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THEME

On a Thermoelastic Laminated Timoshenko Beam with delay term : Well Posedness and Stability results.

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Dedication

I dedicate this work
To my parents who have been my source of inspiration and provided me
with their encouragement, love, understanding and prayers.
To my beloved family KERBANI.
To all those who have been supportive, caring and patient, I dedicate this simple work.

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ملخص

في هذا العمل، ندرس كمرة تيموشينكو رقائقية ذات سلوك حراري مرن تتضمن حد تأخير زمني، حيث يخضع انتقال الحرارة لقانون كاتانيو. نثبت الوضع الجيد للنظام. أما بالنسبة لنتائج الاستقرار، فإننا نبرهن على الانحلال الأسي للنظام، خاصة في حالة تحقق شرط المساواة في الفرضيات.

الكلمات المفتاحية : كمرة تيموشينكو الرقائقية، الوضع الجيد، الانحلال الأسي، حد التأخير.

Résumé

Dans ce travail, nous étudions une poutre stratifiée de Timoshenko thermoélastique avec terme de retard, où la conduction thermique suit la loi de Cattaneo. Nous démontrons le caractère bien posé du système. Pour les résultats de stabilité, nous prouvons la décroissance exponentielle des solutions, particulièrement dans le cas d'égalité dans les hypothèses.

Mots-clés : Poutre de Timoshenko stratifiée ; Problème bien posé ; Décroissance exponentielle ; Terme de retard.

Abstract

In this work, we consider a thermoelastic Laminated Timoshenko beam with delay term, where the heat conduction is given by Cattaneo's law. We establish the well posedness of the system. For stability results, we prove exponential decay of the system, especially in the case of equality in the hypothesis.

Key-words : Laminated Timoshenko Beam ; Well-posedness ; exponential decay ; Delay term.

Contents

Notations

$\mathbb{R}$	Set of real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{Z}$	Set of integers
$C^k([a,b])$	Space of k-times continuously differentiable functions on the interval $[a,b]$
$L^p(\Omega)$	Lebesgue space of p-integrable functions on Ω
$H^k(\Omega)$	Sobolev space of order k on the domain Ω
$L^2(0,T;H^1(\Omega))$	Bochner space of square-integrable functions from $(0,T)$ into $H^1(\Omega)$
$\|\cdot\|_{L^2}$	Norm in the Hilbert space L^2
$\|\cdot\|_{H^1}$	Norm in the Sobolev space H^1
$\langle\cdot,\cdot\rangle$	Inner product in L^2
$\partial_t u$ or u_t	Partial derivative of u with respect to time t
Δu	Laplacian of the function u
A	Linear operator generating a C_0-semigroup
e^{tA}	C_0-semigroup generated by the operator A
$E(t)$	Energy functional at time t
τ	Delay parameter

General Introduction

A Laminated beam system is a structural arrangement composed of multiple layers of materials bonded together to form a single, strong, and durable beam. It is designed to provide enhanced strength, stiffness, and load-bearing capacity compared to traditional solid beams. Laminated beams are widely used in various industries and construction applications where high structural performance is required.

The construction of a Laminated beam involves stacking multiple thin layers of compatible materials, such as wood, metal, or composite materials, and bonding them together using adhesives or other bonding agents. The layers are typically oriented such that the grain or fibers run in different directions, which helps distribute loads evenly and increase overall structural integrity.

The use of different materials in the Laminated beam system allows for the optimization of specific properties. For example, wood layers provide excellent compressive strength, while fiberglass or carbon fiber layers offer high tensile strength and stiffness. By combining these materials, Laminated beams can be tailored to meet specific design requirements, such as increased strength, flexibility, or resistance to environmental factors like moisture or temperature variations.

In summary, Laminated beam systems provide a reliable and versatile solution for structural requirements. By combining different materials in layered configurations, Laminated beams offer enhanced strength, durability, and performance characteristics, making them a preferred choice in modern construction and engineering projects.

The best contribution in this field is the work of Hansen et al. [5], who developed a Laminated beam model that describes the vibrations in a structure consisting of two layered identical beams of uniform thickness stuck together by an adhesive (of negligible thickness), in such a way that a slip is permitted while they are continuously in contact with each other. In the absence of interfering forces, the system of the model takes the following form :

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ I\rho s_{tt} - Ds_{xx} + G(\psi - w_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t = 0, \end{cases} \tag{0.1}$$

In modeling in the biological, physical, and social sciences, it is sometimes necessary to take account of time delays inherent in the phenomena. The inclusion of delays explicitly in the equations is often a simplification or idealization that is introduced because a detailed description of the underlying processes is too complicated to be modeled mathematically or because some of the details are unknown. In these cases, a key question arises : How does the qualitative behavior of the system depend on the form and magnitude of the delays ? In this paper, we shall examine how we can apply the distributed delay term to the Laminated beam model.

The latter was introduced for the first time by Hansen and Spies . This research is particularly relevant due to the widespread applicability of Laminated materials in the industry. Hansen and Spies presented a mathematical model of two-layer beams with structural damping due to the inter-slip, which was obtained by considering the adhesive layer's behavior and its interaction with the beam layers. Their work highlights the importance of incorporating time delays and damping effects to accurately capture the dynamic behavior of Laminated beam systems.

There are some results related to Laminated beam equations that study global existence and stability from the relevant system. In addition, by adding appropriate damping effects such as frictional damping (boundary) viscoelastic, or internal damping. Whereas, if we add linear damping terms to two of the three equations, System (1.1) is exponentially stable under the assumption (equal wave speeds) $\frac{\rho}{I\rho} = \frac{G}{D}$. But if we add these conditions in all equations, the system decays exponentially without assuming equal wave speeds. As for thermoelastic Laminated Timoshenko beam, there are few results including the work of Liu et al. and Apalara. In a previous study, the authors considered the following Laminated beams with past history :

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x + \theta_x = 0, \\ I\rho(3\omega - \psi)_{tt} - D(3\omega - \psi)_{xx} + \int_0^t g(s)(3\omega - \psi)_{xx}(t-s)ds - G(\psi - \varphi_x) - \theta = 0, \\ I\rho\omega_{tt} - D\omega_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\gamma\omega + \frac{4}{3}\beta\omega_t = 0, \\ k\theta_t - \tau\theta_{xx} + \varphi_{tx} + (3\omega - \psi)_t = 0. \end{cases} \tag{0.2}$$

They established the global well posedness of solutions to the system and the stability of the system. If $\beta \neq 0$, they proved the exponential and polynomial stabilities depending on the behavior of the kernel function g and if $\beta = 0$, they established exponential stability in case of equal wave speeds assumption, and exponential stability does not exist in case of nonequal wave speeds assumption. Apalara considered a Laminated beam with second sound of the following system :

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x = 0, \\ I\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \omega_x) + \delta\theta_x = 0, \\ I\rho s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t = 0, \\ \rho_3\theta_t + q_x + \delta(3s - \psi)_{tx} = 0, \\ \tau q_t + \alpha q + \theta_x = 0. \end{cases} \tag{0.3}$$

They established the global well posedness and proved exponential and polynomial stabilities depending on the parameter χ. Feng considered a Timoshenko–Coleman–Gurtin system and studied the long-time dynamics of the system. And the study of Aouadi et al., where the authors considered two classes of nonuniform thermoelastic Timoshenko systems and established global well posedness and proved stability results. We can mention two new results of Laminated beams with thermal damping and a result of a coupled hyperbolic equations with a heat equation of second sound. Recently, Feng considered the following system :

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x + \delta\theta_x = 0, \\ I\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \omega_x) = 0, \\ I\rho s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t = 0, \\ \rho_3\theta_t + q_x + \delta\omega_{tx} = 0, \\ \tau q_t + \alpha q + \theta_x = 0. \end{cases} \tag{0.4}$$

They established the global well posedness and stability of systems. Introducing a distributed delay term makes our problem different from those considered so far in the literature. The importance of this term appears in many works, and this is due to the fact that many phenomena depend on their past. Also, its influence on the asymptotic behavior of the solution for the different types of problems such as Timoshenko system, transmission problem, wave equation, and thermoelastic system. In complement to Feng's work and previous works, we are working to prove the well posedness and stability results with distributed delay for the cases of equal and nonequal speeds of wave propagation under appropriate assumptions, and we prove these results using the energy method.

In this work, we are concerned with the following system :

$$\begin{cases} \rho\omega_{tt} + G(\psi - \omega_x)_x = 0, \\ I\rho(\psi - \omega_x)_{tt} - D(\psi - \omega_x)_{xx} - G(\psi - \omega_x) = 0, \\ I\rho s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \frac{\delta}{3}\theta_x + \frac{4}{3}\gamma s + \frac{4}{3}\mu_1 s_t + \frac{4}{3}\mu_2 s_t(x, t-\tau) = 0, \\ \rho_3\theta_t - k\theta_{xx} + q_x + \delta s_{tx} = 0, \\ \beta q_t + \alpha q + \theta_x = 0, \end{cases} \tag{0.5}$$

where $(x,t) \in (0,1)\times(0,\infty)$, with the Neumann-Dirichlet boundary conditions

$$\omega_x(0,t) = \omega_x(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad t \geq 0,$$

$$s(1,t) = s(0,t) = \theta(0,t) = \theta(1,t) = q(0,t) = q(1,t) = 0, \quad t \geq 0,$$

and the initial data

$$\omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad \psi(x,0) = \psi_0(x),$$

$$\psi_t(x,0) = \psi_1(x), \quad s(x,0) = s_0(x), \quad s_t(x,0) = s_1(x),$$

$$s_t(x, t-\tau) = f_0(x, t-\tau), \quad (x,t) \in (0,1) \times (0,\tau),$$

$$\theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x).$$

Here ρ, G, $I\rho$, D, γ, β, δ, ρ_3, α and k are positive constants, and $\mu_1 > 0$ and μ_2 is a real number satisfying :

$$\begin{aligned} &(\text{H1}) \quad \mu_1 = |\mu_2|. \\ &(\text{H2}) \quad \chi = \frac{G}{\rho} - \frac{D}{I_\rho} = 0 \end{aligned} \tag{0.6}$$

with $x \in (0,1)$ and $t \geq 0$. Hence ρ, $I\rho$, G, D, β, and γ are density, mass moment of inertia, shear stiffness, flexural rigidity, adhesive damping parameter, and adhesive stiffness respectively. The parameters δ is The coupling coefficient between mechanical stress and thermal gradient, and α is The thermal conductivity or thermal damping coefficient. The parameters μ_1 and μ_2 in the system represent damping effects, each with a distinct role. The term μ_1 corresponds to the internal damping coefficient or viscous damping parameter, which accounts for the immediate damping effect proportional to the velocity s_t. On the other hand, μ_2 represents the delay damping coefficient, which introduces a damping effect that depends on the velocity at a previous time $t-\tau$, where τ is the time delay. This term models a memory or hereditary effect in the system, reflecting how past behavior influences the current damping. Together, μ_1 and μ_2 are essential for capturing both instantaneous and time-delayed damping mechanisms, which are critical for accurately describing the dynamic behavior of systems with memory-dependent damping. The variables $w = w(x,t)$ and $\psi = \psi(x,t)$ represent the transverse displacement of the beam from its equilibrium position and the rotation angle, respectively. The term $3s - \psi$ denotes the effective rotation angle, and $s = s(x,t)$ is proportional to the amount of slip along the interface. The first two equations of the system are derived based on the Timoshenko beam theory, while the third equation describes the dynamics of the slip. If $s(x,t)$ is identically zero, the standard Timoshenko system is recovered. Additionally, if $\beta \neq 0$, the adhesion at the interface produces a restorative force to counteract the interfacial slip. In the absence of adhesive damping (i.e., $\beta = 0$), the third equation describes the dynamics of slip in a coupled Laminated beam without structural damping.

In this work, we consider a thermoelastic Laminated Timoshenko beam model with a constant delay term, where the heat conduction is governed by Cattaneo's law. The system couples thermal and mechanical effects and involves memory through the delay in the feedback.

The manuscript is organized as follows :

In Chapter 1, we recall some basic notions and fundamental results from functional analysis that are necessary for the mathematical treatment of the problem. These include properties of Banach and Hilbert spaces, Sobolev and Bochner spaces, essential inequalities, and semigroup theory. This background provides the necessary tools for establishing well-posedness and stability.

In Chapter 2, we study the well-posedness of the delayed thermoelastic system. To handle the delay term, we introduce an appropriate transformation and rewrite the system as an abstract evolution equation. Using semigroup theory and elliptic regularity results, we prove that the system operator is maximal dissipative. Hence, by the Lumer–Phillips theorem, we obtain the existence and uniqueness of solutions.

In Chapter 3, we investigate the exponential stability of the system. We first establish some technical lemmas and then construct an appropriate Lyapunov functional equivalent to the energy. By applying the energy method and the multiplier technique, we derive an exponential decay rate for the energy of the system in the case of equal wave speeds.

Overall, this work provides a rigorous mathematical framework for the analysis of thermoelastic Timoshenko beams with delay effects, covering both existence and stability aspects.

Chapitre 1

Preliminaries

In this chapter, we recall some basic knowledge in functional analysis, most of which will be used in the subsequent chapter. The reader can easily find the details in the related literature.see, e.g [24] [25] [26] [27] [5]

1.1 Functional Analysis Results

1.1.1 Normed and Banach Spaces

Proposition 1 *The dual space X' equipped with the norm $\|f\|_{X'} = \sup_{\|u\| \leq 1} |f(u)|$ is a Banach space.*

We verify the norm properties and completeness :

(a) **Non-negativity**: $\|f\|_{X'} \geq 0$ by definition, and $\|f\|_{X'} = 0$ implies $f(u) = 0$ for all $u \in X$, hence $f = 0$.

(b) **Homogeneity**: For $\alpha \in \mathbb{K}$,

$$\|\alpha f\|_{X'} = \sup_{\|u\| \leq 1} |\alpha f(u)| = |\alpha| \sup_{\|u\| \leq 1} |f(u)| = |\alpha| \|f\|_{X'}.$$

(c) **Triangle inequality**: For $f, g \in X'$,

$$\|f + g\|_{X'} \leq \sup_{\|u\| \leq 1} (|f(u)| + |g(u)|) \leq \|f\|_{X'} + \|g\|_{X'}.$$

— **Completeness**: Let $\{f_n\}$ be Cauchy in X'. For each $u \in X$, $\{f_n(u)\}$ is Cauchy in $\mathbb{K}$ since

$$|f_n(u) - f_m(u)| \leq \|f_n - f_m\|_{X'} \|u\| \to 0.$$

Define $f(u) = \lim_{n\to\infty} f_n(u)$. Then :
- f is linear (by linearity of limits).
- f is bounded since $\|f_n\|$ is uniformly bounded (Cauchy sequences are bounded).
- $\|f_n - f\|_{X'} \to 0$ by the Cauchy property.

1.1.2 Weak and Weak-* Topologies

Remark 1 *If $x_n \rightharpoonup x$ and $x_n \rightharpoonup y$ in X, then $x = y$.*

For all $f \in X'$, we have :

$$f(x) = \lim_{n\to\infty} f(x_n) = f(y).$$

By the Hahn-Banach theorem, if $x \neq y$, there exists $f \in X'$ such that $f(x) \neq f(y)$. Thus, $x = y$.

1.1.3 Hilbert Spaces

Theorem 1 (Weak Sequential Compactness) *Every bounded sequence in a Hilbert space H has a weakly convergent subsequence.*

- If $\dim H < \infty$, weak convergence is equivalent to strong convergence (all norms are equivalent).
- If $\dim H = \infty$:
  - Let $\{x_n\}$ be bounded in H. By separability, H has a countable orthonormal basis $\{e_k\}$.
  - For each k, $\{\langle x_n, e_k\rangle\}$ is bounded. Extract a diagonal subsequence x_{n_j} such that $\langle x_{n_j}, e_k\rangle \to \alpha_k$ for all k.
  - Define $x = \sum_{k=1}^{\infty} \alpha_k e_k$. Then $x_{n_j} \rightharpoonup x$ since for all $y \in H$,

$$\langle x_{n_j}, y\rangle = \sum_{k=1}^{\infty} \langle x_{n_j}, e_k\rangle\langle e_k, y\rangle \to \sum_{k=1}^{\infty} \alpha_k \langle e_k, y\rangle = \langle x, y\rangle.$$

Theorem 2 (Lax-Milgram) *Let $a : H \times H \to \mathbb{R}$ be a bounded, coercive bilinear form. For any $f \in H'$, there exists a unique $u \in H$ such that $a(u, v) = \langle f, v\rangle$ for all $v \in H$.*

- For fixed u, the map $v \mapsto a(u, v)$ is a bounded linear functional. By Riesz representation, there exists $Au \in H$ such that $a(u, v) = \langle Au, v\rangle$.

- Coercivity implies $\|Au\| \geq \alpha\|u\|$, so A is injective with closed range.
- If Range$(A) \neq H$, there exists $w \neq 0$ such that $\langle Au, w\rangle = 0$ for all u. But $a(w,w) \geq \alpha\|w\|^2 > 0$, a contradiction. Thus, A is bijective.
- The solution is $u = A^{-1}f$.

1.1.4 Sobolev Spaces

Theorem 3 (Sobolev Embedding) *For $1 \leq p < n$, $W^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ where $p^* = \frac{np}{n-p}$.*

- For $u \in C_c^\infty(\Omega)$, the Gagliardo-Nirenberg-Sobolev inequality gives :

$$\|u\|_{L^{p^*}} \leq C\|\nabla u\|_{L^p}.$$

- For general $u \in W^{1,p}(\Omega)$, approximate by smooth functions and use density.

1.1.5 Bochner Spaces

Proposition 2 *The dual of $L^p(0,T;X)$ is $L^q(0,T;X')$ for $1 \leq p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$.*

- For $F \in L^q(0,T;X')$, define $\Lambda_F \in (L^p(0,T;X))'$ by :

$$\Lambda_F(f) = \int_0^T \langle F(t), f(t)\rangle_{X',X}\, dt.$$

  Hölder's inequality gives $\|\Lambda_F\| \leq \|F\|_{L^q(0,T;X')}$.
- Conversely, for $\Lambda \in (L^p(0,T;X))'$, construct F via :

$$\langle F(t), x\rangle_{X',X} = \Lambda(\mathbf{1}_{[0,t]} \otimes x),$$

  where $\mathbf{1}_{[0,t]} \otimes x$ denotes the function $s \mapsto x$ on $[0,t]$ and 0 elsewhere.

1.2 Some useful inequalities

In this section, we shall recall some inequalities which will be used in the subsequent chapters.

1.2.1 Young's Inequality

Theorem 4 *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad a, b > 0.$$

Consider the function $f(x) = \frac{x^p}{p} + \frac{y^q}{q} - xy$ for fixed $y > 0$. Taking the derivative with respect to x and setting it to zero gives $x^{p-1} = y$. Substituting $x = y^{1/(p-1)}$ and using $\frac{1}{p} + \frac{1}{q} = 1$, we find $f(x) \geq 0$, which proves the inequality.

Theorem 5 (Young's inequality with ϵ) *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$ab \leq \epsilon \frac{a^p}{p} + \frac{1}{\epsilon^{q/p}} \frac{b^q}{q}, \quad a, b > 0.$$

Apply the standard Young's inequality to $(\epsilon^{1/p} a)$ and $(b/\epsilon^{1/p})$:

$$ab = (\epsilon^{1/p} a)(b/\epsilon^{1/p}) \leq \frac{(\epsilon^{1/p} a)^p}{p} + \frac{(b/\epsilon^{1/p})^q}{q} = \epsilon \frac{a^p}{p} + \frac{1}{\epsilon^{q/p}} \frac{b^q}{q}.$$

Corollary 1 *Let* $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. *Then*
(i) $a^{1/p} b^{1/q} \leq \frac{a}{p} + \frac{b}{q}$.
(ii) $a^{1/p} b^{1/q} \leq \frac{a}{p\epsilon^{1/q}} + \frac{b\epsilon^{1/p}}{q}, \forall \epsilon > 0$.
(iii) $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b, \quad 0 < \alpha < 1$.

(i) Substitute a and b in the standard Young's inequality with $a^{1/p}$ and $b^{1/q}$. (ii) Use Young's inequality with ϵ similarly. (iii) Set $p = 1/\alpha$ and $q = 1/(1-\alpha)$ in (i).

1.2.2 Hölder's Inequality

Theorem 6 *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then if* $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, *we have*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}.$$

Without loss of generality, assume $\|f\|_p = \|g\|_q = 1$. By Young's inequality,

$$|f(x)g(x)| \leq \frac{|f(x)|^p}{p} + \frac{|g(x)|^q}{q}.$$

Integrating over Ω gives the result.

Theorem 7 (Generalized Hölder inequality) *Let* $1 \leq p_1, \ldots, p_m \leq \infty$, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = 1$, *then if* $f_k \in L^{p_k}(\Omega)$ *for* $k = 1, \ldots, m$, *we have*

$$\int_\Omega |f_1 \cdots f_m| \, dx \leq \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\Omega)}.$$

[proof.] Proceed by induction on m, using the base case $m = 2$ (standard Hölder) and applying Hölder to the product $f_1 \cdots f_{m-1}$ and f_m.

Remark 2 *We have the corresponding weighted Hölder inequality of the integral form. Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, $\omega(x) > 0$ *on* Ω. *Then*

$$\int_\Omega |fg|\omega(x) \, dx \leq \left(\int_\Omega |f(x)|^p \omega(x) \, dx \right)^{1/p} \left(\int_\Omega |g(x)|^q \omega(x) \, dx \right)^{1/q}.$$

[proof.] Apply the standard Hölder inequality to the functions $f\omega^{1/p}$ and $g\omega^{1/q}$.

1.2.3 Poincaré's Inequality

Theorem 8 *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $f \in H_0^1(\Omega)$. *Then there is a positive constant* C *such that*

$$\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}, \quad \forall f \in H_0^1(\Omega).$$

[proof.] By density, it suffices to prove for $f \in C_c^\infty(\Omega)$. Extend f by zero outside Ω and use the Fourier transform or the fundamental theorem of calculus along lines to bound $|f|$ by $|\nabla f|$, then integrate.

Theorem 9 *Let* Ω *be a bounded domain of* C^1 *in* $\mathbb{R}^n$. *There is a positive constant* C, *such that for any* $f \in H^1(\Omega)$,

$$\|f - \tilde{f}\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)},$$

where $\tilde{f} = \frac{1}{|\Omega|} \int_\Omega f(x) \, dx$ *is the integral average of* f *over* Ω, *and* $|\Omega|$ *is the volume of* Ω.

[proof.] Assume the contrary and use Rellich-Kondrachov compactness to derive a contradiction, exploiting that $\|\nabla f_k\|_{L^2} \to 0$ implies f_k is Cauchy in L^2.

Theorem 10 *Under the assumption of Theorem 15, for any $f \in H^1(\Omega)$, we have*

$$\|f\|_{L^2(\Omega)} \leq C\left(\|\nabla f\|_{L^2(\Omega)} + \left|\int_\Omega f\,dx\right|\right).$$

[proof.] Decompose f into its average and the fluctuation $f - \tilde{f}$, then apply the previous Poincaré inequality to the fluctuation.

1.2.4 Cauchy-Schwarz Inequality

Theorem 11 *Let $f \in L^2(\Omega)$, $g \in L^2(\Omega)$, we have*

$$\int_\Omega |fg|\,dx \leq \left(\int_\Omega |f(x)|^2\,dx\right)^{1/2}\left(\int_\Omega |g(x)|^2\,dx\right)^{1/2}.$$

[proof.] This is the special case $p = q = 2$ of Hölder's inequality. Alternatively, consider $\int(|f| + t|g|)^2 \geq 0$ for all $t \in \mathbb{R}$ and minimize the quadratic in t.

1.3 Basic theory of semigroups

In this section, we recall some basic knowledge in semigroups, most of which will be used in the subsequent chapters. A general reference to this topic is [27], [5].

1.3.1 C0-Semigroups of Linear Operators

Definition 1 (Semigroups) *Let X be a Banach space, the one-parameter family $S(t)$, $0 \leq t < \infty$ from X to X is called a semigroup if*

(i) $S(0) = I$ (where I is the identity operator on X).

(ii) $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$ (the semigroup property).

Definition 2 *The linear operator A defined by*

$$D(A) = \left\{x \in X : \lim_{t\to 0^+} \frac{S(t)x - x}{t} \text{ exists}\right\}$$

and

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t} = \frac{d}{dt}S(t)x\bigg|_{t=0} \quad \text{for all } x \in D(A)$$

is called the infinitesimal generator of the semigroup $S(t)$. $D(A)$ is called the domain of A.

Definition 3 (C0-Semigroups) *A semigroup $S(t)$, $0 \leq t < \infty$, from X to X is called a strongly continuous semigroup of bounded linear operators if*

$$\lim_{t \to 0^+} S(t)x = x \quad \text{for all } x \in X,$$

or

$$\lim_{t \to 0^+} \|S(t)x - x\| = 0 \quad \text{for all } x \in X.$$

i.e., $S(t)$ is a $C0$-semigroup.

Definition 4 *A semigroup $S(t)$, $0 \leq t < \infty$, is called a semigroup of contractions if there exists a constant $\alpha > 0$ ($0 < \alpha < 1$) such that for all $t > 0$,*

$$\|S(t)x - S(t)y\| \leq \alpha \|x - y\|, \quad \text{for all } x, y \in X.$$

1.3.2 Hille-Yoshida Theorem

Definition 5 *An unbounded linear operator $A : D(A) \subset H \to H$ is said to be monotone if it satisfies*

$$\langle Av, v \rangle \geq 0 \quad \forall v \in D(A).$$

It is called maximal monotone if, in addition, $R(I + A) = H$, i.e.,

$$\forall f \in H, \exists u \in D(A) \text{ such that } u + Au = f.$$

Proposition 3 *Let A be a maximal monotone operator. Then*

(1) $D(A)$ is dense in H.

(2) A is a closed operator.

(3) For every $\lambda > 0$, $(I + \lambda A)$ is bijective from $D(A)$ onto H, $(I + \lambda A)^{-1}$ is a bounded operator, and $\|(I + \lambda A)^{-1}\|_{L(H)} \leq 1$.

Theorem 12 (Hille-Yoshida) *Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$, there exists a unique function*

$$u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A))$$

satisfying

$$\begin{cases} \frac{du}{dt} + Au = 0 \quad \text{on } [0, +\infty), \\ u(0) = u_0. \end{cases}$$

Theorem 13 (Lumer-Phillips theorem) *Let A be a linear operator defined on a linear subspace $D(A)$ of the Banach space X. Then A generates a contraction semigroup if and only if*

(1) $D(A)$ is dense in H.

(2) A is dissipative, and

(3) $(A - \lambda_0 I)$ is surjective for some $\lambda_0 > 0$, where I denotes the identity operator.

An operator satisfying the last two conditions is called maximally dissipative.

Chapitre 2

Well-posedness of the System

In this chapter, we study the well-posedness of the following system of a thermoelastic Laminated Timoshenko beam with a delay term :

2.1 Statement of the Problem

We consider a system of Laminated beam with delay term given by :

$$\begin{cases} \rho\,\omega_{tt} + G(\psi - \omega_x)_x = 0, & (x,t) \in (0,1) \times (0,\infty). \\ I\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - \omega_x) = 0, & (x,t) \in (0,1) \times (0,\infty). \\ I\rho\, s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \frac{\delta}{3}\theta_x + \frac{4}{3}\gamma s + \frac{4}{3}\mu_1 s_t + \frac{4}{3}\mu_2 s_t(x, t-\tau) = 0, & (x,t) \in (0,1) \times (0,\infty). \\ \rho_3\theta_t - k\theta_{xx} + q_x + \delta s_{tx} = 0, & (x,t) \in (0,1) \times (0,\infty). \\ \beta q_t + \alpha q + \theta_x = 0, & (x,t) \in (0,1) \times (0,\infty). \end{cases} \tag{2.1}$$

with the following Neumann-Dirichlet boundary conditions :

$$\begin{aligned} &\omega_x(0,t) = \omega_x(1,t) = \psi(0,t) = \psi(1,t) = 0, \quad t \geq 0, \\ &s(0,t) = s(1,t) = \theta(0,t) = \theta(1,t) = q(0,t) = q(1,t) = 0, \quad t \geq 0, \end{aligned} \tag{2.2}$$

and the initial data :

$$\begin{aligned} &\omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad \psi(x,0) = \psi_0(x), \\ &\psi_t(x,0) = \psi_1(x), \quad s(x,0) = s_0(x), \quad s_t(x,0) = s_1(x), \\ &s_t(x,t-\tau) = f_0(x,t-\tau), \quad (x,t) \in (0,1) \times (0,\tau), \\ &\theta(x,0) = \theta_0(x), \quad q(x,0) = q_0(x). \end{aligned} \tag{2.3}$$

2.2 Well-posedness

In this section, we state the well-posedness results of a Laminated beam with constant delay feedback. Firstly, we proceed by introducing the following new variable as defined based on the delayed system (2.1):

$$y(x,\varrho,t) = s_t(x, t-\tau\varrho), \quad (x,\varrho,t) \in (0,1)\times(0,1)\times(0,\infty).$$

Then, we obtain the following system :

$$\begin{cases} \tau\, y_t(x,\varrho,t) + y_\varrho(x,\varrho,t) = 0, \\ y(x,0,t) = s_t(x,t), \quad (x,t) \in (0,1)\times(0,\infty). \end{cases} \tag{2.4}$$

Consequently, the problem is equivalent to the following system :

$$\begin{cases} \rho\,\omega_{tt} + G(\psi - \omega_x)_x = 0, \\ I\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - \omega_x) = 0, \\ I\rho\, s_{tt} - Ds_{xx} + G(\psi - \omega_x) + \frac{\delta}{3}\theta_x + \frac{4}{3}\gamma s + \frac{4}{3}\mu_1 s_t + \frac{4}{3}\mu_2 y(x,1,t) = 0, \\ \rho_3\theta_t - k\theta_{xx} + q_x + \delta s_{tx} = 0, \\ \beta q_t + \alpha q + \theta_x = 0, \\ \tau\, y_t(x,\varrho,t) + y_\varrho(x,\varrho,t) = 0. \end{cases} \tag{2.5}$$

with the new initial condition :

$$y(x,\varrho,0) = f_0(x,-\tau\varrho), \quad (x,\varrho) \in (0,1)\times(0,1).$$

Meanwhile, from (2.5) - (2.3) , it follows that

$$\frac{d^2}{dt^2}\int_0^1 \omega(x,t)\,dx = 0.(3.7)$$

So, by solving equation (3.7) and using the initial data, we get :

$$\int_0^1 \omega(x,t)\,dx = t\int_0^1 \omega_1(x)\,dx + \int_0^1 \omega_0(x)\,dx.$$

Consequently, if we define the shifted function :

$$\tilde{\omega}(x,t) = \omega(x,t) - t\int_0^1 \omega(x)\,dx - \int_0^1 \omega_0(x)\,dx, (2.6)$$

then we have :

$$\int_0^1 \tilde{\omega}(x,t)\,dx = 0, \quad \forall t \geq 0.$$

Therefore,the use of Poincare's inequality for $\tilde{\omega}$ is justified.In addition,simple substitution shows that $(\tilde{\omega}, \psi, s, \theta, q)$ satisfies System (2.5) Henceforth,we work with $\tilde{\omega}$ instead of ω but write ω for simplicity of notation. First,we introduce the vector function

$$U := (\omega, \omega_t, 3s - \psi, (3s - \psi)_t, s, s_t, \theta, q, y)^T$$

and the new dependent variable $v = \omega_t, u = \psi_t, \varphi = s_t$, then System (2.5) can be written as follows :

$$\begin{cases} U_t = \mathcal{A}U, \\ U(0) = U_0 = (\omega_0, \omega_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, \theta_0, q_0, f_0)^T, \end{cases} \tag{2.7}$$

and we have the energy space $\mathcal{H}$ given by ;

$$\begin{aligned}\mathcal{H} =& H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2(0,1) \\ & \times L^2((0,1) \times (0,1))\end{aligned}$$

where

$$\begin{aligned} L^2_*(0,1) &= \left\{ \phi \in L^2(0,1) \,\middle|\, \int_0^1 \phi(x)\,dx = 0 \right\}, \\ H^1_*(0,1) &= H^1(0,1) \cap L^2_*(0,1), \\ H^2_*(0,1) &= \left\{ \phi \in H^2(0,1) \,\middle|\, \phi_x(1) = \phi_x(0) = 0 \right\}. \end{aligned}$$

the domain of $\mathcal{A}$ is given by

$$D(A) = \left\{ U \in \mathcal{H} \,\middle|\, \begin{array}{l} \omega \in H^2_* \cap H^1_*, \quad v \in H^1_*, \\ 3s - \psi, s, \theta \in H^2 \cap H^1_*, \quad (3\varphi - u), \varphi, q \in H^1_0, \\ y, y_\varrho \in L^2((0,1) \times (0,1) \times (0,\tau), \quad y(x,0,t) = \varphi(x,t) \end{array} \right\}.$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is the linear operator defined by

$$AU = \begin{pmatrix} v \\ -\frac{G}{\rho}(\psi - \omega_x)_x \\ 3\varphi - u \\ \frac{1}{I\rho}\left[D(3s-\psi)_{xx} + G(\psi - \omega_x)\right] \\ \varphi \\ \frac{1}{I\rho}\left[-Ds_{xx} + G(\psi - \omega_x) + \frac{4}{3}\gamma s + \frac{4}{3}\mu_1 \phi + \frac{4}{3}\mu_2 y(x,1,t) + \frac{\delta}{3}\theta_x\right] \\ -\frac{1}{\rho_3}\left[q_x - k\theta_{xx} + \delta\varphi_x\right] \\ \frac{1}{\beta}\left[-\alpha q - \theta_x\right] \\ -\frac{1}{\tau} y_\varrho \end{pmatrix} \tag{2.8}$$

for any

$$U = (\omega, \omega_t, 3s - \psi, (3s-\psi)_t, s, s_t, \theta, q, y)^T$$

$$\hat{U} = \left(\hat{\omega}, \hat{\omega}_t, 3\hat{s} - \hat{\psi}, (3\hat{s} - \hat{\psi})_t, \hat{s}, \hat{s}_t, \hat{\theta}, \hat{q}, \hat{y}\right)^T$$

we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} \langle U, \hat{U}\rangle_H =& \rho \int_0^1 \omega_t \hat{\omega}_t \, dx + I\rho \int_0^1 (3\varphi - u)(3\hat{\varphi} - \hat{u})\, dx + I_\rho \int_0^1 \varphi \hat{\varphi}\, dx \\ &+ G \int_0^1 (\psi - \omega_x)(\hat{\psi} - \hat{\omega}_x)\, dx + \rho_3 \int_0^1 \theta \hat{\theta}\, dx + \beta \int_0^1 q\hat{q}\, dx \\ &+ D \int_0^1 (3s - \psi)_x (3\hat{s} - \hat{\psi})_x \, dx + \frac{4}{3}\gamma \int_0^1 s\hat{s}\, dx + D \int_0^1 s_x \hat{s}_x \, dx \\ &+ k \int_0^1 \theta_x \hat{\theta}_x \, dx + \frac{4}{3}\tau|\mu_2| \int_0^1 \int_0^1 y(x,\varrho)\hat{y}(x,\varrho)\, d\varrho\, dx. \end{aligned} \tag{2.9}$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we can give the following existence result

Theorem 14 *Let $U_0 \in \mathcal{H}$ and assume that (0.6) holds.Then there exists a unique solution $U \in C(\mathbb{R}^+, \mathcal{H})$ of the problem (2.7). Furthermore, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in \mathcal{C}(\mathbb{R}^+, \mathcal{D}(\mathcal{A})) \cap \mathcal{C}^1(\mathbb{R}^+, \mathcal{H})$$

[proof] First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U_0 \in \mathcal{D}(\mathcal{A})$

and using (0.6). we have

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_H = &- \frac{4}{3}\mu_1 \int_0^1 \varphi^2\, dx - \frac{4}{3}\mu_2 \int_0^1 \int_0^1 yy_\varrho\, d\varrho\, dx \\ &- \alpha \int_0^1 q^2\, dx - k \int_0^1 \theta_x^2\, dx \end{aligned} \tag{2.10}$$

For the third term of the right-hand side of (2.10), we have

$$\begin{aligned} -\int_0^1 \int_0^1 yy_\varrho\, d\varrho\, dx &= -\int_0^1 \int_0^1 \frac{d}{d\varrho}(y^2) d\, d\varrho\, dx \\ &= -\frac{1}{2}\int_0^1 y^2(x,1,t)\, dx + \frac{1}{2}\int_0^1 y^2(x,0,t)\, dx \end{aligned} \tag{2.11}$$

By using Young's inequality, we get :

$$-\mu_2 \int_0^1 \varphi y(x,1,t)\, dx \leq \frac{|\mu_2|}{2} \int_0^1 \varphi^2 dx + \frac{|\mu_2|}{2} \int_0^1 y^2(x,1,t)\, dx \tag{2.12}$$

Substituting (2.11), (2.12) into (2.10), using the fact that $y(x,0,t) = \varphi(x,t)$ and (0.6), we obtained

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \leq -\alpha \int_0^1 q^2\, dx - k \int_0^1 \theta_x^2\, dx \leq 0 \tag{2.13}$$

Hence, $\mathcal{A}$ is dissipative.
Next, We prove that the operator A is maximal. It is sufficient to show that the operator $(\mathrm{Id}-\mathcal{A})$ is surjective. Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}$, we prove that there exists a unique $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9)^T \in \mathcal{D}(\mathcal{A})$ such that

$$(\mathrm{Id} - \mathcal{A})V = F. \tag{2.14}$$

This is

$$\begin{cases} v_1 - v_2 = f_1, \\ v_2 - Gv_{1xx} + 3Gv_{5x} - Gv_{3x} = \rho f_2, \\ v_3 - v_4 = f_3, \\ I_\rho v_4 - Dv_{3xx} + Gv_5 - Gv_3 - Gv_{1x} = I_\rho f_4, \\ v_5 - v_6 = f_5, \\ (I_\rho + \frac{4}{3}\mu_1)v_6 - Dv_{5xx} - Gv_3 - Gv_{1x} - \frac{\delta}{3}v_{7x} + (3G + \frac{4}{3}\gamma)v_5 + \frac{4}{3}\mu_2 y(x,1,t) = I_r ho f_6, \\ \rho_3 v_7 + v_{8x} + \delta v_{6x} - kv_{7xx} = \rho_3 f_7, \\ (\beta+\alpha)v_8 + v_{7x} = f_8, \\ \tau y(x,\varrho,t) + y_\varrho(x,\varrho,t) = \tau f_9. \end{cases} \tag{2.15}$$

We note that the last equation in (2.15) with $y(x,0,t) = \varphi(x,t)$ has unique solution given by

$$y(x,\varrho,t) = e^{-\varrho\tau}\varphi(x,t) + \tau e^{\varrho\tau}\int_0^\varrho e^{\sigma\tau} f_9(x,\sigma,t)\,d\sigma \tag{2.16}$$

then

$$y(x,1,t) = e^{-\tau}\varphi(x,t) + \tau e^{\tau}\int_0^1 e^{\sigma\tau} f_9(x,\sigma,t)\,d\sigma \tag{2.17}$$

we have

$$v_2 = v_1 - f_1, v_4 = v_3 - f_3, v_6 = v_5 - f_5 \tag{2.18}$$

Inserting (2.16), (2.17) and (2.18) in $(2.15)_2$, $(2.15)_4$, $(2.15)_6$ and $(2.15)_7$, we get

$$\begin{cases} \rho v_1 - Gv_{1xx} - Gv_{3x} + 3v_{5x} = h_1 \\ (I_\rho + G)v_3 - Dv_{3xx} - 3Gv_5 + Gv_1 = h_2 \\ \mu_3 v_5 - Dv_{5xx} - Gv_3 - Gv_{1x} + \frac{5}{3}v_{7x} = h_3 \\ \rho_3 v_7 + v_{8x} + \delta v5x - kv_{7xx} = h_4 \\ (\beta+\alpha)v_8 + v_{7x} = h_5 \end{cases} \tag{2.19}$$

where

$$\begin{cases} \mu_3 = I_\rho + \frac{4}{3}\mu_1 + \frac{4}{3}\gamma + \frac{4}{3}\mu_2 e^{-\tau} \\ h_1 = \rho(f_1 + f_2) \\ h_2 = I_\rho(f_3 + f_4) \\ h_3 = I_\rho(f_5 + f_6) + \frac{4}{3}(\mu_2 e^{-\tau}) f_5 - \frac{4}{3}\tau\mu_2 e^{\tau} \int_0^1 e^{\tau\sigma} f_9(x,\sigma,t)\, d\sigma + \frac{4}{3}\mu_1 f_5 \\ h_4 = \rho_3 f_7 - \delta f_{5x} \\ h_5 = \beta f_8 \end{cases} \tag{2.20}$$

We multiply (2.19) by $\hat{v_1}, \hat{v_3}, \hat{v_5}, \hat{v_7} and \hat{v_8}$respectively, and integrate their sum over (0,1) to get the following variational formulation :

$$B((v_1, v_3, v_5, v_7, v_8), (\hat{v_1}, \hat{v_3}, \hat{v_5}, \hat{v_7}, \hat{v_8})) = \Gamma(\hat{v_1}, \hat{v_3}, \hat{v_5}, \hat{v_7}, \hat{v_8}) \tag{2.21}$$

where

$$B(H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times L^2(0,1))^2 \to \mathbb{R}$$

$$\begin{aligned} B((v_1, v_3, v_5, v_7, v_8), (\hat{v_1}, \hat{v_3}, \hat{v_5}, \hat{v_7}, \hat{v_8})) = & \rho \int_0^1 v_1\hat{v}_1\, dx + G\int_0^1 v_{1x}\hat{v}_{1x}\, dx - G\int_0^1 v_{3x}\hat{v}_1\, dx \\ & - (I_\rho + G)\int_0^1 v_3\hat{v}_3\, dx + D\int_0^1 v_{3x}\hat{v}_{3x}\, dx - 3G\int_0^1 v_5\hat{v}_3\, dx \\ & + G\int_0^1 v_{1x}\hat{v}_3\, dx + 9G\int_0^1 v_5\hat{v}_5\, dx + 3D\int_0^1 v_{5x}\hat{v}_{5x}\, dx \\ & - 3G\int_0^1 v_3\hat{v}_5\, dx - 3G\int_0^1 v_{1x}\hat{v}_5\, dx + \rho_3\int_0^1 v_7\hat{v}_7\, dx \\ & + k\int_0^1 v_{7x}\hat{v}_{7x}\, dx + (\beta + \alpha)\int_0^1 v_8\hat{v}_8\, dx \end{aligned} \tag{2.22}$$

and

$$\Gamma(H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times L^1_0(0,1))^2 \to \mathbb{R}$$

$$\Gamma(\hat{v_1}, \hat{v_3}, \hat{v_5}, \hat{v_7}, \hat{v_8}) = \int_0^1 h_1\hat{v_1}\,dx + \int_0^1 h_2\hat{v_3}\,dx + \int_0^1 h_3\hat{v_5}\,dx + \int_0^1 h_4\hat{v_7}\,dx + \int_0^1 h_5\hat{v_8}\,dx \tag{2.23}$$

Now, for$V \in H^1_*(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times H^1_0(0,1) \times L^1_0(0,1)$, equipped with the norm

$$\begin{aligned} ||(v_1, v_3, v_5, v_7, v_8)|\,|^2_V = {} & ||(-v_{1x} - v_3 + 3v_5)||^2_2 + ||v_1||^2_2 + ||v_8||^2_2 \\ & + ||v_{3x}||^2_2 + ||v_{5x}||^2_2 + ||v_7||^2_2 + ||v_{7x}||^2_2 \end{aligned} \tag{2.24}$$

then we have

$$\begin{aligned} B((v_1, v_3, v_5, v_7, v_8), (v_1, v_3, v_5, v_7, v_8)) = {} & \rho \int_0^1 v_1^2\,dx + G\int_0^1 v_{1x}^2\,dx - G\int_0^1 v_{3x}v_1\,dx \\ & - (I_\rho + G)\int_0^1 v_3^2\,dx + D\int_0^1 v_{3x}^2\,dx - 3G\int_0^1 v_5v_3\,dx \\ & + G\int_0^1 v_{1x}v_3\,dx + 9G\int_0^1 v_5^2\,dx + 3D\int_0^1 v_{5x}^2\,dx \\ & - 3G\int_0^1 v_3v_5\,dx - 3G\int_0^1 v_{1x}v_5\,dx + \rho_3\int_0^1 v_7^2\,dx \\ & + k\int_0^1 v_{7x}^2\,dx + (\beta+\alpha)\int_0^1 v_8^2\,dx \\ & \le \rho\int_0^1 v_1^2\,dx + (\beta+\alpha)\int_0^1 v_8^2\,dx + D\int_0^1 v_{3x}^2\,dx \\ & + 3D\int_0^1 v_{5x}^2\,dx + G\int_0^1 (-v_{1x} - v_3 + 3v_5)^2\,dx \end{aligned} \tag{2.25}$$

Thus, for some $M_0 > 0$

$$B((v_1, v_3, v_5, v_7, v_8), (v_1, v_3, v_5, v_7, v_8)) \le M_0||(v_1, v_3, v_5, v_7, v_8)||^2_V.$$

Thus, B is coercive. Consequently, using the $Lax - Milgram$ theorem, we conclude that (2.1) has a unique solution.

$$\begin{aligned} & v_1 \in H^1_*(0,1) \\ & v_3, v_5 \in H^1_0(0,1) \\ & v_7 \in H^1_0(0,1) \\ & v_8 \in L^2(0,1) \end{aligned} \tag{2.26}$$

Substituting v_1, v_3, v_5, v_5, v_7 and v_8 into (2.17), (2.18) respectivly, we have

$$\begin{aligned} &v_2 \in H^1_*(0,1) \\ &v_4, v_6 \in H^1_0(0,1) \\ &y, y_\varrho \in L^2((0,1),(0,1),(0,\tau)) \end{aligned} \tag{2.27}$$

Let $v_1 \in H^1_0(0,1)$ and denote

$$\hat{v_1} = v_1(x) - \int_0^1 v_1(\epsilon)\, d\epsilon.$$

which gives us $\hat{v_1} \in H^1_*(0,1)$. Now we replace $(\hat{v_1}, \hat{v_3}, \hat{v_5}, \hat{v_7}, \hat{v_8})$ by $(\hat{v_1}, 0, 0, 0, 0)$ in (2.19) to obtain

$$\rho \int_0^1 v_1 \hat{v}_1 \, dx + G \int_0^1 (-v_{1x} - v_3 + 3v_5)(-\hat{v}_{1x})\, dx + \delta(\beta+\alpha) \int_0^1 v_8 \hat{v}_1 = \int_0^1 h_1 \hat{v}_1 \, dx \tag{2.28}$$

We get

$$G v_{1xx} = \rho v_1 - G v_{3x} + 3G v_{5x} - h_1 \in L^2(0,1) \tag{2.29}$$

Thus

$$v_1 \in H^2(0,1)$$

Moreover, (3.36) also holds for any $\phi \in C^1([0,1])$. Then, by using integration by parts, we obtain :

$$G v_{1x}(1)\phi(1) - G v_{1x}(0)\phi(0) + G \int_0^1 v_{5x}\phi \, dx + (\beta+\alpha) \int_0^1 v_8 \phi \, dx = 0.$$

Then, we get for any $\phi \in C^1([0,1])$:

$$\int_0^1 v_{1xx}\phi \, dx + \int_0^1 v_8 \phi \, dx - = 0.$$

Because ϕ is arbitrary and

$$\int_0^1 v_1 \phi \, dx = G \int_0^1 v_{3x} \phi \, dx,$$

we deduce that :

$$v_{1x}(0) = v_{1x}(1) = 0,$$

hence $v_1 \in H^2(0,1)$. Using similar arguments as above, we obtain :

$$v_3, v_5 \in H^2(0,1) \cap H_0^1(0,1), \quad v_7 \in H_0^1(0,1), \quad v_8 \in H^1(0,1).$$

Finally, the application of regularity theory for linear elliptic equations guarantees the existence of a unique $U \in D(A)$ such that equation (2.14) is satisfied. Consequently, we conclude that A is a maximal dissipative operator.

Hence, by the Lumer–Phillips theorem, we have the well-posedness result. This completes the proof.

Chapitre 3

Stability of the solutions

In this chapter, we state the stability results and then prove them using the energy method. We establish an exponential stability result for the considered system in case of equal wave speeds.

3.1 Technical lemmas

In this section, we state and prove some technical lemmas which are fundamental in the proof of our stability result. We use the multiplier technique to establish stability results for the energy of the solution of system (3.1) . This requires constructing a suitable Lyapunov functional equivalent to energy as we elaborate in the subsequent section.

Lemma 1 *the energy functional defined by*

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^1 \bigg[\rho v^2 + I_\rho(3s-\psi)_t^2 + 3I_\rho s_t^2 + G(\psi-\omega_x)^2 + \rho_3\theta^2 + \beta q^2 + D(3s-\psi)_x^2 \\ & + 4\gamma s^2 + 3Ds_x^2 + 4\tau|\mu_2|\int_0^1 y^2(x,\varrho,t)\,d\varrho\bigg]\,dx \end{aligned} \tag{3.1}$$

satisfies

$$E'(t) \le -\alpha\int_0^1 q^2\,dx - k\int_0^1 \theta_x^2 \le 0 \tag{3.2}$$

Where $\alpha, k > 0$ *.*

[proof.] Multiplying the equations of system (2.1) by $\omega_t, (3s-\psi)_t, s_t, \theta$ and q, respectively, we get

$$\begin{cases} \rho\omega_{tt}\omega_t - G(\psi-\omega_x)_x\omega_t = 0, \\ I(3s-\psi)_{tt}(3s-\psi)_t + D(3s-\psi)_{xx}(3s-\psi)_t - G(\psi-\omega_x)(3s-\psi)_t = 0, \\ I_\rho s_{tt}s_t + Ds_{xx}s_t + G(\psi-\omega_x)s_t + \frac{\delta}{3}\theta_x s_t - \frac{4}{3}\gamma s s_t - \frac{4}{3}\mu_1 s_t^2 - \frac{4}{3}\mu_2 y(x,1,t)s_t = 0, \\ \rho_3\theta_t\theta - k\theta_{xx}\theta + q_x\theta + 3\delta s_{tx}\theta = 0, \\ \beta q_t q + \alpha q^2 + \theta_x q = 0, \\ \tau y_t(x,\varrho,t) + y_{x_1}(x,\varrho,t) = 0. \end{cases} \tag{3.3}$$

then, we integrate over (0,1), using integration by parts, and using (2.2), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \Bigg(\rho\omega - t^2 + I_\rho(\psi-\omega_x)_t^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2 \\ &\quad + D(3s-\psi)_x^2 + G(\psi-\omega_x)^2 + \rho_3\theta^2 + \beta q^2\Bigg)\,dx \\ &\quad = -4\mu_1\int_0^1 s_t^2\,dx - \alpha\int_0^1 q^2\,dx - k\int_0^1 \theta_x^2\,dx \\ &\quad - 4\mu_2\int_0^1 s_t y(x,1,t)\,dx. \end{aligned} \tag{3.4}$$

Using Young's inequality, we arrive at

$$\mu_2\int_0^1 s_t y(x,1,t)dx \leq \frac{|\mu_2|}{2}\int_0^1 s_t^2 dx + \frac{|\mu_2|}{2}\int_0^1 y^2(x,1,t)dx \tag{3.5}$$

Now, multiplying the last equation in (3.6) by $|\mu_2|\,y$ and integrating the result over $(0,1)\times(0,1)$, we get :

$$\begin{aligned}\frac{\tau\left|\mu_2\right|}{2}\frac{d}{dt}\int_0^1\int_0^1 y^2(x,\varrho,t)d\varrho dx &= -\left|\mu_2\right|\int_0^1\int_0^1 yy_\varrho d\varrho dx\\ &= -\frac{1}{2}\left|\mu_2\right|\int_0^1\int_0^1\frac{d}{d\varrho}\left(y^2\right)d\varrho dx\\ &= -\frac{1}{2}\left|\mu_2\right|\int_0^1 y^2(x,1,t)dx+\frac{1}{2}\left|\mu_2\right|\int_0^1 y^2(x,0,t)dx\\ &= -\frac{1}{2}\left|\mu_2\right|\int_0^1 y^2(x,1,t)dx+\frac{1}{2}\left|\mu_2\right|\int_0^1 s_t^2(x,t)dx\end{aligned} \tag{3.6}$$

From (3.1), (3.3), (3.4) and (3.5) we get (3.2).
Then, by (0.6), we have

$$E'(t)\leq-\alpha\int_0^1 q^2dx-k\int_0^1\theta_x^2\,dx\leq 0(3.7)$$

Then, we obtain E is a nonincreasing function.

Lemma 2 *The functional*

$$F_1(t):=F_1(t)=-\rho\int_0^1\omega\omega_t\,dx+\rho\int_0^1\omega_t\left(\int_0^x\psi(y)\,dy\right)\,dx(3.8)$$

satisfies

$$F_1'(t)\leq\frac{-\rho}{2}\int_0^1\omega_t^2\,dx+G\int_0^1\left(\psi-\omega_x\right)^2\,dx+9\rho\int_0^1 s_t^2\,dx+\rho\int_0^1(3s-\psi)_t^2\,dx(3.9)$$

[proof.] Direct computations using integration by parts, we have :

$$\begin{aligned}F_1'(t)=&G\int_0^1\omega(\psi-\omega_x)_xdx-\rho\int_0^1\omega_t^2dx-G\int_0^1(\psi-\omega_x)_x\left(\int_0^x\psi(y)\,dy\right)\\ =&-G\int_0^1\omega_x(\psi-\omega_x)dx-\rho\int_0^1\omega_t^2\,dx+G\int_0^1(\psi-\omega_x)_x\psi\,dx\\ &+\rho\int_0^1\omega_t\left(\int_0^x\psi_t(y)\,dy\right)\,dx\\ =&-\rho\int_0^1\omega_t^2dx+G\int_0^1(\psi-\omega_x)^2dx+\rho\int_0^1\omega_t\left(\int_0^x(\psi_t+3s_t-3s_t)(y)dy\right)dx\end{aligned} \tag{3.10}$$

Using Young's, Cauchy-Schwarz, and poincare's inequalities, we obtain (3.9).

Lemma 3

$$F_2(t) := -I_\rho \int_0^1 (3s-\psi)_t\,(3s-\psi)_t\; dx(3.11)$$

satisfies :

$$F_2'(t) \leq -\, I_\rho \int_0^1 (3s-\psi)_t^2 + \left(D + \frac{DG}{2}\right)\int_0^1 (3s-\psi)_x^2\,dx + \frac{G^2}{2D}\int_0^1 (\psi-\omega_x)^2\,dx \tag{3.12}$$

[proof.] Differentiating $F_2(t)$ with respect to t and using integration by parts, we have :

$$\begin{aligned} F_2'(t) &= -I_\rho \int_0^1 (3s-\psi)_t^2\,dx - I_\rho \int_0^1 (3s-\psi)_{tt}(3s-\psi)\,dx \\ &= -I_\rho \int_0^1 (3s-\psi)_t^2\,dx - D\int_0^1 (3s-\psi)_{xx}(3s-\psi)\,dx - G\int_0^1 (\psi-\omega_x)(3s-\psi)\,dx \end{aligned} \tag{3.13}$$

Using Young's and Cauchy-Schwarz inequalities, we obtain (3.12).

Lemma 4

$$F_3(t) := 3I_\rho \int_0^1 s_t s\,dx + 2\mu_1 \int_0^1 s_t^2\,dx + 3\rho\int_0^1 \omega_t\left(\int_0^x s(y)\,dy\right)\,dx(3.14)$$

satisfies :

$$\begin{aligned} F_3'(t) \leq &-\frac{3D}{2}\int_0^1 s_x^2\,dx - 2\gamma\int_0^1 s^2\,dx + \varepsilon_1\int_0^1 \omega_t^2\,dx + C(\varepsilon_1)\int_0^1 s_t^2\,dx \\ &+ C\int_0^1 y^2(x,1,t)\,dx \end{aligned} \tag{3.15}$$

[Proof.] Differentiating $F_3(t)$ with respect to t and using integration by parts, we have :

$$\begin{aligned}F_3'(t) &= 3I_\rho \int_0^1 s_t s\,dx + 3I_\rho \int_0^1 s_t^2\,dx + 4\mu_1 \int_0^1 s_t s\,dx \\ &+ 3\rho \int_0^1 \omega_t t \left(\int_0^x s(y)\,dy\right) dx + 3\rho \int_0^1 \omega_t \left(\int_0^x s_t(y)\,dy\right) dx \\ &= 3D \int_0^1 s_{xx} s\,dx - \delta \int_0^1 \theta_x s\,dx - 4\gamma \int_0^1 s^2\,dx - 4\mu_2 \int_0^1 sy(x,1,t)\,dx \\ &+ 3I_\rho \int_0^1 s_t^2\,dx + 3\rho \int_0^1 \omega_t \left(\int_0^x s_t(y)\,dy\right) dx\end{aligned} \tag{3.16}$$

using Young's and Cauchy-Schwarz inequalities, we obtain (3.15).

Lemma 5

$$F_4(t) := -\int_0^1 (3s-\psi)_t \omega_x\,dx - \int_0^1 (3s-\psi)_x \omega_t\,dx + 3\int_0^1 (3s-\psi) s\,dx \tag{3.17}$$

satisfies :

$$\begin{aligned}F_4'(t) \leq &-\frac{D}{2I_\rho}\int_0^1 (3s-\psi)_x^2\,dx + \varepsilon_2 \int_0^1 (3s-\psi)_t^2\,dx + C(\varepsilon_2)\int_0^1 s^2\,dx \\ &+ C\int_0^1 (\psi_t - \omega_x)^2\,dx\end{aligned} \tag{3.18}$$

[Proof.] Differentiating $F_4(t)$ with respect to t and using integration by parts, we get :

$$\begin{aligned}F_4'(t) &= -\int_0^1 (3s-\psi)_{tt}\omega_x\,dx - \int_0^1 (3s-\psi)_x \omega_{tt}\,\omega_x\,dx + 3\int_0^1 (3s-\psi)_{tt} s\,dx \\ &+ 3\int_0^1 (3s-\psi)_t s_t\,dx \\ &= -\frac{D}{I_\rho}\int_0^1 (3s-\psi)_{xx}\omega_x\,dx - \frac{G}{I_\rho}\int_0^1 (\psi-\omega_x)\omega_x\,dx + \frac{G}{\rho}\int_0^1 (3s-\psi)(\psi-\omega_x)_x\,dx \\ &+ \frac{3D}{I_\rho}\int_0^1 (3s-\psi)_{xx} s\,dx + \frac{3G}{I_\rho}\int_0^1 (\psi-\omega_x) s\,dx + 3\int_0^1 (3s-\psi)_t s_t\,dx\end{aligned} \tag{3.19}$$

Using Young's and Cauchy-Schwarz inequalities, we obtain (3.18).

Lemma 6

$$F_5(t) = \int_0^1 (\psi - \omega_x)s_t\,dx - \int_0^1 \omega_t s_t\,dx \tag{3.20}$$

satisfies :

$$F_5'(t) \leq \frac{-G}{2I_\rho}\int_0^1 (\psi-\omega_x)^2\,dx + c\int_0^1 s^2\,dx + c(\varepsilon_3)\int_0^1 s_t^2\,dx + \varepsilon\int_0^1 (3s-\psi)_t^2\,dx - c\int_0^1 y^2(x,1,t)\,dx \tag{3.21}$$

[proof.]

$$\begin{aligned}
F_5'(t) &= \int_0^1 (\psi-\omega_x)_t s_t\,dx + \int_0^1 (\psi-\omega_x)s_{tt}\,dx - \int_0^1 \omega_{tt}s_t\,dx - \int_0^1 \omega_t s_{tt}\,dx \\
&= \int_0^1 (\psi-\omega_x)_t s_t\,dx - \frac{D}{I_\rho}\int_0^1 (\psi-\omega_x)s_{xx}\,dx - \frac{G}{I_\rho}\int_0^1 (\psi-\omega_x)^2\,dx - \frac{\gamma}{3I_\rho}\int_0^1 (\psi-\omega_x)\theta_x\,dx \\
&\quad - \frac{4\gamma}{3I_\rho}\int_0^1 (\psi-\omega_x)s\,dx - \frac{4\mu_1}{3I_\rho}\int_0^1 (\psi-\omega_x)s_t\,dx - \frac{4\mu_2}{3I_\rho}\int_0^1 (\psi-\omega_x)y(x,1,t)\,dx \\
&\quad + \frac{G}{\rho}\int_0^1 (\psi-\omega_x)_x s_x\,dx \quad - \int_0^1 \omega_t s_{tx}\,dx + \int_0^1 (3s_t - (3s-\psi)_t)\,s_t\,dx
\end{aligned} \tag{3.22}$$

Using Young's and Cauchy-Schwarz inequalities and (0.6), we obtain (3.21).

Lemma 7

$$F_6(t) = 3\rho_3 I_\rho \int_0^1 s_t\left(\int_0^x \theta(y)\,dy\right)dx \tag{3.23}$$

satisfies :

$$\begin{aligned}
F_6'(t) \leq &-\frac{3I_\rho\delta}{2}\int_0^1 s_t^2\,dx + c\int_0^1 q^2\,dx + c(\varepsilon_8,\varepsilon_9,\varepsilon_{10},\varepsilon_{11})\int_0^1 \theta_x^2\,dx + \varepsilon_8\int_0^1 s_x^2\,dx \\
&+ \varepsilon_9\int_0^1 (\psi-\omega_x)^2\,dx + \varepsilon_{10}\int_0^1 s^2\,dx + \varepsilon_{11}\int_0^1 y^2(x,1,t)\,dx
\end{aligned} \tag{3.24}$$

[proof.]

$$
\begin{aligned}
&F_6'(t) = 3\rho_3 I_\rho \int_0^1 s_{tt} \left(\int_0^x \theta dy \right) dx + 3\rho_3 I_\rho \int_0^1 s_t \left(\int_0^x \theta_t dy \right) dx \\
&= \rho_3 \int_0^1 \left[3Ds_{xx} - 3G(\psi - \omega_x) - \delta\theta_x - 4\gamma s - 4\mu_1 s_t - 4\mu_2 y(x,1,t) \right] \\
&\int_0^x \theta(y)\, dy\, dx + 3I_\rho \int_0^1 s_t \left[-q - \delta s_t \right] dx \\
&= -3I_\rho \delta \int_0^1 s_t^2\, dx - 3I_\rho \int_0^1 s_t q\, dx - 3D\rho_3 \int_0^1 s_x \theta - 3G\rho_3 \int_0^1 (\psi - \omega_x) \left(\int_0^x \theta dy \right) dx \\
&+ \rho_3 \delta \int_0^1 \theta^2 dx - 4\gamma\rho_3 \int_0^1 s \left(\int_0^x \theta dy \right) dx - 4\rho_3 \mu_1 \int_0^1 s_t \left(\int_0^x \theta dy \right) dx \\
&- 4\rho_3 \mu_2 \int_0^1 y(x,1,t) \left(\int_0^x \theta dy \right) dx
\end{aligned}
\tag{3.25}
$$

Using Young's and Cauchy-Schwarz inequalities, we obtain (3.24).

Lemma 8

$$F_7(t) := \int_0^1 \int_0^1 e^{-2\varrho\tau} y^2(x,\varrho,t) d\varrho dx (3.26)$$

satisfies

$$F_7'(t) \leq -2F_7(t) - \eta_1 \int_0^1 y^2(x,1,t)\, dx + \frac{1}{\tau} \int_0^1 s_t^2\, dx (3.27)$$

[proof.]

$$
\begin{aligned}
F_7'(t) &= \frac{-2}{\tau} \int_0^1 \int_0^1 e^{-2\varrho\tau} y_\varrho y\, d\varrho\, dx \\
&= -2 \int_0^1 \int_0^1 e^{-2\varrho\tau} y^2 d\varrho dx - \frac{1}{\tau} \int_0^1 \int_0^1 \frac{d}{d\varrho} \left(e^{-2\varrho\tau} y^2 \right) d\varrho\, dx \\
&= -2F_7(t) - \frac{1}{\tau} \int_0^1 \left[e^{-2\tau} y^2(x,1,t) - y^2(x,0,t) \right] dx \\
&= -2F_7(t) - \frac{1}{\tau e^{2\tau}} \int_0^1 y^2(x,1,t)\, dx + \frac{1}{\tau} \int_0^1 s_t^2\, dx
\end{aligned}
\tag{3.28}
$$

finally, setting $\eta_1 = \frac{1}{\tau e^{2\tau}}$ we obtain (3.27)

3.2 Exponential stability

in this section, we study the exponential stability of systems.

Theorem 15 *Assume (0.6), there exist positive constants λ_1 and λ_2 such that the energy functional given by (3.1) satisfies*

$$E(t) \leq \lambda_2 e^{-\lambda_1 t}, \quad \forall t \geq 0(3.29)$$

[proof.] We define a Lyapunov functional

$$\mathcal{L}(t) := NE(t) + \sum_{i=1}^{7} N_i F_i(t)(3.30)$$

where N and $N_i, i = 1, \ldots, 7$, are positive constants to be selected later. By differentiating
(3.30) and using (3.2), (3.9), (3.12), (3.15), (3.18), (3.21), (3.24) and (3.27), we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{D}{2I\rho} - \left(D + \frac{Dc_*}{2}\right) N_2\right] \int_0^1 (3s-\psi)_x^2 dx \\
& - \left[\frac{3I_\rho \delta}{2} N_6 - 9\rho N_1 - c(\varepsilon_1) N_3 - c(\varepsilon_2) N_4 - c(\varepsilon_3) N_5\right] \int_0^1 s_t^2 dx \\
& - \left[I_\rho N_2 - \rho N_1 - \varepsilon_2 N - 4 - \varepsilon_3 N_5\right] \int_0^1 (3s-\psi)_t^2 dx \\
& - \left[\frac{G}{2I_\rho} N_5 - GN_1 - \frac{G^2}{2D} N_2 - cN_4 - \epsilon_5 N_6\right] \int_0^1 (\psi - \omega_x)^2 \, dx \\
& - \left[\frac{\rho}{2} N_1 - \epsilon_1 N_3\right] \int_0^1 \omega_t^2 dx \qquad (3.31) \\
& - \left[kN - cN_5 - c(\varepsilon_4, \varepsilon_5, \varepsilon_6, \varepsilon_7) N_6\right] \int_0^1 \theta_x^2 dx \\
& - \left[\alpha N - cN_6\right] \int_0^1 q^2 dx \\
& - \left[2\gamma N_3 - cN_5 - \varepsilon_6 N_6\right] \int_0^1 s^2 dx - \left[\frac{3D}{2} N_3 - \varepsilon_4 N_6\right] \int_0^1 s_x^2 dx \\
& - \left[\eta_1 N_7 - cN_3 - cN_5 - \varepsilon_7 N_6\right] \int_0^1 y^2(x,1,t) dx - 2N_7 F_7(t)
\end{aligned}$$

By setting

$$\begin{aligned}
&\epsilon_1 = \frac{\rho N_1}{4N_3}, \quad \epsilon_2 = \frac{I_\rho N_2}{4N_4}, \quad \epsilon_3 = \frac{I_\rho N_2}{4N_5}, \quad \epsilon_4 = \frac{3DN_3}{4N_6}, \\
&\epsilon_5 = \frac{GN_5}{4I_\rho N_6}, \quad \epsilon_6 = \frac{\gamma N_3}{N_6}, \quad \varepsilon_7 = \frac{\eta_1 N_7}{2N_6}
\end{aligned}$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \left[\frac{D}{2I\rho}N_4 - \left(D + \frac{Dc_*}{2}\right)N_2\right]\int_0^1 (3s-\psi)_x^2 dx \\
& - \left[\frac{3I_\rho\delta}{2}N_6 - 9\rho N_1 - c(N_1,N_3)N_3 - c(N_2,N_4)N_4 - c(N_2,N_5)N_5\right]\int_0^1 s_t^2 dx \\
& - \left[\frac{I_\rho}{2}N_2 - \rho N_1\right]\int_0^1 (3s-\psi)_t^2 dx \\
& - \left[\frac{G}{4I_\rho}N_5 - GN_1 - \frac{G^2}{2D}N_2 - cN_4\right]\int_0^1 (\psi-\omega_x)^2 dx \\
& - \left[\frac{\rho}{2}N_1\right]\int_0^1 \omega_t^2 dx \\
& - [kN - cN_5 - c(N_3,N_6,N_5,N_3,N_7)N_6]\int_0^1 \theta_x^2 dx \\
& - [\alpha N - cN_6]\int_0^1 q^2 dx \\
& - [\gamma N_3 - cN_5]\int_0^1 s^2 dx - \left[\frac{3D}{4}N_3\right]\int_0^1 s_x^2 dx \\
& - \left[\frac{\eta_1}{2}N_7 - cN_3 - cN_5\right]\int_0^1 y^2(x,1,t)dx - 2N_7F_7(t)
\end{aligned} \tag{3.32}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. We fixed $N_1 > 0$

$$\alpha_0 = \frac{\rho}{4}N_1 > 0$$

and we choose N_2 large enough such that

$$\alpha_1 = \frac{I_\rho}{2}N_2 - \rho N_1 > 0$$

then, we choose N_4 large enough such that

$$\alpha_2 = \frac{D}{2I\rho}N_4 - \left(D + \frac{Dc_*}{2}\right)N_2 > 0$$

then, we choose N_5 large enough such that

$$\alpha_3 = \frac{G}{4I_\rho}N_5 - GN_1 - \frac{G^2}{2D}N_2 - cN_4 > 0$$

then, we choose N_3 large enough such that

$$\alpha_4 = \gamma N_3 - cN_5 > 0,$$

$$\alpha_5 = \frac{3D}{4} N_3 > 0$$

then, we choose N_7 large enough such that

$$\alpha_6 = \frac{\eta_1}{2} N_7 - cN_3 - cN_5 > 0$$

then, we choose N_6 large enough such that

$$\alpha_7 = \frac{3I_\rho \delta}{2} N_6 - 9\rho N_1 - c(N_1, N_3)N_3 - c(N_2, N_4)N_4 - c(N_2, N_5)N_5 > 0$$

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \alpha_2 \int_0^1 (3s - \psi)_x^2 dx - \alpha_3 \int_0^1 (\psi - \omega_x)^2 \, dx - \alpha_7 \int_0^1 s_t^2 dx \\
& - \alpha_6 \int_0^1 y^2(x, 1, t) dx - \alpha_1 \int_0^1 (3s - \psi)_t^2 dx \\
& - (kN - c) \int_0^1 \theta^2 dx - \alpha_5 \int_0^1 s_x^2 dx - [\alpha N - c] \int_0^1 q^2 dx - \alpha_0 \int_0^1 \omega_t^2 dx \\
& - \alpha_4 \int_0^1 s^2 dx + 2N_7 \int_0^1 \int_0^1 y^2(x, \varrho, t) \, d\varrho \, dx.
\end{aligned} \tag{3.33}$$

On the other hand, if we let

$$\mathcal{T}(t) = \sum_{i=1}^{7} F_i(t).$$

Exploiting Young's, Cauchy-Schwarz, and Poincaré inequalities, we get

$$\begin{aligned}
|\mathcal{T}(t)| \leq & c \int_0^1 \left(\omega_t^2 + (3s - \psi)_t^2 + (3s - \psi)_x^2 + (\psi - \omega_x)^2 + s^2 + s_x^2 + s_t^2 \right) dx \\
& + c \int_0^1 \left(\theta^2 + q^2 \right) dx + c \int_0^1 \int_0^1 y^2(x, \varrho, t) d\varrho dx,
\end{aligned}$$

then,

$$|\mathcal{T}(t)| \leq cE(t)$$

Consequently, we obtain

$$|\mathcal{T}(t)| = |\mathcal{L}(t) - NE(t)| \leq cE(t)$$

that is

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (N+c)E(t)$$

Now, by choosing N large enough such that

$$N - c > 0, \quad \alpha N - c > 0, \quad kN - c > 0.$$

and the fact that

$$-\int_0^1 \theta_x^2 \, dx \leq -c \int_0^1 \theta^2 \, dx,$$

and exploiting (2.5) give

$$\mathcal{L}'(t) \leq -k_2 E(t), \tag{3.34}$$

for some $k_2 > 0$, and

$$c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t), \quad \forall t \geq 0,$$

for some $c_1, c_2 > 0$, we have

$$\mathcal{L}(t) \sim E(t),$$

then

$$\mathcal{L}'(t) \leq -\lambda_1 \mathcal{L}(t). \tag{3.35}$$

where $\lambda_1 = \frac{k_2}{c_2}$.
Finally, a simple integration of (3.35), we obtain (3.29). Then the proof is complete.

Conclusion

In this work, we studied a Thermoelastic Laminated Timoshenko beam with a delay term, where we began by proving the well-posedness of the system by using the energy method and under various suitable assumptions. The first was for equal delay coefficients ($\mu_1 = |\mu_2|$), and the second was for equal wave speeds ($\chi = 0$). Regarding stability, we proved the exponential decay of the solutions by constructing a suitable Lyapunov function.

We have some general observations that could serve as a starting point for future work :

(1) In the case of unequal wave speeds ($\chi \neq 0$), the stability is polynomial, and we can see the work [5].

(2) In the general case ($\mu_1 > |\mu_2|$), the stability is exponential and we dispense with the function $F_7(t)$.

(3) Among the goals of this work are the thermal effects, so we assume that the heat capacity conductivity is present ($k\theta_{xx}, k > 0$) and this helps with stability.

(4) In the absence of the heat capacity conductivity ($k = 0$), the stability becomes more complex and this is an open-ended work that can be attempted.

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