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Mathematical Result for Heat and Mass Transfer in Porous Media.

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Dedication .

“To my mother and father, who never stopped believing in me.”

“For my teacher, who saw potential in a quiet student and fanned the spark into a flame.”

“For my husband, who always pushes me to reach for the stars.”

“For my friends, who taught me the true meaning of friendship.”

Samira MAHBOUB.

ملخّص.

في هَذَا العمل، نثبت وجود نتيجة حل ضعيف لمسأَلة غير خطية تنمذج مشكلة القيمة الحدودية لإِنتقَال الحرَارة و الكتلة في وسط مسَامي غير متجَانس يتكون من ثلَاث معَادلَات غير خطية، اِلأَول إِهليليجي، و الثَاني هو قطع مكَافئ منحط، و الثَالث هو قطع مكَافئ مع وجود جزء مكَافئ يعتمد علَى إِحدَى الدوَال غير المعروفة.

كلمَات مفتَاحية : الوسط المسَامي، اِلقطع المكَافئ المنحط، إِنتقال الحرَارة و الكتلة، اِلمعَادلَات غير الخطية، اِلمعَادلَات الإِهليليجية.

Résumé:

Dans ce travail, nous prouvons un résultat d'existence d'une solution faible d'une initiale non linéaire problème de valeurs limites modélisant le transfert de chaleur et de masse dans un milieu poreux hétérogène. Le système se compose de trois équations non linéaires ; le premier est elliptique, le second est parabolique dégénéré, le troisième est parabolique avec le terme parabolique ayant un coefficient dépendant d'une des fonctions inconnues.

Mots clés:
milieux poreux, dégénérés paraboliques, transferts de chaleur et de masse, équations non linéaires, équations elliptiques.

Abstract:

In this work, we prove an existence result of weak solution of a nonlinear initial boundary value problem modeling heat and mass transfer in an heterogeneous porous medium. The system consists of three nonlinear equations; the first is elliptic, the second is parabolic degenerate, the third is parabolic with the parabolic term having a coefficient depending on one of the unknown functions.

Key words: porous media, parabolic degenerate, heat and mass transfer, nonlinear equations, elliptic equation.

Contents

Notations.

Here are the main notations used in this work:

- Ω : domain of $\mathbb{R}^d$, $d = 1, 2, ...$
- $\Gamma = \partial\Omega$: domain boundary Ω.
- $\frac{\partial}{\partial \mathbf{n}} = \nabla \cdot \mathbf{n}$: derived in the direction normal to the boundary Γ, outgoing from Ω.
- $\mathbf{x} = x, (x, y), ...$: space variable in $\Omega \subset \mathbb{R}^d$.
- t, T: temporary variable and final time, in ($\mathbf{s}$).
- $\operatorname{div} \mathbf{u} = \nabla \cdot \mathbf{u} = \sum\limits_{i=1}^{d} \frac{\partial u_i}{\partial x_i}$, $\mathbf{u} = (u_1, u_2, \ldots, u_d)$.

Functional spaces:

- $L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \mid f \text{ measurable and } \displaystyle\int_\Omega |f(x)|^p dx < \infty \right\}$ where $p \in [1, +\infty[$.
- $L^\infty(\Omega) = \Big\{ f : \Omega \to \mathbb{R} \mid f$ measurable and $\exists\ c \geq 0$ such that $|f(x)| \leq c$, a.e. sur $\Omega \Big\}$.
- $\mathcal{M}(\Omega)$ = space of measurable functions on Ω.
- $W^{m,p}(\Omega) = \{w \in L^p(\Omega) \mid D^\alpha w \in L^p(\Omega),\ \alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d,\ |\alpha| := \sum\limits_{i=1}^{d} \alpha_i \leq m\}$ where $D^\alpha w = \dfrac{\partial^{\alpha_1 + \cdots + \alpha_d}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}} w$.
- $H^m(\Omega) := W^{m,2}(\Omega)$, Sobolev space of order m.
- $V = \{v \in H^1(\Omega) \mid v = 0 \text{ sur } \Gamma_w\}$.

- $C^m(\Omega) := \{f : \Omega \longrightarrow \mathbb{R} \mid D^\alpha f$ continues in $\Omega, |\alpha| := \sum\limits_{i=1}^{d} \alpha_i \leq m\}$.
- $C_0^\infty(\Omega) = \mathcal{D}(\Omega)$:space of functions of $C^\infty(\Omega)$ with compact support in Ω.
- For X a Banach space, we note $L^p(0,T;X)$, the space defined by: $L^p(0,T;X) = \{f : (0,T) \to X \mid$ measurable with $\int_0^T \| f \|_X^p \ dt < \infty\}$.

Introduction.

To study the displacement of oil by hot water in a heterogeneous porous medium, we proposed [4] the following nonlinear system:

$$(\mathcal{S}_0) \qquad \begin{cases} \operatorname{div} \overrightarrow{V} = 0, \quad -\overrightarrow{V} = K\nabla P + \overrightarrow{f} \\ m(x)\dfrac{\partial S}{\partial t} = -\operatorname{div} \overrightarrow{V_1}, \quad -\overrightarrow{V_1} = K_0 a \nabla S - b\overrightarrow{V} + \overrightarrow{F} \\ \ell(x,S,\theta)\dfrac{\partial \theta}{\partial t} + \overrightarrow{V} \cdot \nabla\theta = \operatorname{div}(\Lambda\nabla\theta) \end{cases}$$

in a region $\Omega_T = \Omega\times]0,T[\subset \mathbb{R}^3 \times \mathbb{R}_+$, where Ω is a bounded domain, $T > 0$ a real number; $a(x,S,\theta) \geq 0$, $b(S)$, $\ell(x,S,\theta)$, $\overrightarrow{F}(x,S,\theta)$, and $\overrightarrow{f}(x,S,\theta)$, and $m(x)$ are given functions; $K(x,S,\theta)$, $K_0(x)$, and $\Lambda(S,\theta)$ are positive definite tensors. The unknown functions $P(x,t)$, $S(x,t)$, and $\theta(x,t)$ are respectively the reduced pressure, the reduced saturation, which must satisfy $0 \leq S \leq 1$, and the temperature. The function a vanishes for $S = 0$ and $S = 1$, this means that the second equation is degenerate. To the system $(\mathcal{S})$ are associated the boundary and initial conditions ($\partial\Omega = \Gamma_0 \cup \Gamma_1 \cup \Gamma_2$)

$$(\mathcal{BIC}) \qquad \left\{ \begin{array}{rl} \overrightarrow{V}(P,S,\theta)\cdot\vec{n} \;\; = & \overrightarrow{V_1}(P,S,\theta)\cdot\vec{n} = \Lambda\nabla\theta\cdot\vec{n} = 0, \\ & \qquad\qquad (x,t) \in \Gamma_{0T} \doteq \Gamma_0 \times [0,T]; \\ \overrightarrow{V}\cdot\vec{n} = Q(x,t), & \overrightarrow{V_1}\cdot\vec{n} = bQ(x,t), \quad (x,t) \in \Gamma_{1T}, \\ & \qquad\qquad Q \quad \text{given}; \\ S = S^0(x,t), & P = P^0(x,t), \quad (x,t) \in \Gamma_{2T}, \quad \text{and} \\ & \theta = \theta^0(x,t), \quad (x,t) \in \Gamma_{1T} \cup \Gamma_{2T}; \\ S = S^0(x), & \theta = \theta^0(x) \quad \text{for} \quad t = 0 \quad \text{and} \quad x \in \Omega. \end{array} \right.$$

Assuming that ℓ depends only of x: $\ell(x,S,\theta) = \ell(x)$ we proved in [4] the existence of a bounded weak solution to the above system with the mentioned boundary and initial conditions.

In this work we will study the same system with the same boundary and initial conditions but with ℓ depending on x and S as well. Let us recall the definition of this functional coefficient.

The function ℓ is given by

$$\begin{aligned}\ell(x,S,\theta) &= \rho_1 c_1\{(1-S_1^0-S_2^0)S+S_1^0\}\overline{m}\\ &\quad+\rho_2 c_2\{1-(1-S_1^0-S_2^0)S-S_1^0\}\overline{m}+\rho_3 c_3(1-\overline{m}),\end{aligned}$$

where $c_i(\theta)=(\partial e_i)/\partial\theta$ is the specific heat capacity of the component i at temperature θ. Here e_i is the internal specific energy of the component i, $i=1,2,3$; the subscript 3 stands for the porous matrix.

We observe that ℓ depends on x, S, and θ.

We are interested in the temperature range $[0,300]$ where c_1 and c_2 are nearly constant [6]. Then we may assume that ℓ does not depend on θ. In this case, we can write ℓ as

$$\ell(x,S)=\ell_1(x)S+\ell_0(x),$$

where

$$\begin{aligned}\ell_1(x) &= (\varrho_1 c_1-\varrho_2 c_2)(1-S_1^0-S_2^2)\overline{m}(x),\\ \ell_0(x) &= [(\varrho_1 c_1-\varrho_2 c_2)S_1^0+\varrho_2 c_2-\varrho_3 c_3]\overline{m}(x)+\varrho_3 c_3.\end{aligned}$$

The system to study is thus the following

$$(\mathcal{S})\quad\begin{cases}\operatorname{div}\overrightarrow{V}=0,\quad -\overrightarrow{V}=K\nabla P+\overrightarrow{f}\\ m(x)\dfrac{\partial S}{\partial t}=-\operatorname{div}\overrightarrow{V_1},\quad -\overrightarrow{V_1}=K_0 a\nabla S-b\overrightarrow{V}+\overrightarrow{F}\\ \ell(x,S)\dfrac{\partial\theta}{\partial t}+\overrightarrow{V}\cdot\nabla\theta=\operatorname{div}(\Lambda\nabla\theta),\quad \ell(x,S)=\ell_1(x)S+\ell_0(x),\end{cases}$$

with boundary and initial conditions $(\mathcal{BIC})$.

From now on, we will refer to the system $(\mathcal{S})$ with the conditions $(\mathcal{BIC})$ by problem $(\mathcal{P})$.

Chapter 1

Galerkin method of mathematical model.

1.1 Resolution by Galerkin method of the model problem.

Let

$$\begin{cases} -\Delta u = f(u) \text{ in } \Omega \\ u \in H_0^1(\Omega), \end{cases}$$

with $f \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$. This involves solving the variational problem

$$\forall v \in H_0^1(\Omega), \qquad \int_\Omega \nabla u \nabla v dx = \int_\Omega f(u) v dx. \tag{1.1}$$

The solution is done in several steps
First we consider a Galerkin basis of $H_0^1(\Omega)$,

Lemma 1.1. *Let $H_0^1(\Omega)$ there exists a countable free family $\{w_i\}_{i\in\mathbb{N}}$, $w_i \in H_0^1(\Omega)$ such that the finite linear combinations of w ,are dense in $H_0^1(\Omega)$. Such a "sequence"exists because $H_0^1(\Omega)$ is separable (see Brezis , P 150) .This lemma is expressed equivalently if $W_i = \mathrm{vect}\langle w_1, w_2, ..., w_i\rangle$,is the space generated by the first i vectors, then $\cup_i^\infty = W_i$ is dense in $H_0^1(\Omega)$. (w_i Galerkin space).*

Remark 1.1.1. *The lemma is true if we replace $H_0^1(\Omega)$ by separable Banach V space of infinite dimension. firsly Galerkin step: Let's show the existence for this problem in finite dimension.*

Lemma 1.2. *For any* $i \in \mathbb{N}^*$ *,the variational problem: find* $u_i \in W_i$ *such that:*

$$\forall v \in W_i, \qquad \int_\Omega \nabla u_i \nabla v dx = \int_\Omega f(u_i) v dx \tag{1.2}$$

admits at least one solution.

Proof. We provide W_i, with a scaiar product inherited from $L^2(\Omega)$

$$[u,v] = \int_\Omega uvdx$$

and we identify W_i, of finite dimension,and its dual via this scalar product.

$$(u,v) \longrightarrow a(u,v) = \int_\Omega \nabla u \nabla v dx \tag{1.3}$$

is a bilinear from on W_i, By Riesz's theorem , there exists a linear map

$$A_i \in \zeta(W_i) \qquad \text{such that:} \qquad a(u,v) = [A_i(u), v] \tag{1.4}$$

as W_i, is of finite dimension, this application is continuous. Likewise there exists $F_i : W_i \to W_i$ to for any couple (u,v)

$$\int_\Omega [f(u)v]dx = [F_i(u), v] \tag{1.5}$$

it is enough to take $F_i = \Pi_i \circ f$, with Π_i : the orthogonal projection of L^2 on W_i; this non linear map is continuous (composed of continuous maps)(we use caratheodoy's theorem here). the problem 1.2 is there fore written

$$\forall v \in W_i, \qquad [A_i(u_i), v] = [F_i(u_i), v]. \tag{1.6}$$

Now let P_i be the function

$$\begin{cases} P_i : W_i & \longrightarrow & W_i \\ u & \longrightarrow & P_i(u) = A_i(u) - F_i(u) \end{cases} \tag{1.7}$$

. the equation 1.6 is equivalent to

$$P_i(u_i) = 0. \tag{1.8}$$

□

To resolve this problem.we will apply the following lemma

Lemma 1.3. *Let* E *be a Euclidean space (dimension* $E < +\infty$*)and let* $P : E = (\mathbb{R}^m) \longrightarrow E = (\mathbb{R}^m)$ *continuous such that* $\exists p > 0$*, for which any point* x *on the sphere of radius* p *satisfies* $P(x) \cdot x \geq 0$*. there then exists a point* $x_0, \|x_0\| \leq p$ *with* $P(x_0) = 0$*.*

Proof. we mast therefore calculate $[P_i(u), u]$ on a sphere. By definition of the scalar product on W, and by use of poincare's inequality, we have

$$\begin{cases} [P_i(u), u] & = \int P_i(u)udx = a(u,u) - \int_\Omega f(u)udx \\ & \geq \|\nabla u\|^2_{L^2(\Omega)} - \|f\|_{L^\infty(\mathbb{R})}|\Omega|^{\frac{1}{2}}\|u\|_{L^2(\Omega)} \\ & \geq \|\nabla u\|^2_{L^2(\Omega)} - C_\Omega\|f\|_{L^\infty(\mathbb{R})}|\Omega|^{\frac{1}{2}}|\nabla u\|_{L^2(\Omega)} \\ & = \|\nabla u\|_{L^2(\Omega)}(\|\nabla u\|_{L^2(\Omega)} - C_\Omega\|f\|_{L^\infty(\mathbb{R})}|\Omega|^{\overline{2}}). \end{cases}$$

where $|\Omega|$ is the measure of Ω and C_Ω is the constant of poincare's inequality dane as soon as

$$\|u\|_{H_0^1} \geq M|\Omega|^{\frac{1}{2}}C_\Omega^{\frac{1}{2}}.$$

so

$$[P_i(u), u] \geq 0.$$

Now all the norms are equivalent on W, which is of finite dimension, so there exists two strictly ppositive real numbers C_0 and C_1 such that

$$C_0\|\nabla u\|_2 \leq C_1\|\nabla u\|_2$$

By multipying (*) by C_0, we obtain

$$C_0\|u\|_{H_0^1} \geq C_0M|\Omega|^{\frac{1}{2}}C_\Omega^{\frac{1}{2}}$$

From (*) and (***) we have

$$C_0\|u\|_{H_0^1} \geq C_0M|\Omega|^{\frac{1}{2}}C_\Omega^{\frac{1}{2}} \leq \|u\|_2.$$

So by taking

$$p = C_0M|\Omega|^{\frac{1}{2}}C_\Omega^{\frac{1}{2}},$$

we have

$$[P_i(u), u] \geq 0, \qquad p \geq C_0M|\Omega|^{\frac{1}{2}}C_\Omega^{\frac{1}{2}},$$

and by Lemma 3 the problem admit's a solution u_i in $H_0^1(\Omega)$. □

2^{nd} step: a priori estimation

Lemma 1.4. *the sequence u_i, is bounded in $H_0^1(\Omega)$.*

Proof. we repeat the previous calculation

$$a(u_i, u_i) = \int_\Omega f(u_i)u_i dx \leq C_\Omega \|f\|_{L^\infty(\mathbb{R})} |(\Omega)^{\frac{1}{2}} |\|\nabla u_i\|_{L^2(\Omega)}. \tag{1.9}$$

therefore

$$\|\nabla u\|_{L^2(\Omega)} \leq C_\Omega \|f\|_{L^\infty(\mathbb{R})} |(\Omega)|^{\frac{1}{2}}, \tag{1.10}$$

which shows Lemma 4. ☐

Passage to the limit

Lemma 1.5. *Any weakly convergent subsequence of the sequence u_i converges to a solution of problem 1.1.*

Proof. Let u_p subsequence $u_i \to u$ on $H_0^1(\Omega)$ and by Rellich's theorem , we have therefore

$$u_i \to u \text{ on } L^2(\Omega) \text{ stongly.}$$

therefore , caratheodory's theorem implies

$$f(u_i) \to f(u) \text{ dans } L^2(\Omega) \text{ stongly.}$$

We fixe a_j, the sequence W_i is increasing for all $i \geq j$, $w_j \in W_i$. Consequently , we multiply 1.2 by in test function w_j we obtain:

$$\int_\Omega [\nabla u_i \nabla w_j] dx = \int \Omega [f(u_i) w_j)] dx, \tag{1.11}$$

$$\nabla u_i \to \nabla u \text{ on } L^2(\Omega) weakly \tag{1.12}$$

$$\int \Omega [\nabla u_i, \nabla w_j] dx \to \int \Omega [\nabla u \cdot \nabla w_j] dx, \tag{1.13}$$

like $f(u_i) \to f(u)$ in strong $L^2(\Omega)$. On the other hand , we also have

$$\int \Omega [f(u_i) w_j] dx \to \int \Omega [f(u) w_j] dx$$

Therefore,for all $j \in (\mathbb{N}^*)$,

$$\int_\Omega [\nabla u \nabla w_j] dx = \int_\Omega [f(u) w_j] dx,$$

as this equation is linear with respect to w_j, it remains true for finite linear combinations of tey w_j,

$$\forall w \in \cup_{i=1}^\infty W_i, \int_\Omega [\nabla u \nabla w] dx = \int_\Omega [f(u) w] dx.$$

Finally $\cup_{i=1}^\infty = W_i$ is dense in $H_0^1(\Omega)$. For everything $w \in H_0^1(\Omega)$, $\exists z_i \in W_i$, $z_i \to w$, in$H_0^1(\Omega)$ strongly. ☐

We apply (1.13) with $w = z_i$, and we pass to the limit when $i \to +\infty$ without difficulty to conclude that u is the solution to problem 1.1.

1.2 superposition operators the application of Galerkin's method.

Operators of the type $u \to f(u)$ are called superposition operators or Nemystky operators.

1.2.1 Superposition operators in $L^2(\Omega)$.

we have seen that under reasonable technical assumptions, these operators continuously send strong $L^2(\Omega)$ into strong $L^2(\Omega)$ by caratheodory's theorem. this result does not persistin $L^2(\Omega)$ weakly.

Proposition 1.2.1. *Let* (Ω) *and* f *as in caratheody's theorem the map* f *is sequentially continuous from* $L^2(\Omega)$ *weak into* $L^2(\Omega)$ *weak if and only if* f *is affine .*

Proof. Let $Q \subset \Omega$ a cube. By changing coordinates , we can always set $Q =]0,1[^N$ with $a, b \in \mathbb{R}$ and $0 < \theta < 1$. We define a sequence of functions of $L^2(\Omega)$ by

$$\begin{cases} u_n(x) = 0, & \text{if } x \in Q \\ v_n(x_1) & \text{if non,} \end{cases}$$

$$\begin{cases} v_n(t) = a, & \frac{[nt]}{n} \leq t \leq \frac{[nt]+0}{n}, \\ b, & \frac{[nt]+0}{n} < t < \frac{[nt]+1}{n}, \end{cases}$$

Where $[s]$ designates the integer part $s.$ of s The sequence oscillates between the values a and a all the more quickly as n is large. The crazy sequence , with similar properties in fact

$$\begin{cases} f \circ u_n(x) = f(0) & \text{if} \, x \in Q \\ w_n(x_1) & \text{if non,} \end{cases}$$

$$\begin{cases} w_n(t) = f(a), & \frac{[nt]}{n} \leq t \leq \frac{[ny]+0}{n}, \\ f(b), & \frac{[nt]+0}{n} < t < \frac{[nt]+1}{n}. \end{cases}$$

The sequences u_n and ($f \circ u_n$) for are bounded in $L^2(\Omega)$. They therefore each contain a weakly convergent subsequence. In the following we will not distinguish these sequences, because it is clear that the entire sequences converge, by the uniqueness of the limit. we have there fore lau and foung $u_n \to u$ and $f \circ u_n \to g$ and it is a question of identifying u and g. First of all it is clear that:

$$\begin{cases} u_x = 0 & \text{if } x \in Q \\ v(x_1) & \text{if non,} \end{cases}$$

We vest the weak limit in $L^2(0,1)$ of the sequence v_n. Indeed, the space of functions of x_1, mulles outside of Q. is a closed one of $L^2(\Omega)$ Isometric to $L^2(0,1)$. We therefore return in this way to dimension 1. Consider any scus-interval $[t_1, t_2]$ of $[0,1]$ we deduce from the weak convergence of the sequence v_n that:

$$\int_{[t_1,t_2]} [v_n(t)]dt \to \int_{[t_1,t_2]} [v(t)]dt.$$

Let's calculate $\int_{[t_1,t_2]} [v_n(t)]dt$:

$$\begin{aligned} \int_{[t_1,t_2]} [v_n(t)]dt &= \int_{[t_1 \frac{[nt_1]M}{n}, t_2]} [v_n(t)]dt + \sum_{[nt_2]+1],[[nt_2]-1]} \int_{[\frac{k}{n}, \frac{kM}{n}]} [v_n(t)]dt + \int_{[\frac{nt_2-1}{t_2}[v_n(t)]} dt, \\ \int_{[\frac{k}{n}, \frac{k+1}{n}]} [v_n(t)]dt &= \frac{\theta a + (1-\theta)b}{n}, \end{aligned}$$

and

$$\int_{[t_1,t_2]} [v_n(t)]dt \to (t_2 - t_1)(\theta a + (1-\theta)b,$$

so he comes

$$\frac{1}{t_2 - t_1} \int_{[t_1,t_2]} [v_n(t)]dt = \theta a + (1-\theta)b$$

if to limit $t_2 \to t_2$

$$v(t) = \theta a + (1-\theta)b \qquad \text{a.e.}$$

the same reasoning shows that

$$\begin{cases} g(x) = f(0) & \text{if } x \in Q \\ \theta f(a) + (1-\theta)f(b) & \text{if non.} \end{cases}$$

Consequently, a necessary condition for f to be sequentially continuous(done a fortiori continuous) from L^2 weak is that $f(u)$. for all a, b, 0

$$f(\theta a + (1-\theta)b) = \theta f(a) + (1-\theta)f(b), \tag{1.14}$$

in other words must be affine. □

On the other hand, the situation differs in H.

Theorem 1.2.1. *Let T be Lipschitzian of $\mathbb{R}$ in $\mathbb{R}$ of class C^1 piecewise and having only a finite number of non-differentiability points. then the map $u \to T(u)$ is*

(i) continues from $H^1(\Omega)$ strong into $H^1(\Omega)$ strongly.

(ii) sequentially continues from weak $H^1(\Omega)$ into weak $H^1(\Omega)$.

Proof. Let us mention some simple but very u seful relation ships conceming the positive and negative parts; if $u \in L^2(\Omega)$, we have

$$\begin{aligned} u = u^+ - u^-, &\quad |u| = u^+ + u^-, \\ u^+ = \chi_{u>0} u = X_{u\geq 0} u, &\quad u^- = -\chi_{u<0} u = -\chi_{u\leq 0} u, \\ \nabla u = \nabla u^+ - \nabla u^-, &\quad \nabla |u| = \nabla u^+ + \nabla u^-, \\ \nabla u^+ = \chi_{u>0} \nabla u = \chi_{u\geq 0} \nabla u, &\end{aligned}$$

and Fanalogue for u^- Finally ,note that $|\nabla u^+| \cdot |\nabla u^-| = 0$ a.e.
We define T_k a the truncation at height k by

$$\begin{cases} T_k(t) = t, & \text{if } |t| \leq k, \\ \frac{kt}{|t|}, & \text{if } |t| > k. \end{cases}$$

Truncation at height k is an approximation of identity in various spaces. □

Chapter 2

Generalized Solutions of System $(\mathcal{P})$.

Let$(\mathcal{P})$ be the problem

$$(\mathcal{P})\begin{cases} \operatorname{div}\overrightarrow{V}=0, \quad -\overrightarrow{V}=K\nabla P+\overrightarrow{f} \\ m(x)\dfrac{\partial S}{\partial t}=-\operatorname{div}\overrightarrow{V_1}, \quad -\overrightarrow{V_1}=K_0a\nabla S-b\overrightarrow{V}+\overrightarrow{F} \\ \ell(x,S)\dfrac{\partial\theta}{\partial t}+\overrightarrow{V}\cdot\nabla\theta=\operatorname{div}(\Lambda\nabla\theta), \quad \ell(x,S)=\ell_1(x)S+\ell_0(x), \\ \overrightarrow{V}(P,S,\theta)\cdot\vec{n}=\overrightarrow{V_1}(P,S,\theta)\cdot\vec{n}=\Lambda\nabla\theta\cdot\vec{n}=0, \qquad (x,t)\in\Gamma_{0T}\doteq\Gamma_0\times[0,T]; \\ \overrightarrow{V}\cdot\vec{n}=Q(x,t),\overrightarrow{V_1}\cdot\vec{n}=bQ(x,t), \quad (x,t)\in\Gamma_{1T}, \qquad Q \quad \text{given}; \\ S=S^0(x,t),P=P^0(x,t), \quad (x,t)\in\Gamma_{2T}, \quad \text{and} \\ \theta=\theta^0(x,t), \quad (x,t)\in\Gamma_{1T}\cup\Gamma_{2T}; \\ S=S^0(x),\theta=\theta^0(x) \quad \text{for} \quad t=0 \quad \text{and} \quad x\in\Omega. \end{cases}$$

2.1 Assumptions on the Data of the Problem $(\mathcal{P})$.

We assume that the data $m(x)$, $\ell(x)$, $K_0(x)$, $K(x,S,\theta)$, $\overrightarrow{f}(x,S,\theta)$, $\overrightarrow{F}(x,S,\theta)$, and $\Lambda(S,\theta)$ are defined for

$$(x,S,\theta)\in\overline{\Omega}_\star=\overline{\Omega}\times[0,1]\times\mathbb{R}_+$$

and satisfy the conditions:

(i) $\star$ K_0, K, and Λ are symmetric,

- $\star$ $M^{-1} \leq \Big(\ell_0; \ell_1; m; (K_0\xi,\xi)/|\xi|^2; (K\xi,\xi)/|\xi|^2; (\Lambda\xi,\xi)/|\xi|^2\Big) \leq M$,
uniformly in $\overline{\Omega}_\star$, where M is a positive constant,
- $\star$ the function b depends only on S with $b_S \doteq \dfrac{db}{dS}$ continuous on $[0,1]$,
- $\star$ the tensors $K(x,S,\theta)$, $\Lambda(S,\theta)$ are continuous with respect to S and θ, and

(ii) $\star$ the function a is defined on $(x,S,\theta) \in \Omega \times [0,1] \times \mathbb{R}_+$ and has the form

$$a(x,S,\theta) = a_0(S)\frac{\overline{P_c}(x)}{\mu_2(\theta)}$$

with a_0 a continuous function on $[0,1]$ such that

$$a_0(0) = a_0(1) = 0, \quad a_0(S) > 0, \ \forall S \in]0,1[.$$

The function $\overline{P_c}$ is measurable on Ω and the function μ_2 is continuous on $[0,\infty[$ with

$$M^{-1} \leq \Big(\overline{P_c}(x); \mu_2(\theta)\Big) \leq M \quad \text{for} \quad x \in \Omega \quad \text{and} \quad \theta \geq 0.$$

$\star$ $\overrightarrow{f}(x,S,\theta)$ and $\overrightarrow{F}(x,S,\theta)$ are measurable in x and continuous in θ and S with

$$M^{-1}a_0 \leq \Big(\|\overrightarrow{f}\|_{\mathbb{R}^3}; \|\overrightarrow{F}\|_{\mathbb{R}^3}\Big) \leq Ma_0 \quad \text{on} \quad \overline{\Omega}_\star.$$

The boundary data $P^0(x,t)$, $S^0(x,t)$, and $\theta^0(x,t)$, are assumed to be defined not only on the boundary but also on $\overline{\Omega}_T = \overline{\Omega} \times [0,T]$ and satisfy, with $Q(x,t)$ for $(x,t) \in \Gamma_{2T}$, the conditions:

(iii) $\Big(\Big\|\dfrac{\partial\theta^0}{\partial t}, \dfrac{\partial S^0}{\partial t}\Big\|_{2,\Omega_T}; \ \|\nabla\theta^0; \nabla S^0\|_{2,\Omega_T}; \ \|P^0; \nabla P^0\|_{2,\infty,\Omega_T}; \ \|P^0\|_{\infty,\Gamma_{2T}}; \quad \|Q\|_{q,\infty,\Gamma_{1T}}\Big) \leq M$, where $q > 2$.
Here $\|\cdot\|_{2,\Omega_T}$ denotes the $L^2(\Omega_T)$-norm, $\|\cdot\|_{2,\infty,\Omega_T}$ the $L^\infty(0,T;L^2(\Omega))$-norm, and $\|Q\|_{q,\infty,\Gamma_{1T}}$ the $L^\infty(0,T;L^q(\Gamma_{1T}))$-norm.

(iv) $0 < \theta^0(x,t) \leq M_1$, a.e. in Ω_T, $M_1 \in \mathbb{R}_+$, and $0 \leq \delta_0 \leq S^0(x,t) \leq 1 - \delta_1$, a.e. in Ω_T with δ_0 and δ_1 two numbers in $\mathbb{R}_+$.

2.2 Notations.

Let us put

$$
\begin{array}{rcl}
I &=&]0,T[,\\
\mathcal{H} &=& W_2^1(\Omega,\Gamma_2) = H^1_{\Gamma_2}(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_2\},\\
\mathcal{U} &=& W_2^1(\Omega,\Gamma_1\cup\Gamma_2) = H^1_{\Gamma_1\cup\Gamma_2}(\Omega) = \{v \in H^1(\Omega) \mid v = 0 \quad \text{on} \quad \Gamma_1\cup\Gamma_2\},\\
&& \mathcal{H} \quad \text{and} \quad \mathcal{U} \quad \text{are equipped with the gradient norm,}
\end{array}
$$

For X a Banach space, $H^1(I;X) = \left\{ v \in L^2(I;X) \,\middle|\, v' = \dfrac{dv}{dt} \in L^2(I;X) \right\}\cdot$

Definition 2.2.1. A triplet (P,S,θ), of measurable and bounded functions in Ω_T, is called a generalized solution of system $\mathcal{P}$ if:

(a) $0 \le S(x,t) \le 1$ and $0 \le \theta(x,t)$ a.e. in Ω_T.

(b) $P \in L^\infty(I;\mathcal{H}) + P^0$, $u(S) \in L^2(I;\mathcal{H}) + S^0$ where u is the function defined by $u(s) = \int_0^s a_0(\xi)\, d\xi$, $mS_t \in L^2(I;\mathcal{H}^\star)$; $\theta \in L^2(I;\mathcal{U}) + \theta^0$ and $\ell\theta_t \in L^2(I;\mathcal{U}^\star)$.

(c) $\forall \psi \in \mathcal{H}$; $\forall \varphi \in H^1(I;\mathcal{H})$, $\varphi(x,T) = 0$, $x \in \Omega$; $\forall \eta \in H^1(I;\mathcal{U}) \cap L^\infty(\Omega_T)$, $\eta(x,T) = 0$, $x \in \Omega$, we have

(c.1) $(\overrightarrow{V}, \nabla\psi)_\Omega = -(K\nabla P + \overrightarrow{f}, \nabla\psi)_\Omega = (Q,\psi)_{\Gamma_1}$, a.e. in $]0,T[$;

(c.2) $\displaystyle\int_I \langle mS_t, \varphi\rangle_{\mathcal{H}^\star,\mathcal{H}}\, dt - (\overrightarrow{V_1}, \nabla\varphi)_{\Omega_T} + (bQ,\varphi)_{\Gamma_{1T}} = 0$;

(c.3) $\displaystyle\int_I \langle \ell\theta_t, \eta\rangle_{\mathcal{U}^\star,\mathcal{U}}\, dt + (\overrightarrow{V}, \eta\nabla\theta)_{\Omega_T} + (\Lambda\nabla\theta, \nabla\eta)_{\Omega_T} = 0$;

(c.4) $\forall \varphi \in L^2(I;\mathcal{H}) \cap W^{1,1}(I;L^1(\Omega))$, $\forall \eta \in L^2(I;\mathcal{U}) \cap W^{1,1}(I;L^1(\Omega))$ with $\varphi(x,T) = \eta(x,T) = 0$, $x \in \Omega$, we have

$$
\begin{array}{rcl}
\displaystyle\int_I \langle mS_t, \varphi\rangle\, dt &+& \displaystyle\int_I (m(S-S^0), \varphi_t)_\Omega\, dt = 0,\\[2ex]
\displaystyle\int_I \langle \ell\theta_t, \eta\rangle\, dt &+& \displaystyle\int_I (\ell(\theta-\theta^0), \eta_t)_\Omega\, dt = 0.
\end{array}
$$

Here $\langle\cdot,\cdot\rangle_{\mathcal{H}^\star,\mathcal{H}}$ is the scalar product for the duality $\mathcal{H}^\star$, $\mathcal{H}$; $S_t = \partial_t S = \partial S/\partial t$; $(\cdot,\cdot)_{\Omega_T}$, $(\cdot,\cdot)_\Omega$, $(\cdot,\cdot)_{\Gamma_1}$ denote the scalar product of $L^2(\Omega_T)$, $L^2(\Omega)$, and $L^2(\Gamma_1)$ respectively.

Remark 2.2.1. *Since* $\overrightarrow{V}_1 = -(K_0 a \nabla S - b\overrightarrow{V} + \overrightarrow{F})$, *and*

$$(b\overrightarrow{V}, \nabla\varphi)_{\Omega_T} = (bQ, \varphi)_{\Gamma_{1T}} - (b_S \overrightarrow{V} \cdot \nabla S, \varphi)_{\Omega_T},$$

here $b_S = (db)/dS$, $(\mathbf{c}.\mathbf{2})$ *may be written as*

$$(\mathbf{c}.\mathbf{2})' \quad \int_I \langle mS_t, \varphi\rangle_{\mathcal{H}^\star,\mathcal{H}}\, dt + (K_0 a \nabla S + \overrightarrow{F}, \nabla\varphi)_{\Omega_T} + (b_S \overrightarrow{V} \cdot \nabla S, \varphi)_{\Omega_T} = 0.$$

Our main result is the following:

Theorem 2.2.1. If conditions (i)-(iv) are fulfilled problem $(\mathcal{P})$ has at least one generalized solution, i.e., there exists at least one triplet (P, S, θ) satisfying condition (a), (b), and (c) of Definition 2.2.1 in which (c.2) has been replaced by $(\mathbf{c}.\mathbf{2})'$.

To prove this theorem we will construct a triplet with the required properties; this needs several steps:

2.3 Regularization.

We extend the coefficients of identities $(\mathbf{c}.\mathbf{1})$, $(\mathbf{c}.\mathbf{2})'$, and $(\mathbf{c}.\mathbf{3})$ by putting

$$f_*(x,\theta,S) = \begin{cases} f(x,S,\theta), & (x,S,\theta) \in \Omega_\star; \\ f(x,0,\theta), & S \leq 0, \quad \theta \geq 0; \\ f(x,0,0), & S \leq 0, \quad \theta \leq 0; \\ f(x,S,0), & S \in [0,1], \quad \theta \leq 0; \\ f(x,1,0), & S \geq 1, \quad \theta \leq 0; \\ f(x,1,\theta), & S \geq 1, \quad \theta \geq 0. \end{cases}$$

Also, we put

$$a_\varepsilon(x,S,\theta) = a_*(x,S,\theta) + \varepsilon, \quad \varepsilon > 0; \quad \overrightarrow{V_\kappa} = \frac{\overrightarrow{V}}{1+\kappa|\overrightarrow{V}|}, \quad \kappa > 0.$$

Then we have

$$\|\overrightarrow{V_\kappa}\|_{\mathbb{R}^3} \leq \|\overrightarrow{V}\|_{\mathbb{R}^3} \quad \text{and} \quad \|\overrightarrow{V_\kappa}\|_{\mathbb{R}^3} \leq \frac{1}{\kappa}\cdot \tag{2.1}$$

2.4 Generalized Solutions of the Regularized Problem ($\mathcal{P}_{\varepsilon\kappa}$)

Definition 2.4.1. A triplet (P, S, θ), of measurable and bounded functions in Ω_T, is called a generalized solution of system $(\mathcal{P}_{\varepsilon\kappa})$ if:

($\boldsymbol{\alpha}$) P, S, and θ are in $V_2(\Omega_T)$ and $\nabla P \in L^{2,\infty}(\Omega_T)$.

($\boldsymbol{\beta}$) P, S, and θ satisfy conditions **(b)** of Definition 2.2.1.

($\boldsymbol{\gamma}$) For all ψ, φ, and η chosen as in **(c)** we have

($\boldsymbol{\gamma}$.1)
$$(\overrightarrow{V}, \nabla\psi) = (Q, \psi)_{\Gamma_1}, \quad \text{a.e. in} \quad]0, T[;$$

($\boldsymbol{\gamma}$.2)
$$\int_I \langle mS_t, \varphi\rangle_{\mathcal{H}^\star,\mathcal{H}}\,dt + (K_0 a_\varepsilon \nabla S + \overrightarrow{F}_*, \nabla\varphi)_{\Omega_T} + ((b_S)_* \overrightarrow{V}_\kappa \cdot \nabla S, \varphi)_{\Omega_T} = 0;$$

($\boldsymbol{\gamma}$.3)
$$\int_I \langle \ell\theta_t, \eta\rangle_{\mathcal{U}^\star,\mathcal{U}}\,dt + (\overrightarrow{V}_\kappa, \eta\nabla\theta)_{\Omega_T} + (\Lambda_* \nabla\theta, \nabla\eta)_{\Omega_T} = 0,$$

($\boldsymbol{\gamma}$.4) $\forall\varphi \in L^2(I; \mathcal{H}) \cap W^{1,1}(I; L^1(\Omega))$, $\forall\eta \in L^2(I; \mathcal{U}) \cap W^{1,1}(I; L^1(\Omega))$ with $\varphi(x, T) = \eta(x, T) = 0$, $x \in \Omega$, we have

$$\int_I \langle mS_t, \varphi\rangle\,dt + \int_I (m(S - S^0), \varphi_t)_\Omega\,dt = 0,$$
$$\int_I \langle \ell\theta_t, \eta\rangle\,dt + \int_I (\ell(\theta - \theta^0), \eta_t)_\Omega\,dt = 0.$$

Here

$$V_2(\Omega_T) = \left\{u \,\middle|\, u, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \frac{\partial u}{\partial x_3} \in L^2(\Omega_T); \right.$$
$$\left. \operatorname*{vrai\,max}_{0\le t\le T} \|u(\cdot, t)\|_{2,\Omega} + \|\nabla u\|_{2,\Omega_T} < +\infty \right\}.$$

We are going to investigate first the regularized problem, then, passing to the limit with respect to h and ε, the proof of the existence of one generalized solution of the degenerate problem $(\mathcal{P})$ is achieved.

Theorem 2.4.1. If conditions (i)-(iv) of Theorem 2.2.1 are fulfilled problem $(\mathcal{P}_{\varepsilon\kappa})$ has at least one generalized solution, in the sense of Definition 2.4.1.

Below C (with or without a subscript) indicates a generic constant independent of the number α (defined in Section 2.5 below), which will probably takes on different values in different occurrences.

2.5 Time discretization of system ($\mathcal{P}_{\varepsilon\kappa}$).

In order to prove the existence of a generalized solution of system ($\mathcal{P}_{\varepsilon\kappa}$) we proceed as follows: We approximate the time derivative by time discretization, that is, we replace for instance $\partial\theta/\partial t$ by the backward difference quotient $\partial_t^\alpha\theta$. More precisely, for each positive integer n, divide the time interval $I =]0, T]$ into $N = 2^n$ subintervals of equal length $\alpha = T/N = 2^{-n}T$. Set $t_j = j\alpha$ and $I_j = (t_{j-1}, t_j]$ for an integer j, $1 \le j \le N$. Denote the time difference operator by

$$\partial_t^\alpha \xi(t) = \frac{\xi(t+\alpha) - \xi(t)}{\alpha},$$

for any function $\xi(t)$ and constant $\alpha \in \mathbb{R}^\star$. Also, for any linear space H, define

$$\mathcal{L}^\alpha(H) = \{v \in L^\infty(I; H) \mid v \ \text{ is constant in time on each subinterval } \ I_j \subset I\}.$$

For notational convenience, for $v^\alpha \in \mathcal{L}^\alpha(H)$, set $v^{j\alpha} \equiv v^\alpha|_{I_j} = v^\alpha(t_j)$. We set also $v^{j'\alpha}$ for $v^\alpha(t_{j-1})$. If $w = w(x,t)$ is a function, the average in time over I_j is

$$w_\alpha(x,t) = \frac{1}{\alpha}\int_{I_j} w(x,\tau)\, d\tau, \quad t \in I_j. \tag{2.2}$$

The value of $w_\alpha(\cdot)$ on the interval I_j is denoted $w_{\alpha j}(\cdot)$.

Definition 2.5.1. A discrete time solution is a triplet of functions

$$P^\alpha \in \mathcal{L}^\alpha(\mathcal{H}) + P^0_\alpha, \quad S^\alpha \in \mathcal{L}^\alpha(\mathcal{H}) + S^0_\alpha, \quad \text{and} \quad \theta^\alpha \in \mathcal{L}^\alpha(\mathcal{U}) + \theta^0_\alpha$$

satisfying the following integral identities

$$\begin{aligned} (K^{j'\alpha}\nabla P^{j\alpha} \quad &+ \quad \overrightarrow{f}^{j'\alpha}, \nabla\psi)_\Omega = -(Q_\alpha, \psi)_{\Gamma_1}, \quad t \in I_j, && (2.3)\\ & \qquad\qquad\qquad\qquad j = 1, \ldots, N, \qquad \forall\psi \in \mathcal{H};\\ \int_I (m\partial_t^{-\alpha} S^\alpha, \varphi)_\Omega\, dt \quad &+ \quad \int_I (K_0\, a_\varepsilon^\alpha\, \nabla S^\alpha + \overrightarrow{F_*}^\alpha, \nabla\varphi)_\Omega\, dt && (2.4)\\ &+ \quad \int_I (b_S^\alpha \overrightarrow{V_\kappa}^\alpha \cdot \nabla S^\alpha, \varphi)_\Omega\, dt = 0, \quad \forall\varphi \in \mathcal{L}^\alpha(\mathcal{H});\\ \int_I (\ell^\alpha \partial_t^{-\alpha}\theta^\alpha, \eta)_\Omega\, dt \quad &+ \quad \int_I (\overrightarrow{V_\kappa}^\alpha, \eta\nabla\theta^\alpha)_\Omega\, dt && (2.5)\\ &+ \quad \int_I (\Lambda_*^\alpha \nabla\theta^\alpha, \nabla\eta)_\Omega\, dt = 0, \quad \forall\eta \in \mathcal{L}^\alpha(\mathcal{U}), \end{aligned}$$

where $\theta^\alpha(\cdot, t-\alpha) = \theta^0(\cdot)$, $S^\alpha(\cdot, t-\alpha) = S^0(\cdot)$ when $0 < t \le \alpha$. Here

$$\begin{aligned}
\ell^\alpha &= \ell(x, S^\alpha), \quad \overrightarrow{V_\kappa}^\alpha = \overrightarrow{V_\kappa}(P^\alpha, S^\alpha, \theta^\alpha), \quad \Lambda_*^\alpha = \Lambda_*(S^\alpha, \theta^\alpha),\\
K^\alpha &= K(x, S^\alpha, \theta^\alpha), \quad a_\varepsilon^\alpha = a_\varepsilon(x, S^\alpha, \theta^\alpha), \quad \overrightarrow{F_*}^\alpha = \overrightarrow{F_*}(x, S^\alpha, \theta^\alpha),\\
b_S^\alpha &= b_S(S^\alpha), \quad \overrightarrow{f}^\alpha = \overrightarrow{f}(x, S^\alpha, \theta^\alpha), \quad \text{and}\\
K^{j'\alpha} &= K^{j'\alpha}(\cdot) = K\Big(\cdot, S^\alpha(\cdot, (j-1)\alpha), \theta^\alpha(\cdot, (j-1)\alpha)\Big).
\end{aligned}$$

Remark 2.5.1. *Let us write, for example, the integral identity (2.4) in another (equivalent) way. If we take the test function in the form $\chi_{I_j}(t)\varphi(x)$ with χ_{I_j} the characteristic function of the interval $I_j =]j'\alpha, j\alpha] =]t_{j-1}, t_j]$ and φ a function in the space $\mathcal{H}$, we obtain*

$$\int_{I_j} \Big(m \frac{S^\alpha(x,t) - S^\alpha(x, t-\alpha)}{\alpha}, \varphi\Big)_\Omega dt \ + \ \int_{I_j} (K_0\, a_\varepsilon^\alpha \nabla S^\alpha + \overrightarrow{F_*}^\alpha, \nabla\varphi)_\Omega \, dt + \int_{I_j} (b_S^\alpha \overrightarrow{V_\kappa}^\alpha \cdot \nabla S^\alpha, \varphi)_\Omega \, dt = 0.$$

Since $S^\alpha(\cdot, t)$ is constant with respect to t on the interval I_j, equals $S^\alpha(\cdot, t_j)$, and the same is true for P^α, θ^α, we get the integral identity (where $S^{j\alpha}$ denotes $S^\alpha(\cdot, t_j) = S^\alpha(\cdot, j\alpha)$, we do the same with P and θ, and denote for instance $K^{j\alpha} = K(x, S^{j\alpha}, \theta^{j\alpha})$):

$$\begin{aligned}
(mS^{j\alpha}, \varphi)_\Omega \ &+ \ \alpha(K_0\, a_\varepsilon^{j\alpha} \nabla S^{j\alpha} + \overrightarrow{F_*}^{j\alpha}, \nabla\varphi)_\Omega \\
&+ \alpha(b_S^{j\alpha} \overrightarrow{V_\kappa}^{j\alpha} \cdot \nabla S^{j\alpha}, \varphi)_\Omega = (mS^{j'\alpha}, \varphi)_\Omega, \quad \forall\, \varphi \in \mathcal{H}.
\end{aligned} \tag{2.6}$$

Now, choosing in (2.5) the test function $\chi_{I_j}(t)\eta(x)$ with η a function in the space $\mathcal{U}$, we obtain

$$\begin{aligned}
(\ell^{j\alpha}\theta^{j\alpha}, \eta)_\Omega + \alpha(\overrightarrow{V_\kappa}^{j\alpha}, \eta\nabla\theta^{j\alpha})_\Omega \ &+ \ \alpha(\Lambda_*^{j\alpha} \nabla\theta^{j\alpha}, \nabla\eta)_\Omega \\
&= (\ell^{j\alpha}\theta^{j'\alpha}, \eta)_\Omega, \quad \forall \eta \in \mathcal{U}.
\end{aligned} \tag{2.7}$$

2.6 Maximum principles about discrete time solutions.

We will prove maximum principles for $j = 1$, i.e. for $(S^{1\alpha}, \theta^{1\alpha})$. It is obvious that the same techniques show that similar results are true for $(S^{j\alpha}, \theta^{j\alpha})$, $j = 2, \ldots, N$.

Theorem 2.6.1. Suppose conditions (i–iv) are fulfilled. If $(P^{1\alpha}, S^{1\alpha}, \theta^{1\alpha})$ is a time discrete solution of $(\mathcal{P}_{\varepsilon\kappa})$ at the time level $t_1 = \alpha$, then

$$0 \le S^{1\alpha}(x) \le 1 \quad \textbf{a.e. in} \quad \Omega, \tag{2.8}$$

$$0 \le \underline{\theta}^0 = \min_{\overline{\Omega}_T} \theta^0(x,t) \le \theta^{1\alpha}(x) \le \overline{\theta}^0 = \max_{\overline{\Omega}_T} \theta^0(x,t) \quad \textbf{a.e. in} \quad \Omega \tag{2.9}$$

Proof — To prove the right-hand side of inequality in (2.8), we consider the Lipschitzian function $G(t) = (t-1)^+$ and take in (2.6), written for $j = 1$, the test function $\varphi = G(S^{1\alpha})$ which is in $\mathcal{H}$. We get

$$\begin{aligned} (mS^{1\alpha}, G(S^{1\alpha}))_\Omega \quad &+ \quad \alpha(K_0\, a_\varepsilon^{1\alpha}\, \nabla S^{1\alpha} + \overrightarrow{F_*}^{1\alpha}, \nabla G(S^{1\alpha}))_\Omega \\ &+ \quad \alpha(b_S^{1\alpha} \overrightarrow{V_\kappa}^{1\alpha} \cdot \nabla S^{1\alpha}, G(S^{1\alpha}))_\Omega = (mS^0, G(S^{1\alpha}))_\Omega. \end{aligned}$$

Putting $\Omega^{1\alpha}_{S_+} = \{x \in \Omega \mid S^{1\alpha}(x) > 1\}$, we get

$$\begin{aligned} (mS^{1\alpha}, G(S^{1\alpha}))_\Omega \quad &= \quad \int_{\Omega^{1\alpha}_{S_+}} m(x) S^{1\alpha}(S^{1\alpha} - 1)\, dx \\ &= \quad \int_{\Omega^{1\alpha}_{S_+}} m(S^{1\alpha} - 1)^2\, dx + \int_{\Omega^{\alpha 1}_{S_+}} m(S^{1\alpha} - 1)\, dx \end{aligned}$$

and

$$\begin{aligned} \alpha(K_0\, a_\varepsilon^{1\alpha}\, \nabla S^{1\alpha}, \nabla G(S^{1\alpha}))_\Omega \quad &= \quad \alpha \int_{\Omega^{1\alpha}_{S_+}} K_0\, a_\varepsilon^{1\alpha}\, \nabla S^{1\alpha} \cdot \nabla S^{1\alpha}\, dx \\ &\ge \quad \frac{\alpha\varepsilon}{M} \int_{\Omega^{1\alpha}_{S_+}} |\nabla S^{1\alpha}|^2\, dx. \end{aligned}$$

Since $\overrightarrow{F_*}^{1\alpha} = \overrightarrow{F_*}(x, S^{1\alpha}, \theta^{1\alpha}) = 0$ for $S^{1\alpha} > 1$, we have

$$(\overrightarrow{F_*}^{1\alpha}, \nabla G(S^{1\alpha}))_\Omega = \int_{\Omega^{1\alpha}_{S_+}} \overrightarrow{F_*}^{1\alpha} \cdot \nabla G(S^{1\alpha})\, dx = 0.$$

Also, we have

$$\begin{aligned} \alpha|(b_S^{1\alpha} \overrightarrow{V_\kappa}^{1\alpha} \cdot \nabla S^{1\alpha}, G(S^{1\alpha}))_\Omega| \quad &\le \quad \frac{\alpha M}{\kappa} \int_{\Omega^{1\alpha}_{S_+}} |\nabla S^{1\alpha}||S^{1\alpha} - 1|\, dx \\ (mS^0, G(S^{1\alpha}))_\Omega \quad &= \quad \int_{\Omega^{1\alpha}_{S_+}} mS^0(S^{\alpha 1} - 1)\, dx \end{aligned}$$

The above computations and estimates lead to

$$\begin{aligned}\int_{\Omega^{1\alpha}_{S_+}} m(S^{1\alpha}-1)^2\,dx \quad &+ \quad \int_{\Omega^{1\alpha}_{S_+}} m(1-S^0)(S^{1\alpha}-1)\,dx \\ &+ \quad \frac{\alpha\varepsilon}{M}\int_{\Omega^{1\alpha}_{S_+}} |\nabla S^{1\alpha}|^2\,dx \\ &\leq \quad \frac{\alpha M}{\kappa}\int_{\Omega^{1\alpha}_{S_+}} |\nabla S^{1\alpha}|(S^{1\alpha}-1)\,dx\end{aligned} \tag{2.10}$$

Let us now use the elementary inequality

$$ab \leq \frac{r}{2}a^2 + \frac{1}{2r}b^2, \quad \forall a, b \geq 0, \quad r > 0.$$

We have

$$\int_{\Omega^{1\alpha}_{S_+}} |\nabla S^{1\alpha}|(S^{1\alpha}-1)\,dx \leq \frac{r}{2}\int_{\Omega^{1\alpha}_{S_+}} |\nabla S^{1\alpha}|^2\,dx + \frac{1}{2r}\int_{\Omega^{1\alpha}_{S_+}} (S^{1\alpha}-1)^2\,dx$$

Choosing $r = \varepsilon\kappa/M^2$ and plugging into inequality (2.10), we get

$$\begin{aligned}\int_{\Omega^{1\alpha}_{S_+}} \left[m - \frac{\alpha M^3}{2\varepsilon\kappa^2}\right](S^{1\alpha}-1)^2\,dx \quad &+ \quad \int_{\Omega^{1\alpha}_{S_+}} m(1-S^0)(S^{1\alpha}-1)\,dx \\ &+ \frac{\alpha\varepsilon}{2M}\int_{\Omega^{1\alpha}_{S_+}} |\nabla S^{1\alpha}|^2\,dx \leq 0.\end{aligned} \tag{2.11}$$

Since (according to (**i**))

$$\left[m - \frac{\alpha M^3}{2\varepsilon\kappa^2}\right] \geq \left[\frac{1}{M} - \frac{\alpha M^3}{2\varepsilon\kappa^2}\right] \quad \text{a.e. in} \quad \Omega,$$

for $\alpha > 0$ sufficiently small (i.e. for $0 < \alpha < 2\varepsilon\kappa^2/M^4$, here ε and κ are fixed), all the terms on the left of inequality (2.11) are nonnegative, this implies that $S^{1\alpha} \leq 1$ a.e. in Ω. The same argument shows that S^α is less than 1 on each time level t_j, $j = 1, ..., N$. Therefore, $S^\alpha(x,t) \leq 1$ a.e. in Ω_T.

Now, to prove the left-hand side of inequality in (2.8), we consider the function $G(t) = t_- = \frac{1}{2}(|t| - t)$ and choose in (2.6) as test function $\varphi = G(S^{1\alpha}) = S^{1\alpha}_-$. We get

$$\begin{aligned}(mS^{1\alpha}, S^{1\alpha}_-)_\Omega \quad &+ \quad \alpha(K_0\, a^{1\alpha}_\varepsilon\,\nabla S^{1\alpha} + \overrightarrow{F}^{1\alpha}_*, \nabla S^{1\alpha}_-)_\Omega \\ &+ \alpha(b^{1\alpha}_S \overrightarrow{V}^{1\alpha}_\kappa \cdot \nabla S^{1\alpha}, S^{1\alpha}_-)_\Omega = (mS^0, S^{1\alpha}_-)_\Omega.\end{aligned} \tag{2.12}$$

Putting $\Omega^{1\alpha}_{S_-} = \{x \in \Omega \mid S^{1\alpha}(x) < 0\}$, we have

$$\begin{aligned}(mS^{1\alpha}, S^{1\alpha}_-)_\Omega &= -\int_{\Omega^{1\alpha}_{S_-}} m(S^{1\alpha})^2\, dx,\\ (mS^0, S^{1\alpha}_-)_\Omega &= -\int_{\Omega^{1\alpha}_{S_-}} mS^0 S^{1\alpha}\, dx \geq 0,\end{aligned}$$

and

$$\begin{aligned}\alpha(K_0\, a^{1\alpha}_\varepsilon\, \nabla S^{1\alpha}, \nabla S^{1\alpha}_-)_\Omega &= \alpha \int_{\Omega^{1\alpha}_{S_-}} K_0\, a^{1\alpha}_\varepsilon \nabla S^{1\alpha} \cdot \nabla S^{1\alpha}_-\, dx\\ &= -\alpha \int_{\Omega^{1\alpha}_{S_-}} K_0\, a^{1\alpha}_\varepsilon \nabla S^{1\alpha} \cdot \nabla S^{1\alpha}\, dx.\end{aligned}$$

Since $\overrightarrow{F}^{1\alpha}_* = 0$ for $S^{1\alpha} < 0$, we have

$$(\overrightarrow{F}^{1\alpha}_*, \nabla S^{1\alpha}_-)_\Omega = -\int_{\Omega^{1\alpha}_{S_-}} \overrightarrow{F}^{1\alpha}_* \cdot \nabla S^{1\alpha}\, dx = 0.$$

Also, we have

$$\alpha(b^{1\alpha}_S \overrightarrow{V}^{1\alpha}_\kappa \cdot \nabla S^{1\alpha}, S^{1\alpha}_-)_\Omega = -\alpha \int_{\Omega^{1\alpha}_{S_-}} b^{1\alpha}_S \overrightarrow{V}^{1\alpha}_\kappa \cdot \nabla S^{1\alpha}\, S^{1\alpha}\, dx,$$

with

$$\left| \int_{\Omega^{1\alpha}_{S_-}} b^{1\alpha}_S \overrightarrow{V}^{\alpha 1}_\kappa \cdot \nabla S^{1\alpha}\, S^{1\alpha}\, dx \right| \leq \frac{M}{\kappa} \int_{\Omega^{1\alpha}_{S_-}} |\nabla S^{1\alpha}|\, |S^{1\alpha}|\, dx$$

Putting together the previous equalities, we can rewrite the relation (2.12) as

$$\begin{aligned}-\int_{\Omega_{S^{1\alpha}_-}} m(S^{1\alpha})^2\, dx &- \alpha \int_{\Omega_{S^{1\alpha}_-}} K_0\, a^{1\alpha}_\varepsilon\, \nabla S^{1\alpha} \cdot \nabla S^{1\alpha}\, dx\\ &- \alpha \int_{\Omega_{S^{1\alpha}_-}} b^{1\alpha}_S \overrightarrow{V}^{1\alpha}_\kappa \nabla S^{1\alpha}\, S^{1\alpha}\, dx = -\int_{\Omega_{S^{1\alpha}_-}} mS^0 S^{1\alpha}\, dx \geq 0.\end{aligned}$$

Thus,

$$\begin{aligned}\int_{\Omega_{S^{1\alpha}_-}} m(S^{1\alpha})^2\, dx &+ \alpha \int_{\Omega_{S^{1\alpha}_-}} K_0 a^{1\alpha}_\varepsilon\, \nabla S^{1\alpha} \cdot \nabla S^{1\alpha}\, dx\\ &+ \alpha \int_{\Omega_{S^{1\alpha}_-}} b^{1\alpha}_S \overrightarrow{V}^{1\alpha}_\kappa \nabla S^{1\alpha}\, S^{1\alpha}\, dx \leq 0.\end{aligned}$$

Using the coerciveness of K_0, the definition of a_ε the above estimate on the last term, we get the inequality

$$\begin{aligned}\int_{\Omega_{S^{1\alpha}_-}} m(S^{1\alpha})^2\,dx &+ \frac{\alpha\varepsilon}{M}\int_{\Omega_{S^{1\alpha}_-}} |\nabla S^{1\alpha}|^2\,dx \\ &\le \frac{\alpha M}{\kappa}\int_{\Omega_{S^{1\alpha}_-}} |\nabla S^{1\alpha}|\,|S^{1\alpha}|\,dx \\ &\le \frac{\alpha M}{\kappa}\int_{\Omega_{S^{1\alpha}_-}} \left(\frac{r}{2}|\nabla S^{1\alpha}|^2 + \frac{1}{2r}|S^{1\alpha}|^2\right)dx\end{aligned}$$

Choosing $r = \varepsilon\kappa/M^2$ and continuing as above by choosing α sufficiently small we get

$$\int_{\Omega^{1\alpha}_{S_-}} (S^{1\alpha})^2\,dx \le 0.$$

Then, $0 \le S^{1\alpha}$ a.e. in Ω. This finishes the proof of estimates (2.8).

To prove the right-hand side of (2.9), we take in (2.7) as test function $\eta = G(\theta^{1\alpha})$ with $G(t) = (t - \overline{\theta}^0)^+$ and $\overline{\theta}^0 = \max_{\overline{\Omega}_T} \theta^0(x,t)$. We get

$$\begin{aligned}(\ell^{1\alpha}\theta^{1\alpha}, G(\theta^{1\alpha}))_\Omega &+ \alpha(\Lambda^{1\alpha}_*\nabla\theta^{1\alpha}, \nabla G(\theta^{1\alpha}))_\Omega \\ &= \alpha(\overrightarrow{V_\kappa}^{1\alpha}, G(\theta^{1\alpha})\nabla\theta^{1\alpha})_\Omega + (\ell^{1\alpha}\theta^0, G(\theta^{1\alpha}))_\Omega.\end{aligned} \tag{2.13}$$

Putting $\Omega^{1\alpha}_{\theta+} = \{x \in \Omega \mid \theta^{1\alpha} > \overline{\theta}^0\}$, we can write

$$\begin{aligned}(\ell^{1\alpha}\theta^{1\alpha}, G(\theta^{1\alpha}))_\Omega &= \int_{\Omega^{1\alpha}_{\theta+}} \ell^{1\alpha}(x)\theta^{1\alpha}(\theta^{1\alpha} - \overline{\theta}^0)\,dx \\ &= \int_{\Omega^{1\alpha}_{\theta+}} \ell^{1\alpha}(x)(\theta^{1\alpha} - \overline{\theta}^0)^2\,dx + \int_{\Omega^{1\alpha}_{\theta+}} \ell^{1\alpha}(x)\overline{\theta}^0(\theta^{1\alpha} - \overline{\theta}^0)\,dx.\end{aligned}$$

Since

$$\alpha(\Lambda^{1\alpha}_*\nabla\theta^{1\alpha}, \nabla G(\theta^{1\alpha}))_\Omega \ge \frac{\alpha}{M}\int_{\Omega_{\theta+}} |\nabla\theta^{1\alpha}|^2\,dx$$

and

$$\begin{aligned}\alpha(\overrightarrow{V_\kappa}^{1\alpha}, G(\theta^{1\alpha})\nabla\theta^{1\alpha})_\Omega &\le \frac{\alpha}{\kappa}\int_{\Omega_{\theta+}} (\theta^{1\alpha} - \overline{\theta}^0)\,|\nabla\theta^{1\alpha}|\,dx \\ &\le \frac{\alpha}{\kappa}\int_{\Omega_{\theta+}} \left\{\frac{r}{2}|\nabla\theta^{1\alpha}|^2 + \frac{1}{2r}(\theta^{1\alpha} - \overline{\theta}^0)^2\right\}dx, \quad (r > 0)\end{aligned}$$

with

$$\begin{aligned}(\ell^{1\alpha}\theta^0, G(\theta^{1\alpha}))_\Omega \quad &- \quad \int_{\Omega^{1\alpha}_{\theta+}} \ell^{1\alpha}(x)\overline{\theta}^0(\theta^{1\alpha}-\overline{\theta}^0)\,dx \\ &= \int_{\Omega^{1\alpha}_{\theta+}} \ell^{1\alpha}(x)(\theta^0-\overline{\theta}^0)(\theta^{1\alpha}-\overline{\theta}^0)\,dx \le 0,\end{aligned}$$

the relation (2.13) leads to

$$\int_{\Omega^{1\alpha}_{\theta+}} \left(\ell^{1\alpha}(x) - \frac{\alpha}{2\kappa r}\right)(\theta^{1\alpha}-\overline{\theta}^0)^2\,dx + \left(\frac{\alpha}{M}-\frac{\alpha r}{2\kappa}\right)\int_{\Omega_{\theta+}} |\nabla\theta^{1\alpha}|^2\,dx \le 0.$$

choosing $r = \kappa/M$ and noting that (see the hypothesis (**i**) and estimate (2.8)):

$$\ell^{1\alpha}(x) - \frac{\alpha M}{2\kappa^2} \quad = \quad \ell_0(x)S^{1\alpha} + \ell_1(x) - \frac{\alpha M}{2\kappa^2} \ge \frac{1}{M} - \frac{\alpha M}{2\kappa^2}$$

we get, for α sufficiently small, $\int_{\Omega^{1\alpha}_{\theta+}} (\theta^{1\alpha}-\overline{\theta}^0)^2\,dx \le 0$ which implies that $\theta^{1\alpha}(x) \le \overline{\theta}^0$ a.e. in Ω.

To prove the left-hand side of (2.9), we take in (2.7) as test function $\eta = G(\theta^{1\alpha})$ with $G(t) = (\underline{\theta}^0 - t)^+$ and $\underline{\theta}^0 = \min\limits_{\overline{\Omega}_T} \theta^0(x,t)$. Putting $\Omega^{1\alpha}_{\theta_-} = \{x \in \Omega \mid \theta^{1\alpha}(x) < \underline{\theta}^0\}$, we get, after multiplying and transposing:

$$\begin{aligned}\int_{\Omega^{1\alpha}_{\theta_-}} \ell^{1\alpha}(x)(\underline{\theta}^0-\theta^{1\alpha})^2\,dx \quad &+ \quad \alpha\int_{\Omega^{1\alpha}_{\theta_-}} \Lambda^{1\alpha}_*\nabla\theta^{1\alpha}\cdot\nabla\theta^{1\alpha}\,dx \\ &= \quad -\alpha\int_{\Omega^{1\alpha}_{\theta_-}} \overrightarrow{V_\kappa}^{1\alpha}\cdot\nabla\theta^{1\alpha}\ (\underline{\theta}^0-\theta^{1\alpha})\,dx \\ &\qquad -\int_{\Omega^{1\alpha}_{\theta_-}} \ell^{1\alpha}\underline{\theta}^0(\underline{\theta}^0-\theta^{1\alpha})\,dx \\ &\qquad -\int_{\Omega^{1\alpha}_{\theta_-}} \ell^{1\alpha}\theta^0(\underline{\theta}^0-\theta^{1\alpha})\,dx \\ &\le \quad -\alpha\int_{\Omega^{1\alpha}_{\theta_-}} \overrightarrow{V_\kappa}^{1\alpha}\cdot\nabla\theta^{1\alpha}\ (\underline{\theta}^0-\theta^{1\alpha})\,dx \\ &\le \quad \frac{\alpha}{\kappa}\int_{\Omega^{1\alpha}_{\theta_-}} \left[\frac{r}{2}|\nabla\theta^{1\alpha}|^2 + \frac{1}{2r}(\underline{\theta}^0-\theta^{1\alpha})^2\right]dx\end{aligned}$$

continuing as above we arrive to the result that

$$\int_\Omega [(\underline{\theta}^0-\theta^{1\alpha})^+]^2\,dx \le 0$$

which implies $\underline{\theta}^0 \le \theta^{1\alpha}$ a.e. in Ω. This finishes the proof of estimates (2.9).

Chapter 3

Galerkin method for Heat and Mass Transfer in porous media.

3.1 Galerkin's approximations of discrete time solutions.

The time discretization leads thus to elliptic problems, which can be solved by a Galerkin's procedure, see for instance [1, 2, 3]. Let us determine solution at the time level $t_1 = \alpha$. For this we choose linearly independent functions $\{e_i\}_{i=1}^{\infty} \subset \mathcal{H}$ and $\{f_i\}_{i=1}^{\infty} \subset \mathcal{U}$, such that the subspace spanned by these functions is dense in $\mathcal{H}$ and $\mathcal{U}$ respectively. We are looking for functions

$$\begin{aligned} P^{1\alpha}(\cdot) &\in \mathcal{H} + P^0_{\alpha 1}(\cdot), \\ S^{1\alpha}_d(x) &= \sum_{i=1}^{d} \sigma^1_i e_i(x) + S^0_{\alpha 1}(x), \quad \text{and} \quad \theta^{1\alpha}_d(x) = \sum_{i=1}^{d} \tau^1_i f_i(x) + \theta^0_{\alpha 1}(x), \end{aligned}$$

with σ_i^1 and τ_i^1 unknown real coefficients such that the following identities hold true

$$\begin{aligned}
(K^{1'\alpha}\nabla P^{1\alpha} &+ \overrightarrow{f}^{1'\alpha}, \nabla\psi)_\Omega = -(Q_{\alpha 1}, \psi)_{\Gamma_1}, \quad \forall\psi \in \mathcal{H}; && (3.1)\\
\int_\Omega m\frac{S_d^{1\alpha} - S^0}{\alpha}\varphi\, dx &+ \int_\Omega (K_0 a_{\varepsilon d}^{1\alpha}\nabla S_d^{1\alpha} + \overrightarrow{F}_{*d}^{1\alpha})\cdot\nabla\varphi\, dx && (3.2)\\
&+ \int_\Omega b_{Sd}^{1\alpha}\overrightarrow{V}_{\kappa d}^{1\alpha}\cdot\nabla S_d^{1\alpha}\,\varphi\, dx = 0,\\
&\qquad\qquad \forall\varphi \in \mathcal{H}_d = \operatorname{span}[e_1, \dots, e_d];\\
\int_\Omega \ell^0\,\frac{\theta_d^{1\alpha} - \theta^0}{\alpha}\,\eta\, dx &+ \int_\Omega \overrightarrow{V}_{\kappa d}^{1\alpha}\cdot\nabla\theta_d^{1\alpha}\,\eta\, dx && (3.3)\\
&+ \int_\Omega \Lambda_{*d}^{1\alpha}\nabla\theta_d^{1\alpha}\cdot\nabla\eta\, dx = 0,\\
&\qquad\qquad \forall\eta \in \mathcal{U}_d = \operatorname{span}[f_1, \dots, f_d],
\end{aligned}$$

where $\ell^0 = \ell(x, S^0) = \ell_0(x)S^0(x) + \ell_1(x)$ and

$$\begin{aligned}
&K^{1'\alpha} = K(x, S^0, \theta^0), \quad \overrightarrow{f}^{1'\alpha} = \overrightarrow{f}(x, S^0, \theta^0), \quad a_{\varepsilon d}^{1\alpha} = a_\varepsilon(x, S_d^{1\alpha}, \theta_d^{1\alpha}),\\
&b_{Sd}^{1\alpha} = b_S(S_d^{1\alpha}), \quad \overrightarrow{F}_{*d}^{1\alpha} = \overrightarrow{F}_*(x, S_d^{1\alpha}, \theta_d^{1\alpha}), \quad \overrightarrow{V}_{\kappa d}^{1\alpha} = \overrightarrow{V}_\kappa(x, S_d^{1\alpha}, \theta_d^{1\alpha}),\\
&\text{and} \quad \Lambda_{*d}^{1\alpha} = \Lambda_*(x, S_d^{1\alpha}, \theta_d^{1\alpha}).
\end{aligned}$$

Remark 3.1.1. *In fact, we determine only the Galerkin's approximations at every time level* t_j*,* $j = 1, \dots, N$ *and take* $(P_d^\alpha, S_d^\alpha, \theta_d^\alpha)$ *constant on each time interval* $I_j =]t_{j-1}, t_j]$ *for* $j = 1, \dots, N$*.*

3.2 Proof of existence of Galerkin's approximations.

Here, for simplicity, it is assumed that $\frac{T}{\alpha}$ is an integer. Giving the couple $(S_{\alpha 1}^0, \theta_{\alpha 1}^0)$, we can determine the triplet $(P_d^\alpha, S_d^\alpha, \theta_d^\alpha)(t)$ inductively for $t \in](j-1)\alpha, j\alpha]$, $j = 1$,..., N as a solution of an elliptic system. Before carrying out the proof, we give the following.

Remark 3.2.1. *The finite dimensional space* $\mathcal{H}_d$ *(or* $\mathcal{U}_d$*) is equipped with the three (equivalent) norms defined, for* $v = \sum_{i=1}^d \alpha_i e_i \in \mathcal{H}_d$*, by*

$$\|v\|_{\mathbb{R}^d} = \left[\sum_{i=1}^d \alpha_i^2\right]^{\frac{1}{2}}, \quad \|v\|_{2,\Omega} = \left[\int_\Omega |v|^2\, dx\right]^{\frac{1}{2}}, \quad \|v\|_{\mathcal{H}} = \left[\int_\Omega |\nabla v|^2\, dx\right]^{\frac{1}{2}}.$$

Let us explain the first step of the existence of Galerkin's approximations. Let therefore take $j = 1$ and suppose that (S^0, θ^0) is known -the initial conditions. By injecting S^0 and θ^0 in the equation of pressure, we have to find $P^{1\alpha}$ such that

$$\begin{aligned} P^{1\alpha} &\in \mathcal{H} + P^0_{\alpha 1} \\ \int_\Omega \Big[K(x, S^0(x), \theta^0(x))\nabla P^{1\alpha}(x) &+ \overrightarrow{f}(x, S^0(x), \theta^0(x))\Big] \cdot \nabla\psi(x)\,dx \\ &= -\int_{\Gamma_1} Q_{\alpha 1}(\sigma)\psi\,d\sigma, \quad \forall\psi \in \mathcal{H}. \end{aligned}$$

This problem is coercive (by assumption **(i)**) and using the trace theorem we can use the Lax-Milgram Lemma to see that it possesses an unique solution $P^{1\alpha}$ defined on Ω.

Now, to determine $S^{1\alpha}_d$ and $\theta^{1\alpha}_d$, we use a corollary of the Brouwer's Fixed Point Theorem. We need some notations. If $\boldsymbol{\beta} = (\beta_1, \dots, \beta_d)$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_d)$ are two vectors in $\mathbb{R}^d$, the vector $\sum\limits_{i=1}^{d} \beta_i e_i(x)$ of $\mathcal{H}_d$ and the vector $\sum\limits_{i=1}^{d} \gamma_i f_i(x)$ of $\mathcal{U}_d$ are denoted $v_{\boldsymbol{\beta}}$ and $w_{\boldsymbol{\gamma}}$ respectively.

Let us now consider the mapping

$$\mathbb{R}^d \times \mathbb{R}^d \ni (\boldsymbol{\beta}, \boldsymbol{\gamma}) \longmapsto \Pi(\boldsymbol{\beta}, \boldsymbol{\gamma}) = (\mathbf{b}, \mathbf{c}) \in \mathbb{R}^d \times \mathbb{R}^d$$

where the $2d$ parameters are the unknown coefficients given by

$$\begin{aligned} b_i &= \int_\Omega m\frac{v_{\boldsymbol{\beta}} - S^0}{\alpha}\,e_i\,dx \\ &\quad + \int_\Omega [K_0(x)a_\varepsilon(x, v_{\boldsymbol{\beta}}, w_{\boldsymbol{\gamma}})\nabla v_{\boldsymbol{\beta}} + \overrightarrow{F_*}(x, v_{\boldsymbol{\beta}}, w_{\boldsymbol{\gamma}})] \cdot \nabla e_i\,dx \\ &\qquad + \int_\Omega b_S(v_{\boldsymbol{\beta}})\overrightarrow{V_\kappa}(P^{1\alpha}, v_{\boldsymbol{\beta}}, w_{\boldsymbol{\gamma}}) \cdot \nabla v_{\boldsymbol{\beta}}\,e_i\,dx, \quad i = 1, \dots, d; \\ c_i &= \int_\Omega \ell(x, S^0)\frac{w_{\boldsymbol{\gamma}} - \theta^0}{\alpha}\,f_i\,dx \\ &\qquad + \int_\Omega \overrightarrow{V_\kappa}(P^{1\alpha}, v_{\boldsymbol{\beta}}, w_{\boldsymbol{\gamma}}) \cdot \nabla w_{\boldsymbol{\gamma}}\;f_i\,dx \\ &\qquad\qquad + \int_\Omega \Lambda_*(v_{\boldsymbol{\beta}}, w_{\boldsymbol{\gamma}})\nabla w_{\boldsymbol{\gamma}} \cdot \nabla f_i\,dx, \quad i = 1, \dots, d. \end{aligned}$$

The operator Π is continuous. In fact, if $\{(\boldsymbol{\beta}_q, \boldsymbol{\gamma}_q)\}_{q=1}^{\infty}$ is a sequence of $\mathbb{R}^d \times \mathbb{R}^d$ converging in this space to $(\boldsymbol{\beta}, \boldsymbol{\gamma})$, the functional sequence $\{(v_{\boldsymbol{\beta}_q}, w_{\boldsymbol{\gamma}_q})\}_{q=1}^{\infty}$ is converging a.e. in Ω and in $(\mathcal{H}, \mathcal{U})$ to $(v_{\boldsymbol{\beta}}, w_{\boldsymbol{\gamma}})$. Now, using the assumptions on the functional coefficients of our system $(\mathcal{P})$ and the dominated convergence theorem, we see that

$$\lim_{q\to\infty}\Pi(\boldsymbol{\beta}_q,\boldsymbol{\gamma}_q)=\Pi(\boldsymbol{\beta},\boldsymbol{\gamma}).$$

Let us denote $(\cdot\mid\cdot)$ the "natural" scalar product of $\mathbb{R}^{2d}$ and compute

$$(\Pi(\boldsymbol{\beta},\boldsymbol{\gamma})\mid(\boldsymbol{\beta},\boldsymbol{\gamma})) \;=\; \sum_{i=j}^{d} b_i\beta_i+\sum_{i=1}^{d} c_i\gamma_i=\Sigma_1+\Sigma_2,$$

where

$$\begin{aligned}\Sigma_1 \;=\;& \int_\Omega m\frac{v_\beta-S^0}{\alpha}\,v_\beta\,dx\\ &+\int_\Omega[K_0(x)a_\varepsilon(x,v_\beta,w_\gamma)\nabla v_\beta+\overrightarrow{F_*}(x,v_\beta,w_\gamma)]\cdot\nabla v_\beta\,dx\\ &\qquad+\int_\Omega b_S(v_\beta)\overrightarrow{V_\kappa}(P^{1\alpha},v_\beta,w_\gamma)\cdot\nabla v_\beta\,v_\beta\,dx,\end{aligned}$$

and

$$\begin{aligned}\Sigma_2 \;=\;& \int_\Omega \ell(x,S^0)\frac{w_\gamma-\theta^0}{\alpha}\,w_\gamma\,dx\\ &+\int_\Omega\overrightarrow{V_\kappa}(P^{1\alpha},v_\beta,w_\gamma)\cdot\nabla w_\gamma\,w_\gamma\,dx+\int_\Omega\Lambda_*(v_\beta,w_\gamma)\nabla w_\gamma\cdot\nabla w_\gamma\,dx.\end{aligned}$$

Using the assumptions (**i**)–(**iii**), we obtain the estimates:

$$\begin{aligned}\int_\Omega m\frac{v_\beta-S^0}{\alpha}v_\beta\,dx \;&=\; \frac{1}{\alpha}\int_\Omega mv_\beta^2\,dx-\frac{1}{\alpha}\int_\Omega mS^0v_\beta\,dx\\ &\geq\; \frac{1}{M\alpha}\|v_\beta\|_{2,\Omega}\big[\|v_\beta\|_{2,\Omega}-M^2\|S^0\|_{2,\Omega}\big];\\ \int_\Omega K_0\bar{a}(x,v_\beta,w_\beta)\nabla v_\beta\cdot\nabla v_\beta\,dx \;&\geq\; \frac{\varepsilon}{M}\|v_\beta\|^2_{\mathcal{H}};\\ \int_\Omega|\overrightarrow{F_*}(x,v_\beta,w_\gamma)\cdot\nabla v_\beta|\,dx \;&\leq\; \Big(\int_\Omega|\overrightarrow{F_*}|^2\,dx\Big)^{\frac{1}{2}}\|v_\beta\|_{\mathcal{H}}\\ &\leq\; C_1\|v_\beta\|_{\mathcal{H}},\quad C_1=\text{const.};\end{aligned}$$

and

$$\begin{aligned}\int_\Omega|b_S(v_\beta)\overrightarrow{V_\kappa}\cdot\nabla v_\beta\,v_\beta|\,dx \;&\leq\; \frac{C_2}{\kappa}\int_\Omega|\nabla v_\beta|\,|v_\beta|\,dx\\ &\leq\; \frac{C_3}{\kappa}\|v_\beta\|^2_{\mathcal{H}}\quad\text{(Poincaré's inequality).}\end{aligned}$$

Similarly, using assumptions (**i**)–(**iii**), we obtain the estimate

$$\begin{aligned}\int_\Omega \ell(x,S^0)\frac{w_\gamma-\theta^0}{\alpha}w_\gamma\,dx &= \frac{1}{\alpha}\int_\Omega \ell(x,S^0)w_\gamma^2\,dx-\frac{1}{\alpha}\int_\Omega \ell(x,S^0)\theta^0 w_\gamma\,dx\\ &\geq \frac{1}{M\alpha}\|w_\gamma\|_{2,\Omega}\Big[\|w_\gamma\|_{2,\Omega}-M^2\|\theta^0\|_{2,\Omega}\Big];\end{aligned}$$

also

$$\begin{aligned}\Big|\int_\Omega \overrightarrow{V_\kappa}(P_d^\alpha,v_\beta,w_\gamma)\cdot\nabla w_\gamma\, w_\gamma\,dx\Big| &\leq \frac{1}{\kappa}\int_\Omega |\nabla w_\gamma|\,|w_\gamma|\,dx\\ &\leq \frac{1}{\kappa}\|\nabla w_\gamma\|_{2,\Omega}\,\|w_\gamma\|_{2,\Omega}\\ &\leq \frac{C_4}{\kappa}\|w_\gamma\|_{\mathcal{U}}^2;\end{aligned}$$

and

$$\int_\Omega \Lambda_*(v_\beta,w_\gamma)\nabla w_\gamma\cdot\nabla w_\gamma\,dx \geq \frac{1}{M}\int_\Omega \|\nabla w_\gamma\|^2\,dx=\frac{1}{M}\|w_\gamma\|_{\mathcal{U}}^2.$$

The previous estimates lead to

$$\Sigma_1\geq \frac{1}{M\alpha}\|v_\beta\|_{2,\Omega}\Big[\|v_\beta\|_{2,\Omega}-M^2\|S^0\|_{2,\Omega}\Big]+\frac{\varepsilon}{M}\|v_\beta\|_{\mathcal{H}}^2-C_1\|v_\beta\|_{\mathcal{H}}-\frac{C_3}{\kappa}\|v_\beta\|_{\mathcal{H}}^2$$

and

$$\Sigma_2\geq \frac{1}{M\alpha}\|w_\gamma\|_{2,\Omega}\Big[\|w_\gamma\|_{2,\Omega}-M^2\|\theta^0\|_{2,\Omega}\Big]+\frac{1}{M}\|w_\gamma\|_{\mathcal{U}}^2-\frac{C_4}{\kappa}\|w_\gamma\|_{\mathcal{U}}^2.$$

Using the equivalence of norms in a finite-dimensional space, the previous estimate prove the existence of positive constants such that

$$(\Pi(\boldsymbol{\beta},\boldsymbol{\gamma})\mid(\boldsymbol{\beta},\boldsymbol{\gamma}))\geq \frac{C_5}{\alpha}\|(\boldsymbol{\beta},\boldsymbol{\gamma})\|_{\mathbb{R}^{2d}}^2-C_6\|(\boldsymbol{\beta},\boldsymbol{\gamma})\|_{\mathbb{R}^{2d}}^2-C_7\|(\boldsymbol{\beta},\boldsymbol{\gamma})\|_{\mathbb{R}^{2d}}.$$

In each step, we get an estimate like the previous. For $(\boldsymbol{\beta},\boldsymbol{\gamma})\in\mathbb{R}^{2d}$, for $\alpha>0$ sufficiently small, the right side is positive.

If α is small enough independent of d, the second term is non-negative, hence Π has a zero, that is $(S_d^\alpha,\theta_d^\alpha)(t)$ for $t\in](j-1)\alpha,j\alpha]$, exists. This proves that the triplet $(P_d^\alpha,S_d^\alpha,\theta_d^\alpha)(t)$ can be determined inductively for $t\in](j-1)\alpha,\alpha j]$.

3.3 Proof of Theorem 2.4.1.

From Corollaries 2.1 and 2.2, the pressure equation in the weak sense of Definition 2.1 can be easily seen to hold since $\bigcup_{n=1}^{\infty}\mathcal{L}^{\alpha}(\mathcal{H})$ is dense in $L^{\infty}(I;\mathcal{H})$ (let us recall that $\alpha = T/N = T/2^n$). Also, it follows from (2.4) that

$$\begin{aligned}\lim_{\alpha\downarrow 0}\int_I (m\partial_t^{-\alpha}S^{\alpha},\varphi)_{\Omega}\,dt \quad &+\quad \int_I (K_0\,a_{\varepsilon}\,\nabla S+\overrightarrow{F},\nabla\varphi)_{\Omega}\,dt \\ &+\quad \int_I (b_S\overrightarrow{V_{\kappa}}\cdot\nabla S,\varphi)_{\Omega}\,dt=0,\quad \forall\varphi\in\bigcup_{n=1}^{\infty}\mathcal{L}^{\alpha}(\mathcal{H})\end{aligned}\tag{3.4}$$

For any $\varphi \in L^2(I;\mathcal{H})$, $\varphi_{\alpha}\in\mathcal{L}^{\alpha}(\mathcal{H})$, where $\varphi_{\alpha}(x,t)=\frac{1}{\alpha}\int_{I_j}v(x,\tau)\,d\tau$, $t\in I_j$. Because $S^{\alpha}(\cdot,t)$ is constant over each interval $I_j =]t_{j-1},t_j]$ (equals $S(\cdot,t_j)$), we observe that

$$\int_I (m\partial_t^{-\alpha}S^{\alpha},\varphi)_{\Omega}\,dt=\int_I (m\partial_t^{-\alpha}S^{\alpha},\varphi_{\alpha})_{\Omega}\,dt.$$

Then, the identity (2.4), can be written as

$$\begin{aligned}\int_I (m\partial_t^{-\alpha}S^{\alpha},\varphi_{\alpha})_{\Omega}\,dt \quad &=\quad -\int_I (K_0\,a_{\varepsilon}\,\nabla S+\overrightarrow{F},\nabla\varphi_{\alpha})_{\Omega}\,dt \\ &\qquad -\int_I (b_S\overrightarrow{V_{\kappa}}\cdot\nabla S,\varphi_{\alpha})_{\Omega}\,dt.\end{aligned}$$

This implies that

$$\left|\int_I (m\partial_t^{-\alpha}S^{\alpha},\varphi)_{\Omega}\,dt\right|\le C\|\varphi\|_{L^2(I;\mathcal{H})},\quad \forall\,\varphi\in L^2(I;\mathcal{H}).\tag{3.5}$$

The sequence $\{m\partial_t^{-\alpha}S^{\alpha}\}$ is thus bounded in $L^2(I;\mathcal{H}')$. Consequently, for a subsequence, $\{m\partial_t^{-\alpha}S^{\alpha}\}$ converges weakly in $L^2(I;\mathcal{H}')$. For $\varphi\in\mathcal{D}(I;\mathcal{H})$ and $\alpha>0$ small enough, using formula (**??**), we see that

$$\begin{aligned}\int_I \Big(m(\cdot)\partial_t^{-\alpha}S^{\alpha}(\cdot,t),\varphi(\cdot,t)\Big)_{\Omega}\,dt \quad &=\quad -\int_0^{T-\alpha}(mS^{\alpha},\partial_t^{\alpha}\varphi)\,dt \\ &\xrightarrow[\alpha\downarrow 0]{}\quad -\int_I (mS,\partial_t\varphi)_{\Omega}\,dt=\int_I\langle mS_t,\varphi\rangle\,dt,\end{aligned}$$

as a distribution. Therefore, $m\partial_t S^{\alpha}\rightharpoonup m\partial_t S$ weakly in $L^2(I;\mathcal{H}')$. Combining these results, the saturation equation holds in the weak sense of Definition 2.4.1 since $\bigcup_{n=1}^{\infty}\mathcal{L}^{\alpha}(\mathcal{H})$ is dense in $L^2(I;\mathcal{H})$.

Finally, if $v\in L^2(I;\mathcal{H})\cap W^{1,1}(I;L^1(\Omega))$ with $\varphi(x,T)=0$, we find that

$$\int_I (m\partial_t^{-\alpha}S^{\alpha},\varphi)_{\Omega}\,dt \quad +\quad \int_0^{T-\alpha}(m[S^{\alpha}-S^0],\partial_t^{\alpha}\varphi)_{\Omega}\,dt=\frac{1}{\alpha}\int_{T-\alpha}^{T}(m[S^{\alpha}-S^0],\varphi)_{\Omega}\,dt,$$

which yields the last equation in Definition 2.4.1, and thus the proof of Theorem 2.4.1 is complete.

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