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THEME

Étude de la stabilité d'un système élastique poreux

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Abstract

In this memory we cosidered two elastic systems with the presence of diferent mechanisms of dissipation. In chapter 1 we recall some basic knowledge in functionel anlysis. In chapter 2 ,we consider one dimensional porous elastic system with delay term and we proof the exponential decay results the solutions. In chapter 3, we study a one dimensional porous-elastic system with the presence o both memory and distributed delay term. Using the weel known energy method with Lyapunov functionals approach. we proof a general decay result. .

Keywords : Porous-elastic system, memory term, destributed delay term, stability, delay term, energy method.

ملخص

في هذه المذكرة ندرس استقرار الحل لجملتين مساميتين، وقد قسمنا المذكرة إلى ثلاث فصول.

نذكر في الفصل الأول ببعض المفاهيم الأساسية في التحليل الدالي. نقوم في الفصل الثاني بإثبات الاستقرار الأسي لحل جملة مسامية خطيةأحادية البعد بإضافة حد التأخير وذلك باستعمال طريقة الطاقة.

أما في الفصل الثالث ندرس استقرار حل جملة مسامية زائدية بإضافة حدي الذاكرة و التأخير بتوزيع و ذلك باستخدام طريقة الطاقة.

الكلمات المفتاحية: جملة مسامية، استقرار، حد التأخير، حد الذاكرة، طريقة الطاقة، حد التأخير بتوزيع.

Résumé

Dans ce mémoire nous avons considéré deux systèmes élastiques avec la présence de différents mécanismes de dissipation. Dans le chapitre 1, nous rappelons quelques connaissances de base en analyse fonctionnelle. Dans le chapitre 2, nous considérons un système élastique poreux unidimensionnel avec un terme de retard et nous prouvons les résultats de la décroissance exponentielle des solutions. Dans le chapitre 3, nous étudions un système unidimensionnel poreux avec des termes de mémoire et de retard distribué. En utilisant la méthode d'énergie avec l'approche des fonctionnelles de Lyapunov. Nous prouvons un résultat général de décroissance

Mots clés : Système élastique poreux, terme de retard distribué, mémoire infinie, stabilité, méthode d'énergie, terme de retard.

Dedication

I dedicate this work

To my parents who have been my source of inspiration and provided me with their encouragement, love, understanding and prayers.

To my beloved family *HOUARI* & *ATTIA*.

To all those who have been supportive, caring and patient, I dedicate this simple work.

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Abdelhak ATTIA

Contents

Introduction

The subjet of this memory is the study of the stability of two porous-elastic problems, The systems that we treated here are the following :

The theory of porous materials is an important generalization of the classical theory of elasticity for the treatment of porous solids in which the skeletal materials is thermoelas- tic and the interstices are void of material. This theory deals with materials containing small pores or voids. The basic premise underlying this theory is the concept that the bulk density is the product of two fields, the matrix material density field and the volume fraction field. This representation of the bulk density introduces an additional degree of kinematic freedom in the theory and was employed previously by Goodman and Cowin [33] to overcome the failure of the classical theory of elasticity to describe the deformation produced by the microstructure contribution. The theory of granular materials developed by Goodeman and Cowin [33], equally valid for porous materials, was motivated by physical grounds. In this theory they introduced a higher order stress and body force to account for energy flux and energy supply associated with the time rate of volume fraction. Terms of this type are also contained in the higher order elasticity theories developed by Mindlin [41], Toupin [66] and Green and Rivlin [34].

Nunziato and Cowin [46] employed the same balance equations developed by Goodman and Cowin [33] and presented a nonlinear theory for the behavior of porous solids. This theory admits both finite deformations and nonlinear constitutive relations. Jaric and Goluboviç [26] and Jariç and Rankoviç [27] studied the nonlinear theory of thermoelastic materials with voids. Cowin and Nunziato [9] developed a linear theory of elastic materials with voids to study mathematically the mechanical behavior of porous solids. An extension of this theory to linear thermoelastic bodies was proposed by Ie san [29]. In addition, Ie san [28],[30] added the microtemperature elements to this theory.

On the basis of micromorphic continua theory, Grot [35] developed a theory of thermodynamics of elastic material with inner structure whose microelements, in addition to microdeformations, possess microtemperatures. The importance of materials with microstructure has been demonstrated by the huge number of papers appeared in different fields of applications such as petroleum industry, material science, biology and many others.

Since this type of material has both microscopic and macroscopic structures, scientists have investigated the coupling and how strong it is. In addition, an increasing interest has been paid by mathematicians to analyze the longtime behavior of the solutions of thermoe-lastic and porous problems. One of the first studies, in this sense, was the thermoelastic coupling proposed by Slemrod [64]. As a result it was seen that in the one-dimensional case the solutions decay exponentially. Since then, many problems were studied by considering different dissipation mechanisms at the microscopic and/or the macroscopic levels. Many papers have been published where the authors tried to determine the type, as well as, the rate of decay of solutions in porous elasticity with voids.

ourmain results this memory can be summarised as follows :

Chapter 1. In this chapter, we recall some basic knowledge in functional analysis, most of wich will be used in the subsequent chapter .

Chapter 2. In this chapter, we consider one-dimensional porous-elastic system with decay term in the second equation and using the energy method, we have proved the exponential decay result of the solutions.

Chapter 3. In this chapter, we consider a one-dimensional porous-elastic system with the presence of both memory and distributed decay terms in the second equation . Using the well know energy methode combined with lyaponov functionals approch, we a general decay result of the solution.

Chapter 1

Preliminaries

In this chapter, we recall some basic knowledge in fonctional analysis, moste of wich will be used in the subseqent chapter. The reader can easily find the detailed in the related literature, see, e.g.,[1], [9], [51], [68]

1.1 Functional Spaces

We denote by $\mathbb{R}^n$ the Euclid space, $\Omega \subset \mathbb{R}^n$ is bonded smoth domain,$C^k(\Omega)$ is the k^{th} differentiable continuous function space in Ω, $C^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space in Ω, $C_c^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space with compact support in Ω

Definition 1.1. *Let X be a vector space over the filed $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then a semi-norm on X is a function $\|.\| : X \longrightarrow \mathbb{R}$, such that :*
a) $\|x\| \geqslant 0$ for all $x \in X$,
b) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$,
c) $\|x+y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$.
A norm on X is a semi-norm which also satisfies :
d) $\|x\| = 0 \Rightarrow x = 0$. A vector space X together with a norm $\|.\|$ is called a normed linear space, a normed vector space or simply, a normed space.

Definition 1.2. *Let X be a vector space over the filed $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then a semi-norm on X is a function $\|.\| : X \longrightarrow \mathbb{R}$, such that :*
a) $\|x\| \geq 0$ for all $x \in X$,
b) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$,

c) $\|x+y\| \leqslant \|x\|+\|y\|$ *for all* $x, y \in X$.
A norm on X *is a semi-norm wich also satisfies :*
d) $\|x\|=0 \Rightarrow x=0$. *A vector space* X *toghether with a norm* $\|.\|$ *is called a normed linear space, a normed vector space or simply, a normed space.*

Definition 1.3. *(Convergent and Cauchy sequences). Let* X *be a normed space, and let* $\{x_n\}_{n\in\mathbb{N}}$ *be a sequence of elements of* X.
a) $\{x_n\}_{n\in\mathbb{N}}$ *converges to* $x \in X$ *if*

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

i.e. if

$$\forall \varepsilon > 0; \exists N > 0, \forall n \geq N, \|x_n - x\| < \varepsilon.$$

b) $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence if*

$$\forall \varepsilon > 0; \exists N > 0, \forall m, n \geq N, \|x_m - x_n\| < \varepsilon.$$

Normed spaces in which every Cauchy sequence is convergent are called complet normed spaces. In general a normed space is not complete.

Definition 1.4. *(Banach Spaces). A normed spaces is called a Banach space if it is complet i.e. if any Cauchy sequence inside the space converges to a point of the space. Its dual space* X' *is the linear space of all contnuous linear functional* $f : X \to \mathbb{R}$.

Proposition 1.1. X' *equipped with the norm* $\|.\|_{X'}$ *defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}$$

is olso a Banach space.

Remark 1.1. *From* X' *we construct the bidual or secend dual* $X'' = (X')'$. *Futhermore, with each* $u \in X$ *we can define* $\varphi(u) \in X''$ *by* $\varphi(u)(f) = f(u)$, $f \in X'$, *this satisfies clearly* $\|\varphi(x)\| \leq \|u\|$. *Moreover, for each* $u \in X$ *there is an* $f \in X'$ *with* $f(u) = \|u\|$ *and* $\|f\| = 1$, *so it follows that* $\|\varphi(x)\| = \|u\|$.

Definition 1.5. *Since* φ *is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X onto a closed subspace of X'', we denote this by

$$X \hookrightarrow X''.$$

Definition 1.6. *if φ (in the above definition) is onto X'' we say X is reflexive, $X \approxeq X''$*

1.1.1 Weak and weak star topologies:

1.1.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analyse, known as Hilbert spaces. Then, we most give some impotant result on these spaces here.

Definition 1.7. *A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 1.1. *Let $(x_n)_{n\in\mathbb{N}}$ is a bounded squence in the Hilbrt sâce H, then it possess a subsequence which converges in the weak topology of H.*

Theorem 1.2. *In the Hilbrt space, all sequence which converges in the weak topology is bounded.*

Theorem 1.3. *Let $(x_n)_{n\in\mathbb{N}}$ be sequence which converges to x, in the weak topology and $(y_n)_{n\in\mathbb{N}}$ is an other squence which converge weakly to y, then*

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

Proposition 1.2. *Let X and Y be tow Hilbert space, let $(x_n)_{n\in\mathbb{N}} \in X$ be a sequence which conveges weakly to $x \in X$, let $A \in \mathcal{L}(X, Y)$. Then, the sequence $(A(x_n))_{n\in\mathbb{N}}$ converges to $A(x)$ in the weak topology of Y.*

Theorem 1.4. *(The Lax-Milgram Theorem)*
Let X be a Hilbert space and let $a : X \times X \to \mathbb{R}$ be a bilinear functional. Asume that there existe tow constants $C < \infty, \alpha > 0$ such that:
(i) $|a(u, v)| \leq C\|u\|.\|v\|$ *for all* $(u, v) \in X \times X$ *(continuity);*
(ii) $a(u, u) \geq \alpha\|u\|^2$ *for all* $u \in X$ *(coerciveness).*
Then, for evry $f \in X^*$ (the dual space of X), there exists a unique $u \in X$ such that $a(u, v) = \langle f, v\rangle$ for all $v \in X$.

1.1.3 $L^p(\Omega)$ spaces

Definition 1.8. *Let* $1 \leq p \leq \infty$, *and let* Ω *be on open domain in* $\mathbb{R}^n$, $n \in \mathbb{N}$. *Define the standard Lebesgue space* $L^p(\Omega)$ *by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{N} : f \ \textit{is measurable and} \int_\Omega |f(x)|^p dx < \infty \right\}$$

Notation 1 : for $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

. If $p = \infty$, we have

$$L^p(\Omega) = \{ f : \Omega \to \mathbb{R} : f \ \textit{is measurable and there exists } C \ \textit{suche that}, |f(x)| \leq C \ \textit{in} \ \Omega \}$$

Notation 2 : Let $1 \leq p \leq \infty$,we denote by q the conjugate of p i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 1.5. *It is well known that* $L^p(\Omega)$ *supplied with the norm* $\|.\|_p$ *is a Banach space, for all* $1 \leq p \leq \infty$

Remark 1.2. *In particularly, when* $p = 2$, $L^2(\Omega)$ *equipped with the inner product*

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) g(x) dx,$$

is a Hilbert space .

Theorem 1.6. *For* $1 < p < \infty$, $L^p(\Omega)$ *is reflexive space.*

1.1.4 Sobolev space $W^{m,p}(\Omega)$

Definition 1.9.

i) Let $m \in \mathbb{N}$ *and* $p \in [0, \infty]$. *The* $W^{m,p}(\Omega)$ *is the space of all* $f \in L^p(\Omega)$, *defined as*

$$W^{m,p}(\Omega) = \{ f \in L^p(\Omega), \textit{such that } \partial^\alpha f \in L^p(\Omega) \textit{for all } \alpha \in \mathbb{N}^m \}$$

such that $|\alpha| = \sum_{j=1}^n \alpha_j \leq m$ *where,* $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} .... \partial_n^{\alpha_n}$.
ii) if $f \in W^{m,p}(\Omega)$, *we define its norm to be*

$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} (\sum_{|\alpha|<k} \int_\Omega |D^\alpha f|^p dx)^{\frac{1}{p}} & ; (1 \leq p < \infty), \\ \sum_{|\alpha|<k} ess \sup |D^\alpha f| & ; (p = \infty) \end{cases}$$

Definition 1.10. *We denote by*

$$W_0^{m,p}(\Omega)$$

the closure of $C_0^\infty(\Omega)$ *in* $W^{m,p}(\Omega)$

Remark 1.3. *i) if* $p = 2$ *we usully write*

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Supplied with the norm

$$\|f\|_{H^m} = \left(\sum_{|\alpha| \leq m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}}$$

The letter H *is used, since - as we will see -* $H^m(\Omega)$ *is a Hilber space.*
with usual scalar product

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx$$

Note that $H^0(\Omega) = L^2(\Omega)$

Theorem 1.7. .
1. $H^m(\Omega)$ *supplied with inner product* $\langle ., . \rangle_{H^m(\Omega)}$ *is Hilbert space.*
2. *If* $m \geq m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$.

Theorem 1.8. *Assume that* Ω *is an open domain in* $\mathbb{R}^n, n \geq 1$, *with smooth boundary* Γ. *Then,*
i) if $1 \leq p \leq n$, *we have* $W^{1,p} \subset L^q(\Omega)$, *for every* $q \in [p, p^*]$, *where* $p^* = \dfrac{np}{n-p}$.
ii) if $p = n$ *we have* $W^{1,p} \subset L^q(\Omega)$, *for every* $q \in [p, \infty)$.
iii) if $p > n$ *we have* $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, *where* $\alpha = \dfrac{p-n}{p}$.

1.1.5 $L^p(0,T,X)$ space

Definition 1.11. *Let X be a Banach space, denote by $L^p(0,T,X)$ the space of measurable functions*

$$\begin{aligned} f :]0,T[&\to X \\ t &\mapsto f(t) \end{aligned}$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt\right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \qquad 1 \leq p < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t\in]0,T[} ess\|f(t)\|_X$$

Theorem 1.9. *$L^p(0,T,X)$ equipped with the norm $\|.\|_{L^p(0,T,X)}$ is a Banach space .*

Proposition 1.3. *Let X be a reflexive Banach space, X' it's dual, and $1 \leq p < \infty$, $1 \leq q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0,T,X)$ is identify algebraically and topologically with $L^q(0,T,X')$*

1.2 Some useful inequalities

In this section, we shall recall some inqualities which will be used in the supsequent chapters.

1.2.1 Young inequalities

Theorem 1.10. *Let $1 < p,q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, a,b > 0$$

Theorem 1.11. *(Young inequality with ε) Let $1 < p,q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,then*

$$ab \leq \varepsilon\frac{a^p}{p} + \frac{1}{\varepsilon^{\frac{q}{p}}}\frac{b^q}{q}, a,b > 0$$

The Young inequality has several variants in the following.

Corollary 1. *Let* $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. *Then*

i) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \dfrac{a}{p} + \dfrac{b}{q}$.

ii) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \dfrac{a}{p\varepsilon^{\frac{1}{q}}} + \dfrac{b\varepsilon^{\frac{1}{p}}}{q} \qquad ,\forall \varepsilon > 0.$

iii) $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \qquad ,0 < \alpha < 1.$

1.2.2 Holder inequalities

Theorem 1.12. *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,
then if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, *we have*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} . \|g\|_{L^q(\Omega)}$$

Theorem 1.13. *(Generalized Holder inequality) Let* $1 \leq p_1, ..., p_m \leq \infty$,
$\frac{1}{p_1} + ... + \frac{1}{p_m} = 1$, *then if* $f_k \in L^{p_k}(\Omega)$ *for* $k = 1, ..., m$, *we have*

$$\int_{\Omega} |f_1....f_m| dx \leq \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\Omega)}$$

Remark 1.4. *We have the corresponding weighted Holder inequality of the integral form. Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$,$\omega(x) > 0$ *on* Ω. *Then*

$$\int_{\Omega} |fg| \omega(x) dx \leq \left(\int_{\Omega} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |g(x)|^q \omega(x) dx \right)^{\frac{1}{q}}.$$

1.2.3 Minkowski inequality

Theorem 1.14. *Assume* $1 \leq p \leq \infty$, $f, g \in L^p(\Omega)$, *then*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

If $0 \neq p < 1$, *then*

$$\|f + g\|_{L^p(\Omega)} \geq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

In the applications, the integral form the Minkowski inequality is used frequentely.

1.2.4 Poincaré inequality

In this subsection, we shall recall the Poincar inequality in different forms.

Theorem 1.15. . *Let Ω be a bounded domain in $\mathbb{R}^n$ and $f \in H_0^1(\Omega)$. Then there is a positive constant C such that*

$$\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}, \qquad \forall f \in H_0^1(\Omega)$$

Theorem 1.16. *Let Ω be a bounded domain of C^1 in $\mathbb{R}^n$. There is a positive constant C, such that for any $f \in H_0^1(\Omega)$.*

$$\|f - \tilde{f}\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$$

Where $\tilde{f} = \frac{1}{|\Omega|}\int_\Omega f(x)dx$ is the integral average of f over Ω, and $|\Omega|$ is the volume fo Ω.

Theorem 1.17. *Under assumption of Theorem (1.16) for any $f \in H^1(\Omega)$, we have*

$$\|f\|_{L^2(\Omega)} \leq C\left(\|\nabla f\|_{L^2(\Omega)} + |\int_\Omega f dx|\right).$$

1.3 Notion of semigroups

In this section, we recall some basic knowledge in semigroups,most of whiche will be used in the subsequent chapters. Ageneral reference to this topic is [?],

1.3.1 C_0-semigroups of Linear Operators

Definition 1.12. *(semigroups)*
Let X be a Banach space, the one-parametre family $S(t), 0 \leq t < \infty$ from X to X is called a semigroups if
(i) $S(0) = I$ (I is the identity operateur on X),
(ii) $S(t+s) = S(t) \circ S(s)$ for every $t, s \geq 0$ (the semigroup property).

Definition 1.13. *The linear operator A defined by*

$$D(A) = \left\{x \in X : \lim_{t \to 0^+} (S(t)x - x)/t \quad \textit{exists}\right\}$$

and

$$Ax = \lim_{t\to 0^+}(S(t)x - x)/t = \frac{d(S(t)x)}{dt}|_{t=0} \quad for \;\; all \quad x \in D(A)$$

is called the infinitesimal generator of the semigroup $S(t)$, $D(A)$ *is called the domain of* A.

Definition 1.14. *(C_0-Semigroups).*
A semigroup $S(t)$,$0 \le t < \infty$,*from* X *to* X *is called a strong continuous semigroup of bounded lineaar operators if*

$$\lim_{t\to 0^+} S(t)x = x \quad for \;\; all \quad x \in X,$$

or

$$\lim_{t\to 0^+} \|S(t)x - x\| = 0 \quad for \;\; all \quad x \in X.$$

i.e $S(t)$ C_0-*semigroup.*

Definition 1.15. *A semigroup* $S(t)$,$0 \le t < \infty$ *is called a semigroup of contraction if there exists a constant* $\alpha > 0$ $(0 < \alpha < 1)$ *such that for all* $t > 0$,

$$\|S(t)x - S(t)y\| \le \alpha\|x - y\|, \quad for \;\; all \quad x, y \in X.$$

1.3.2 Hille-Yoshida Theorem

Definition 1.16. *An unbonded linear operator* $A : D(A) \subset H \to H$[1] *is said to be monotone*[2] *if it satisfies*

$$\langle Av, v\rangle \ge 0 \qquad \forall v \in D(A).$$

It is called maximal monotone if, in addition; $R(I + A) = H$ *i.e*

$$\forall f \in H \;\; \exists u \in D(A) \quad such \;\; that \;\; u + Au = f.$$

Proposition 1.4. *Let* A *be a maximal monotone operator. Then*
1. $D(A)$ *is dense in* H.
2. A *is closed operator.*
3. For evry $\lambda > 0, (I + \lambda A)$ *is bijective from* $D(A)$ *onto* H, $(I + \lambda A)^{-1}$ *is a bounded operator, and* $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \le 1$.

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[1] H denotes a Hilbert space

[2] Some authors say that A is accretive or $-A$ is dissipative.

Theorem 1.18. *(Hille-Yosida) Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad on [0, +\infty) \\ u(0) = u_0. \end{cases}$$

Chapter 2

General decay of solutions in one-dimensional porous-elastic system with delay term

2.1 Introduction and motivation

The elasticity problems are very interesting to figure out this decade. Many authors from various fields have investigated these pertinent problems, and they were exactly attracted by the temporal decay behavior of the solutions. This interest has given many results that can be found in the literature. In the one-dimensional case, the conbination of the elastic equations with thermal consequences causes a negative exponential to control the decay of solutions (Jiang and Racke, 2000; Quintanilla and Racke, 2003; Slemrod, 1981).

Originally the one-dimensional porous-elastic model has been studied by different authors as follows:

$$\begin{cases} \rho_0 u_{tt} = \mu u_{xx} + \beta \varphi_x, \text{ in } (0,l) \times (0,L) \\ \rho_0 k \varphi_{tt} = \alpha \varphi_{tt} - \beta u_x - \tau \varphi_t - \xi \varphi, \text{ in } (0,l) \times (0,L) \, . \end{cases}$$

The first contribution in this direction was in 2003 by Quintanilla [50]. To be more precised, which was developed by Goodman and Cowin in [32], they showed an yield of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to the usual elastic effects, the materials with voids possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This latter concept was introduced in the pioneered work of Nunziato and Cowin in [45], when they advanced nonlinear theory of elastic materials with voids. The importance of such materials could not be over-emphasized as it has resulted in the huge number of papers published in different fields of human endeavors most importantly, in petroleum industry, material science, soil mechanics, foundation engineering, powder technology, biology and others. We invite the reader to [17, 18] and the references therein for more details. The basic evolution equations for one-dimensional theories of porous materials with memory effect are given by

$$\begin{cases} \rho u_{tt} = T_x \\ J\phi_{tt} = H_x + G, \end{cases} \tag{2.1}$$

where T is the stress tensor, H is the equilibrated stress vector, and G is the equilibrated body force. The variables u and ϕ are the displacement of the solid elastic material, and the volume fraction, respectively.

The constitutive equations are:

$$\begin{cases} T = \mu u_x + b\phi \\ H = \delta\phi_x - \int_0^t g\left(t-s\right)\phi_x\left(s\right)ds \\ G = -bu_x - \xi\phi \end{cases} \tag{2.2}$$

Tijani A. Apalara is substituting (2.2) into (2.1) is concerned

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \text{ in } (0,1)\times(0,\infty) \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g\left(t-s\right)\phi_{xx}\left(x,s\right)ds = 0, \text{ in } (0,1)\times(0,\infty) \end{cases} \tag{2.3}$$

A porous-elastic system with memory term and Neumann-Dirichlet boundary conditions where g is the relaxation function it has been proved a general decay result, for more detail (see [8]).

In [50], Quintanilla considered a one-dimensional linear equations of an homogeneous and isotropic porous elastic solid

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x, x\in(0,L), t>0 \\ J\phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \tau\phi_t, x\in(0,L), t>0 \end{cases} \tag{2.4}$$

with initial and mixed boundary conditions, and supposed that the damping in the porous equation $(-\tau\phi_t)$ is not p enough to obtain an exponential decay but only a slow (nonexponential) decay can be obtained. To improve this decay, several other damping mechanisms were considered.

In [14], Casas and Quintanilla have considered the following system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x - \beta\theta_x, x \in (0, L), \quad t > 0 \\ J\phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi + m\theta - \tau\phi_t, x \in (0, L), \quad t > 0 \\ c\theta_t = k\theta_{xx} - \beta u_{xt} - m\phi_t, x \in (0, L), \quad t > 0, \end{cases} \tag{2.5}$$

where θ is the temperature difference, with initial and Dirichlet-Neumann boundary conditions. They applied the semigroup theory and the method advanced by Liu and Zheng in[10] to establish the exponential decay of the solutions.

Later, with $\tau = 0$ (absence of porous dissipation), in [15] the same authors have proposed that the heat effect alone is not strong sufficient to bring about an exponential decay but only a slow decay could be established. However, the heat effect together with micro-temperature created an exponential decay result. Similarly, when $\tau = 0$ and γu_{xxt} is added to the first equation in (3.1), in[52] Pamplona et al. have proved that the system lacks exponential stability. However, by taking some regular initial data, a polynomial stability is obtained. Also, for $\tau = 0$.

In[63] Soufyane et al. have considered (3.1) with the following boundary conditions:

$$\begin{cases} u(0, t) = \phi(0, t) = \theta(0, t) = \theta(L, t) = 0, t \geq 0, \\ u(L, t) = -\int_0^t g_1(t - s)\left[\mu u_x(L, s) + b\phi(L, s)\right] ds, t \geq 0, \\ \phi(L, t) = -\delta\int_0^t g_2(t - s)\phi_x(L, s)\, ds, t \geq 0, \end{cases}$$

where g_1 and g_2 are positive decreasing functions. They obtained a general decay result, in which the usual exponential and polynomial decay rates are just special cases. We refer the reader to [38, 40, 53, 60, 61] and the references therein for more results. The viscoelastic damping (see [24] for details) is (according to the Boltzmann Principle) represented by a memory term in the form of a convolution which arises in the constitutive equation between the stress $\sigma(x, t)$ and the strain $\epsilon(x, t)$

$$\sigma(x, t) = \epsilon(x, t) + \int_0^t g(t - s)\epsilon(x, s)\, ds.$$

This type of viscoelastic dissipation (see [8] for details) could be said to coincide to viscosity with null initial history because it is assumed that the strains have been zero for $-\infty < t < 0$ or, equivalently, if any past strains have occurred sufciently long ago that the efect is trivial In other words, there will be a time prior to which all the strains which have previously occurred will have a trivial contribution. Thus, an experiment generally starts at some time $(t = 0)$ when the material is free of stresses.

2.2 Statement of the problem and main results

In this work, we are interested in the following problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \text{ in } (0,1) \times (0,\infty) \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\phi_t(x,t) + \mu_2\phi_t(x,t-\tau) = 0, \text{ in } (0,1) \times (0,\infty), \end{cases} \tag{2.6}$$

a porous-elastic system with delay term acting only on the porous equation together with the initial data

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ \phi(x,0) = \phi_0(x), \ \phi_t(x,0) = \phi_1(x), x \in (0,1), \tag{2.7}$$

and Neumann-Dirichlet boundary conditions

$$u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, t \geq 0 \tag{2.8}$$

Here, u is the longitudinal displacement,ϕ is the volume fraction of the solid elastic material, and $\rho, \mu, b, J, \delta, \xi$ are constitutive constants which are positive with μ, ξ, b satisfying $\mu\xi > b^2$, $\tau > 0$ represents the time delay, μ_1 and μ_2 are positive constituve constants. Our aim is to establish an explicit and a general decay rate result for the energy of system (2.6) in case of the same speed of propagation in the two equations of the system, that is

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \tag{2.9}$$

We introduce the new variable

$$z(x,\rho,t) = \phi_t(x, t - \tau\rho), \ x \in (0,1), \ \rho \in (0,1), \tau \in (0,1), t > 0.$$

Then, we obtain

$$\tau z_t(x,\rho,t)+z_\rho(x,\rho,t)=0,\ x\in(0,1),\ \rho\in(0,1),\ \tau\in(0,1),t>0 \tag{2.10}$$

Our purpose in this work is to give a general decay result of solutions in one dimensional porous-elastic system with memory and delay term for which exponential and polynomial decay results cases, our result is new and improves previous results in the literature.

Time-delay arised in many applications because most phenomena naturally depend-not only on the present state but also on some past occurrences. In recent years,the control of PDEs withtime delay effects has become an activearea of research,see for example [63] and references therein. In many cases it was shown that delay is a source of instability unless additional condition or control terms are used,see [22] Therefore, the stability issue of systems with delay is of theoretical and practical great importance. It is well know that, in the single wave equation, if $\mu_2=0$ that is, in absence of a decay, the energy of system exponentially decays (see [8]) On the contrary, if $\mu_1=0$, that is, there exists only the delay part in the interior, the system becomes unstable (see [22]). It is shown that a small delay in a boundary control can turn such a well-behaved hyperbolic system into a wild one and therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [44, 48, 67]). In what follows, we consider $(u,\ \phi)$ to be a solution of system (2.6)-(2.8) with the regularity needed to justify the calculations in this paper. Meanwhile, from (2.6) and (2.8), it follows that

$$\frac{d^2}{dt^2}\int_0^1 u(x,t)\,dx=0 \tag{2.11}$$

So, by solving (3.8) and using the initial data of u, we get

$$\int_0^1 u(x,t)\,dx=t\int_0^1 u_1(x)\,dx+\int_0^1 u_0(x)\,dx$$

Consequently, if we let

$$\bar{u}(x,t)=u(x,t)-t\int_0^1 u_1(x)\,dx-\int_0^1 u_0(x)\,dx, \tag{2.12}$$

we get

$$\int_0^1 \bar{u}(x,t)\,dx = 0, \forall t \geq 0.$$

Therefore, the use of Poincare's inequality for $\bar{u}$ is justified. In addition, simple substitution shows that $\left(\bar{u},\phi\right)$ satisfies system (2.6) with initial data for $\bar{u}$ given as

$$\bar{u_0}(x) = u_0(x) - \int_0^1 u_0(x)\,dx \text{ and } \bar{u_1}(x) = u_1(x) - \int_0^1 u_1(x)\,dx.$$

Henceforth, we work with $\bar{u}$ instead of u but write u for simplicity of notation. We introduce the following spaces:

$$H = H_*^1(0,1) \times L_*^2(0,1) \times H^1(0,1) \times L^2(0,1)$$

and

$$\widetilde{H} = \Phi_0 \in \left[H_*^2(0,1) \cap H_*^1(0,1)\right] \times H_*^1(0,1) \times \left[H^2(0,1) \cap H_0^1(0,1)\right] \times H_0^1(0,1),$$

where

$$\begin{aligned} L_*^2(0,1) &= \left\{\psi \in L^2(0,1) : \int_0^1 \psi(x)\,dx = 0\right\}, H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1), \\ H_*^2(0,1) &= \left\{\psi \in H^2(0,1) : \psi_x(0) = \psi_x(1) = 0\right\}. \end{aligned}$$

For $\Phi = (u, u_t, \phi, \phi_t)$, we have the following existence and regularity result

Proposition 2.1. *For all $\Phi_0 \in H$, the system (2.6)-(2.8) has a unique global (weak) solution*

$$u \in C\left(\mathbb{R}^+; H_*^1(0,1)\right) \cap C^1\left(\mathbb{R}^+; L_*^2(0,1)\right), \ \phi \in C\left(\mathbb{R}^+; H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}^+; L^2(0,1)\right).$$

Moreover, if $\Phi_0 \in \widetilde{H}$, then the solution satisfies

$$\begin{aligned} u &\in L^\infty\left(\mathbb{R}^+; H_*^1(0,1) \cap H_*^1(0,1)\right) \cap W^{1,\infty}\left(\mathbb{R}^+; H_*^1(0,1)\right) \cap W^{2,\infty}\left(\mathbb{R}^+; L_*^2(0,1)\right) \\ \phi &\in L^\infty\left(\mathbb{R}^+; H^2(0,1) \cap H_0^1(0,1)\right) \cap W^{1,\infty}\left(\mathbb{R}^+; H_0^1(0,1)\right) \cap W^{2,\infty}\left(\mathbb{R}^+; L^2(0,1)\right). \end{aligned}$$

2.3 General Decay for $\mu_2 < \mu_1$

In this section, we state and prove our decay result for the energy of the system (2.6)-(2.8) by using the multiplier technique. To achieve our main goal, we need the following lemmas.

For ξ satisfying

$$\tau\mu_2 < \zeta < \tau\left(2\mu_1 - \mu_2\right). \tag{2.13}$$

Lemma 2.1. *The energy functional E, defined by*

$$\begin{aligned} E\left(t\right) &= \frac{1}{2}\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right] dx \\ &\quad +\frac{\zeta}{2}\int_0^1\int_0^1 z^2\left(x,\rho,t\right) d\rho dx \end{aligned} \tag{2.14}$$

From (2.13), we get

$$E'\left(t\right) \leq -\left(\mu_1 - \frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 \phi_t^2 dx - \left(\frac{\zeta}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 z^2\left(x,1,t\right) dx \leq 0. \tag{2.15}$$

Proof. Multiplying the first equation of (2.6) by u_t and the second equation by ϕ_t then integration by parts over $(0,1)$, and using (2.8), we get

$$\begin{aligned} \frac{dE\left(t\right)}{dt} &\leq -\left(\mu_1 - \frac{\zeta}{2\tau}\right)\int_0^1 \phi_t^2 dx - \frac{\zeta}{2\tau}\int_0^1 z^2\left(x,1,t\right) dx \\ &\quad -\mu_2\int_0^1 \phi_t z\left(x,1,t\right) dx. \end{aligned} \tag{2.16}$$

Now, using Young's inéquality, (3.17) can be rewritten as;

$$\frac{dE\left(t\right)}{dt} \leq -\left(\mu_1 - \frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 \phi_t^2 dx - \left(\frac{\xi}{2\tau} - \frac{\mu_2}{2}\right)\int_0^1 z^2\left(x,1,t\right) dx \leq 0.$$

□

Lemma 2.2. *The functional*

$$F_1\left(t\right) := J\int_0^1 \phi_t\phi dx + \frac{\mu_1}{2}\int_0^1 \phi^2 dx$$

satisfies

$$\begin{aligned} F_1'(t) &\leq -\delta\int_0^1 \phi_x^2 dx - b\varepsilon_1 \int_0^1 u_x^2 dx + \left(\frac{1}{2\varepsilon_1} - \xi\right)\int_0^1 \phi^2 dx \\ &+\varepsilon_1 \int_0^1 \phi_t^2 dx + \frac{\mu_2}{2\varepsilon_1}\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{2.17}$$

Proof. Direct computation using integration by parts and Young's inequality, for

$$\begin{aligned} F_1'(t) &= -\delta\int_0^1 \phi_x^2 dx - b\int_0^1 u_x\phi dx - \xi\int_0^1 \phi^2 dx \\ &+\mu_2\int_0^1 \phi_t\phi_t(x,t-\tau)\,dx, \end{aligned} \tag{2.18}$$

we obtain the estimate (2.17). □

Lemma 2.3. *The functional*

$$F_2(t) := b\int_0^1 \phi_x u_t dx + b\int_0^t u_x\phi_t dx$$

satisfies, for any $\varepsilon_2 > 0$,

$$\begin{aligned} F_2'(t) &\leq \left(-\frac{b^2}{J} + 2\varepsilon_2 - \frac{b\xi\varepsilon_2}{J}\right)\int_0^1 u_x^2 dx + \left(\frac{b\xi c}{2J\varepsilon_2} + \frac{b^2}{\rho}\right)\int_0^1 \phi_x^2 \\ &+\frac{1}{2\varepsilon_2}\int_0^1 \phi_t^2 dx + \frac{1}{2\varepsilon_2}\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{2.19}$$

Proof. Differentiating D_2, taking into account (2.6) and using integrating by parts together with the boundary conditions, we obtain

$$\begin{aligned} F_2'(t) &= -\frac{b^2}{J}\int_0^t u_x^2 dx - \frac{b\xi}{J}\int_0^1 u_x\phi dx + b\left(\frac{\mu}{\rho} - \frac{\delta}{J}\right)\int_0^1 u_{xx}\phi_x dx \\ &+\frac{b^2}{\rho}\int_0^t \phi_x^2 dx + \int_0^1 u_x\phi_t dx + \int_0^1 u_x\phi_t(x,t-\tau)\,dx. \end{aligned}$$

Young's and Poincare's inequality, give the result (2.19). □

Lemma 2.4. *The functional*

$$F_3(t) := -\rho\int_0^1 u_t u dx$$

satisfies

$$F_3'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{3\mu}{2} \int_0^1 u_x^2 dx + c \int_0^1 \phi^2 dx. \tag{2.20}$$

Proof. Direct computation gives

$$F_3'(t) = -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b \int_0^1 u_x \phi dx.$$

Estimate (2.20) easily follows by using Young's inequality. □

Lemma 2.5. *The functional*

$$F_4(t) := -\rho \int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\, d\rho dx \tag{2.21}$$

satisfies

$$F_4'(t) \leq -F_4(t) - \frac{c_1}{2\tau} \int_0^1 z^2(x,1,t)\, dx + \frac{1}{2\tau} \int_0^1 \phi_t^2(x,t)\, dx, \tag{2.22}$$

where c_1 is a positive constant.

Proof. Differentiating (2.21) with respect to t and using the equation (2.10), we obtain (2.22). □

Theorem 2.1. *Assume that $\mu_2 < \mu_1$, Then there exist two positive constants α and β like the energy functional given by (3.14) satisfies*

$$E(t) \leq \alpha e^{-\beta t}, \ \forall t \geq 0. \tag{2.23}$$

Proof. We define the Lyapunov functional by

$$\mathcal{L}(t) := NE(t) + N_1 F_1(t) + N_2 F_2(t) + F_3(t) + F_4(t) \tag{2.24}$$

where N, N_1 and N_2 are positive constants to be selected later.

By differentiating (3.40) and using (3.15), (2.17), (2.19) and (2.20), we have

$$\begin{aligned} \mathcal{L}'(t) \leq\ & \left[N_2\left(\frac{b\xi c}{2J\varepsilon_2}+\frac{b^2}{\rho}\right)-N_1\delta\right]\int_0^1 \phi_x^2 dx \\ & +\left[-N\left(\mu_1+\frac{\zeta}{2\tau}+\frac{\mu_2}{2}\right)+N_1\varepsilon_1+N_2\frac{1}{2\varepsilon_2}+\frac{1}{2\tau}\right]\int_0^1 \phi_t^2 dx \\ & +\left[-N_1 b\varepsilon_1+N_2\left(-\frac{b^2}{J}+2\varepsilon_2-\frac{b\xi\varepsilon_2}{J}+\frac{3\mu}{2}\right)\right]\int_0^1 u_x^2 dx \\ & +\left[-N\left(\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)+N_1\frac{\mu_2}{2\varepsilon_2}+N_2\frac{1}{2\varepsilon_2}-\frac{c_1}{2\tau}\right]\int_0^1 z^2(x,1,t)\,dx \\ & +\left[N_1\left(\frac{1}{2\varepsilon_1}-\xi\right)+c\right]\int_0^1 \phi^2 dx-\rho\int_0^1 u_t^2 dx \end{aligned}$$

By setting

$$\varepsilon_1=\frac{1}{\xi},\ \varepsilon_2=\frac{b^2}{4\delta},$$

and we choose N_1 and N_2 are large enough such that

$$\begin{aligned} \alpha_1 &= N_2\left(\frac{b^2}{4\delta}+\frac{b^3\zeta}{4\delta^2}\right)+N_1\frac{b}{\zeta}-\frac{3\mu}{2}>0,\ \alpha_2=-\frac{1}{2\varepsilon_1}+\xi>0, \\ \alpha_3 &= -N_2\left(\frac{b\xi c}{2J\varepsilon_2}+\frac{b^2}{\rho}\right)+N_1\delta>0,\ \alpha_4=N\left(\mu-\frac{c_1}{2\tau}+\frac{\mu_2}{2}\right)-\frac{1}{2\varepsilon_2}-\varepsilon_2, \\ \alpha_5 &= \rho,\ N\left(\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)-N_1\frac{\mu_2}{2\varepsilon_2}-N_2\frac{1}{2\varepsilon_2}+\frac{c_1}{2\tau}. \end{aligned}$$

Then, we obtain

$$\begin{aligned} \mathcal{L}'(t) \leq\ & -\alpha_1\int_0^1 u_x^2 dx-\alpha_2\int_0^1 \phi^2 dx-\alpha_3\int_0^1 \phi_x^2 dx-\alpha_4\int_0^1 \phi_t^2 dx \\ & -\alpha_5\int_0^1 u_t^2 dx-\alpha_6\int_0^1 z^2(x,1,t)\,dx. \end{aligned}$$

Let

$$|\mathcal{H}(t)| \leq N_1F_1(t)+N_2F_2(t)+F_3(t)+F_4(t).$$

Hence

$$\begin{aligned} H\left(t\right) &\leq N_1\left(J+b\right)\int_0^1 \phi_t^2 dx + N_1\left(J+\frac{\mu_1}{2}\right)\int_0^1 \phi^2 dx \\ &+N_2 b\int_0^1 \phi_x^2 dx + N_2\left(b+\rho\right)\int_0^1 u_t^2 dx \\ &+N_2\left(b+\rho c\right)\int_0^1 u_x^2 dx + \rho\int_0^1 \left|u_t u\right| dx + F_4\left(t\right) \end{aligned} \tag{2.25}$$

Consequently, there exists a positive constant C such that

$$\left|H\left(t\right)\right| \leq CE\left(t\right),$$

then, we have

$$\left|H\left(t\right)\right| = \left|\mathcal{L}\left(t\right) - NE\right| \leq CE\left(t\right). \tag{2.26}$$

Now, by choosing N large enought such that

$$N - c > 0,$$

and exploiting (3.42) we give

$$\mathcal{L}'\left(t\right) \leq -k_1\mathcal{L}\left(t\right) \tag{2.27}$$

A simple integration of (3.43) over $(0, t)$ leads to

$$\mathcal{L}\left(t\right) \leq \mathcal{L}\left(0\right)e^{-\lambda t}, \forall t \geq 0. \tag{2.28}$$

Consequently, (3.39) is established by virture of (2.26) and (2.28). ☐

Chapter 3

General Decay of Solutions in One-Dimensional Porous-Elastic with Memory and Distributed Delay Term

3.1 Introduction

Researchers from various feilds were interested in elasticity problems, and they have been mainely attracted by the qualitative studies of different type of this problems and many results can be found in the literature. In the one-dimensional case, for instance, the combination of the elastic equations with thermal consequences causes a negative exponential to control the decay of solution.
The one-dimensional porous-elastic model is given by

$$\begin{aligned} \rho_0 u_{tt} &= \mu u_{xx} + \beta \varphi_x, \text{ in } (0,l) \times (0,L) \ , \\ \rho_0 k \varphi_{tt} &= \alpha \varphi_{tt} - \beta u_x - \tau \varphi_t - \xi \varphi, \text{ in } (0,l) \times (0,L) \ , \end{aligned}$$

and it has been studied by many authors. The first contribution in this direction was obtained by [54], to be more precise, which was developed in [31], the authors showed that the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into theory of elastic solids with voids. In addition to the usual elastic effects, the materials with voids possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This concept was introduced in the pioneered work in [43], when the authors have advanced nonlinear theory of elastic materials with voids (See [12],[13]). The basic evolution equations for one-dimensional theories of porous materials with memory effect are given by

$$\rho_0 u_{tt} = T_x, J\phi_{tt} = H_x + G, \tag{3.1}$$

where T is the stress tensor, H is the equilibrated stress vector and G is the equilibrated body force. The variables u and $\phi$are the displacement of the solid elastic material and the volume fraction, respectively. The constitutive equatin are

$$T = \mu u_x + b\phi, H = \delta\phi_x - \int_0^t g(t-s)\phi_x(s)ds, G = -bu_x - \xi\phi. \tag{3.2}$$

In [54] substituting (3.2) into (3.1) is concerned

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, in(0,1) \times (0,\infty), \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\phi_{xx}(x,s)ds = 0, in(0,1) \times (0,\infty). \end{cases} \tag{3.3}$$

A porous-elastic system with memory term and Neumann-Dirichlet boundary conditions where g is the relaxation function it has been proved a general decay result, for more detail (see [3]).
Quintanilla in [54] considere

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x, x \in (0,L), t > 0, \\ J\phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi - \tau\phi_t, x \in (0,L), t > 0, \end{cases} \tag{3.4}$$

with initial and mixed boundary conditions and supposed that the damping in the porous equation $(-\tau\theta_t)$ is not enough to obtain an exponential decay but only a slow decaycan be obtained.
To improve this decay several other damping mechanisms were considered. In Casas and Quintanilla have considered

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\phi_x - \beta\theta_x, x \in (0,L), t > 0, \\ J\phi_{tt} = \delta\phi_{xx} - bu_x - \xi\phi + m\theta - \tau\phi_t, x \in (0,L), t > 0, \\ c\theta_t = k\theta_{xx} - \beta u_{xt} - m\phi_t, x \in (0,L), t > 0, \end{cases} \tag{3.5}$$

where θ is the temperature difference with initial and Dirichlet-Neumann boundary conditions. The authors applied the semigroup theory and the method proposed and developed in [71] to establish the exponential decay of the solutions. Later, with $\tau = 0$, in [11] the same authors have proposed that the heat effect alone is not strong sufficient to bring an exponential decay but only a slow decay could be establiched. However, the heat effect together with micro-temperature created an exponential decay result. However, by taking some regular initial data a polynomial stability is obtained. Also, for $\tau = 0$, problem (3.1) was considered in [58]with the following boundary conditions

$$\begin{cases} u(0,t) = \phi(0,t) = \theta(0,t) = \theta(L,t) = 0, t \geq 0, \\ u(L,t) = -\int_0^t g_1(t-s)(\mu u_x(L,s) + b\phi(L,s))ds, t \geq 0, \\ \phi(L,t) = -\delta \int_0^t g_2(t-s)\phi_x(L,s)ds, t \geq 0, \end{cases}$$

where g_1 and g_2 are positive decreasing functions. They obtained a general decay result in which the usual exponential and polynommial decay rates are just special cases. ([6],[56],[55],[37],[58] and the references therein).

The viscoelastic damping is represented by a memory term in the form of a convolution which arises in the constitutive equation between the stress $\delta(x,t)$ and the strain $\epsilon(x,t)$ (See [21],[3])

$$\delta(x,t) = \epsilon(x,t) + \int_0^t g(t-s)\epsilon(x,s)ds.$$

This type of viscoelastic dissipation could be said to coincide to viscosity with null initial history because it is assumed that the strains have been zero for $-\infty < t < 0$or, equivalently, if any past strains have occurred sufficiently long ago that the effect is trivial. In other words, there will be a time prior to which all the strains which have previously occurred will have a trivial contribution.
Thus, an experiment generally starts at some time ($t = 0$) when the material is free of stresses.

We must mention the pioneer works recently published by [4], the author considered a one-dimensional porous thermo-elastic system which memory effects and proved a general decay result, for which exponential and polynomial decay results are special cases, depending only on the kernel of the memory effects. The obtained result were established irrespective of the wave speeds of the system (See [2],[5]). In [25] the authors investigated a porous thermo-elastic systeme where the heat conduction is given by Cattaneo's law and where the energy associated wuth the solution is not necessary positive. They introduced a stability number and proved an exponential and polynomial decay results.

0ur purpose in this work is to give a general decay result of solutions in one dimensional porous-elastic system with memory and distributed delay term, our result is new and improves previous results in the literature.

Let $H = (0,1)(\tau_1, \tau_2)(0,\infty)$, in the present work, we are interested in the following problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\phi_{xx}(x,s)ds + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} \mu_2(s)\phi_t(x,t-s)ds = 0, \end{cases}$$

where

$$(x,s,t) \in H$$

As in [42], taking the following new variable

$$z(x,p,s,t) = \phi_t(x,t-sp),$$

then we obtain

$$\begin{cases} sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ z(x,0,s,t) = \phi_t(x,t), \end{cases}$$

Consequently, the problemis equivalent to

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g(t-s)\phi_{xx}(x,s)ds + \mu_1\phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\phi_t(x,t-s)ds = 0, \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0. \end{cases} \tag{3.6}$$

where

$$(x,\rho,s,t) \in (0,1)H.$$

The system with memory and delay term acting only on the porous equation together with the initial data

$$\begin{cases} u(x,0) = u_0(x), u_t(x,0) = u_1(x), \\ \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), x \in (0,1), \end{cases} \tag{3.7}$$

and Neumann-Dirichlet boundary conditions

$$u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, t \geq 0. \tag{3.8}$$

Here, u is the longitudinal displacement, ϕ is the volume fraction of the solid elastic material and $\rho, \mu, b, J, \delta, \xi$ are positive constants with μ, ξ, b satisfying $\mu\xi > 0$ the integral represents the memory and delay term with $\tau_1, \tau_2 > 0$ are a time delay, μ_1 is positive constant, μ_2 is an L^∞ function and g is the relaxation function satisfying (H1) $g \in C^1(R_+, R_+)$ is a non-increasing function satisfying

$$g(0) > 0, b - \int_0^\infty g(s)ds = l > 0. \tag{3.9}$$

(H2) there exists a positive non-increasing differentiable function $v \in (R_+, R_+)$satisfying

$$g'(t) \leq -v(t)g(t), t \geq 0. \tag{3.10}$$

(H3) $\mu_2 : [\tau_1, \tau_2] \to R$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds \leq \mu_1. \tag{3.11}$$

Time delays arise in many applications because most phenomena naturally depend not only on the present state but also on some past occurrences in recent years, the control of PDEs with time delay effects has become an active area of research, see for example [59] and references therein. In many cases it was shown that delay is a source of instability unless additional condition or control terms are used, see [20] therefore, the stability issue of systems with delay is of theoretical and practical great importance.

It is well know that, in the single wave equation, if$\mu_2 = 0$ that is, in absence of a decay, the energy of system exponentially decays (see[4]) on the contrary, if $\mu_1 = 0$, that is, there exists only the delay part in the interior, the system becomes unstable (see[20]). It is shown that a small delay in a boundary control can turn such a well-behaved hyperbolic system into a wild one and therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see[42],[47]).

In what follows, we consider (u, ϕ) to be a solution of system (3.6)-(3.8) with the regularity needed to justify the calculations in this paper. We specify Section 2 to the statements and prove of our stability result. We use c throughout this paper to denote a generic positive constant. Meanwhile, from (3.6) and (3.8), itfollows that

$$\frac{d^2}{dt^2} \int_0^1 u(x,t)dx = 0. \tag{3.12}$$

So by solving (3.12) and using the initial data of u, we get

$$\int_0^1 u(x,t)dx = t\int_0^1 u_1(x)dx + \int_0^1 u_0(x)dx.$$

consequently, if we let

$$\bar{u}(x,t) = u(x,t) - t\int_0^1 u_1(x)dx - \int_0^1 u_0(x)dx, \tag{3.13}$$

we get

$$\int_0^1 \bar{u}(x,t)dx = 0, \forall t \geq 0.$$

Therefore, the use of Poincare's inequality for $\bar{u}$ is justified. In addition, simple substitution shows that $(\bar{u},\, \phi)$ satisfies system (3.6) with initial datafor $\bar{u}$ given as

$$\bar{u_0} = u_0(x) - \int_0^1 u_0(x)dx and \bar{u_1} = u_1(x) - \int_0^1 u_1(x)dx.$$

Henceforth, we work with $\bar{u}$ instead of u but write u for simplicity of notation .

3.2 Main result

In this section, we state and prove our decay result for the energy of the system (3.6)-(3.8) using the multiplier. We need the folowing lemmas.

Lemma 3.1. *The energy functional E, defined by*

$$\begin{aligned} E(t) =& \frac{1}{2}\left[\int_0^1 \rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \left(\delta - \int_0^t g(s)ds\right)\phi_x^2 + \xi\phi^2 + 2bu_x\phi\right] dx \\ &+ \frac{1}{2}g\circ\phi_x + \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx, \end{aligned} \tag{3.14}$$

satisfies

$$E'(t) = \frac{1}{2}g'\circ\phi_x - \frac{1}{2}g(t)\int_0^1 \phi_x^2 dx - \left(\mu_1 - \int_{\tau_1}^{\tau_2}\mu_2(s)ds\right)\int_0^1 \phi_t^2 dx, \tag{3.15}$$

and

$$E'(t) \leq \frac{1}{2}g'\circ\phi_x - \eta_0\int_0^1 \phi_t^2 dx \leq 0, \tag{3.16}$$

where

$$\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2} \mu_2(s)ds \geq 0$$

and

$$g \circ v = \int_0^1 \int_0^t g(t-s)(v_x(t) - v_x(s))^2 dsdx.$$

Proof. Multiplying the first equation of (3.6) by u_t and the second equation by ϕ_t, then integration by parts over (0,1), and using (3.8) we get

$$\begin{aligned} &\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_0^1 \left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \xi\phi^2 + 2bu_x\phi\right] dx - \int_0^1 \phi_{xt} \int_0^t g(t-s)\phi_x(s)dsdx \\ &+\mu_1 \textstyle\int_0^1 \phi_t^2 dx + \int_0^1 \phi_t \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,s,t)dsdx = 0. \end{aligned} \tag{3.17}$$

The last term in the left hand side of (3.17) is estimated as follows.

$$\begin{aligned} &-\textstyle\int_0^1 \phi_{xt} \int_0^t g(t-s)\phi_x(s)dsdx \\ &= \textstyle\int_0^1 \phi_{xt} \int_0^t g(t-s)(\phi_x(t) - \phi_x(s))dsdx - \int_0^t g(s)ds \int_0^1 \phi_{xt}\phi_x dx \\ &= \tfrac{1}{2}\tfrac{d}{dt} g \circ \phi_x - \tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_0^t g(s)ds \int_0^1 \phi_x^2 dx - \tfrac{1}{2} g' \circ \phi_x + \tfrac{1}{2} g(t) \int_0^1 \phi_x^2 dx. \end{aligned} \tag{3.18}$$

Now, muliplying the last equation in (3.6) by $z|\mu_2(s)|$, and integrating the result over $(0,1) \times (\tau_1, \tau_2)$

$$\begin{aligned} &\frac{d}{dt}\frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,\rho,s,t)dsd\rho dx \\ &\qquad = -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z z_\rho(x,\rho,s,t)dsd\rho dx \\ &\qquad = -\frac{1}{2}\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \frac{d}{d\rho} z^2(x,\rho,s,t)dsd\rho dx \\ &\qquad = \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)|(z^2(x,0,s,t) - z^2(x,1,s,t)dsdx) \\ &\qquad = \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds \int_0^1 \phi_t^2 dx - \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dsdx. \end{aligned} \tag{3.19}$$

Now, using Young's inequality, we have

$$\begin{aligned} E'(t) &\leq \frac{1}{2} g' \circ \phi_x - \frac{1}{2} g(t) \int_0^1 \phi_x^2 dx - (\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2|(s)ds) \int_0^1 \phi_t^2 dx \\ &\leq \frac{1}{2} g' \circ \phi_x - (\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2|(s)ds) \int_0^1 \phi_t^2 dx, \end{aligned} \tag{3.20}$$

then, by (H3), there exists a positive constant η_0 such that

$$E'(t) \leq \frac{1}{2}g' \circ \phi_x - \eta_0 \int_0^1 \phi_t^2 dx, \tag{3.21}$$

then we obtain E is a non-increasing function.

Lemma 3.2. *The functional*

$$D_1(t) := J \int_0^1 \phi_t \phi dx + \frac{b\rho}{\mu} \int_0^1 \phi \int_0^x u_t(y) dy dx, \tag{3.22}$$

satisfies

$$\begin{aligned} D_1'(t) \leq & -\frac{l}{2} \int_0^1 \phi_x^2 dx - \mu_3 \int_0^1 \phi^2 dx + \epsilon_1 \int_0^1 u_t^2 dx + c(1 + \frac{1}{\epsilon_1}) \int_0^1 \phi_t^2 dx \\ & + cg \circ \phi_x + c \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t) ds dx, \end{aligned} \tag{3.23}$$

where $\mu_3 = \xi - \frac{b^2}{\mu} \geq 0$.

Proof. Direct computation using integration by parts and Young's inequality, for $\epsilon_1 \geq 0$, yields

$$\begin{aligned} D'(t) = & -\delta \int_0^1 \phi_x^2 dx - (\xi - \frac{b^2}{\mu}) \int_0^1 \phi^2 dx + J \int_0^1 \phi_t^2 dx + \frac{b\rho}{\mu} \int_0^1 \phi_t \int_0^x u_t(y) dy dx \\ & + \int_0^1 \phi_x \int_0^1 g(t-s) \phi_x(s) ds dx - \mu_1 \int_0^1 \phi_t \phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, 1, s, t) ds dx \\ \leq & -\delta \int_0^1 \phi_x^2 dx - (\xi - \frac{b^2}{\mu}) \int_0^1 \phi^2 dx - c(1 + \frac{1}{\epsilon_1}) \int_0^1 \phi_t^2 dx + \epsilon_1 \int_0^1 (\int_0^x u_t(y) dy)^2 dx. \\ & + \int_0^1 \phi_x \int_0^t g(t-s) \phi_x ds dx - \mu_1 \int_0^1 \phi_t \phi dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, 1, s, t) ds dx. \end{aligned} \tag{3.24}$$

By Cauchy-Schwartz inequality, it is clear that

$$\int_0^1 (\int_0^x u_t(y) dy)^2 dx \leq \int_0^1 \left(\int_0^1 u_t dx \right)^2 dx \leq \int_0^1 u_t^2 dx.$$

So, etimate (3.24) becomes

$$\begin{aligned}D'(t) \leq & -\delta\int_0^1 \phi_x^2dx - (\xi - \frac{b^2}{\mu})\int_0^1 \phi^2dx + c(1+\frac{1}{\epsilon_1})\int_0^1 \phi_t^2dx + \epsilon_1\int_0^1 u_t^2dx \\ & -\mu_1\int_0^1 \phi_t\phi dx - \int_0^1 \phi\int_{\tau_1}^{\tau_2} |\mu_2(s)|z(x,1,s,t)dsdx \\ & + \int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx.\end{aligned} \tag{3.25}$$

The last term in the RHS of (3.25) is estimated as dollows :

$$\begin{aligned}&\int_0^1 \phi_x\int_0^t g(t-s)\phi_x(s)dsdx \\ &= \int_0^t g(s)ds\int_0^1 \phi_x^2 - \int_0^1 \phi_x\int_0^t g(t-s)(\phi_x(t)-\phi_x(s))dsdx \\ &\leq (\delta_1 + \int_0^t g(s)ds)\int_0^1 \phi_x^2dx + \frac{1}{4\delta_1}(\int_0^t g(s)ds)g\circ\phi_x,\end{aligned} \tag{3.26}$$

where we have used Caychy-Schwartz, Young's and poincare's inequalities, for $\delta_1, \epsilon_2, \epsilon_3 \geq 0$.

By substituting (3.26) into (3.24), we obtain

$$\begin{aligned}D'(t) \leq & (\delta - \int_0^t g(s)ds - \delta_1 - \mu_1 c\delta_2 - \mu_1 c\delta_3)\int_0^1 \phi_x^2dx - \xi - \frac{b^2}{\mu}\int_0^1 \phi^2dx \\ & + \epsilon_1\int_0^1 u_t^2dx + (c(1+\frac{1}{\epsilon_1}) + \frac{\mu_1}{4\delta_2})\int_0^1 \phi_t^2dx + + \frac{1}{4\delta_1}(\int_0^t g(s)ds)g\circ\phi_x \\ & + \frac{1}{4\delta_3}\int_0^t\int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,s,t)dsdx.\end{aligned}$$

Bearing in mind that $\mu\xi \geq b^2$ and using the fact that $\delta - \int_0^t g(s)ds \geq l$, and letting $\delta_1 = \frac{1}{6}, \delta_2 = \delta_3 = \frac{1}{6c\mu_1}$, we obtain estimate (3.23.)

In the foolwing lemma, we use the essentiel hypothesis that the wave speeds of the system are equal

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \tag{3.27}$$

Lemma 3.3. *Assume that (H1) and (3.27) hold. Then the functional*

$$D_2(t) := \int_0^1 \phi_x u_t dx + \int_0^1 \phi_x u_x dx - \frac{\rho}{\mu J}\int_0^1 u_t\int_0^t g(t-s)\phi_x(s)dsdx,$$

satisfies, for any $\epsilon_2 > 0$

$$\begin{aligned} D_2'(t) \leq &- \frac{b}{2J}\int_0^1 u_x^2 dx + c(1+\frac{1}{\epsilon_2})\int_0^1 \phi_x^2 dx + c\epsilon_2 \int_0^1 u_t^2 dx \\ &+ c\int_0^1 \phi_t^2 + cg\circ\phi_x - \frac{c}{\epsilon_2} g'\circ\phi_x + c\mu_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dx. \end{aligned} \tag{3.28}$$

Proof . By differentiating D_2, then using (3.6), integation by parts, and (3.8) we obtain

$$\begin{aligned} D_2'(t) = &- \frac{b}{J}\int_0^1 u_x^2 dx + (\frac{\delta}{J} - \frac{\mu}{\rho})\int_0^1 u_x\phi_{xx}dx + \frac{b}{\rho}\int_0^1 \phi_x^2 dx - \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx \\ &- \frac{b}{J}\int_0^1 u_x\phi dx - \frac{b}{\mu J}\int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)dsdx \\ &- \frac{\rho}{\mu J}\int_0^1 u_t \int_0^t g'(t-s)\phi_x(s)dsdx \\ &- \frac{\mu_1}{J}\int_0^1 \phi_t u_x dx - \frac{1}{J}\int_0^1 u_x \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dsdx. \end{aligned} \tag{3.29}$$

In what follows, we estimate the last five terms in the rieght hand side of (3.29), using Young's, Cauchy-Schwartz, and poincae's inequalities. For $\delta_4, \delta_5, \epsilon_2 \geq 0$, we have

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \leq \frac{\xi}{J}\delta_4 \int_0^1 u_x^2 dx + \frac{\xi}{4J\delta_4}\int_0^1 \phi^2 dx.$$

By letting $\delta_4 = \frac{b}{6\xi}$, using Poincar's inequality, we get

$$-\frac{\xi}{J}\int_0^1 u_x\phi dx \leq \frac{b}{6J}\int_0^1 u_x^2 + c\int_0^1 \phi^2 dx, \tag{3.30}$$

$$\begin{aligned} &\frac{b}{\mu J}\int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)dsdx \\ &\qquad = \frac{b}{\mu J}\int_0^1 \phi_x \int_0^t g(t-s)(\phi_x(t) - \phi_x(s))dsdx - \frac{b}{\mu J}\int_0^t g(s)ds \int_0^1 \phi_x^2 dx \\ &\qquad \leq (\delta_5 - \frac{b}{\mu J})\int_0^t g(s)ds \int_0^1 \phi_x^2 dx + \frac{c}{\delta_5} g\circ\phi_x. \end{aligned}$$

By letting $\delta_5 = \frac{b}{\mu J}$ we get

$$-\frac{b}{\mu J}\int_0^1 \phi_x \int_0^t g(t-s)\phi_x(s)dsdx \leq cg\circ\phi_x, \tag{3.31}$$

$$\begin{aligned}
&-\frac{\rho}{\mu J}\int_0^1 u_t \int_{0)}^t g'(t-s)\phi_x(s)dsdx\\
&= \frac{b}{\mu j}\int_0^1 u_t \int_0^t g'(t-s)(\phi_x(t)-\phi_x(s))dsdx - \frac{b}{\mu J}\int_0^t g'(s)ds\int_0^1 u_t\phi_x dx\\
&\leq \frac{\rho\epsilon_2}{2\mu J}\int_0^1 u_t^2dx + \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx - \frac{\rho g(t)}{\mu J}\int_0^1 u_t\phi_x dx\\
&+\frac{\rho}{2\mu j\epsilon_2}\int_0^1 g'(s)ds\int_0^1\int_0^t g'(t-s)(\phi_x(t)-\phi_x(s))^2dsdx\\
&\leq \frac{\rho\epsilon_2}{\mu J}\int_0^1 u_t^2dx + \frac{\rho}{2\mu J\epsilon_2}(\int_0^1 g'(s)ds)g'\circ\phi_x + \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx\\
&+\frac{\rho g(t)}{2\mu J\epsilon_2}\int_0^1 u_t\phi_x dx\\
&\leq \frac{\rho\epsilon_2}{\mu J}\int_0^1 u_t^2dx + \frac{\rho}{2\mu J\epsilon_2}(\int_0^1 g'(s)ds)g'\circ\phi_x + \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx\\
&+\frac{\rho (g(t))^2}{2\mu J\epsilon_2}\int_0^1 \phi_x^2 dx\\
&\leq c\epsilon_2\int_0^1 u_t^2dx + \frac{c}{\epsilon_2}\int_0^1 \phi_x^2dx + \frac{\rho g(0)}{\mu J}\int_0^1 u_t\phi_x dx - \frac{c}{\epsilon_2}g'\circ\phi_x,
\end{aligned} \tag{3.32}$$

$$-\frac{\mu_1}{J}\int_0^1 \phi_t u_x dx \leq \frac{\mu_1\delta_6}{2J}\int_0^1 \phi_t^2dx + \frac{\mu_1}{2J\delta_6}\int_0^1 u_x^2dx, \tag{3.33}$$

$$\begin{aligned}
&\frac{1}{J}\int_0^1 u_x \int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,d,t)dsdx\\
&\leq \frac{\delta_7\mu_1}{2J}\int_0^1 u_x^2dx + \frac{1}{2J\delta_7}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,s,t)ds
\end{aligned} \tag{3.34}$$

The replacement of (3.31)-(3.34) into (3.29), and by letting $\delta_6 = \delta_7 = \frac{b}{4\mu_1}$, bearing in the mind (3.27) yields (3.28).

Lemma 3.4. *The functionnal*

$$D_3(t) := -\rho \int_0^1 u_t u dx,$$

satisfies

$$D_3'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{3\mu}{2} \int_0^1 u_x^2 dx + c \int_0^1 \phi_x^2 dx. \tag{3.35}$$

Proof. Direct computations give

$$D_3'(t) = -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b \int_0^1 u_x \phi dx.$$

Estimat (3.35) easily follows by using Young's and Pioncaré inequalities.

$$\begin{aligned} D_3'(t) &\leq -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b\epsilon \int_0^1 u_x^2 dx + \frac{b}{4\epsilon} \int_0^1 \phi^2 dx \\ &\leq -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b\epsilon \int_0^1 u_x^2 dx + \frac{bc}{4\epsilon} \int_0^1 \phi_x^2 dx. \end{aligned}$$

by letting $\epsilon = \frac{\mu}{2b}$, we obtain (3.35).

Now, let us introduce the folling functional used by

Lemma 3.5. *The functional*

$$D_4(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-s\rho} |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx,$$

satisfies

$$\begin{aligned} D_4'(t) \leq -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x, \rho, s, t) ds d\rho dx + \mu_1 \int_0^1 \phi_t^2 dx \\ - \eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2|(s) z^2(x, 1, s, t) ds dx. \end{aligned} \tag{3.36}$$

where η_1 is a positive constant.

proof ,By differentiating D_4, with respect to t and using the last equation in (H3), we have

$$\begin{aligned} D_4'(t) =& -2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}|\mu_2(s)|zz_\rho(x,\rho,s,t)dsd\rho dx \\ =& -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx \\ & -\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\left[e^{-s}z^2(x,1,s,t)-z^2(x,0,s,t)\right]dsdx. \end{aligned}$$

Using the fact that $z(x,0,s,t)=\phi_t(x,t)$, and $e^{-s}\le e^{-s\rho}\le 1$, for all $0<\rho<1$, we obtain

$$\begin{aligned} D_4'(t) =& -\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)ds,d\rho dx \\ & -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,s,t)dsdx+\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_0^1\phi_t^2dx. \end{aligned}$$

Because $-e^{-s}$is a increasing function, we have $-e^-s\le -e^{-\tau_2}$, for all ss $\in[\tau_1,\tau_2]$. Finally, detting $\eta_1=e^{-\tau_2}$ and recalling (H3), we obtain (3.36). We are now ready to prove the main result.

Theorem 3.1. *assume(H1), (H2), (H3) and (3.27) hold. Then, for any $t-\tau>0$, there exist positive constants α andβ such that the energy funtional given by (3.14) satisfies*

$$E(t)\le \alpha e^{-\beta\int_{t_0}^t v(s)ds}, t>t_0. \tag{3.37}$$

proof. We define a Lyapunov functional

$$\mathcal{L} := NE(t)+N_1D_1(t)N_2D_2(t)D_3(t)+N_4D_4(t), \tag{3.38}$$

Where $N, N_1, N_2,$ and N_4 are positive constants to be selected later. By differentiating

(3.38) and using (3.14), (3.23), (3.28), (3.35), (3.36), we have

$$\begin{aligned}
\mathcal{L}' \leq & -\left[\frac{lN_1}{2} - cN_2(1+\frac{1}{\epsilon_2}) - c\right]\int_0^1 \phi_x^2 dx - [\rho - N_1\epsilon_1 - N_2 c\epsilon_2]\int_0^1 u_t^2 dx \\
& - \left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx - \left[\eta_0 N - cN_1(1+\frac{1}{\epsilon_1}) - N_2 c - \mu_1 N_4\right]\int_0^1 \phi_t^2 dx \\
& - N_1\mu_3 \int_0^1 \phi^2 dx - [N_4\eta_1 - cN_1 - cN_2]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2|(s) z^2(x,1,s,t) ds dx \\
& + c\,[N_1 + N_2]\, g\circ\phi_x + \left[\frac{N}{2} - \frac{cN_2}{\epsilon_2}\right] g'\circ\phi_x \\
& - N_4\eta_1 \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx.
\end{aligned}$$

By setting

$$\epsilon_1 = \frac{\rho}{4N_1}, \epsilon_2 = \frac{\rho}{4cN_2}$$

We obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{lN_1}{2} - cN_2(1+N_2) - c\right]\int_0^1 \phi_x^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx \\
& - \left[\frac{bN_2}{2J} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx - [\eta_0 N - cN_1(1+N_1) - cN_2 - \mu_1 N_4]\int_0^1 \phi_t^2 dx \\
& - N_1\mu_3 \int_0^1 \phi^2 dx - [N_4\eta_1 - cN_1 - cN_2]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2|(s) z^2(x,1,s,t) ds dx \\
& + c[N_1 + N_2] g\circ\phi_x + \left[\frac{N}{2} - cN_2^2\right] g'\circ\phi_x \\
& - N_4\eta_1 \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,s,t) ds d\rho dx.
\end{aligned}$$

Next, we carfully choose our constants so that the terms inside the brackets are positive. We choose N_2 larde enough such that

$$\alpha = \frac{bN_2}{2J} - \frac{3\mu}{2} > 0,$$

then We choose N_2 large enough such that

$$\alpha_2 = \frac{lN_1}{4} - cN_2(1+N_2) - c \geq 0,$$

then We choose N_2 larde enough such that

$$N_4\eta_1 - cN_1 - cN_2 > 0,$$

thus, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq & -\alpha_2\int_0^1 \phi_x^2 dx - \alpha_0\int_0^1 \phi^2 dx - \frac{\rho}{2}\int_0^1 u_t^2 dx - \alpha_1\int_0^1 u_x^2 dx - [\eta_0 N - c]\int_0^1 \phi_t^2 dx\\ & \left[\frac{N}{2} - c\right] g' \circ \phi_x + cg\circ\phi_x - \alpha_3\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)dsdx\\ & -\alpha_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2|(s)z^2(x,\rho,s,t)dsd\rho dx,\end{aligned} \tag{3.39}$$

where $\alpha_0 = \mu_3 N_1 = \left(\xi - \frac{b^2}{\mu}\right) N_1$. On the other hand, if we let

$$l(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t),$$

then

$$\begin{aligned}|\tilde{\mathcal{L}}(t)| \leq & JN_1\int_0^1 |\phi\phi_t| dx + N_2\int_0^1 |\phi_x u_t + u_x\phi_t - \frac{\rho}{\mu J}u_t\int_0^t g(t-s)\phi_x(s)ds| dx\\ & \frac{b\rho N_1}{\mu}\int_0^1 \left|\phi\int_0^x u_t(y)dy\right| dx + \rho\int_0^1 |u_t u| dx\\ & N_4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho dx.\end{aligned}$$

Exploiting Yound's, Cauchy-Schwartz, and Poincaré inequalities, we obtain

$$\begin{aligned}|\tilde{\mathcal{L}}(t)| &\leq c\int_0^1 (u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2 + \phi^2)dx + cg\circ\phi_x\\ & +c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,s,t)dsd\rho\\ & \leq cE(t).\end{aligned}$$

Consequently, we obtain

$$|\mathcal{L}(t)| = |\mathcal{L}(t) - NE(t) \leq cE(t)|,$$

that is

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (N+c)E(t). \tag{3.40}$$

Now, by choosing N lard enough such that

$$\frac{N}{2} - c > 0, N - c > 0, N\eta_0 - c > 0,$$

and exploiting (3.14), estimates (3.39) and (3.40), respectively, give

$$\mathcal{L}'(t) \leq -k_1 E(t) + k_2 g \circ \phi_x, \forall t \geq t_0, \tag{3.41}$$

and

$$c_2 E(t) \leq \mathcal{L}(t) \leq c_3 E(t), \forall t \geq 0, \tag{3.42}$$

for some $k_1, k_2, c_2, c_3 > 0$. By multiplying (3.41) by $v(t)$, we obtain

$$v(t)\mathcal{L}'(t) \leq -k_1 v(t) E(t) + k_2 v(t) g \circ \phi_x, \forall t \geq t_0. \tag{3.43}$$

The final term in (3.43) is estimated as following, using (3.10), we have

$$\begin{aligned} v(t) g \circ \phi_x &= v(t) \int_0^1 \int_0^t g(t-s)(\phi_x(t) - \phi_x(s))^2 ds dx \\ &\leq \int_0^1 \int_0^t v(t-s) g(t-s)(\phi_x(t) - \phi_x(s))^2 ds dx \\ &\leq -\int_0^1 \int_0^t g'(t-s)(\phi_x(t) - \phi_x(s))^2 ds dx = -g' \circ \phi_x \\ &\leq -2E'(t). \end{aligned} \tag{3.44}$$

Thus, (3.43) becomes

$$v(t)\mathcal{L}'(t) \leq -k_1 v(t) E(t) - 2k_2 E'(t), \forall t \geq t_0,$$

which can be rewritten as

$$(v(t)\mathcal{L}(t) + 2k_2 E(t))' - v'(t)\mathcal{L}(t) \leq -k_1 v(t) E(t), \forall t \geq t_0,$$

using the fact that $v'(t) \leq 0, \forall t \geq 0$, we have

$$(v(t)\mathcal{L}(t) + 2k_2 E(t))' \leq -k_1 v(t) E(t), \forall t \geq t_0.$$

By exloiting (3.42), we notice that

$$R(t)=v(t)l(t)+2k_2E(t)\sim E(t). \tag{3.45}$$

Consequently, for some positive constant λ, we obtain

$$R'(t)\geq -\lambda R(t)v(t), \forall t\leq t_0. \tag{3.46}$$

A simple integration of (3.46) over (t_0,t)leads to

$$R(t)\geq R(t_0)e^{-\lambda\int_{t_0}^{t}v(s)ds}, \forall t\leq t_0. \tag{3.47}$$

Consequently, (3.37) is estabished by virtue of (3.42), (3.47).

Remark 3.1. *We give some examples to illustrate the energy decay rates obtained by*

Examples : We consider the three different examples If $g(t)=\beta_1e^{-\beta_2t}$ then $g'(t)=-v(t)g(t)$, where $v(t)=\beta_2$, then

$$E(t)\leq c_0e^{-\beta_2c_1t}, \forall t\geq 0,$$

If $\frac{\beta_1}{(1+t)^{\beta_2+1}}$ then $g'(t)=-v(t)g(t)$, where $v(t)=\frac{\beta_2+1}{1+t}$ then

$$E(t)\leq \frac{c_0}{(1+t)^{(\beta_2+1)c_1}}, \forall t\leq 0,$$

If $g(t)=\frac{\beta_1}{(e^{t(\frac{\pi}{2}-arctgt)}\sqrt{1+t^2})^{\beta_2}}$, then $g'(t)=-v(t)g(t)$, where $v(t)=\beta_2(\frac{\pi}{2}-arctgt)$, then

$$E(t)\leq \frac{c_0}{c_1(e^{t(\frac{\pi}{2}-arctgt)}\sqrt{1+t^2})^{\beta_2})}, \forall t\geq 0.$$

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