\in\mathbb{Z}} |\mathcal{H}_0 f_j|^q \right)^{1/q} \right\|_{L^p} \\ &= \left\| \left(\sum_{j\in\mathbb{Z}} |U_0 f_j|^q \right)^{1/q} \right\|_{L^p} \\ &\leq C \left\| \left(\sum_{j\in\mathbb{Z}} |f_j|^q \right)^{1/q} \right\|_{L^p} \end{aligned} \tag{3.34}$$

for $2 \leq p,q < 5/2$. The duality implies the result for $5/3 < p,q \leq 2$. This completes the proof of Theorem 3.3.

□

For more details on the proof of Theorem 3.3 see [22].

3.2 Main results

The principal theorem of this chapter is the following, where we will prove the boundedness of the Hilbert transform on Lizorkin-Triebel spaces. We deal with plane curves of the form $\Gamma(t) = (t, \gamma(t))$, where $\gamma(t)$ is a convex function for $t \geq 0$.

Theorem 3.4. *Let $\gamma(t)$ be a continuous odd function, twice continuously differentiable, increasing and convex for $t \geq 0$. Suppose that γ'' is monotone for $t > 0$ and that there exists $C > 0$ so that $\gamma'(t) \leq Ct\gamma''(t)$ for $t > 0$. Then $\mathcal{H}$ is bounded on $F^s_{p,q}(\mathbb{R}^2)$ for all $s \in \mathbb{R}$, and all p, q with $5/3 < p, q < 5/2$.*

Remark 3.4. *Notice that $\left(F^{s_0}_{p,q_0}(\mathbb{R}^n), F^{s_0}_{p,q_1}(\mathbb{R}^n)\right)_{\theta,q} = B^s_{p,q}(\mathbb{R}^n)$ for $0 < \theta < 1$ and $s = (1-\theta)s_0 + \theta s_1$, we therefore may obtain Theorem 2.9, for some p, q, by applying Theorem 3.4 and the interpolation result above. However, we have seen in Chapter 2 that* *since $\mathcal{H}$ is a convolution operator, the Besov $B^s_{p,q}$-boundedness of $\mathcal{H}$ can be easily obtained by its L^p-boundedness, which does not rely on the Lizorkin-Triebel $F^s_{p,q}$-boundedness of $\mathcal{H}$ (see Subsection 3.2.1 below for its proof).*

3.2.1 Proof of Theorem 3.4

From Theorem 3.3, it is well known that $\mathcal{H}$ satisfies the following vector valued inequality:

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{H}f_j|^q\right)^{1/q}\right\|_{L^p}\leqslant c_{p,q}\left\|\left(\sum_{j\in\mathbb{Z}}|f_j|^q\right)^{1/q}\right\|_{L^p}, \qquad (3.35)$$

for all p,q with $5/3<p,q<5/2$, and all $f_j\in L^p(\mathbb{R}^2)$.

Now, by the definition of $F^s_{p,q}(\mathbb{R}^n)$ and the convolution formula of $\mathcal{H}$ we have

$$\begin{aligned}\|\mathcal{H}f\|_{F^s_{p,q}(\mathbb{R}^2)} &:= \left\|\left(\sum_{j=0}^{\infty}2^{jsq}|(\Phi_j*\mathcal{H}f)(x)|^q\right)^{1/q}\right\|_{L^p}\\ &= \left\|\left(\sum_{j=0}^{\infty}2^{jsq}|(\Phi_j*K*f)(x)|^q\right)^{1/q}\right\|_{L^p}\end{aligned}$$

By the convolution properties and the formula (3.35) we have

$$\begin{aligned}\|\mathcal{H}f\|_{F^s_{p,q}(\mathbb{R}^2)} &= \left\|\left(\sum_{j=0}^{\infty}|K*2^{js}(\Phi_j*f)(x)|^q\right)^{1/q}\right\|_{L^p}\\ &= \left\|\left(\sum_{j=0}^{\infty}|\mathcal{H}(2^{js}\Phi_j*f)(x)|^q\right)^{1/q}\right\|_{L^p}\\ &\le C\left\|\left(\sum_{j=0}^{\infty}|(2^{js}\Phi_j*f)(x)|^q\right)^{1/q}\right\|_{L^p}\\ &\le C\left\|\left(\sum_{j=0}^{\infty}2^{jsq}|(\Phi_j*f)(x)|^q\right)^{1/q}\right\|_{L^p}\\ &= C\|f\|_{F^s_{p,q}(\mathbb{R}^2)}.\end{aligned}$$

The theorem is thus proved.

□

In the context of Theorem 3.4 and by applying Theorem 3.1 we have the following result:

Theorem 3.5. *Let $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be a well-curved curve in $\mathbb{R}^n$ with $\Gamma(0) = 0$. Then there exists $c > 0$, such that*

$$\|\mathcal{H}f\|_{F^s_{p,q}(\mathbb{R}^n)} \leq c\|f\|_{F^s_{p,q}(\mathbb{R}^n)}, \tag{3.36}$$

for all $s \in \mathbb{R}$, and all p, q with $1 < p, q < \infty$, $n \geq 2$.

And in the context of Theorem 3.4 and by applying Theorem 3.2 we have the following result:

Theorem 3.6. *(1) Let $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be a polynomial function. Then there exists $c > 0$, such that*

$$\|\mathcal{H}f\|_{F^s_{p,q}(\mathbb{R}^n)} \leq c\|f\|_{F^s_{p,q}(\mathbb{R}^n)},$$

for all $s \in \mathbb{R}$, and all p, q with $1 < p, q < \infty$, $n \geq 2$.

(2) Let the image of $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be of finite type at 0. Then there exists $c > 0$, such that

$$\|\mathcal{H}f\|_{F^s_{p,q}(\mathbb{R}^n)} \leq c\|f\|_{F^s_{p,q}(\mathbb{R}^n)},$$

for all $s \in \mathbb{R}$, and all p, q with $1 < p, q < \infty$, $n \geq 2$.

Chapter 4

Boundedness of the Hilbert Transform on Localized Lizorkin-Triebl Spaces

In this chapter, we will study the boundedness of the Hilbert transform on localized Lizorkin-Triebl spaces $(F^s_{p,q})_{\ell^r}$ in the $\ell^r(\mathbb{Z}^n)$ norm for $r = p$, see for instance [15, 17, 27], and for more theory on localization, we refer the reader to [7, 34, 36].

4.1 Pointwise multipliers for the Besov and the Lizorkin-Triebel spaces

We recall the definition of the pointwise multipliers space of a Banach space E (in this thesis $E = B^s_{p,q}(\mathbb{R}^n)$ or $E = F^s_{p,q}(\mathbb{R}^n)$), denoted by $M(E)$, this is the set of all functions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f \cdot g\|_E \leq c\|g\|_E \quad (\forall g \in E).$$

We say that f is a multiplier of E, we denote by $\|f\|_{M(E)}$ the supremum of the number $\|f \cdot g\|_E$ taken over all $g \in E$ such that $\|g\|_E \leq 1$. i.e.

$$\|f\|_{M(E)} := \sup\{\|f \cdot g\|_E : g \in C^\infty \cap E, \|g\|_E \leq 1\}.$$

$M(E)$ is a Banach space. We recall that $M(E)$ is isometrically invariant by translation, if $\tau_a f \in M(E)$ then

$$\|\tau_a f\|_{M(E)} = \|f\|_{M(E)}.$$

We refer the reader to [36, p.213] and [41] for the following embedding ($1 \leq p, q \leq \infty$), we need respectively:

$$B^s_{\infty,q} \hookrightarrow M(F^s_{p,q}). \tag{4.1}$$

Concerning the properties of $M(B^s_{p,q})$ and $M(F^s_{p,q})$ see [34, 36, 41].

4.2 Characterization of the localized Lizorkin-Triebel spaces

For the definition of the localized Lizorkin-Triebel space, denoted by $(F^s_{p,q})_{\ell^r}$, we need to introduce a new resolution of unity: Let $\psi \in \mathcal{D}(\mathbb{R}^n)$, supported in the ball $\mathbb{B}(0, R)$ with $R > \sqrt{n}$ and satisfying

$$\sum_{k\in\mathbb{Z}^n} \psi(x-k) = 1 \qquad (\forall x \in \mathbb{R}^n). \tag{4.2}$$

Indeed, Let $g \in \mathcal{D}(\mathbb{R}^n)$ be a positive function which equals 1 on the cube $|x_i| \leqslant 2^{-1}(i = 1, \cdots, n)$. Then the function $G(x) = \sum_{k\in\mathbb{Z}^n} g(x-k)$ is C^∞ on $\mathbb{R}^n$, bounded and $G(x) \geqslant 1$, (since for every x, there exist $k \in \mathbb{Z}^n$, such that $|x_i - k_i| \leq \frac{1}{2}$ (for all $i = 1, \cdots, n$) and so $g(x-k) = 1$). we also have $G(x-k) = G(x), \forall k \in \mathbb{Z}^n, \forall x \in \mathbb{R}^n$. The function $\dfrac{1}{G(x)}$ is C^∞ on $\mathbb{R}^n$. we just pose

$$\psi(x) = \frac{g(x)}{G(x)}.$$

□

We consider the set $\mathcal{A}$ as the class of all the functions $\psi \in \mathcal{D}(\mathbb{R}^n)$ satisfying

$$\sum_{k\in\mathbb{Z}^n} \psi(x-k) = 1, \textit{ for all } x \in \mathbb{R}^n. \tag{4.3}$$

Definition 4.1. *[7] A **B**anach **S**pace of **D**istributions, denoted by (**B.S.D**) in $\mathcal{D}'(\mathbb{R}^n)$, is a vector subspace E of $\mathcal{D}'(\mathbb{R}^n)$ equipped with a complete norm $\|\cdot\|_E$ such that the canonical injection $E \hookrightarrow \mathcal{D}'$ is continuous.*

Remark 4.1. *Let E be a Banach space of distributions (B.S.D). We associate on the space E the following hypothesis.*

(1) *Translation invariance; if we denote τ_x the operator given by $\tau_x f(t) = f(t-x)$, then τ_x is an isometric of E;*

(2) *Localization invariance; for all $f \in E$ and all $\psi \in \mathcal{D}(\mathbb{R}^n)$, we have that $\psi \cdot f \in E$.*

Let $\psi \in \mathcal{D}(\mathbb{R}^n)$. The notion of localized is defined by $f_x = \tau_x \psi \cdot f$, it follows immediately from the hypotheses (1) and (2) that the family $(f_x)_{x\in\mathbb{R}^n}$ is bounded in E.

Definition 4.2. *Let $1 \leq r < \infty$, and E be a Banach space of distributions, Let $\psi \in \mathcal{A}$. The space $(E)_{\ell^r}$ is the set of functions $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$\|f\|_{(E)_{\ell^r}} = \left(\sum_{k\in\mathbb{Z}^n} \|\psi(\cdot - k)f\|_E^r\right)^{1/r} < \infty. \tag{4.4}$$

In the case where $r = \infty$, the expression (4.4) should be interpreted as

$$\|f\|_{(E)_{\ell^\infty}} = \sup_{k\in\mathbb{Z}^n} \|\psi(\cdot - k)f\|_E < \infty. \tag{4.5}$$

The norm defined in (4.4) or (4.5) is independent of the chosen system (4.3).

Definition 4.3. *Let E be a Banach space of distributions, E is localizable in the ℓ^r-norm $(1 \leq r \leq \infty)$, if there exists a function $\psi \in \mathcal{A}$ and a constant $c \geq 1$ such that:*

$$c^{-1}\|f\|_E \leq \left(\sum_{k\in\mathbb{Z}^n} \|\psi(\cdot - k)f\|_E^r\right)^{1/r} \leq c\|f\|_E.$$

Theorem 4.1. *[7] Let $1 \leq p \leq \infty$. Then the space L^p is localizable in the ℓ^p norm.*

Proof. It suffices to show that the operator

$$T_\varphi : f \longmapsto \{\varphi(\cdot - k)f\}_{k\in\mathbb{Z}^n}$$

is an isomorphism of $L^p(\mathbb{R}^n)$ on a closed subspace of $(L^p)_{\ell^p}$ for a well chosen function $\varphi \in \mathcal{D}(\mathbb{R}^n)$.

For that, we introduce the operator

$$S_\psi((f_k)_{k\in\mathbb{Z}^n}) = \sum_{k\in\mathbb{Z}^n} \psi(\cdot - k)f_k, \tag{4.6}$$

where for a suitable choice of $\psi \in \mathcal{D}(\mathbb{R}^n)$, S_ψ is a right inverse of T_φ.

Since φ has a compact support, there exists $c = c(\varphi) > 0$ such that, for all $x \in \mathbb{R}^n$, we have

$$\sum_{k\in\mathbb{Z}^n} |\varphi(x-k)| \leq c.$$

We have also

$$\begin{aligned} \|T_\varphi f\|_{(L^1)_{\ell^1}} &= \sum_{k\in\mathbb{Z}^n} \int_{\mathbb{R}^n} |\varphi(x-k)||f(x)|dx \\ &\leq c\|f\|_1. \end{aligned} \tag{4.7}$$

On the other hand we have

$$\begin{aligned} \|T_\varphi f\|_{(L^\infty)_{\ell^\infty}} &= \sup_{k\in\mathbb{Z}^n, x\in\mathbb{R}^n} |\varphi(x-k)f(x)| \\ &\leq \|\varphi\|_\infty \|f\|_\infty. \end{aligned} \tag{4.8}$$

By applying the Riesz-Thorin's theorem to (4.7) and (4.8), then the operator T_φ is continuous from L^p in $(L^p)_{\ell^p}$. Otherwise we have

$$\left(\sum_{k\in\mathbb{Z}^n} \|\varphi(\cdot - k)f\|_p^p\right)^{1/p} \leq c\|f\|_p. \tag{4.9}$$

The continuity of S_ψ from $(L^p)_{\ell^p}$ in L^p is proved in the same way, we have

$$\begin{aligned} \left\|S_\psi\left((f_k)_{k\in\mathbb{Z}^n}\right)\right\|_1 &\leq \sum_{k\in\mathbb{Z}^n} \|\psi(\cdot - k)f_k\|_1 \quad (\textit{Hölder}) \\ &\leq \|\psi\|_\infty \sum_{k\in\mathbb{Z}^n} \|f_k\|_1 \\ &= \|\psi\|_\infty \|(f_k)_{k\in\mathbb{Z}^n}\|_{(L^1)_{\ell^1}}. \end{aligned} \tag{4.10}$$

On the other hand we have

$$\|S_\psi\left((f_k)_{k\in\mathbb{Z}^n}\right)\|_\infty \leq \left(\sup_{k\in\mathbb{Z}^n} \|f_k\|_\infty\right) \sup_{x\in\mathbb{R}^n} \left(\sum_{k\in\mathbb{Z}^n} |\psi(x-k)|\right), \tag{4.11}$$

where, φ and ψ can be any compact support functions.

Now, suppose that $\varphi \in \mathcal{A}$, and choose ψ so that $\psi = 1$ on the support of φ, so we have

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}^n} \varphi(\cdot - k)f \\ &= \sum_{k\in\mathbb{Z}^n} \psi(\cdot - k)\cdot\varphi(\cdot - k)\cdot f \\ &= S_\psi\left((\varphi(\cdot - k)\cdot f)_{k\in\mathbb{Z}^n}\right), \end{aligned}$$

and the continuity of S_ψ from $(L^p)_{\ell^p}$ in L^p gives

$$\| f \|_p \leq c \left(\sum_{k\in\mathbb{Z}^n} \|\varphi(\cdot - k) f\|_p^p \right)^{1/p} . \tag{4.12}$$

The inequalities (4.9) and (4.12) imply that the space L^p is localizable in the ℓ^p norm. □

4.2.1 Localisation of Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$

Proposition. 4.1. *For all $R > 0$ there exists a constant $c = c(n, R)$ such that for any family $(f_k)_{k\in\mathbb{Z}^n}$ of functions carried respectively by the balls $|x - k| \leq R$, we have*

$$\left\| \sum_{k\in\mathbb{Z}^n} f_k \right\|_p \leq c \left(\sum_{k\in\mathbb{Z}^n} \|f_k\|_p^p \right)^{1/p} .$$

Proof. The proof is immediate if we notice that

$$\sum_{k\in\mathbb{Z}^n} f_k = \sum_{k\in\mathbb{Z}^n} \psi(\cdot - k) f = S_\psi((f_k)_{k\in\mathbb{Z}^n})$$

where $\psi \in \mathcal{D}(\mathbb{R}^n)$ is chosen so that $\psi = 1$ on the ball $|x| \leq R$, (see (4.6) for the definition of the operator S_ψ). □

Proposition. 4.2. *[7] If the function $\theta \in \mathcal{S}(\mathbb{R}^n)$ is not null on the support of ψ, (see (4.2) for the definition of ψ). Then we have:*

$$\| f \|_{(E)_{\ell^p}} \sim \left(\sum_{k\in\mathbb{Z}^n} \|\theta(\cdot - k) f\|_E^p \right)^{1/p} .$$

Proof. Firstly, we can write $\psi = g.\theta$ where $g \in \mathcal{D}(\mathbb{R}^n)$ and

$$\|\psi(\cdot - k) f\|_E \leq \|g\|_{M(E)} \|\theta(\cdot - k) f\|_E ,$$

so

$$\|f\|_{(E)_{\ell^p}} \leq c \left(\sum_{k\in\mathbb{Z}^n} \|\theta(\cdot - k) f\|_E^p \right)^{1/p} .$$

On the other hand

$$\begin{aligned} \|\theta(\cdot - k) f\|_E &\leq \sum_{k'\in\mathbb{Z}^n} \|\psi(\cdot - k')\theta(\cdot - k) f\|_E \\ &\leq \sum_{k'\in\mathbb{Z}^n} \|\lambda(\cdot - k')\theta(\cdot - k)\|_{M(E)} \|\psi(\cdot - k') f\|_E , \end{aligned} \tag{4.13}$$

where $\lambda \in \mathcal{D}(\mathbb{R}^n)$ and $\lambda = 1$ on the support of ψ. By a change of variables, and the fact that $\|\cdot\|_{M(E)}$ is invariant by translation, we have

$$\|\lambda(\cdot - k')\theta(\cdot - k)\|_{M(E)} = \|\lambda(\cdot - (k - k'))\theta\|_{M(E)}\,. \tag{4.14}$$

By combining (4.13), (4.14) and the discrete young inequality, we get the result. □

Lemma 4.3. *[15] Let $s \in \mathbb{R}, 1 \leq p \leq \infty, 1 \leq q \leq \infty$ and $1 \leq r \leq \infty$. There exists $c = c(n) > 0$, such that for any $Q > 0$ and any family $(f_k)_{k\in\mathbb{Z}^n}$ of functions, carried respectively by the balls $|x - k| \leq Q$, we have,*

$$\left\|\sum_{k\in\mathbb{Z}^n} f_k\right\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^r}} \leq cQ^n \left(\sum_{k\in\mathbb{Z}^n} \|f_k\|^r_{F^s_{p,q}(\mathbb{R}^n)}\right)^{1/r}.$$

Proof. Let us pose $m = 2[R+Q]+1 \in \mathbb{N}$. for all k *and* k' *in* $\mathbb{Z}^n$, the intersection of the balls $|x-k| \leq Q$ *and* $|x-k'| \leq R$ is empty as soon as $|k-k'| \geq m$, whatever $k' \in \mathbb{Z}^n$, we have

$$\begin{aligned}
\left\|\sum_{k\in\mathbb{Z}^n} f_k\right\|_{(F^s_{p,q})_{\ell^r}} &= \left(\sum_{k'\in\mathbb{Z}^n} \left\|\psi(\cdot - k')\sum_{k\in\mathbb{Z}^n} f_k\right\|^r_{F^s_{p,q}}\right)^{1/r} \\
&= \left(\sum_{k'\in\mathbb{Z}^n} \left\|\sum_{k\in\mathbb{Z}^n} \psi(\cdot - k')f_k\right\|^r_{F^s_{p,q}}\right)^{1/r}.
\end{aligned}$$

We estimate $\sum_{k\in\mathbb{Z}^n} \psi(\cdot - k')f_k$ in the norm of $F^s_{p,q}$, then by the inequality of hölder we obtain

$$\begin{aligned}
\left\|\sum_{k\in\mathbb{Z}^n} \psi(\cdot - k')f_k\right\|_{F^s_{p,q}} &= \left\|\sum_{|k-k'|\leq m} \psi(\cdot - k')f_k\right\|_{F^s_{p,q}} \\
&\leq \sum_{|k-k'|\leq m} \|\psi(\cdot - k')f_k\|_{F^s_{p,q}} \\
&\leq \left(\sum_{|k-k'|\leq m} 1\right)^{1/r'} \left(\sum_{|k-k'|\leq m} \|\psi(\cdot - k')f_k\|^r_{F^s_{p,q}}\right)^{1/r},
\end{aligned} \tag{4.15}$$

and as

$$\sum_{k\in\mathbb{Z}^n, |k-k'|\leq m} 1 = \sum_{k'-m}^{k'+m} 1 \sum_{k'-m}^{k'+m} 1 \cdots \sum_{k'-m}^{k'+m} 1 = \underbrace{(2m)\cdots(2m)}_{n \text{ times}} = (2m)^n,$$

so

$$\sum_{k\in\mathbb{Z}^n, |k-k'|\leq m} 1 \leq (2m+1)^n \leq cQ^n.$$

we replace in relation (4.15) it becomes

$$\left\| \sum_{k \in \mathbb{Z}^n} \psi(\cdot - k') f_k \right\|_{F^s_{p,q}} \leq C Q^{n/r'} \left(\sum_{k \in \mathbb{Z}^n, |k-k'| \leq m} \| \psi(\cdot - k') f_k \|^r_{F^s_{p,q}} \right)^{1/r} .$$

Let us show that there is $C > 0$, such that

$$\sup_{k' \in \mathbb{Z}^n} \| \psi(\cdot - k') f_k \|_{F^s_{p,q}} \leq C \| f \|_{F^s_{p,q}} . \tag{4.16}$$

To show (4.16) it suffices to apply (4.1)

$$B^s_{\infty,q} \hookrightarrow M(F^s_{p,q}).$$

Let us return to the series $\sum_{k \in \mathbb{Z}^n} f_k$, using (4.16) we obtain

$$\begin{aligned} \left\| \sum_{k \in \mathbb{Z}^n} f_k \right\|_{(F^s_{p,q})_{\ell^r}} &\leq c_1 Q^{n/r'} \left(\sum_{k' \in \mathbb{Z}^n} \sum_{|k-k'| \leq m} \| \psi(\cdot - k') f_k \|^r_{F^s_{p,q}} \right)^{1/r} \\ &\leq c_2 Q^{n/r'} \left(\sum_{k \in \mathbb{Z}^n} \| f_k \|^r_{F^s_{p,q}} \sum_{|k-k'| \leq m} 1 \right)^{1/r} \\ &\leq c_3 Q^n \left(\sum_{k \in \mathbb{Z}^n} \| f_k \|^r_{F^s_{p,q}} \right)^{1/r} , \end{aligned}$$

□

the Lemma 4.3 is fully proven.

Remark 4.2. *The norm of $(F^s_{p,q})_{\ell^r}$ is independent of the chosen of ψ, this is an emidiate consequence of Lemma 4.3, Indeed:*
Let ψ_1, ψ_2 in $\mathcal{D}(\mathbb{R}^n)$ carried respectively by the balls $|x| < R_1$, $|x| < R_2$ such that $\psi_1 \in \mathcal{A}, \psi_2 \in \mathcal{A}$, we denote by $\|\cdot\|^{(1)}_{(F^s_{p,q})_{\ell^r}}$, $\|\cdot\|^{(2)}_{(F^s_{p,q})_{\ell^r}}$ the associated norms respectively of ψ_1, ψ_2. Then for all f we have

$$\begin{aligned} \| f \|^{(1)}_{(F^s_{p,q})_{\ell^r}} &= \left\| \sum_{k \in \mathbb{Z}^n} \psi_2(\cdot - k) f \right\|^{(1)}_{(F^s_{p,q})_{\ell^r}} \\ &\leq c R_2^n \| f \|^{(2)}_{(F^s_{p,q})_{\ell^r}} ; \end{aligned}$$

in the same way, we have

$$\| f \|^{(2)}_{(F^s_{p,q})_{\ell^r}} \leq c R_1^n \| f \|^{(1)}_{(F^s_{p,q})_{\ell^r}} .$$

Lemma 4.4. *For any λ_1 verifying $0 < \lambda_1 < 1/2$, there exists $c > 0$ such that for any sequence $(f_k)_{k\in\mathbb{Z}^n}$ of elements of E, carried respectively by $|x-k| < \lambda_1$, we have*

$$\left(\sum_{k\in\mathbb{Z}^n} \|f_k\|_E^r\right)^{1/r} \leq c\, \|f\|_{(E)_{\ell^r}} .$$

Proof. Let λ_2 a real verifying $0 < \lambda_1 < \lambda_2 < 2\lambda_1$. Consider the function $\theta \in \mathcal{D}(\mathbb{R}^n)$ carried by $[-\lambda_2, \lambda_2]^n$, such that $\theta = 1$ on $[-\lambda_1, \lambda_2]^n$. If $f = \sum_{k\in\mathbb{Z}^n} f_k$, then

$$\begin{aligned}
\|\theta(\cdot-k)f\|_E &= \|f_k\|_E \\
&\leq \sum_{k'\in\mathbb{Z}^n} \|\psi(\cdot-k')\theta(\cdot-k)f\|_E \\
&\leq \sum_{k'\in\mathbb{Z}^n} \|\lambda(\cdot-k')\theta(\cdot-k)\|_{M(E)}\, \|\psi(\cdot-k')f\|_E\,,
\end{aligned}$$

where $\lambda \in \mathcal{D}(\mathbb{R}^n)$ and $\lambda = 1$ on the support of ψ. We use then, the Young's inequality in $\ell^r(\mathbb{Z}^n)$ we get exactly

$$\left(\sum_{k\in\mathbb{Z}^n} \|f_k\|_E^r\right)^{1/r} \leq c\, \|f\|_{(E)_{\ell^r}} .$$

□

4.3 convergent series in Besov and Lizorkin-Triebel spaces

This section is devoted to Yamazaki type estimates. (See [45]).

Proposition. 4.5. *Let $s \in \mathbb{R}$ and $\mu > 1$.*

(i) There exists $c > 0$ such that for any function $\theta \in C^\infty(\mathbb{R}^n)$ with support in $\mu^{-1} \leq |\xi| \leq \mu$ and any sequence of distributions $(f_j)_{j\in\mathbb{N}}$ defined by $\widehat{f_j}(\xi) = \theta(2^{-j}\xi)\widehat{f}(\xi)$ with $f \in \mathcal{S}'(\mathbb{R}^n)$. We have

$$\left(\sum_{j\geq 0} 2^{sjq}\|f_j\|_p^q\right)^{\frac{1}{q}} \leq c \sup_{|\alpha|\leq[\frac{n}{2}]+1} \|\theta^{(\alpha)}\|_\infty \|f\|_{B^s_{p,q}(\mathbb{R}^n)}. \tag{4.17}$$

(ii) *There exists* $c > 0$ *such that for any sequence of functions* $(f_j)_{j\in\mathbb{N}}$ *, where* $supp\,\mathcal{F}f_j \subset \{\xi : \mu^{-1}2^j \leq |\xi| \leq \mu 2^j\}$*, for all* $j \in \mathbb{N}$*, we have*

$$\left\|\sum_{j\geq 0} f_j\right\|_{B^s_{p,q}(\mathbb{R}^n)} \leq c\left(\sum_{j\geq 0} 2^{sjq}\|f_j\|_p^q\right)^{\frac{1}{q}}. \qquad (4.18)$$

(iii) *For all* $a > 1$*, there exists* $c > 0$ *such that for any sequence of functions* $(f_j)_{j\in\mathbb{N}}$ *, where* $supp\,\mathcal{F}f_j \subset \{\xi : a^{-1}2^j \leq |\xi| \leq a2^j\}$*, for all* $j \in \mathbb{N}$*, we have*

$$\left\|\sum_{j\geq 0} f_j\right\|_{F^s_{p,q}(\mathbb{R}^n)} \leq c\left\|\left(\sum_{j\geq 0} 2^{sjq}|f_j|^q\right)^{\frac{1}{q}}\right\|_p. \qquad (4.19)$$

Proof. (i) For proving (4.17). We start from the series

$$f_j = \sum_{k\geq 0} \Delta_k f_j,\ \forall j \in \mathbb{N},$$

we obtain

$$f_j = \sum_{k\geq 0}(2^{kn}\mathcal{F}^{-1}\theta(2^k\cdot)) * \Delta_k f_j,$$

with $supp\,\theta(2^{-k}\cdot) \cap supp\widehat{f_j} \neq \emptyset$ for all $|j-k| \leq N$, where $N = 2 + [\frac{\ln\mu}{\ln 2}]$.

We have

$$(1+|y|^2)^m\mathcal{F}^{-1}\theta(y) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\cdot y}(I-\Delta_x)^m\theta(x)dx,$$

then by the Cauchy-Schwartz inequality and the Bessel-Parseval equality we have

$$\begin{aligned}\|\mathcal{F}^{-1}\theta\|_1 &\leq \left(\int_{supp\ \theta}|(I-\Delta_x)^{\frac{m}{2}}\theta(x)|^2dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^n}(1+|y|^2)^{-m}dy\right)^{\frac{1}{2}}\\ &\leq c\sup_{|\alpha|\leq m}\|\theta^{(\alpha)}\|_\infty,\ \textit{where}\ m = [\frac{n}{2}]+1.\end{aligned}$$

By Young's inequality we deduce

$$\begin{aligned}\|f_j\|_p &\leq c(\theta)\sum_{j-N\leq k\leq j+N}\|\Delta_k f\|_p\\ &\leq c(\theta)\sum_{k=j}^{\infty}\|\Delta_{k-N} f\|_p.\end{aligned}$$

We can then distinguish three cases:

The first case: $s > 0$**.**

Proposition 1.5 gives the inequality

$$\begin{aligned}\left(\sum_{j\geq 0} 2^{sjq}\|f_j\|_p^q\right)^{\frac{1}{q}} &\leq c_1(\theta)\left(\sum_{k\geq 0} 2^{skq}\|\Delta_{k+N}f\|_p^q\right)^{\frac{1}{q}}\\ &\leq 2^{-sN}c_1(\theta)\|f\|_{B^s_{p,q}(\mathbb{R}^n)}, \quad (N=2+[\frac{\ln\mu}{\ln 2}]).\end{aligned}$$

The second case: $s<0$**.**

In a similar way we have

$$\begin{aligned}2^{sj}\|f_j\|_p &\leq c(\theta)2^{sj}\sum_{j-N\leq k\leq j+N}\|\Delta_k f\|_p\\ &\leq c(\theta)2^{-sN}2^{s(j+N)}\sum_{0\leq k\leq j+N}2^{-sk}(2^{sk}\|\Delta_k f\|_p).\end{aligned}$$

Proposition 1.5 allows us to conclude.

The third case: $s=0$**.** By the Hölder inequality , we have

$$\|f_j\|_p\leq c(\theta)\left(\sum_{j-N\leq k\leq j+N}1\right)^{\frac{1}{q'}}\left(\sum_{j-N\leq k\leq j+N}\|\Delta_k f\|_p^q\right)^{\frac{1}{q}}, \quad (N=2+[\frac{\ln\mu}{\ln 2}]),$$

from which we deduce

$$\begin{aligned}\left(\sum_{j\geq 0}\|f_j\|_p^q\right)^{\frac{1}{q}} &\leq c(\theta)(2N+1)^{\frac{1}{q'}}\left(\sum_{j\geq 0}\sum_{j-N\leq k\leq j+N}\|\Delta_k f\|_p^q\right)^{\frac{1}{q}}\\ &= c(\theta)(2N+1)^{\frac{1}{p}+\frac{1}{q'}}\left(\sum_{k\geq 0}\|\Delta_k f\|_p^q\right)^{\frac{1}{q}}\\ &= c_1(\theta)\|f\|_{B^0_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

(ii) Proof of (4.18). We pose

$$f_j=\sum_{k\geq 0}\mathcal{F}^{-1}(\psi(2^{-k}\cdot))*\Delta_k f_j,$$

where $supp\,\psi(2^{-k}\cdot)\cap supp\,\widehat{f_j}\neq\emptyset$ for all $|j-k|\leq N$, with $N=2+[\frac{\ln\mu}{\ln 2}]$, which gives

$$\begin{aligned}2^{ks}\left\|\Delta_k\left(\sum_{j\geq 0}f_j\right)\right\|_p &\leq \sum_{-N\leq\ell\leq N}2^{ks}\|\Delta_k f_{k+\ell}\|_p\\ &\leq c(\psi)\sum_{-N\leq\ell\leq N}2^{ks}\|f_{k+\ell}\|_p.\end{aligned}$$

Hence

$$\begin{aligned}\left(\sum_{k\geq 0}\left(2^{ks}\left\|\Delta_k\left(\sum_{j\geq 0} f_j\right)\right\|_p\right)^q\right)^{\frac{1}{q}} &\leq c \sum_{-N\leq \ell\leq N}\left(\sum_{k\geq 0}\left(2^{ks}\|f_{k+\ell}\|_p\right)^q\right)^{\frac{1}{q}} \\ &\leq c'\left(\sum_{k'\geq 0}\left(2^{k's}\|f_{k'}\|_p\right)^q\right)^{\frac{1}{q}}.\end{aligned}$$

(iii) To prove (4.19) we use the same method as (ii).

□

Proposition. 4.6. *If $s > 0$, we can replace the annulus $a^{-1}2^j \leq |\xi| \leq a2^j$ with the balls $|\xi| \leq a2^j$, in proposition 4.5 (iii).*

Proof. See [28]. □

Proposition. 4.7. *Let $a > \frac{n}{\min(p,q)}$. Then there exists $c > 0$ such that*

$$\left\|\left(\sum_{j=0}^{\infty} 2^{jsq}|\Delta_j^{*,a} f|^q\right)^{\frac{1}{q}}\right\|_p \leq c\|f\|_{F^s_{p,q}(\mathbb{R}^n)},$$

for all $f \in F^s_{p,q}(\mathbb{R}^n)$.

Proof. See [36, p.22] □

Proposition. 4.8. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. then there exists $c > 0$ such that*

$$\left\|\left(\sum_{j=0}^{\infty} |2^{jn}\mathcal{F}^{-1}\theta(2^j\cdot) * f_j|^q\right)^{\frac{1}{q}}\right\|_p \leq c\left\|\left(\sum_{j=0}^{\infty} |f_j|^q\right)^{\frac{1}{q}}\right\|_p,$$

for all $\theta \in \mathcal{S}(\mathbb{R}^n)$ and all sequence of functions $(f_j)_{j=0}^{\infty} \subset \mathcal{S}'(\mathbb{R}^n)$.

Proof. See [36, p.23] □

Proposition. 4.9. *Let $s > 0$ and $\mu \geq 1$. There exists $c > 0$ such that*

$$\left\|\left(\sum_{j=0}^{\infty} |\Delta_j f_j|^q\right)^{\frac{1}{q}}\right\|_p \leq c\mu^s\left\|\left(\sum_{j=0}^{\infty} |f_j|^q\right)^{\frac{1}{q}}\right\|_p,$$

for all sequence of functions $(f_j)_{j=0}^{\infty} \subset \mathcal{S}'(\mathbb{R}^n)$, where supp $\widehat{f_j} \subset \mathbb{B}(0, \mu 2^j)$.

Proof. See [36, p.22] □

Proposition. 4.10. *Let $s \in \mathbb{R}$ and $\mu > 1$. Then there exists $c > 0$ such that*

$$\left\| \left(\sum_{j\geq 0} 2^{qsj} |f_j|^q \right)^{\frac{1}{q}} \right\|_p \leq c(\theta) \|f\|_{F^s_{p,q}(\mathbb{R}^n)}, \tag{4.20}$$

for any function $\theta \in C^\infty(\mathbb{R}^n)$ with support in $\mu^{-1} \leq |\xi| \leq \mu$ and any sequence of distributions $(f_j)_{j\in\mathbb{N}}$ defined by $\widehat{f_j}(\xi) = \theta(2^{-j}\xi)\widehat{f}(\xi)$ with $f \in \mathcal{S}'(\mathbb{R}^n)$.

Proof. By the decomposition

$$f_j = \sum_{k\geq 0} \Delta_k f_j,$$

we obtain

$$f_j = \sum_{k\geq 0} \left(2^{kn}\mathcal{F}^{-1}\theta(2^k\cdot)\right) * \Delta_k f,$$

with *supp* $\theta(2^{-k}\cdot)) \cap$ *supp* $\widehat{f_j} \neq \emptyset$ for $|j-k| \leq N$, where $N = 2 + \left[\frac{\ln \mu}{\ln 2}\right]$.
We obtain

$$\left|(\mathcal{F}^{-1}\theta(2^{-k}\cdot)) * \Delta_k f\right| \leq c(\theta)|\Delta_k^{*,a} f|.$$

Then we have

$$2^{sj}|f_j| \leq c(\theta)2^{sj} \sum_{j-N\leq k\leq j+N} 2^{-sk}2^{sk}|\Delta_k^{*,a} f|.$$

The first case: $s \geq 0$. By Proposition 1.5, we get

$$\left(\sum_{j\geq 0} 2^{sjq}|f_j|^q \right)^{\frac{1}{q}} \leq c(\theta) \left(\sum_{k\geq 0} 2^{skq}|\Delta_k^{*,a} f|^q \right)^{\frac{1}{q}},$$

Proposition 4.7 allows us to conclude.

The second case: $s \leq 0$. In the similar way, we can estimate

$$c(\theta)2^{sj} \sum_{j-N\leq k\leq j+N} 2^{-sk}2^{sk}|\Delta_k^{*,a} f|,$$

by

$$c'(\theta)2^{sj} \sum_{k\leq j} 2^{-sk}2^{sk}|\Delta_k^{*,a} f|,$$

we get the result. □

4.4 Main results

The principal theorem of this chapter is the following:

Theorem 4.2. *Suppose that $\gamma(t)$ is a continuous odd function, twice continuously differentiable, increasing and convex for $t \geq 0$. Suppose also that γ'' is monotone for $t > 0$ and that there exists $C > 0$ so that $\gamma'(t) \leq Ct\gamma''(t)$ for $t > 0$. Then $\mathcal{H}$ is bounded on the localized Lizorkin-Triebel spaces $(F^s_{p,q}(\mathbb{R}^2))_{\ell^p}$, for all p, q with $5/3 < p, q < 5/2$. and $s \in \mathbb{R}$*

To prove Theorem 4.2, we need a few preparations, it suffices to show that the Lizorkin-Triebel spaces is localizable in the ℓ^p norm, for that we have the following theorem.

Theorem 4.3. *[17] Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. $F^s_{p,q}$ and $(F^s_{p,q})_{\ell^p}$ are respectively the Lizorkin-Triebel and the localized Lizorkin-Triebel spaces. Then the space $F^s_{p,q}$ is localizable in the ℓ^p norm.*

Proof. (i) $(F^s_{p,q})_{\ell^p} \hookrightarrow F^s_{p,q}$.

Prove that

$$\|f\|_{F^s_{p,q}} \leq c\|f\|_{(F^s_{p,q})_{\ell^p}}, \ \textit{for } c > 0.$$

From Definition 1.14 in Chapter 1,

$$\|f\|_{F^s_{p,q}} = \left\| \left(\sum_{j=0}^{\infty} 2^{sjq} |\Delta_j f|^q \right)^{\frac{1}{q}} \right\|_p .$$

We put

$$\Delta_j f = \sum_{k \in \mathbb{Z}^n} \tau_k \psi \cdot \Delta_j f,$$

it follows that

$$\|f\|_{F^s_{p,q}} = \left\| \left(\sum_{j=0}^{\infty} \left(\sum_{k \in \mathbb{Z}^n} 2^{sj} |\tau_k \psi \Delta_j f| \right)^q \right)^{\frac{1}{q}} \right\|_p .$$

Since, $1 \leq q$. Then from Minkowski inequality we have

$$\begin{aligned} \|f\|_{F^s_{p,q}} &= \left\| \left\| 2^{sj} \left(\tau_k \psi \Delta_j f \right)_{k \in \mathbb{Z}^n} \right\|_{\ell^q(\ell^1)} \right\|_p \\ &\leq \left\| \left\| 2^{sj} \left(\tau_k \psi \Delta_j f \right)_{k \in \mathbb{Z}^n} \right\|_{\ell^1(\ell^q)} \right\|_p . \end{aligned}$$

Consequently

$$\|f\|_{F^s_{p,q}} \le c\left(\sum_{k\in\mathbb{Z}^n}\left\|\left(\sum_{j=0}^{\infty}2^{sjq}|\tau_k\psi\Delta_j f|^q\right)^{\frac{1}{q}}\right\|_p^p\right)^{\frac{1}{p}}.$$

Hence,

$$\|f\|_{F^s_{p,q}} \le c\left(\sum_{k\in\mathbb{Z}^n}\|\tau_k\psi\cdot f\|^p_{F^s_{p,q}}\right)^{\frac{1}{p}}.$$

Thus,

$$(F^s_{p,q})_{\ell^p} \hookrightarrow F^s_{p,q}.$$

(ii) $F^s_{p,q} \hookrightarrow (F^s_{p,q})_{\ell^p}$.

Now, prove that

$$\|f\|_{(F^s_{p,q})_{\ell^p}} \le c\|f\|_{F^s_{p,q}}, \; \textit{for } c > 0.$$

In the same way, we have

$$\begin{aligned}\left(\sum_{k\in\mathbb{Z}^n}\|\tau_k\psi\cdot f\|^p_{F^s_{p,q}}\right)^{\frac{1}{p}} &= \left(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\psi\sum_{j=0}^{\infty}\Delta_j f\right\|^p_{F^s_{p,q}}\right)^{\frac{1}{p}}\\ &= \left(\sum_{k\in\mathbb{Z}^n}\left\|\sum_{j=0}^{\infty}\Delta_j f\tau_k\psi\right\|^p_{F^s_{p,q}}\right)^{\frac{1}{p}}.\end{aligned}$$

From Proposition 4.5, it follows that

$$\left(\sum_{k\in\mathbb{Z}^n}\|\tau_k\psi\cdot f\|^p_{F^s_{p,q}}\right)^{\frac{1}{p}} \le c\left(\sum_{k\in\mathbb{Z}^n}\left\|\tau_k\psi\left(\sum_{j=0}^{\infty}2^{sjq}|\Delta_j f|^q\right)^{\frac{1}{q}}\right\|_p^p\right)^{\frac{1}{p}} \tag{4.21}$$

Since L^p is a space localizable in the ℓ^p norm, then it holds that

$$\left(\sum_{k\in\mathbb{Z}^n}\|\tau_k\psi\cdot f\|^p_{F^s_{p,q}}\right)^{\frac{1}{p}} \le c\|f\|_{F^s_{p,q}}.$$

Thus,

$$F^s_{p,q} \hookrightarrow (F^s_{p,q})_{\ell^p}.$$

□

For more details on the proof of Theorem 4.3 see [17].

4.4.1 Proof of Theorem 4.2

The proof follows from Theorem 4.3 and by applying Theorem 3.4.

□

Now, in the context of Theorem 4.2 and by applying Theorem 3.5 we have the following result:

Theorem 4.4. *Let $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be a well-curved curve in $\mathbb{R}^n$ with $\Gamma(0) = 0$. Then there exists $c > 0$ such that*

$$\|\mathcal{H}f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}} \leq c\|f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}},$$

for all $s \in \mathbb{R}$, and all p, q with $1 < p, q < \infty$, $n \geq 2$.

And in the context of Theorem 4.2 and by applying Theorem 3.6, we have the following result:

Theorem 4.5. *(1) Let $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be a polynomial function. Then there exists $c > 0$ such that*

$$\|\mathcal{H}f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}} \leq c\|f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}}.$$

for all $s \in \mathbb{R}$, and all p, q with $1 < p, q < \infty$, and $n \geqslant 2$.

(2) Let the image of $\Gamma : \mathbb{R} \to \mathbb{R}^n$ be of finite type at 0. Then there exists $c > 0$ such that

$$\|\mathcal{H}f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}} \leq c\|f\|_{(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}}.$$

for all $s \in \mathbb{R}$, and all p, q with $1 < p, q < \infty$, $n \geqslant 2$.

General Conclusion and Future Research

The objective of this work is to study the boundedness of the Hilbert transform along curves on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$, as well as on Localized Lizorkin-Triebel spaces $(F^s_{p,q}(\mathbb{R}^n))_{\ell^r}$ with $r = p$, on some appropriate curve $\Gamma(t)$ on $\mathbb{R}^n$.

By using the Littlewood-Paley decomposition, we have proved that the boundedness of the Hilbert transform on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ can be obtained by its $L^p(\mathbb{R}^n)$-boundedness, where $s \in \mathbb{R}$, and $p, q \in (1, \infty)$.

In the other hand, we use the boundedness of the Hilbert transform on the spaces of vector-valued functions $L^p(\mathbb{R}^n; \ell^q)$ to prove the boundedness on Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$, and more than on the localized Lizorkin-Triebel spaces $(F^s_{p,q}(\mathbb{R}^n))_{\ell^r}$, whith $r = p$.

The totality of this work is fruitful by results appearing in:

- Chapter 2 is a paper published in Journal of Carpathian Mathematical Publication "Boundedness of the Hilbert Transform on Besov Spaces". (2020), **12** (2), 443-450.(see [24])

- Chapter 3 and Chapter 4 are papers in preparation.

Open questions

- Is the Hilbert transform bounded on the localized Lizorkin-Triebl spaces $(F^s_{p,q}(\mathbb{R}^n))_{\ell^p}$ for $r \neq p$?
- Is the Hilbert transform bounded on the variable Besov spaces $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$, or the variable Lizorkin-Triebel spaces $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$? Where $B^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and $F^{s(\cdot)}_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ are with variable exponents.
- Is the Hilbert transform bounded on the generalized Lizorkin-Triebl spaces $(F^{v}_{p,q}(\mathbb{R}^n)$?

Abstract

In this Thesis, we study the boundedness of the Hilbert transform along curves $\Gamma(t)$ on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, for that, we use the Littlewood-Paley theory to prove that the boundedness on Besov spaces $B^s_{p,q}(\mathbb{R}^n)$ can be obtained by its $L^p(\mathbb{R}^n)$-boundedness, for $s \in \mathbb{R}, p, q \in (1, \infty)$, and $\Gamma(t)$ is an appropriate curve in $\mathbb{R}^n$. Furthermore, we study the boundedness on Lizorkin-Triebel spaces $F^s_{p,q}(\mathbb{R}^n)$, and more than on the Localized Lizorkin-Triebel spaces $(F^s_{p,q}(\mathbb{R}^n))_{\ell^r}$ with $r = p$, where we use the boundedness of such a transform on the spaces of vector-valued functions $L^p(\mathbb{R}^n; \ell^q)$.

Keywords: Singular integrals, Hilbert transform, Littlewood-Paley decomposition, Besov spaces, Lizorkin-Triebel spaces, Pointwise multipliers, Localized Lizorkin-Triebel spaces.

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Résumé.

Dans cette Thèse, nous étudions la continuité de l'opérateur de Hilbert le long d'une courbe $\Gamma(t)$ sur les espaces de Besov $B^s_{p,q}(\mathbb{R}^n)$, pour cela, nous utilisons la théorie de la décomposition de Littlewood-Paley pour prouver que cette continuité sur les espaces de Besov $B^s_{p,q}(\mathbb{R}^n)$ peut être obtenu par sa $L^p(\mathbb{R}^n)$-continuité, où $s \in \mathbb{R}, p, q \in (1, \infty)$ et $\Gamma(t)$ est une courbe appropriée dans $\mathbb{R}^n$. De plus, Nous étudions la continuité sur les espaces de Lizorkin-Triebel $F^s_{p,q}(\mathbb{R}^n)$, ainsi que sur les espaces de Lizorkin-Triebel localisés $(F^s_{p,q}(\mathbb{R}^n))_{\ell^r}$ avec $r = p$, où nous utilisons la continuité de cet opérateur sur les espaces des fonctions $L^p(\mathbb{R}^n; \ell^q)$ à valeurs vectorielles.

Mots-Clés: Intégrales singulières, L'opérateur de Hilbert, La décomposition de Littlewood-Paley, Espaces de Besov, Espaces de Lizorkin-Triebel, Multiplicateurs ponctuels, Espaces de Lizorkin-Triebel localisés.

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