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Présenté par :

YAGOUBI Imane Reghia

THEME

Exponential stability of laminated beam With constant delay feedback

Soutenance publique devant le jury composé de :

Dr. OUCHENANE Djamel	M.C.A	Président
Dr. FARES Yazid	M.C.A	Encadreur
DJERADI Fatima Siham	Doctorante	Co-Encadreur
Dr. RAHMOUNE Abdelaziz	M.C.A	Examinateur

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Dedication

I dedicate this work

To my parents who have been my source of inspiration and provided me with their encouragement , love, understanding and prayers.

To my beloved family *YAGOUBI*.

To all those who have been supportive, caring and patient, I dedicate this simple work.

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YAGOUBI Reghia Imane

Abstract

In this memory, we considered a system of laminated beam with an internal constant delay term in the transverse displacement. We established the global existence and the uniqueness of the solution by using the semi-group theory.

Finally, we studied the asymptotic behavior of solution by using the multiplier method, and we proved the exponential stability of the system under suitable assumptions on delay feedback and coefficients of wave propagation speed.

Keys words : Laminated beam, global existence, constant delay, exponential decay.

ملخص

في هذه المذكرة، اعتبرنا نظامًا للحزمة المصفحة مع تأخير ثابت في الإزاحة المستعرضة. أولا برهنا وجود ووحدانية الحل باستخدام نظرية هيل-يوشيدا Hille-Yosida.

أخيرًا، قمنا بدراسة السلوك المقارب للحل باستخدام طريقة الطاقة، وبرهنا الاستقرار الأسي للنظام وفقًا لافتراضات مناسبة حول ردود الفعل المتأخرة ومعاملات سرعة انتشار الموجة.

الكلمات المفتاحية: شعاع مصفح، وجود ووحدانية، تأخير، استقرار أسي.

Résumé

Dans ce mémoire, nous considérons un système de poutres lamellaires avec un terme de retard constant dans le déplacement transversal. Nous établissons l'existence globale et l'unicité de la solution en utilisant la théorie de semi groupe.

En fin, nous étudions les comportements asymptotiques de solution en utilisant la méthode des multiplicateurs et nous montrons la stabilité exponentielle du système sous des hypothèses appropriées sur le retard et les coefficients d'onde vitesse de propagation.

Mot clés : Poutre lamellaire, existence globale, retard, décroissance exponentielle.

Contents

General introduction

A laminated beam system is a structural arrangement composed of multiple layers of materials bonded together to form a single, strong, and durable beam. It is designed to provide enhanced strength, stiffness, and load-bearing capacity compared to traditional solid beams. Laminated beams are widely used in various industries and construction applications where high structural performance is required.

The construction of a laminated beam involves stacking multiple thin layers of compatible materials, such as wood, metal, or composite materials, and bonding them together using adhesives or other bonding agents. The layers are typically oriented such that the grain or fibers run in different directions, which helps distribute loads evenly and increase overall structural integrity.

The use of different materials in the laminated beam system allows for the optimization of specific properties. For example, wood layers provide excellent compressive strength, while fiberglass or carbon fiber layers offer high tensile strength and stiffness. By combining these materials, laminated beams can be tailored to meet specific design requirements, such as increased strength, flexibility, or resistance to environmental factors like moisture or temperature variations.

In summary, laminated beam systems provide a reliable and versatile solution for structural requirements. By combining different materials in layered configurations, laminated beams offer enhanced strength, durability, and performance characteristics, making them a preferred choice in modern construction and engineering projects.

The best contribution in this fild is the work of Hansen et al.[19], the laminated beam model describes the vibrations in a structure consisting of two layered identical beams of uniform thickness stuck together by an adhesive (of negligible thickness), in such a way that a slip is permitted while they are continuously in contact with each other. In the absence of interfering forces, the system of the model takes the following form

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x = 0, \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, \end{cases} \tag{1}$$

with $x \in (0,1)$ and $t \geq 0$. Hence $\rho, I_\rho, G, D, \beta$, and γ are density, mass moment of inertia, shear stiffness, flexural rigidity, adhesive damping parameter and, adhesive stiffness respectively. Similarly, $w = w(x,t)$ denotes the transverse displacement of the beam from its equilibrium position, $\psi = \psi(x,t)$ is the rotation angle, $3s - \psi$ denotes the effective rotation angle and, $s = s(x,t)$ is proportional to the amount of slip along the interface. The first two equations of (1) are derived on the assumption of Timoshenko beam theory and, the third equation describes the dynamics of the slip. Moreover, if $s(x,t)$ is identically zero, then the standard Timoshenko system is recovered. Furthermore, if $\beta \neq 0$, then the adhesion at the interface produces a restoration comparable force to counteract the interfacial slip. Otherwise, in absence of adhesive damping (i.e. $\beta = 0$)), the third equation of describes the dynamics of slip of coupled laminated beam without structural damping.

Laminated beams have wider applications in engineering as structures are often made out of more than one beam or plate stuck together using the appropriate substance depending on their intended purposes. Among other applications, the closest examples one can think of in recent times are the layered glass gorilla screen protection for smart gadgets, windscreens, among others. Being a controlled system, stability is very important. Thus, many researchers among mathematicians and engineers have focused a lot of attention on the study of well-posedness and more importantly, the stability behavior of this differential model, majorly by exploiting different damping mechanisms introduced to the system. We discus some of the results below. The asymptotic behavior of system (1) with boundary feedback controls of the form

$$\begin{cases} w(0,t) = \psi(0,t) = s(0,t), \quad (\psi - w_x)(1,t) = k_1 w_t(1,t), \\ (3s_x - \psi_x)(1,t) = -k_2(3s_t - \psi_t)(1,t), \end{cases}$$

was studied by Wang et al. [38]. The authors established an exponential stability of the system provided that $r_1 = \sqrt{\frac{\rho}{G}} \neq \sqrt{\frac{I_\rho}{D}} = r_2$,
$k_i \neq r_i (i = 1, 2)$.
Interestingly, Tatar [37] and Mustafa [28] obtained the result in [38] under weaker conditions on the parameters ρ, G, I_ρ, and D. D. Some related results were also obtained by Cao et al [13] with different boundary controls.

Apart from stabilization through boundary damping mechanisms, researchers have considered other interesting damping techniques. For example, Raposo [34] introduced extra linear frictional damping terms in the first two equations of (1) in addition to structural damping, and proved exponential stability without further restrictions. Later Apalara et al. [10] established that a single linear frictional damping in the effective rotation angle is sufficient for exponential decay in case of equal wave speeds. Similarly, in [3], the authors consider system (1) with structural damping, and prove that if it is coupled with boundary feedback controls acting through complementary displacements, then no further dissipation or restrictions on parameters are required for exponential decay, otherwise the assumption of equal wave speeds is necessary.

Regarding dissipation through material damping, for laminated beam with infinity memory, we mention the work in [22], in which with only structural damping and suitable assumptions on the relaxation function, the authors established general exponential decay results in case of equal wave speeds and polynomial stability otherwise. For earlier results concerning stabilization of laminated beam through viscoelastic damping, we refer the reader to [14],[25],[29]. Furthermore, regarding stabilization through thermal effects, we cite the result in[24]. The authors investigated a thermoelastic laminated beam with past history, and proved that in presence of structural damping, the solution decays exponentially and polynomially without any restrictions on the parameters. For a system without structural damping, exponential and polynomial decay of the solution are possible in case of equal wave speeds, otherwise, the system lacks exponential stability. Other interesting results about damping through thermal effects can be found in [5], [6], [18] for thermoelasticity, and [23] for thermoelasticity of type III. In all these works, authors mainly established that the system decays exponentially in the case of equal wave speeds and polynomially otherwise, with and without structural damping.

In control systems, time delays are inherent since propagation and transport of material and/or information are involved. Time delay may manifest in form of lags between the input and processing the output, or lags in attaining or restoring the desired system stability after perturbations due to internal or external factors, among others. To explicitly analyse the delay effect on physical properties especially stability, it is preferred that control systems are modeled and represented by delay differential equations. Although there are isolated cases where that voluntary inclusion of delay may benefit control (see [2]) or may not significantly disturb the general system stability, for instance, in [27], time lags have been established as one of the underlying causes of instability and deterioration of the system performance. For example, consider the following system of wave equation

$$\begin{cases} \varphi_{tt} - \Delta\varphi = 0, & \text{in } \Omega \times (0, \infty), \\ \varphi = 0, & \text{on } \Gamma_0 \times (0, \infty), \\ \frac{\partial\varphi}{\partial\nu} = -\mu_1\varphi_t - \mu_2\varphi_t(x, t - \tau), & \text{on } \Gamma_1 \times (0, \infty), \end{cases} \tag{2}$$

where $\varphi = \varphi(x, t), \Omega \subset \mathbb{R}^2$ is open and bounded having a smooth boundary $\partial\Omega \equiv \Gamma_0 \cup \Gamma_1$ and, $\nu = \nu(x)$ is the unit normal to $\partial\Omega$. It is long established that in the absence of delay ($\mu_2 = 0, \mu_1 > 0$), the system (2) is exponentially stable, see [20],[21],[41] .Whereas, on including delay ($\mu_2 > 0$) , Nicaise et al. [30] established that the solution decays exponentially provided that $\mu_2 < \mu_1$, , and in case of a reversed scenario ($\mu_2 \geq \mu_1$),the authors proved that the system solutions become chaotic by introducing a correlating sequence of delays to the solution. Similar conclusions were reached by [16],[39]. For more works regarding constant time delay effect on stability, we refer the reader to [4],[8],[32],[35] and references therein.

In the dynamic Timoshenko beam model, the amplitude of vibrations of the complementary displacements vanishes due to damping. A constant time delay translates into a forward phase shift increasing early time response, which is seen to cause frequency dispersion in displacements [26]. This may require stronger damping to counteract the longer time needed for decay. This delay effect is inherent in the laminated beam model as it is derived on assumption of Timoshenko beam theory. The presence of structural damping in a laminated beam provides some dissipation, which is sufficient for exponential stability in absence of delay on assumption of equal wave speeds [7],[9]. It is yet to be established if the internal structural damping can still solely stabilize the system in presence of delay, rather authors have chosen other damping mechanisms. For instance, Feng [17], considered a laminated beam with three internal constant delay feedbacks, with help of three external boundary controls and some conditions on the system parameters, he established exponential decay result. Seghour et al. [36] on the other hand, investigated a thermoelastic laminated beam with neutral delay in dynamics of slip equation, and established uniform stability provided $\rho = GI_{\rho}.$. The required dissipation was obtained through thermal effects in addition to linear frictional damping in the transverse displacement.

In this work; we study the well-posedness and exponential decay of the system (2.7). The remainder of this work is organized as follows. The first chapter contains the various key concepts on which our study will be founded, such as the presentation of some reminders and definitions and some fundamental theorems (Sobolev spaces, the Lax-Milgram theorem, the Hille-Yosida theorem, some useful inequalities...), in the second chapter we prove the well-posedness of solution of the system (2.7), and in the last chapter we discuss the exponential stability results.

Chapter 1

Preliminaries

In this chapter, we recall some basic knowledge in fonctional analysis, moste of wich will be used in the subseqent chapter. The reader can easily find the detailed in the related literature, see, e.g. [1], [11], [12], [33], and [40].

1.1 Functional Spaces

We denote by $\mathbb{R}^n$ the Euclid space, $\Omega \subset \mathbb{R}^n$ is bonded smoth domain,$C^k(\Omega)$ is the k^{th} differentiable continuous function space in Ω, $C^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space in Ω, $C_c^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space with compact support in Ω

Definition 1. *Let X be a vector space over the filed $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then a semi-norm on X is a function $\|.\| : X \longrightarrow \mathbb{R}$, such that :*

a) $\|x\| \geqslant 0$ for all $x \in X$,

b) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$,

c) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$.

A norm on X is a semi-norm wich also satisfies :

d) $\|x\| = 0 \Rightarrow x = 0$. A vector space X toghether with a norm $\|.\|$ is called a normed linear *space, a normed vector space or simply, a normed space.*

Definition 2. *(Convergent and Cauchy sequences). Let X be a normed space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.*
a) $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

i.e. if

$$\forall\varepsilon > 0; \exists N > 0, \forall n \geq N, \|x_n - x\| < \varepsilon.$$

b) $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence if

$$\forall\varepsilon > 0; \exists N > 0, \forall m, n \geq N, \|x_m - x_n\| < \varepsilon.$$

Normed spaces in which every Cauchy sequence is convergent are called complet normed spaces. In general a normed space is not complete.

Definition 3. *(Banach Spaces). A normed spaces is called a Banach space if it is complet i.e. if any Cauchy sequence inside the space converges to a point of the space. Its dual space X' is the linear space of all contnuous linear functional $f : X \to \mathbb{R}$.*

Proposition 1. *X' equipped with the norm $\|.\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}$$

is olso a Banach space.

Remark 1. *From X' we construct the bidual or secend dual $X'' = (X')'$. Futhermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u)$, $f \in X'$, this satisfies clearly $\|\varphi(x)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$, so it follows that $\|\varphi(x)\| = \|u\|$.*

Definition 4. *Since φ is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X onto a closed subspace of X'', we denote this by

$$X \hookrightarrow X''.$$

Definition 5. *if φ (in the above definition) is onto X'' we say X is reflexive, $X \approxeq X''$*

1.1.1 The weak and weak star topologies:

Let X be a Banach space and $f \in X'$. Denot by

$$\begin{aligned} \varphi_f : & X \to \mathbb{R} \\ & x \mapsto \varphi_f \end{aligned}$$

When f cover X', we obtain a family $(\varphi_f)_{f \in X'}$ of appmications to X in $\mathbb{R}$.

Definition 6. *The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X for which every $(\varphi_f)_{f \in X'}$ is continuous.*

We will define the topology on X', the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_x : & X' \to \mathbb{R} \\ & f \mapsto \varphi_x(f) = \langle f, x \rangle_{X', X} \end{aligned}$$

Definition 7. *The weak star topology on X' is the weakest topology on X' for wich every $(\varphi_x)_{x \in X'}$ is continuous.*

Remark 2. *Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest then* *the topology $\sigma(X', X'')$, and this later is weakest then the strong topology.*

Definition 8. *A sequence (x_n) in X is weakly convergent to x if and only if*

$$\lim_{n\to\infty} f(x_n) = f(x)$$

for evry $f \in X'$, and this is denoted by $x_n \rightharpoonup x$.

Remark 3. *:*

1.If the weak limit exist, it is unique.

2.If $x_n \to x \in X$ (strongly), then $x_n \rightharpoonup x$ (weakly).

3.If $\dim X < \infty$, then the weak convergent implise the strong convergent.

1.1.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analyse, known as Hilbert spaces. Then, we most give some impotant result on these spaces here.

Definition 9. *A Hilbert space $H$is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 1. *Let $(x_n)_{n\in\mathbb{N}}$ is a bounded squence in the Hilbrt sâce H, then it possess a subsequence which converges in the weak topology of H.*

Theorem 2. *In the Hilbrt space, all sequence which converges in the weak topology is bounded.*

Theorem 3. *Let $(x_n)_{n\in\mathbb{N}}$ be sequence which converges to x, in the weak topology and $(y_n)_{n\in\mathbb{N}}$ is an other squence which converge weakly to y, then*

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

Proposition 2. *Let X and Y be tow Hilbert space, let $(x_n)_{n\in\mathbb{N}} \in X$ be a sequence which conveges weakly to $x \in X$, let $A \in \mathcal{L}(X,Y)$. Then, the sequence $(A(x_n))_{n\in\mathbb{N}}$ converges to $A(x)$ in the weak topology of Y.*

Theorem 4. *(The Lax-Milgram Theorem)*
Let X be a Hilbert space and let $a : X \times X \to \mathbb{R}$ be a bilinear functional. Assume that there existe tow constants $C < \infty, \alpha > 0$ such that:
(i) $|a(u,v)| \leq C\|u\|.\|v\|$ for all $(u,v) \in X \times X$ (continuity);
(ii) $a(u,u) \geq \alpha\|u\|^2$ for all $u \in X$ (coerciveness).
Then, for evry $f \in X^*$ (the dual space of X), there exists a unique $u \in X$ such that $a(u,v) = \langle f, v\rangle$ for all $v \in X$.

1.1.3 The $L^p(\Omega)$ spaces

Definition 10. *Let $1 \leq p \leq \infty$, and let Ω be on open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \text{ is measurable and } \int_\Omega |f(x)|^p dx < \infty \right\}$$

Notation 1 : for $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

. If $p = \infty$, we have

$$L^p(\Omega) = \{ f : \Omega \to \mathbb{R} : f \text{ is measurable and there exists } C \text{ suche that}, |f(x)| \leq C \text{ in } \Omega \}$$

Notation 2 : Let $1 \leq p \leq \infty$,we denote by q the conjugate of p i.e. $\frac{1}{p} + \frac{1}{q} = 1$.

Theorem 5. *It is well known that $L^p(\Omega)$ supplied with the norm $\|.\|_p$ is a Banach space, for all $1 \leq p \leq \infty$*

Remark 4. *In particularly, when $p=2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f,g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

is a Hilbert space .

Theorem 6. *For $1<p<\infty$, $L^p(\Omega)$ is reflexive space.*

1.1.4 The Sobolev space $W^{m,p}(\Omega)$

Definition 11.

i) Let $m\in\mathbb{N}$ and $p\in[0,\infty]$. The $W^{m,p}(\Omega)$ is the space of all $f\in L^p(\Omega)$, defined as

$$W^{m,p}(\Omega)=\{f\in L^p(\Omega), such\ \ that\ \ \partial^\alpha f\in L^p(\Omega) for\ \ all\ \ \alpha\in\mathbb{N}^m\}$$

such that $|\alpha|=\sum_{j=1}^n \alpha_j\leq m$ where, $\partial^\alpha=\partial_1^{\alpha_1}\partial_2^{\alpha_2}....\partial_n^{\alpha_n}$.
ii) if $f\in W^{m,p}(\Omega)$, we define its norm to be

$$\|f\|_{W^{m,p}(\Omega)}=\begin{cases}(\sum_{|\alpha|<k}\int_\Omega|D^\alpha f|^p dx)^{\frac{1}{p}} & ;(1\leq p<\infty),\\ \sum_{|\alpha|<k} ess\sup|D^\alpha f| & ;(p=\infty)\end{cases}$$

Definition 12. *We denote by*

$$W_0^{m,p}(\Omega)$$

the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$

Remark 5. *i) if $p=2$ we usully write*

$$H^m(\Omega)=W^{m,2}(\Omega),\quad H_0^m(\Omega)=W_0^{m,2}(\Omega).$$

Supplied with the norm

$$\|f\|_{H^m} = \left(\sum_{|\alpha|\leq m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}}$$

The letter H *is used, since - as we will see -* $H^m(\Omega)$ *is a Hilber space.*

with usual scalar product

$$\langle u, v\rangle = \sum_{|\alpha|\leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx$$

Note that $H^0(\Omega) = L^2(\Omega)$

Theorem 7. *.*

1. $H^m(\Omega)$ *supplied with inner product* $\langle ., .\rangle_{H^m(\Omega)}$ *is Hilbert space.*
2. *If* $m \geq m'$ *,* $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$*.*

Theorem 8. *Assume that* Ω *is an open domain in* $\mathbb{R}^n, n \geq 1$*, with smooth boundary* Γ*. Then,*
i) if $1 \leq p \leq n$*, we have* $W^{1,p} \subset L^q(\Omega)$*, for every* $q \in [p, p^*]$*, where* $p^* = \dfrac{np}{n-p}$*.*
ii) if $p = n$ *we have* $W^{1,p} \subset L^q(\Omega)$*, for every* $q \in [p, \infty)$ *.*
iii) if $p > n$ *we have* $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$*, where* $\alpha = \dfrac{p-n}{p}$*.*

1.1.5 The $L^p(0, T, X)$ space

Definition 13. *Let* X *be a Banach space, denote by* $L^p(0, T, X)$ *the space of measurable functions*

$$f :]0, T[\to X$$
$$t \mapsto f(t)$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \qquad 1 \leq p < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t\in]0,T[} ess\|f(t)\|_X$$

Theorem 9. $L^p(0,T,X)$ *equipped with the norm* $\|.\|_{L^p(0,T,X)}$ *is a Banach space .*

Proposition 3. *Let* X *be a reflexive Banach space,* X' *it's dual, and* $1 \leq p < \infty$, $1 \leq q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. *Then the dual of* $L^p(0,T,X)$ *is identify algebraically and topologically with* $L^q(0,T,X')$

1.2 Some useful inequalities

In this section, we shall recall some inqualities which will be used in the supsequent chapters.

1.2.1 Young's inequality

Theorem 10. *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,*then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, a, b > 0$$

Theorem 11. *(Young inequality with* ε*) Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,*then*

$$ab \leq \varepsilon \frac{a^p}{p} + \frac{1}{\varepsilon^{\frac{q}{p}}} \frac{b^q}{q}, a, b > 0$$

The Young inequality has several variants in the following.

Corollary 1. *Let* $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. *Then*
i) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.
ii) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p\varepsilon^{\frac{1}{q}}} + \frac{b\varepsilon^{\frac{1}{p}}}{q} \qquad ,\forall \varepsilon > 0.$
iii) $a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \qquad ,0 < \alpha < 1.$

1.2.2 Holder's inequality

Theorem 12. *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

then if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, *we have*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)}.\|g\|_{L^q(\Omega)}$$

Theorem 13. *(Generalized Holder inequality) Let* $1 \leq p_1, ..., p_m \leq \infty$,

$\frac{1}{p_1} + ... + \frac{1}{p_m} = 1$, *then if* $f_k \in L^{p_k}(\Omega)$ *for* $k = 1, ..., m$, *we have*

$$\int_\Omega |f_1....f_m| dx \leq \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\Omega)}$$

Remark 6. *We have the corresponding weighted Holder inequality of the integral form. Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$,$\omega(x) > 0$ *on* Ω. *Then*

$$\int_\Omega |fg|\omega(x)dx \leq \left(\int_\Omega |f(x)|^p \omega(x)dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q \omega(x)dx\right)^{\frac{1}{q}}.$$

1.2.3 Poincaré's inequality

In this subsection, we shall recall the Poincar inequality in different forms.

Theorem 14. . *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $f \in H_0^1(\Omega)$. *Then there is a positive constant* C *such that*

$$\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}, \qquad \forall f \in H_0^1(\Omega)$$

Theorem 15. *Let* Ω *be a bounded domain of* C^1 *in* $\mathbb{R}^n$. *There is a positive constant* C , *such that for any* $f \in H^1(\Omega)$.

$$\|f - \tilde{f}\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$$

Where $\tilde{f} = \frac{1}{|\Omega|}\int_\Omega f(x)dx$ *is the integral average of* f *over* Ω, *and* $|\Omega|$ *is the volume fo* Ω.

Theorem 16. *Under assumption of Theorem (15) for any $f \in H^1(\Omega)$, we have*

$$\|f\|_{L^2(\Omega)} \leq C\left(\|\nabla f\|_{L^2(\Omega)} + |\int_\Omega f dx|\right).$$

1.2.4 Cauchy-Schwarz inequality

Theorem 17. *Let $f \in L^2(\Omega)$, $g \in L^2(\Omega)$, we have*

$$\int_\Omega |fg| dx \leq \left(\int_\Omega |f(x)|^2 dx\right)^{\frac{1}{2}} \left(\int_\Omega |g(x)|^2 dx\right)^{\frac{1}{2}}.$$

1.3 Basic theory of semigroups

In this section, we recall some basic knowledge in semigroups,most of whiche will be used in the subsequent chapters. Ageneral rference to this topic is [12], [11],

1.3.1 C_0-Semigroups of Linear Operators

Definition 14. *(Semigroups)*

Let X be a Banach space, the one-parametre family $S(t), 0 \leq t < \infty$ from X to X is called a Semigroups if

(i) $S(0) = I$ (I is the identity operateur on X).

(ii) $S(t+s) = S(t) + S(s)$ for every $t, s \geq 0$ (the Semigroup property).

Definition 15. *The linear operator A defined by*

$$D(A) = \left\{x \in X : \lim_{t \to 0^+} (S(t)x - x)/t \quad exists\right\}$$

and

$$Ax = \lim_{t \to 0^+} (S(t)x - x)/t = \frac{d(S(t)x)}{dt}|_{t=0} \quad for \quad all \quad x \in D(A)$$

is called the infinitesimal generator of the Semigroup $S(t)$, $D(A)$ is called the domain of A.

Definition 16. *(C_0-Semigroups).*

A Semigroup $S(t)$,$0 \leq t < \infty$,from X to X is called a strong continuous Semigroup of bounded lineaar operators if

$$\lim_{t \to 0^+} S(t)x = x \quad for \quad all \quad x \in X,$$

or

$$\lim_{t \to 0^+} \|S(t)x - x\| = 0 \quad for \quad all \quad x \in X.$$

i.e $S(t)$ C_0-Semigroup.

Definition 17. *A semigroup $S(t)$,$0 \leq t < \infty$ is called a semigroup of contraction if there exists a constant $\alpha > 0$ $(0 < \alpha < 1)$ such that for all $t > 0$,*

$$\|S(t)x - S(t)y\| \leq \alpha \|x - y\|, \quad for \quad all \quad x, y \in X.$$

1.3.2 Hille-Yoshida Theorem

Definition 18. *An unbonded linear operator $A : D(A) \subset H \to H$[1] is said to be monotone[2] if it satisfies*

$$\langle Av, v \rangle \geq 0 \qquad \forall v \in D(A).$$

It is called maximal monotone if, in addition; $R(I + A) = H$ i.e

$$\forall f \in H \quad \exists u \in D(A) \quad such \quad that \quad u + Au = f.$$

Proposition 4. *Let A be a maximal monotone operator. Then*

1. $D(A)$ is dense in H.

2. A is closed operator.

3. For evry $\lambda > 0, (I + \lambda A)$ is bijective from $D(A)$ onto H, $(I + \lambda A)^{-1}$ is a bounded operator, and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.

[1] H denotes a Hilbert space
[2] Some authors say that A is accretive or $-A$ is dissipative.

Theorem 18. *(Hille-Yosida) Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function*

$$u \in C^1([0,+\infty); H) \cap C([0,+\infty); D(A))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad on[0,+\infty) \\ u(0) = u_0. \end{cases}$$

Theorem 19. *(Lumer-Phillips theorem) Let A be a linear operator defined on a linear subspace $D(A)$ of the Banach space X. Then A generates a contraction semi group if and only if*

1. *$D(A)$ is dense in H.*
2. *A is dissipative, and*
3. *$(A - \lambda_0 I)$ is surjective for some $\lambda_0 > 0$, where I denotes the identity operator .*

An operator satisfying the last two conditions is called maximally dissipative.

Chapter 2

Laminated beam system: Well-posedness

2.1 Statement of problem

We consider a system of laminated beam with constant delay term acting on the transverse displacement:

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + \mu w_t(x, t - \tau) = 0, & \text{in } (0,1) \times (0,\infty), \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, & \text{in } (0,1) \times (0,\infty), \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, & \text{in } (0,1) \times (0,\infty), \\ w_t(x, -t) = f_0(x,t), & \text{in } (0,1) \times (0,\tau), \\ w(x,0) = w_0, s(x,0) = s_0, \psi(x,0) = \psi_0, & \text{in } (0,1), \\ w_t(x,0) = w_1, s_t(x,0) = s_1, \psi_t(x,0) = \psi_1 & \text{in } (0,1), \\ w(0,t) = s_x(0,t) = \psi_x(0,t) = 0, & \text{in } (0,\infty), \\ w_x(1,t) = s(1,t) = \psi(1,t) = 0, & \text{in } (0,\infty), \end{cases} \tag{2.1}$$

where $w_0, w_1, \psi_0, \psi_1, s_0, s_1, f_0$ is the initial data which belongs to an appropriate space,

$\tau > 0$ is time delay and the none zero real number μ is the weight of delay. With some restrictions on μ , we prove that the adhesive damping is strong enough to stabilize the system exponentially, even in presence of delay without any other additional damping or boundary controls, provided the assumption of equal wave propagation speed $(GI_\rho = \rho D)$ holds.

2.2 Existence and uniqueness

In this section, we state the well-possedness results of a laminated beam with constant delay feedback. Firstly, we proceed by introducing the following new variable as in [30].

$$z(x,\sigma,t) = w_t(x, t-\tau\sigma) \quad \text{in } (0,1)\times(0,1)\times(0,\infty).$$

It follows directly that z satisfies

$$\tau z_t(x,\sigma,t) + z_\sigma(x,\sigma,t) = 0 \quad \text{in } (0,1)\times(0,1)\times(0,\infty).$$

Consequently, the system (2.1) is equivalent to

$$\begin{cases} \rho w_{tt} + G(\psi - w_x)_x + \mu z(x,1,t) = 0, & \text{in } (0,1)\times(0,\infty), \\ I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, & \text{in } (0,1)\times(0,\infty), \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau z_t(x,\sigma,t) + z_\sigma(x,\sigma,t) = 0, & \text{in } (0,1)\times(0,1)\times(0,\infty), \\ z(x,0,t) = w_t(x,t) & \text{in } (0,1)\times(0,\infty), \\ z(x,\sigma,0) = f_0(x,\tau\sigma), & \text{in } (0,1)\times(0,1), \\ w_t(x,-t) = f_0(x,t), & \text{in } (0,1)\times(0,\tau), \\ w(x,0) = w_0, s(x,0) = s_0, \psi(x,0) = \psi_0, & \text{in } (0,1), \\ w_t(x,0) = w_1, s_t(x,0) = s_1, \psi_t(x,0) = \psi_1 & \text{in } (0,1), \\ w(0,t) = s_x(0,t) = \psi_x(0,t) = 0, & \text{in } (0,\infty), \\ w_x(1,t) = s(1,t) = \psi(1,t) = 0, & \text{in } (0,\infty). \end{cases} \tag{2.2}$$

At this step, in order to define the energy functional of system (2.2), we multiply the first four equations in the system (2.2) by w_t, $(3s_t - \psi_t)$, s_t, and $|\mu|z$, respectively, then integrate over $(0,1)$, we have

$$\int_0^1 \rho w_{tt} w_t dx + G\int_0^1 (\psi - w_x)_x w_t dx + \mu \int_0^1 z(x,1) w_t dx = 0,$$

$$I_\rho \int_0^1 (3s_t - \psi_t)_t (3s_t - w_t) dx - D\int_0^1 (3s_x - \psi_x)_x (3s_t - w_t) dx$$

$$-G\int_0^1 (\psi - w_x)(3s_t - \psi_t) dx = 0,$$

$$3I_\rho \int_0^1 s_{tt} s_t dx - 3D\int_0^1 s_{xx} s_t dx + 3G\int_0^1 (\psi - w_x) s_t \, dx + 4\gamma \int_0^1 s_t s \, dx + 4\beta \int_0^1 s_t^2 dx = 0,$$

And

$$\tau|\mu| \int_0^1 \int_0^1 z z_t(x,\sigma,t) d\sigma dx + |\mu| \int_0^1 \int_0^1 z z_\sigma(x,\sigma,t) d\sigma dx = 0,$$

Summing up, we find

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \rho w_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 I_\rho(3s_t-\psi_t)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 3I_\rho s_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 4\gamma s^2 dx \\
&+\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1 \tau|\mu| z^2(x,\sigma)d\sigma dx + 3D\int_0^1 s_x s_{xt}\, dx + D\int_0^1 (3s_x-\psi_x)(3s_x-\psi_x)_t dx \\
&+G\int_0^1 (\psi-w_x)_x w_t dx - 3G\int_0^1 (\psi-w_x)s_t dx + G\int_0^1(\psi-w_x)\psi_t dx \\
&+3G\int_0^1(\psi-w_x)s_t dx = -4\beta\int_0^1 s_t^2 dx - \mu\int_0^1 z(x,1)w_t dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx \\
&+\frac{|\mu|}{2}\int_0^1 w_t^2 dx,
\end{aligned}
\tag{2.3}
$$

Hence,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t-\psi_t)^2 + D(3s_x-\psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2\right] dx \\
&+\frac{1}{2}\frac{d}{dt}\int_0^1 \left[4\gamma s^2 + \int_0^1 \tau|\mu| z^2(x,\sigma)d\sigma\right] dx + G\int_0^1 (\psi-w_x)(\psi-w_x)_t dx \\
&= -4\beta\int_0^1 s_t^2 dx - \mu\int_0^1 z(x,1)w_t dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx.
\end{aligned}
\tag{2.4}
$$

we end up with,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t-\psi_t)^2 + D(3s_x-\psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2\right] dx \\
&+\frac{1}{2}\frac{d}{dt}\int_0^1 \left[G(\psi-w_x)^2 + \int_0^1 \tau|\mu| z^2(x,\sigma)d\sigma\right] dx = -\mu\int_0^1 z(x,1)w_t dx \\
&-4\beta\int_0^1 s_t^2 dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx.
\end{aligned}
\tag{2.5}
$$

Then, the energy of the solution to the system (2.2) is given by

$$
E(t) = \frac{1}{2}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t-\psi_t)^2 + D(3s_x-\psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2\right] dx \tag{2.6}
$$

$$+\frac{1}{2}\int_0^1 \left[4\gamma s^2 + G(\psi - w_x)^2 + \tau|\mu| \int_0^1 z^2(x,\sigma)d\sigma\right] dx.$$

On the existence, uniqueness, and smoothness of solution of problem (2.2), we introduce the vector function $\Phi = (w, u, \xi, v, s, z)^T; u = w_t, \xi = 3s - \psi, v = \xi_t$, and $y = s_t$ and thereby transform system (2.2) to

$$\begin{cases} \frac{d}{dt}\Phi(t) = \mathcal{A}\Phi(t), \quad t > 0, \\ \Phi(0) = \Phi_0 = (w_0, w_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, f_0)^T, \end{cases} \tag{2.7}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\Phi = \begin{pmatrix} u \\ -\frac{1}{\rho}\left(G(3s - \xi - w_x)_x + \mu z(x,1)\right) \\ v \\ \frac{1}{I_\rho}\left(D\xi_{xx} + G(3s - \xi - w_x)\right) \\ y \\ \frac{1}{I_\rho}\left(Ds_{xx} - G(3s - \xi - w_x) - \frac{4\gamma}{3}s - \frac{4\beta}{3}y\right) \\ -\frac{1}{\tau}z_\sigma(x,\sigma) \end{pmatrix}.$$

We consider the following spaces

$$H_a^1 = \left\{v : v \in H^1(0,1) : v(0) = 0\right\}, \quad H_b^1 = \left\{v : v \in H^1(0,1) : v(1) = 0\right\},$$

And let

$$\mathcal{H} := H_a^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1) \times H_b^1(0,1) \times L^2(0,1)$$

$$\times L^2((0,1) \times (0,1)),$$

be the Hilbert space equipped with the following inner product

$$(\Phi, \tilde{\Phi})_{\mathcal{H}} = \rho\int_0^1 u\tilde{u}dx + G\int_0^1 (3s - \xi - w_x)(3\tilde{s} - \tilde{\xi} - \tilde{w}_x)dx + I_\rho\int_0^1 v\tilde{v}dx$$

$$+3I_\rho \int_0^1 y\tilde{y}dx + D\int_0^1 \xi_x\tilde{\xi}_x dx + 4\gamma \int_0^1 s\tilde{s}dx + 3D\int_0^1 s_x\tilde{s}_x dx$$
$$+\tau|\mu| \int_0^1 \int_0^1 z(x,\sigma)\tilde{z}(x,\sigma)d\sigma dx.$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \Phi \in \mathcal{H} \; | w \in H^2(0,1)\cap H_a^1(0,1), \quad \xi, s \in H^2(0,1)\cap H_b^1(0,1), \\ u \in H_a^1(0,1), \quad v, y \in H_b^1(0,1), z, z_\sigma \in L^2((0,1)\times(0,1)), \\ w_x(1) = \xi_x(0) = s_x(0) = 0 \end{array} \right\}.$$

We observe that $D(\mathcal{A})$ is independent of time $t > 0$. Furthermore, it is obvious that $D(\mathcal{A})$ is dense in $\mathcal{H}$. obvious that D(A) is dense in H.

Now, we present the theorem of existence and uniqueness of solution, then present their proof. To this end we will use the semigroup method and the Lumer-Philips theorem.

Theorem 20. *Let $\Phi_0 \in \mathcal{H}$, , then there exists a unique weak solution $\Phi \in C(\mathbb{R}^+, \mathcal{H})$ of problem (2.7). . Moreover, if $\Phi_0 \in D(\mathcal{A})$, then $\Phi \in C(\mathbb{R}^+, D(\mathcal{A}) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. for any $\Phi \in D(\mathcal{A})$, we use the inner product and the operator $\mathcal{A}$, we have :

$$\begin{aligned} \langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = &- \int_0^1 u\left[G(3s-\xi-w_x)_x + \mu z(x,1)\right]dx \\ &+ G\int_0^1 (3y-v-u_x)(3s-\xi-w_x)dx \\ &+ \int_0^1 v\left[D\xi_{xx} + G(3s-\xi-w_x)\right]dx \\ &+ 3\int_0^1 y\left[Ds_{xx} - G(3s-\xi-w_x) - \frac{4\gamma s}{3} - \frac{4\beta y}{3}\right]dx \\ &+ D\int_0^1 v_x\xi_x dx + 4\gamma\int_0^1 ys dx + 3D\int_0^1 y_x s_x dx \\ &- |\mu| \int_0^1 \int_0^1 z(x,\sigma,t)z_\sigma(x,\sigma,t)d\sigma dx, \end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} = & -G\int_0^1 u(3s-\xi-w_x)_x dx - \mu\int_0^1 uz(x,1)dx \\
& + 3G\int_0^1 y(3s-\xi-w_x)dx - G\int_0^1 v(3s-\xi-w_x)dx \\
& - G\int_0^1 u_x(3s-\xi-w_x)dx + D\int_0^1 v\xi_{xx}dx \\
& + G\int_0^1 v(3s-\xi-w_x)dx + 3D\int_0^1 ys_{xx}dx \\
& - 3G\int_0^1 y(3s-\xi-w_x)dx - 4\gamma\int_0^1 sydx - 4\beta\int_0^1 y^2dx \\
& + D\int_0^1 v_x\xi_x dx + 4\gamma\int_0^1 ysdx + 3D\int_0^1 y_x s_x dx \\
& - |\mu|\int_0^1\int_0^1 z(x,\sigma,t)z_\sigma(x,\sigma,t)d\sigma dx,
\end{aligned}$$

Then, using integration by parts, and boundary condition, we get :

$$\begin{aligned}
\langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} = & -4\beta\int_0^1 y^2dx - \mu\int_0^1 uz(x,1)dx \\
& + G\int_0^1 u(3s-\xi-w_x)_x dx - G\int_0^1 u(3s-\xi-w_x)_x dx \\
& - D\int_0^1 v_x\xi_x dx - 3D\int_0^1 y_x s_x dx + D\int_0^1 v_x\xi_x dx \\
& + 3D\int_0^1 y_x s_x dx - \frac{|\mu|}{2}\int_0^1\int_0^1 \frac{d}{d\sigma}z^2(x,\sigma)d\sigma dx \\
= & -4\beta\int_0^1 y^2dx - \mu\int_0^1 uz(x,1)dx \\
& - \frac{|\mu|}{2}\int_0^1 \left[z^2(x,1)-u^2\right]dx.
\end{aligned} \tag{2.8}$$

Now, using Yong's inquality, we find :

$$-\mu\int_0^1 uz(x,1)dx \le \frac{|\mu|}{2}\int_0^1 u^2dx + \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx, \tag{2.9}$$

Substituting (2.9) in (2.8), we find; for some $c_1 > 0$,

$$\begin{aligned}\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} &\leq -4\beta \int_0^1 y^2 dx + |\mu| \int_0^1 u^2 dx \\ &\leq c_1 \langle \Phi, \Phi\rangle_{\mathcal{H}}\end{aligned}$$

This shows that the operator $\mathcal{A} - c_1 I$ is dissipative .

We have to show now that for all $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7) \in \mathcal{H}$ there exists $\Phi \in D(\mathcal{A})$ such that

$$(I - \mathcal{A})\Phi = F. \tag{2.10}$$

Then in terms of it's components, equation (2.10) becomes :

$$\begin{cases} w - u = f_1, \\ \rho u + G(3s - \xi - w_x)_x + \mu z(x, 1) = \rho f_2, \\ \xi - v = f_3, \\ I_\rho v - D\xi_{xx} - G(3s - \xi - w_x) = I_\rho f_4, \\ s - y = f_5, \\ 3I_\rho y - 3Ds_{xx} + 3G(3s - \xi - w_x) + 4\gamma s + 4\beta y = 3I_\rho f_6, \\ \tau z + z_\sigma = \tau f_7. \end{cases} \tag{2.11}$$

Following the same arguments in Nicaise and Pingotti [30] it follows from equation $(2.11)_7$, that :

$$\tau z + z_\sigma = \tau f_7,$$

First, we solve the homogeneous equation, we have :

$$\tau z + z_\sigma = \tau f_7 \Longrightarrow z(x, \sigma) = ke^{-\sigma\tau},$$

Then, using the variation of constants, we have

$$\tau k(x,\sigma)e^{-\sigma\tau} + k'(x,\sigma)e^{-\sigma\tau} - \tau k(x,\sigma)e^{-\sigma\tau} = \tau f_7,$$

Then,

$$k'(x,\sigma) = \tau e^{\sigma\tau} f_7 \Longrightarrow k(x,\sigma) = \tau\int_0^\sigma e^{\eta\tau} f_7 d\eta + c_1, c \in \mathbb{R}.$$

Hence,

$$z(x,\sigma) = ce^{-\sigma\tau} + \tau e^{-\sigma\tau}\int_0^\sigma e^{\eta\tau} f_7 d\eta,$$

Using the boundary conditions :

$$z(x,\sigma) = u(x)e^{-\sigma\tau} + \tau e^{-\sigma\tau}\int_0^\sigma e^{\eta\tau} f_7 d\eta.$$

It follows that :

$$z(x,1) = u(x)e^{-\tau} + \tau e^{-\tau}\int_0^1 e^{\eta\tau} f_7 d\eta. \tag{2.12}$$

Now we insert (2.12), $u = w - f_1, v = \xi - f_3$, and $y = s - f_5$ into $(2.11)_2$, $(2.11)_4$ and $(2.11)_6$, we obtain

$$\begin{cases} (\mu e^{-\sigma\tau} + \rho)w + G(3s - \xi - w_x)_x = \rho f_1 + \mu e^{-\sigma\tau} f_1 + \rho f_2 \\ -\tau e^{-\sigma\tau}\int_0^1 e^{\eta\tau} f_7 d\eta; \\ I_\rho \xi - D\xi_{xx} - G(3s - \xi - w_x) = I_\rho(f_3 + f_4); \\ (3I_\rho + 4\gamma + 4\beta)s - 3Ds_{xx} + 3G(3s - \xi - w_x) = (3I_\rho + 4\beta)f_5 + \\ (3I_\rho)f_6, \end{cases} \tag{2.13}$$

Then , we multiply $(2.13)_1$, $(2.13)_2$ and $(2.13)_3$ by $\tilde{w}, \tilde{\xi}$, and $\tilde{s}$ respectively and integrate over

(0,1) , we find

$$\begin{cases} \int_0^1 (\mu e^{-\sigma\tau} + \rho) w\tilde{w} dx + G\int_0^1 (3s - \xi - w_x)_x \tilde{w} dx \\ = \int_0^1 \left[(\rho + \mu e^{-\sigma\tau}) f_1 + \rho f_2 - \tau e^{-\sigma\tau} \int_0^1 e^{\eta\tau} f_7 d\eta \right] \tilde{w} dx; \\ I_\rho \int_0^1 \xi\tilde{\xi} dx - D\int_0^1 \xi_{xx}\tilde{\xi} dx - G\int_0^1 (3s - \xi - w_x)\tilde{\xi} dx \\ = I_\rho \int_0^1 (f_3 + f_4)\tilde{\xi} dx; \\ (3I_\rho + 4\gamma + 4\beta) \int_0^1 s\tilde{s} dx - 3D\int_0^1 s_{xx}\tilde{s} dx + 3G\int_0^1 (3s - \xi - w_x)\tilde{s} dx \\ = \int_0^1 \left[(3I_\rho + 4\beta) f_4 + 3I_\rho f_6 \right] \tilde{s} dx. \end{cases} \tag{2.14}$$

To solve (2.14), we introduce the following variational formulation

$$B\left((w, \xi, s), (\tilde{w}, \tilde{\xi}, \tilde{s}) \right) = L(\tilde{w}, \tilde{\xi}, \tilde{s}), \quad \forall (\tilde{w}, \tilde{\xi}, \tilde{s}) \in W, \tag{2.15}$$

Where the Hilbert space $W = H^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)$ is equipped with the norm :

$$\|(w, \xi, s)\|_W^2 = \|(3s - \xi - w_x)\|_2^2 + \|w\|_2^2 + \|\xi_x\|_2^2 + \|s_x\|_2^2;$$

$B : W \times W \longrightarrow \mathbb{R}$ is bilinear from defined by

$$B\left((w, \xi, s), (\tilde{w}, \tilde{\xi}, \tilde{s}) \right) = (\mu e^{-\tau\sigma} + \rho) \int_0^1 w\tilde{w} dx + I_\rho \int_0^1 \xi\tilde{\xi} dx \tag{2.16}$$

$$+ (3I_\rho + 4\gamma + 4\beta) \int_0^1 s\tilde{s} dx + D\int_0^1 \xi_x\tilde{\xi_x} dx + 3D\int_0^1 s_x\tilde{s_x} dx$$

$$+ G\int_0^1 (3s - \xi - w_x)(3\tilde{s} - \tilde{\xi} - \tilde{w_x}) dx,$$

and $L : W \longrightarrow \mathbb{R}$ is the linear form given by

$$L(\tilde{w}, \tilde{\xi}, \tilde{s}) = (\rho + \mu e^{-\tau\sigma}) \int_0^1 f_1\tilde{w} dx + \rho\int_0^1 f_2\tilde{w} dx \tag{2.17}$$

$$- \tau e^{-\tau\sigma} \int_0^1 \tilde{w} \int_0^\sigma f_7 e^{\eta\tau} d\eta dx + I_\rho \int_0^1 (f_3 + f_4)\tilde{\xi} dx$$

$$+\int_0^1 [(3I_\rho + 4\beta)f_4 + 3If_6]\,\tilde{s}dx.$$

We easily show that B and L are continuous. In addition form the definition of B, we have

$$\begin{aligned} B\Big((w,\xi,s),(\tilde{w},\tilde{\xi},\tilde{s})\Big) &= (\rho+\mu e^{-\tau\sigma})\int_0^1 w^2dx + I_\rho\int_0^1 \xi^2 dx \\ &+ (3I_\rho + 4\beta + 4\gamma)\int_0^1 s^2dx + D\int_0^1 \xi_x^2 dx \\ &+ 3D\int_0^1 s_x^2dx + G\int_0^1 (3s-\xi-w_x)^2dx \\ &\geq \alpha\|(w,\xi,s)\|_w^2,\ \alpha>0. \end{aligned} \tag{2.18}$$

Hence B is coercive. Therefore, by using the Lax-Milgram theorem we can obtain that problem (2.15) has a unique solution $(w,\xi,s)\in W$. Now substituning w, ξ, and s into $(2.11)_1$, $(2.11)_3$, $(2.11)_5$ we get $u\in H^1(0,1)$, $v\in H_0^1(0,1)$, and $y\in H_0^1(0,1)$.

Then using the classical elliptic regularity it follows that $\Phi\in D(\mathcal{A})$ therfore $\mathcal{A}$ is surjective. Finally, by using the Lumer-Philips theorem, the well-posedness results of problem (2.7) stated in Theorem 20 are established. This completes the proof.

□

Chapter 3

Laminated beam system: Stability results

In this chapter, we state the stability results, then proof them by using the energy method. We establish an exponential stability result for the considered system in case of equal wave speeds.

3.1 Technical lemmas

In this section, we state and prove some technical lemmas which are fundamental in the proof of our stability result. We use multiplier technique to establish stability results for the energy of the solution of system (2.2) . This requires constructing a suitable Lyapunov functional equivalent to energy as we elaborate in the subsequent section.

Lemma 3.1.1. *If* (w, ψ, s, z) *is a solution of (2.2), then the energy functional* E *defined by (2.6) satisfies*

$$E'(t) \leq -4\beta \int_0^1 s_t^2 dx + |\mu| \int_0^1 x_t^2 dx, \qquad \forall t \geq 0. \tag{3.1}$$

Proof. Multiplying the first three equations in the system(2.2) by $w_t, (3s_t - \psi_t)$ and s_t respectively, then integrating each by parts over $(0,1)$ using the boundary conditions

$$\begin{cases} \rho\int_0^1 w_{tt}w_t dx + G\int_0^1(\psi - w_x)_x w_t dx + \mu\int_0^1 z(x,1)w_t dx = 0, \\ I_\rho\int_0^1(3s_t-\psi_t)_t(3s_t-\psi_t)dx - D\int_0^1(3s_x-\psi_x)_x(3s_t-\psi_t)dx \\ -G\int_0^1(\psi-w_x)(3s_t-\psi_t)dx = 0, \\ 3I_\rho\int_0^1 s_{tt}s_t dx - 3D\int_0^1 s_{xx}s_t dx + 3G\int_0^1(\psi-w_x)s_t dx + 4\gamma\int_0^1 s_t s dx \\ +4\beta\int_0^1 s_t^2 dx = 0, \end{cases} \tag{3.2}$$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \rho w_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 I_\rho(3s_t-\psi_t)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 3I_\rho s_t^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1 4\gamma s^2 dx \\ &+3D\int_0^1 s_x s_{xt} dx + D\int_0^1(3s_x-\psi_x)(3s_x-\psi_x)_t dx + G\int_0^1(\psi-w_x)_x w_t dx \\ &-3G\int_0^1(\psi-w_x)s_t dx + G\int_0^1(\psi-w_x)\psi_t dx + 3G\int_0^1(\psi-w_x)s_t dx \\ &= -4\beta\int_0^1 s_t^2 dx - \mu\int_0^1 z(x,1)w_t dx, \end{aligned} \tag{3.3}$$

that is

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t-\psi_t)^2 + D(3s_x-\psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2\right] dx \\ &+\frac{1}{2}\frac{d}{dt}\int_0^1 4\gamma s^2 dx + G\int_0^1(\psi-w_x)(\psi-w_x)_t dx = -4\beta\int_0^1 s_t^2 dx \\ &-\mu\int_0^1 z(x,1)w_t dx, \end{aligned} \tag{3.4}$$

we finally end up with,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t-\psi_t)^2 + D(3s_x-\psi_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2\right] dx \\ &+\frac{1}{2}\frac{d}{dt}\int_0^1 \left[G(\psi-w_x)^2\right] dx = -\mu\int_0^1 z(x,1)w_t dx - 4\beta\int_0^1 s_t^2 dx. \end{aligned} \tag{3.5}$$

Similarly, multiplying $(2.2)_4$ by $|\mu|z$, , followed by integrating the product over $(0,1)\times(0,1)$

and then using the substitution $z(x,0,t) = w_t$,

$$\begin{aligned}\frac{\tau|\mu|}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\sigma)d\sigma dx &= -|\mu|\int_0^1\int_0^1 zz_\sigma(x,\sigma,t)d\sigma dx\\ &= -\frac{|\mu|}{2}\int_0^1\int_0^1 \frac{d}{d\sigma}z^2(x,\sigma)d\sigma dx\\ &= -\frac{|\mu|}{2}\left[\int_0^1 z^2(x,1,t)dx - \int_0^1 z^2(x,0,t)dx\right]\\ &= -\frac{|\mu|}{2}\left[\int_0^1 z^2(x,1,t)dx - \int_0^1 w_t^2 dx\right],\end{aligned}$$

we obtain,

$$\frac{\tau|\mu|}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\sigma)d\sigma dx = -\frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx. \tag{3.6}$$

Next, merging (3.5) and (3.6, we note that from (2.6),

$$E'(t) = -\mu\int_0^1 z(x,1)w_t dx - 4\beta\int_0^1 s_t^2 dx - \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx. \tag{3.7}$$

We now exploit Young's inequality on the first term of (3.7) to obtain

$$-\mu\int_0^1 z(x,1)w_t dx \le \frac{|\mu|}{2}\int_0^1 z^2(x,1)dx + \frac{|\mu|}{2}\int_0^1 w_t^2 dx. \tag{3.8}$$

Consequently, substituting (3.8) in (3.7) completes the proof of (3.1). □

Lemma 3.1.2. *If (w,ψ,s,z) is a solution of (2.2), then the functional F_1 defined by*

$$F_1(t) := -\rho\int_0^1 ww_t dx + \rho\int_0^1 w_t\int_0^x \psi(y)dydx,$$

For any$\varepsilon_1 > 0$, satisfies the estimate

$$\begin{aligned}\frac{d}{dt}F_1(t) \le -\frac{\rho}{2}\int_0^1 w_t^2 dx + \rho\int_0^1 (3s_t - \psi_t)^2 dx + \varepsilon_1\int_0^1 z^2(x,1)dx\\ +9\rho\int_0^1 s_t^2 dx + \left(\frac{G}{2} + \frac{\mu^2}{4\varepsilon_1}\right)\int_0^1 (\psi - w_x)^2 dx.\end{aligned} \tag{3.9}$$

Proof. DifferentiatingF_1,using the first equation in(2.2), integrating by parts the term containing $(\psi - w_x)_x$ and exploiting the fact that $\psi_t = -(3s_t - \psi_t) + 3s_t$,

$$\begin{aligned}
\frac{d}{dt}F_1(t) &= -\rho\int_0^1 (w_t^2 + w_{tt}w)dx + \rho\int_0^1 w_{tt}\int_0^x \psi(y)dydx \\
&+ \rho\int_0^1 w_t\int_0^x \psi_t(y)dyd \\
&= -\rho\int_0^1 w_t^2 dx - \int_0^1 \rho w_{tt}dx + \rho\int_0^1 w_{tt}\int_0^x \psi(y)dydx \\
&+ \rho\int_0^1 w_t\int_0^x \psi_t(y)dydx \\
&= -\rho\int_0^1 w_t^2 dx + \int_0^1 \left[G(\psi - w_x)_x - \mu z(x,1)\right] wdx \\
&+ \int_0^1 \left[-G(\psi - w_x)_x - \mu z(x,1)\right]\int_0^1 \psi(y)dydx \\
&+ \rho\int_0^1 w_t\int_0^x ((3s_t - \psi_t) + 3s_t)(y)dydx \\
&= -\rho\int_0^1 w_t^2 dx + \int_0^1 G(\psi - w_x)_x wdx - \int_0^1 \mu z(x,1)wdx \\
&- G\int_0^1 (\psi - w_x)_x\int_0^1 \psi(y)dydx - \mu\int_0^1 z(x,1)\int_0^1 \psi(y)dydx \\
&- \rho\int_0^1 w_t\int_0^x (3s_t - \psi_t)(y)dydx + 3\rho\int_0^1 w_t\int_0^x s_t(y)dydx,
\end{aligned}$$

It follows that,

$$\begin{aligned}
\frac{d}{dt}F_1(t) &= -\rho\int_0^1 w_t^2 dx - \rho\int_0^1 w_t\int_0^x (3s_t - \psi_t)(y)dydx \\
&+ 3\rho\int_0^1 w_t\int_0^x s_t(y)dydx - \mu\int_0^1 \left(\int_0^x (\psi(y)dy - w)z(x,1)\right)dx \\
&+ G\int_0^1 (\psi - w_x)\psi dx - G\int_0^1 (\psi - w_x)w_x dx.
\end{aligned}$$

We deduce that,

$$\begin{aligned}
\frac{d}{dt}F_1(t) = -\rho\int_0^1 w_t^2 dx + G\int_0^1 (\psi - w_x)^2 dx - \rho\int_0^1 w_t\int_0^x (3s_t - \psi_t)(y)dydx \\
+3\rho\int_0^1 w_t\int_0^x s_t(y)dydx - \mu\int_0^1 \left(\int_0^x \psi(y)dy - w\right)z(x,1)dx. \qquad (3.10)
\end{aligned}$$

By Young's and Poincaré's inequalities, the last three terms of (3.10) give

$$-\rho\int_0^1 w_t\int_0^x(3s_t-\psi_t)(y)dydx \leq \rho\int_0^1\left(\int_0^x(3s_t-\psi_t)dy\right)^2dx+\frac{\rho}{4}\int_0^1 w_t^2dx$$

$$\leq \rho\int_0^1(3s_t-\psi_t)^2dx+\frac{\rho}{4}\int_0^1 w_t^2dx,$$

$$3\rho\int_0^1 w_t\int_0^x s_t(y)dydx \leq 9\rho\int_0^1\left(\int_0^x s_t(y)dy\right)^2dx+\frac{\rho}{4}\int_0^1 w_t^2dx$$
$$\leq 9\rho\int_0^1 s_t^2dx\frac{\rho}{4}\int_0^1 w_t^2dx,$$

And,

$$-\mu\int_0^1\left(\int_0^x\psi(y)dy-w\right)z(x,1)dx \leq \frac{\mu^2}{4\varepsilon_1}\int_0^1\left(\int_0^x\psi(y)dy-w\right)^2dx$$

$$+\varepsilon_1\int_0^1 z^2(x,1)dx \leq \int_0^1(\psi-w_x)^2dx+\varepsilon_1\int_0^1 z^2(x,1)dx. \tag{3.11}$$

The combination of (3.10)-(3.11) leads to (3.9). □

Lemma 3.1.3. *If (w,ψ,s,z) is a solution of (2.2), then the functional F_2 defined by*

$$F_2(t) := -I_\rho\int_0^1(3s_t-\psi_t)(3s-\psi)dx,$$

Satisfies the estimate

$$\frac{d}{dt}F_2(t) \leq -I_\rho\int_0^1(3s_t-\psi_t)^2dx+\frac{3D}{2}\int_0^1(3s_x-\psi_x)^2dx \tag{3.12}$$

$$+\frac{G^2}{2D}\int_0^1(\psi-w_x)^2dx.$$

Proof. By direct computations, using the second equation in (2.2), we find

$$\frac{d}{dt}F_3(t) = -I_\rho\int_0^1(3s_{tt}-\psi_{tt})(3s-\psi)dx-I_\rho\int_0^1(3s_t-\psi_t)^2dx$$
$$= -\int_0^1[D(3s_{xx}-\psi_{xx})+G(\psi-w_x)](3s-\psi)dx-I_\rho\int_0^1(3s_t-\psi_t)^2dx,$$

Then,

$$\frac{d}{dt}F_3(t) = -D\int_0^1 (3s_{xx} - \psi_{xx})(3s - \psi)dx - G\int_0^1 (\psi - w_x)(3s - \psi)dx$$
$$- I_\rho \int_0^1 (3s_t - \psi_t)^2 dx.$$

Integrating by parts, we obtain,

$$\frac{d}{dt}F_2(t) = -I_\rho \int_0^1 (3s_t - \psi_t)^2 dx + D\int_0^1 (3s_x - \psi_x)^2 dx(3s - \psi)dx. \tag{3.13}$$

$$-G\int_0^1 (\psi - w_x)(3s - \psi)dx.$$

Exploiting Young's and Poincaré's inequalities, we estimate the last term of (3.13) as follows:

$$-G\int_0^1 (\psi - w_x)(3s - \psi)dx \leq \frac{G^2}{2D}\int_0^1 (\psi - w_x)^2 dx + \frac{D}{2}\int_0^1 (3s - \psi)^2 dx$$

$$\leq \frac{G^2}{2D}\int_0^1 (\psi - w_x)^2 dx + \frac{D}{2}\int_0^1 (3s_x - \psi_x)^2 dx. \tag{3.14}$$

Consequently, the relation (3.12) follows directly by substituting (3.14) into (3.13). □

Lemma 3.1.4. *If (w, ψ, s, z) is a solution of (2.2), then the functional F_3 defined by*

$$F_3(t) := 3I_\rho \int_0^1 s_t s dx + 2\beta \int_0^1 s^2 dx + 3\rho \int_0^1 w_t \int_0^x s(y)dydx,$$

For any $\varepsilon_2 > 0$, satisfies the estimate

$$\frac{d}{dt}F_3(t) \leq -3D\int_0^1 s_x^2 dx - 3\gamma\int_0^1 s^2 dx + \varepsilon_2\int_0^1 w_t^2 dx + \frac{9\mu^2}{4\gamma}\int_0^1 z^2(x,1)dx \tag{3.15}$$

$$+ \left(3I_\rho + \frac{9\rho^2}{4\varepsilon_2}\right)\int_0^1 s_t^2 dx.$$

Proof. Differentiating F_3, we find

$$\begin{aligned}\frac{d}{dt}F_3(t) &= 3I_\rho \int_0^1 (s_{tt}s + s_t^2)dx + 4\beta \int_0^1 s_t s dx + 3\rho \int_0^1 w_{tt} \int_0^x s(y)dydx \\ &+ 3\rho \int_0^1 w_t \int_0^x s_t(y)dydx,\end{aligned}$$

Using (2.2), we find

$$\begin{aligned}\frac{d}{dt}F_3(t) &= \int_0^1 (3Ds_{xx} - 3G(\psi - w_x) - 4\gamma s - 4\beta s_t)\, sdx \\ &- 3\int_0^1 [G(\psi - w_x)_x + \mu z(x,1)] \int_0^x s(y)dydx \\ &+ 3\rho \int_0^1 w_t \int_0^x s_t(y)dydx + 4\beta \int_0^1 s_t s dx + 3I_\rho \int_0^1 s_t^2 dx \\ &= 3D \int_0^1 s_{xx} s dx - 3G \int_0^1 (\psi - w_x) s dx - 4\gamma \int_0^1 s^2 dx \\ &- 4\beta \int_0^1 s_t s dx - 3G \int_0^1 (\psi - w_x)_x \int_0^x s(y)dydx \\ &- 3\mu \int_0^1 z(x,1) \int_0^x s(y)dydx + 3\rho \int_0^1 w_t \int_0^x s_t(y)dydx \\ &+ 4\beta \int_0^1 s_t s dx + 3I_\rho \int_0^1 s_t^2 dx,\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}F_3(t) &= -3D \int_0^1 s_x^2 dx - 4\gamma \int_0^1 s^2 dx + 3\rho \int_0^1 w_t \int_0^x s_t(y)dydx \\ &+ 3I_\rho \int_0^1 s_t^2 dx - 3\mu \int_0^1 z(x,1) \int_0^x s(y)dydx \\ &- 3G \int_0^1 (\psi - w_x) s dx + 3G \int_0^1 (\psi - w_x) s dx + 4\beta \int_0^1 s_t s dx \\ &- 4\beta \int_0^1 s_t s dx.\end{aligned}$$

We finally arrive at,

$$\frac{d}{dt}F_3(t) = -3D \int_0^1 s_x^2 dx - 4\gamma \int_0^1 s^2 dx + 3\rho \int_0^1 w_t \int_0^x s_t(y)dydx \tag{3.16}$$

$$+3I_\rho \int_0^1 s_t^2 dx - 3\mu \int_0^1 z(x,1) \int_0^x s(y) dy dx.$$

Using Young's, Poincaré's and Cauchy-Schwarz inequalities, we have

$$-3\mu \int_0^1 z(x,1) \int_0^x s(y) dy dx \le \frac{9\mu^2}{4\gamma} \int_0^1 z^2(x,1) dx + \gamma \int_0^1 \left(\int_0^x s(y) dy \right)^2 dx$$

$$\le \frac{9\mu^2}{4\gamma} \int_0^1 z^2(x,1) dx + \gamma \int_0^1 s^2 dx, \tag{3.17}$$

$$3\rho \int_0^1 w_t \int_0^x s_t(y) dy dx \le \varepsilon_2 \int_0^1 w_t^2 dx + + \frac{9\rho^2}{4\varepsilon_2} \int_0^1 \left(\int_0^x s_t(y) dy \right)^2 dx$$

$$\le \varepsilon_2 \int_0^1 w_t^2 dx + + \frac{9\rho^2}{4\varepsilon_2} \int_0^1 s_t^2 dx, \tag{3.18}$$

For any $\varepsilon_2 > 0$. Estimate (3.15) follows directly by virtue of (3.16)-(3.18). □

The assumption of equal wave speeds $GI_\rho = \rho D$ plays an important role in the next two lemmas.

Lemma 3.1.5. *If (w, ψ, s, z) is a solution of (2.2), then the functional F_4 defined by*

$$F_4(t) := -\int_0^1 (3s_t - \psi_t) w_x dx - \int_0^1 (3s_x - \psi_x) w_t dx + 3 \int_0^1 (3s_t - \psi_t) s dx,$$

For any $\varepsilon_3 > 0$, satisfies the estimate

$$\frac{d}{dt} F_4(t) \le -\frac{D}{2I\rho} \int_0^1 (3s_x - \psi_x)^2 dx + \varepsilon_3 \int_0^1 (3s_t - \psi_t)^2 dx + \frac{9}{\varepsilon_3} \int_0^1 s_t^2 dx \tag{3.19}$$

$$+ \frac{I_\rho \mu^2}{D\rho^2} \int_0^1 z^2(x,1) dx + \left(\frac{G}{I_\rho} + \frac{G^2}{DI_\rho} \right) \int_0^1 (\psi - w_x)^2 dx.$$

Proof. As in the previous Lemmas, we have

$$
\begin{aligned}
\frac{d}{dt}F_4(t) = &-\int_0^1 (3s_{tt}-\psi_{tt})w_x dx - \int_0^1 (3s_t-\psi_t)w_{tx}dx \\
&-\int_0^1 (3s_x-\psi_x)_t w_t dx - \int_0^1 (3s_x-\psi_x)w_{tt}dx \\
&+3\int_0^1 (3s_{tt}-\psi_{tt})s dx + 3\int_0^1 (3s_t-\psi_t)s_t dx,
\end{aligned}
$$

using system (2.2), we get

$$
\begin{aligned}
\frac{d}{dt}F_4(t) = &-\int_0^1 \left[\frac{G}{I_\rho}(\psi-w_x)+\frac{D}{I_\rho}(3s_{xx}-\psi_{xx})\right] w_x dx \\
&-\int_0^1 (3s_t-\psi_t)w_{tx}dx \\
&+\int_0^1 (3s_x-\psi_x)\left[\frac{\mu}{\rho}z(x,1)+\frac{G}{\rho}(\psi-w_x)_x\right] dx \\
&+3\int_0^1 \left[\frac{G}{I_\rho}(\psi-w_x)+\frac{D}{I_\rho}(3s_{xx}-\psi_{xx})\right] s dx + 3\int_0^1 (3s_t-\psi_t)s_t dx \\
= &-\frac{G}{I_\rho}\int_0^1 (\psi-w_x)w_x dx - \frac{D}{I_\rho}\int_0^1 (3s_{xx}-\psi_{xx})w_x dx \\
&-\int_0^1 (3s_t-\psi_t)w_{tx}dx - \int_0^1 (3s_x-\psi_x)_t w_t dx \\
&+\frac{\mu}{\rho}\int_0^1 (3s_x-\psi_x)z(x,1)dx + \frac{G}{\rho}\int_0^1 (3s_x-\psi_x)(\psi-w_x)_x dx \\
&+\frac{3G}{I_\rho}\int_0^1 (\psi-w_x)s dx + \frac{3D}{I_\rho}\int_0^1 (3s_{xx}-\psi_{xx})s dx \\
&+3\int_0^1 (3s_t-\psi_t)s_t dx,
\end{aligned}
$$

Integration by parts the term containing $3s_{xx}-\psi_{xx}$,

$$
\begin{aligned}
\frac{d}{dt}F_4(t) = &-\frac{G}{I_\rho}\int_0^1 (\psi-w_x)w_x dx + \frac{D}{I_\rho}\int_0^1 (3s_x-\psi_x)w_{xx}dx \\
&-\int_0^1 (3s_t-\psi_t)w_{tx}dx - \int_0^1 (3s_x-\psi_x)_t w_t dx \\
&+\frac{\mu}{\rho}\int_0^1 (3s_x-\psi_x)z(x,1)dx + \frac{G}{\rho}\int_0^1 (3s_x-\psi_x)(\psi-w_x)_x dx
\end{aligned}
$$

$$+\frac{3G}{I_\rho}\int_0^1(\psi-w_x)sdx-\frac{3D}{I_\rho}\int_0^1(3s_x-\psi_x)s_xdx+3\int_0^1(3s_t-\psi_t)s_tdx,$$

And exploiting the fact that $w_x=-(\psi-w_x)-(3s-\psi)+3s$, we find

$$\begin{aligned}\frac{d}{dt}F_4(t)=&\frac{G}{I_\rho}\int_0^1(\psi-w_x)^2dx+\frac{G}{I_\rho}\int_0^1(\psi-w_x)(3s-\psi)dx\\&-\frac{3G}{I_\rho}\int_0^1(\psi-w_x)sdx-\frac{D}{I_\rho}\int_0^1(3s_x-\psi_x)(\psi-w_x)_xdx\\&-\frac{D}{I_\rho}\int_0^1(3s_x-\psi_x)^2dx+\frac{3D}{I_\rho}\int_0^1(3s_x-\psi_x)s_xdx\\&+\int_0^1(3s_t-\psi_t)w_{tx}dx-\int_0^1(3s_t-\psi_t)_xw_tdx\\&+\frac{\mu}{\rho}\int_0^1(3s_x-\psi_x)z(x,1)dx+\frac{G}{\rho}\int_0^1(3s_x-\psi_x)(\psi-w_x)_xdx\\&+\frac{3G}{I_\rho}\int_0^1(\psi-w_x)sdx-\frac{3D}{I_\rho}\int_0^1(3s_x-\psi_x)s_xdx\\&+3\int_0^1(3s_t-\psi_t)s_tdx,\end{aligned}$$

We end up with,

$$\frac{d}{dt}F_4(t)=-\frac{D}{I_\rho}\int_0^1(3s_x-\psi_x)^2+\frac{G}{I_\rho}\int_0^1(\psi-w_x)^2dx+3\int_0^1(3s_t-\psi_t)s_tdx$$
$$+\frac{G}{I_\rho}\int_0^1(\psi-w_x)(3s-\psi)dx+\frac{\mu}{\rho}\int_0^1(3s_x-\psi_x)z(x,1)dx. \tag{3.20}$$

Next, Young's and Poincaré's inequalities guarantee the relations

$$3\int_0^1(3s_t-\psi_t)s_tdx\le\varepsilon_3\int_0^1(3s_t-\psi_t)^2dx+\frac{9}{\varepsilon_3}\int_0^1s_t^2dx, \tag{3.21}$$

$$\begin{aligned}\frac{G}{I_\rho}\int_0^1(\psi-w_x)(3s-\psi)dx&\le\frac{G^2}{DI_\rho}\int_0^1(\psi-w_x)^2dx+\frac{D}{4I\rho}\int_0^1(3s-\psi)^2dx\\&\le\frac{G^2}{DI_\rho}\int_0^1(\psi-w_x)^2dx+\frac{D}{4I\rho}\int_0^1(3s_x-\psi_x)^2dx,\end{aligned}$$

and,

$$\frac{\mu}{\rho}\int_0^1 (3s_x-\psi_x)z(x,1)dx \le \frac{D}{4I\rho}\int_0^1 (3s_x-\psi_x)^2dx + \frac{I_\rho \mu^2}{D\rho^2}\int_0^1 z^2(x,1)dx. \tag{3.22}$$

for any $\varepsilon_3 > 0$, . Estimate (3.19) follows directly by substituting (3.21)-(3.22) into (3.20). □

Lemma 3.1.6. *If (w,ψ,s,z) is a solution of (2.2), then the functional F_5 defined by*

$$F_5(t) := \int_0^1 (\psi - w_x)s_t dx - \int_0^1 w_t s_x dx$$

for any $\varepsilon_4, \varepsilon_5 > 0$, satisfies the estimate

$$\begin{aligned}\frac{d}{dt}F_5(t) \le& -\frac{G}{2I_\rho}\int_0^1 (\psi-w_x)^2dx + \varepsilon_4\int_0^1 (3s_t-\psi_t)^2dx + \varepsilon_5\int_0^1 z^2(x,1)dx\\ &+\left(\frac{16\gamma^2}{9I_\rho}+\frac{\mu^2}{4\rho^2\epsilon_5}\right)\int_0^1 s_x^2dx + \left(3+\frac{1}{4\varepsilon_4}+\frac{16\beta^2}{9I_\rho}\right)\int_0^1 s_t^2dx.\end{aligned} \tag{3.23}$$

Proof. Differentiating F_5, then integrating by parts over $(0,1)$ the term containing s_{xt} and, using the substitution $w_{xt} = -(3s_t-\psi_t) - (\psi - w_x)_t + 3s_t$,

$$\begin{aligned}\frac{d}{dt}F_5(t) =& \int_0^1 (\psi-w_x)s_{tt}dx - \int_0^1 w_{tt}s_xdx - \int_0^1 w_t s_{xt}dx\\ &+\int_0^1 (\psi-w_x)_t s_t dx\\ =& \int_0^1 (\psi-w_x)s_{tt}dx - \int_0^1 w_{tt}s_xdx - \int_0^1 (3s_t-\psi_t)s_tdx\\ &-\int_0^1 (\psi-w_x)_t s_tdx + \int_0^1 (\psi-w_x)s_{tt}dx - \int_0^1 w_{tt}s_xdx\\ &+3\int_0^1 s_tdx + \int_0^1 (\psi-w_x)_t s_tdx.\end{aligned}$$

We arrive at,

$$\frac{d}{dt}F_5(t) = \int_0^1 (\psi-w_x)s_{tt}dx - \int_0^1 w_{tt}s_xdx - \int_0^1 (3s_t-\psi_t)s_tdx + 3\int_0^1 s_t^2dx. \tag{3.24}$$

Next, using (3.24), (2.2) and integrating by parts the term containing s_x, we end up with

$$\begin{aligned}\frac{d}{dt}F_5(t) &= \int_0^1(\psi-w_x)s_{tt}dx - \int_0^1 w_{tt}s_x dx - \int_0^1(3s_t-\psi_t)s_t dx + 3\int_0^1 s_t^2 dx\\ &= \int_0^1(\psi-w_x)\left(-\frac{4\beta}{3I_\rho}s_t - \frac{4\gamma}{3I_\rho}s - \frac{3G}{3I_\rho} + \frac{D}{I_\rho}s_{xx}\right)dx\\ &+ \int_0^1\left[\frac{\mu}{\rho}z(x,1)dx + \frac{G}{\rho}(\psi-w_x)_x\right]s_x dx - \int_0^1(3s_t-\psi_t)s_t dx + 3\int_0^1 s_t^2 dx,\end{aligned}$$

that is,

$$\begin{aligned}\frac{d}{dt}F_5(t) &= -\frac{4\beta}{3I_\rho}\int_0^1(\psi-w_x)s_t dx - \frac{4\gamma}{3I_\rho}\int_0^1(\psi-w_x)s dx - \frac{G}{I_\rho}\int_0^1(\psi-w_x)^2 dx\\ &+ \frac{D}{I_\rho}\int_0^1(\psi-w_x)s_{xx}dx + \frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx + \frac{G}{\rho}\int_0^1(\psi-w_x)_x s_x dx\\ &- \int_0^1(3s_t-\psi_t)s_t dx + 3\int_0^1 s_t^2 dx\\ &= -\frac{4\beta}{3I_\rho}\int_0^1(\psi-w_x)s_t dx - \frac{4\gamma}{3I_\rho}\int_0^1(\psi-w_x)s dx - \frac{G}{I_\rho}\int_0^1(\psi-w_x)^2 dx\\ &+ \frac{D}{I_\rho}\int_0^1(\psi-w_x)s_{xx}dx + \frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx - \frac{G}{\rho}\int_0^1(\psi-w_x)s_{xx}dx\\ &- \int_0^1(3s_t-\psi_t)s_t dx + 3\int_0^1 s_t^2 dx\end{aligned}$$

,

we arrive at

$$\frac{d}{dt}F_5(t) = -\frac{G}{I_\rho}\int_0^1(\psi-w_x)^2 dx + 3\int_0^1 s_t^2 dx - \frac{4\gamma}{3I_\rho}\int_0^1 s(\psi-w_x)dx$$

$$-\frac{4\beta}{3I_\rho}\int_0^1 s_t(\psi-w_x)dx - \int_0^1(3s_t-\psi_t)s_t dx + \frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx. \tag{3.25}$$

Exploiting Youngs and Poincaré's inequalities, the last four terms of (3.25) are estimated as follows

$$-\frac{4\gamma}{3I_\rho}\int_0^1 s(\psi-w_x)dx \le \frac{G}{4I_\rho}\int_0^1(\psi-w_x)^2 dx + \frac{16\gamma^2}{9I_\rho}\int_0^1 s^2 dx$$

$$\leq \frac{G}{4I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{16\gamma^2}{9I_\rho}\int_0^1 s_x^2 dx \tag{3.26}$$

$$-\frac{4\beta}{3I_\rho}\int_0^1 s_t(\psi - w_x)dx \leq \frac{G}{4I_\rho}\int_0^1 (\psi - w_x)^2 dx + \frac{16\beta^2}{9I_\rho}\int_0^1 s_t^2 dx$$

$$-\int_0^1 (3s_t - \psi_t)s_t dx \leq \varepsilon_4 \int_0^1 (3s_t - \psi_t)^2 dx + \frac{1}{4\varepsilon_4}\int_0^1 s_t^2 dx$$

$$\frac{\mu}{\rho}\int_0^1 z(x,1)s_x dx \leq \frac{\mu^2}{4\rho^2\varepsilon_5}\int_0^1 s_x^2 dx + \varepsilon_5 \int_0^1 z^2(x,1)dx, \tag{3.27}$$

for any $\varepsilon_4, \varepsilon_5 > 0$. 0. The assertion of the lemma follows from the estimates (3.26)-(3.27) and (3.25). □

Lemma 3.1.7. *If (w, ψ, s, z) is a solution of (2.2), then the functional F_6 defined by*

$$F_6(t) := \tau \int_0^1 \int_0^1 e^{-\sigma\tau} z^2(x,\sigma)d\sigma dx$$

satisfies, for $m_1 > 0$ the estimate:

$$\frac{d}{dt}F_6(t) \leq -m_1 \int_0^1 z^2(x,1)dx - m_1\tau \int_0^1 \int_0^1 z^2(x,\sigma)d\sigma dx + \int_0^1 w_t^2 dx. \tag{3.28}$$

Proof. Differentiate F_6 and use the fourth equation in the system (2.2) and $z(x,0) = w_t$ as follows

$$\frac{d}{dt}F_6(t) = 2\tau \int_0^1 \int_0^1 e^{-\sigma\tau} z_t(x,\sigma)z(x,\sigma)d\sigma dx$$

$$
\begin{aligned}
\frac{d}{dt}F_6(t) &= -2\int_0^1\int_0^1 e^{-\sigma\tau} z(x,\sigma)z_\sigma(x,\sigma)d\sigma dx \\
&= -\int_0^1\int_0^1 \frac{d}{d\sigma}\left[e^{-\sigma\tau}z^2(x,\sigma)\right]d\sigma dx - \tau\int_0^1\int_0^1 e^{-\sigma\tau}z^2(x,\sigma)d\sigma dx \\
&= -\int_0^1 \left[e^{-\tau}z^2(x,1) - z^2(x,0)\right]dx - \tau\int_0^1\int_0^1 e^{-\sigma\tau}z^2(x,\sigma)d\sigma dx \\
&= -\int_0^1 e^{-\tau}z^2(x,1)dx + \int_0^1 w_t^2 dx - \tau\int_0^1\int_0^1 e^{-\sigma\tau}z^2(x,\sigma)d\sigma dx.
\end{aligned}
$$

Observe that, $\forall\sigma\in(0,1)$, the relation $e^{-\tau}\leq e^{-\tau\sigma}\leq 1$ hols. Therefore, for some $m_1 = e^{-\tau}$, we arrive at the estimate(3.28). □

3.2 Exponential stability

In this section, using the lemmas obtained in Section 3.1, we state and prove our main stability results.

Lemma 3.2.1. *Let $N, N_k, k = 1,\cdots,6$, be positive constants. The functional defined by*

$$
\mathcal{L}(t) := NE(t) + \sum_{k=1}^{6} N_k F_k(t), \quad N_k > 0, k = 1,\cdots,6, t\geq 0 \tag{3.29}
$$

satisfies the equivalence relation

$$
c_1E(t) \leq \mathcal{L} \leq c_2E(t), \quad \forall t\geq 0, \tag{3.30}
$$

for some positive constants c_1 and c_2.

Proof. Let $\mathbf{L} = \sum_{k=1}^{6} N_kF_k(t)$.

$$
|\mathbf{L}(t)| = |N_1F_1 + N_2F_2 + N_3F_3 + N_4F_4 + N_5F_5 + N_6F_6|
$$

$$
\begin{aligned}
|\mathbf{L}(t)| \leq \rho N_1 \int_0^1 |ww_t| dx + \rho N_1 \int_0^1 \left| w_t \int_0^x \psi(y) dy \right| dx \\
+ I_\rho N_2 \int_0^1 |(3s_x - \psi_x)(3s_t - \psi_t)| dx + 3 I_\rho N_3 \int_0^1 |s_t s| dx \\
+ 2\beta N_3 \int_0^1 s^2 dx + 3\rho N_3 \int_0^1 \left| w_t \int_0^x \psi(y) dy \right| dx \\
+ 3N_4 \int_0^1 |(3s_t - \psi_t) s| dx + N_4 \int_0^1 |(3s_t - \psi_t) w_x| dx \\
+ N_4 \int_0^1 |(3s_x - \psi_x) w_t| dx + N_5 \int_0^1 |(\psi - w_x) s_t| \, dx \\
+ N_5 \int_0^1 |w_t s_x| \, dx + \tau N_6 \int_0^1 \int_0^1 e^{-\sigma\tau} z^2(x,\sigma) d\sigma dx.
\end{aligned}
$$

Exploiting Young's, Poincaré's, Cauchy-Schwarz inequalities, (2.6), accompanied with the fact that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$ and $e^{-\sigma\tau} \leq 1$ for all $\sigma \in (0,1)$, we deduce that for some positive constant η.

$$
|\mathbf{L}(t)| \leq \eta \int_0^1 \left[w_t^2 + (3s_t - \psi_t)^2 + (3s_x - \psi_x)^2 + s_t^2 + s_x^2 + s^2 + (\psi - w_x)^2 \right] dx
$$

$$
+ \eta \int_0^1 \int_0^1 z^2(x,\sigma) d\sigma dx \leq \eta E(t).
$$

It is easy to observe that, from (3.29) that $|\mathcal{L}(t) - NE(t)| \leq \eta E(t)$, which is equivalent to

$$
(N - \eta) E(t) \leq \mathcal{L}(t) \leq (N + \eta) E(t),
$$

and hence the relation (3.30) follows by taking N large enough. □

At this point, we're in position to prove our stability result which reads as follows.

Theorem 21. *Let (w, ψ, s, z) be a solution of (2.2), and suppose that $GI_\rho = \rho D$, there exist a positive number $\bar{\mu}$ such that if $|\mu| < \bar{\mu}$, then the energy $E(t)$of (2.2) defined by (2.6) vanishes exponentially as t approaches infinity, i.e.*

$$E(t) \leq ae^{-bt}, \quad t \geq 0, \tag{3.31}$$

for some positive constants a and b.

Proof. We proceed by differentiating (3.29), then substitute for functionals F_1 to F_6 using estimates (3.9), (3.12), (3.15), (3.19), (3.23) and (3.28) respectively. Setting

$$N_1 = 1, \quad \varepsilon_1 = \varepsilon_5 = \mu^2, \quad \varepsilon_2 = \frac{\rho}{4N_3}, \quad \varepsilon_3 = \frac{I_\rho N_2}{2N_4}, \quad N_6 = |\mu|,$$

$$\begin{aligned}
\mathcal{L}'(t) \leq & NE' + \sum_{k=1}^{6} N_k' F_k'(t) \\
\leq & -\left[4\beta N - 9\rho N_1 - \left(3I_\rho + \frac{9\rho^2}{4\varepsilon_2}\right) N_3 - \frac{9}{\varepsilon_3} N_4 \right.\\
& \left. - \left(3 + \frac{1}{4\varepsilon_4} + \frac{16\beta^2}{9I_\rho}\right)\right] \int_0^1 s_t^2 dx - 3\gamma N_3 \int_0^1 s^2 dx \\
& - [I_\rho N_3 - \rho N_1 - \varepsilon_3 N_4 - \varepsilon_4 N_5] \int_0^1 (3s_t - \psi_t)^2 dx \\
& - \left[\frac{DN_4}{2I_\rho} - \frac{3DN_2}{2}\right] \int_0^1 (3s_x - \psi_x)^2 dx \\
& - \left[3DN_3 - \left(\frac{16\gamma^2}{9I_\rho} + \frac{\mu^2}{4\rho^2\varepsilon_5}\right) N_5\right] \int_0^1 s_x^2 dx \\
& - \left[-\frac{G^2 N_2}{2D} - \left(\frac{G}{I_p} + \frac{G^2}{DI_p}\right) N_4 + \frac{GN_5}{2I_\rho}\right] \int_0^1 (\psi - w_x)^2 dx \\
& + \left[\varepsilon_1 N_1 + \frac{9\mu^2 N_3}{4\gamma} + \frac{I_\rho \mu^2 N_3}{D\rho^2} + \varepsilon_5 N_5 - m_1 N_6\right] \int_0^1 z^2(x,1) dx \\
& - m_1 \tau |\mu| \int_0^1 \int_0^1 z^2(x,\sigma) d\sigma dx - \left[\frac{\rho}{2} - |\mu| N - \varepsilon_2 N_3 - N_6\right] \int_0^1 w_t^2 dx,
\end{aligned}$$

we end up with

$$
\begin{aligned}
\mathcal{L}'(t) \leq & - \left[4\beta N - c_3 - c_3 N_3(1+N_3) - \frac{cN_4^2}{N_2} - c_3 N_5 \left(1 + \frac{1}{\varepsilon_4}\right)\right] \int_0^1 s_t^2 dx \\
& - 3\gamma N_3 \int_0^1 s^2 dx - \left[\frac{I_\rho}{2} N_2 - \rho - \varepsilon_4 N_5\right] \int_0^1 (3s_t - \psi_t)^2 dx \\
& - \left[\frac{DN_4}{2I_\rho} - \frac{3DN_2}{2}\right] \int_0^1 (3s_x - \psi_x)^2 dx - [3DN_3 - c_3 N_4] \int_0^1 s_x^2 dx \\
& - \left[\frac{GN_5}{2I_\rho} - c_3 - c_3 N_2 - c_3 N_4\right] \int_0^1 (\psi - w_x)^2 dx \\
& - |\mu| \left[m_1 - |\mu| \left(1 + \frac{9N_3}{4\gamma} + \frac{I_\rho N_4}{D\rho^2} + N_5\right)\right] \int_0^1 z^2(x,1) dx \\
& - m_1 \tau |\mu| \int_0^1 \int_0^1 z^2(x,\sigma) d\sigma dx - \left[\frac{\rho}{4} - |\mu|(N+1)\right] \int_0^1 w_t^2 dx,
\end{aligned}
$$

for some $c_3 > 0$. Next, we choose N_2 large enough such that

$$k := \frac{I_\rho}{2} N_2 - \rho > 0.$$

Fixing N_2 permits to choose N_4 large enough such that

$$\frac{DN_4}{2I_\rho} - \frac{3DN_2}{2} > 0.$$

With N_2 and N_4 fixed, we can easily choose N_5 large enough such that

$$\frac{GN_5}{2I_\rho} - c_3 - c_3 N_2 - c_3 N_4 > 0.$$

We pick ε_4 adequately small and N_3 sufficiently large such that

$$k - \varepsilon_4 N_5 > 0 \quad \text{and} \quad 3DN_3 - c_3 N_5 > 0,$$

respectively. Next, we select N sufficiently large such that (3.30) remains valid and that

$$4\beta N - c_3 - c_3 N_3(1+N_3) - \frac{cN_4^2}{N_2} - c_3 N_5 \left(1 + \frac{1}{\varepsilon_4}\right) > 0.$$

Finally, pick

$$\bar{\mu} = \min \left\{ \frac{m_1}{\left(1 + \frac{9N_3}{4\gamma} + \frac{I_\rho N_4}{D\rho^2} + N_5\right)}, \frac{\rho}{4(N+1)} \right\}$$

to we end up with

$$\begin{aligned} \mathcal{L}'(t) \leq & -\alpha \int_0^1 \left[w_t^2 + s_t^2 + (3s_t - \psi_t)^2 + (s_x - \psi_x)^2 + s_x^2 + s^2 \right] dx \\ & -\alpha \int_0^1 \left[(\psi - w_x)^2 + z^2(x,1) + \int_0^1 z^2(x,\sigma) d\sigma \right] dx \end{aligned}$$

for some $\alpha > 0$. By the virtue of (2.6), it is clear that for some $\alpha_0 > 0$,

$$\mathcal{L}'(t) \leq -\alpha_0 E(t), \quad \forall \geq 0. \tag{3.32}$$

It then follows directly from (3.30) and (3.32) that

$$\mathcal{L}'(t) \leq -b\mathcal{L}(t), \quad \forall \geq 0, \tag{3.33}$$

where $b = \frac{\alpha_0}{c_2}$. A simple integration of (3.33) over $(0,t)$ yields

$$\mathcal{L}'(t) \leq \mathcal{L}(0)e^{-bt}, \quad \forall \geq 0. \tag{3.34}$$

Consequently, the assertion of the relation (3.31) follows from (3.34) and (3.30) with $a = c_2E(0)/c_1$. □

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