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Sur les opérateurs Lipschitz p-sommants et les opérateurs Lipschitz p-nucléaires

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On Lipschitz p-summing operators and Lipschitz p-nuclear operators

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Dedicate

I dedicate this work to:
My dear parents,
All my family,
All my friends,
Anyone interested in the development of knowledge.

List of Publications:

1. Amar Belacel and Khedidja Bey, Strongly Lipschitz up-nuclear operators, Moroccan J. of Pure and Appl. Anal. **5**(1) (2019), 22-30.
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ملخص

في هذِهِ الأَطرُوحة قمنَا بتعريف فِئَة جَديدَة مِنَ المثَالِيَات اللّيبشيزية " المؤُثرَات ليبشيز $up-$ النَوَوِيَة القَوِية " $(1 \leq p < \infty)$ و قَدَمنَا مبرهنة التفكيك المُميزة لهَا بَالَإِضَافة إِلَى خصَائِص أَسَاسية أخرَى و درسنَا العلَاقَات التي تربط بينهَا وَ المؤُثرَات الَأخرَى، مِن بَين النَتَائِج أيضَا نُمَيز المؤُثر القَرِين بَالنسبة لِهَذِهِ المؤُثرَات. قمنَا أيضا بدرَاسَة تَسَاوِي القيَاس وَ قَدَمنَا تَوصيفَا كَامِلَا بَالنسبة لِلمؤُثرَات اللّيبشيز $p-$ النَوَوِيَة القَوِية المعرَفة مِن طَرف شِن و زينغ.

كلمَات مفتَاحية: المؤُثرَات $p-$ جمعية، مؤُثرَات ليبشيز $p-$ جمعية، المؤُثرَات $p-$ النَوَوِيَة القَوِية، المؤُثرَات ليبشيز $p-$ النَوَوِيَة القَوِية، مبرهنة التركيب، مبرهنة الإِحتوَاء، مبرهنة التفكيك.

Résumé

Dans cette thèse, nous avons introduit une nouvelle classe d'idéal d'opérateurs de Lipschitz appelé "Opérateurs de Lipschitz fortement up-nucléaires" $(1 \leq p < \infty)$ et on a étudié leurs propriétés fondamentales: propriété d'idéal et le théorème de factorisation pour cette classe. Entre autres résultats, nous avons caractérisé leurs conjugués également nous avons étudié les isométries de la classe des opérateurs fortement Lipschitz p-nucléaires définis par Chen et Zheng, nous avons donné une caractérisation complète de ces isométries.

Mots clés: Opérateur p-sommant, opérateur Lipschtiz p-sommant, opérateurs p-nucléaires, opérateurs fortement Lipschitz p-nucléaires, Propriété d'Idéal et Théorèmes de l' inclusion, Composition et Factorisation.

Abstract

In this thesis, we introduced a new class of Lipschitz operator ideal called ”Strongly Lipschitz up-nuclear operators” ($1 \leq p < \infty$) and studied their important properties ideal property and factorization theorem for this class. Among other results we characterized their conjugates also we studied the onto isometries of the class of Strongly Lipschitz p-nuclear operators defainded by Chen and Zheng we gave a full characterization of such isometries.

Keywords: P-summing operators, Lipschitz p-summing operators, p-nuclear operators, Strongly Lipschitz p-nuclear operators, Properties of Ideal, Theorems of Inclusion, Composition and Factorization.

CONTENTS

Table of Notations

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$	The field of real or complex numbers
p^*	The conjugate of the number p; $\frac{1}{p} + \frac{1}{p^*} = 1$
E, F, G	Banach spaces
E^*	The topological dual of E
B_{E^*}	The closed unit ball of E^*
$L(E, F)$	The set of all linear operators from E into F
$\mathcal{L}(E, F)$	The set of all continuous linear operators from E into F
$\mathcal{K}$	The set of all compact linear operators
$\mathcal{L}_f$	The set of all finite rank linear operators
w	The weak topology
w^*	The weak* topology
$L_p(\mu)$	Lebesgue space
X, Y, Z	Metric spaces
$Lip_0(X, Y)$	The set of all Lipschitz operators between X and Y that vanish at 0.

$X^\sharp = Lip_0(X, \mathbb{K})$	The Lipschitz dual of the pointed metric space X
$\mathcal{M}(X)$	The linear space of all molecules on the metric space X
χ_A	The characteristic function of the set A
$m_{x\acute{x}}$	The molecule defined by $m_{x\acute{x}} = \chi_x - \chi_{\acute{x}}$, for $x, \acute{x} \in X$
Æ(X)	The Arens-Eells space of X
T^*	The adjoint of linear operator T
$T^\sharp$	The Lipschitz adjoint of Lipschitz operator T
T_L	The linearization of Lipschitz operator T
$T^t = T^\sharp \backslash F^*$	The Lipschitz transpose of Lipschitz map from $T : X \longrightarrow F$
$J_p(1 \leq p < \infty)$	The canonical inclusion map defined from $C(K)$ to $L_p(\mu)$
i_E	The isometric embedding, $i_E : E \longrightarrow C(E^*)$ given by $i_E(x) := \langle x, . \rangle$
Π_p	The set of all linear p-summing operators
Π_p^L	The set of all Lipschitz p-summing operators $1 \leq p < \infty$
$\mathcal{N}_p$	The set of all p-nuclear linear operators $1 \leq p < \infty$
$\mathcal{N}_p^L$	The set of all Lipschitz p-nuclear operators $1 \leq p < \infty$
$\mathcal{D}_p$	The set of all strongly p-summing operators $1 < p \leq \infty$
$\mathcal{D}_p^L$	The set of all strongly Lipschitz p-summing operators $1 < p \leq \infty$
$Lip_{0\mathcal{K}}$	The set of all Lipschitz compact operators
$Lip_{0\mathcal{W}}$	The set of all Lipschitz weakly compact operators
$Lip_{0\mathcal{F}}$	The set of all Lipschitz finite rank operators

INTRODUCTION

The class of p-nuclear operators due to the work of Grothendieck [16] at first for $p = 1$ in the middle of the last century (1955). Later on A. Persson and A. Pietsch [27] generalized this concepts for $1 \leq p < \infty$. The fundamental properties were established concerning this class: the domination-factorization theorem, ideal property, inclusion-composition theorem. The theory of p-nuclear operators the utmost interest in the study of many different problems in the theory of operator ideals [28]. This topic was comprehensive studied and it was generalized by many ways like: Cohen Strongly p-nuclear operators [10], also the rihgt p-nuclear operators [24] as well the onto isometries of p-nuclear operators was characterized by Khalil and Yousef [22]... .

Famer and Johnson in [13] introduced a new class of Lipschitz operator ideal called Lipschitz p-summing operators and proved their basic results, where the domain of such operators is a metric space that need not be a normed space. Since then several works have developed related to this class in the sens of creating a theory analogous to the ideal Banach operator. Various operators ideals have been studied by many authors for instance the absolute p-summing operators and their variants [2, 18, 33]... . Other variants of p-intgrale and p-nuclear operators [4, 5, 8]... .

This thesis is made up of four chapters, which are linked together, to study the onto isometries of Lipschitz p-nuclear operators and to define the strongly Lipschitz up-nuclear operators.

Chapter 1 and **chapter 2** are devoted to reminder of some basic concepts, in particular all the definitions and results that will be used in **chapter 3** and **chapter 4** for that we started with definitions and properties elementary to define the important theorems of Pietsch on domination and factorization, as well as three examples illustrating these operators.

In **chapter 3**, we introduce the notion of strongly Lipschitz up-nuclear operators. Among other results, we prove an analog of the factorization theorem for these classes and characterize their conjugates.

In **chapter 4**, we study the onto isometries of the space of strongly Lipschitz p-nuclear operators, introduced by D. Chen and B. Zheng. We give some new results about such isometrics and we focus, in particular, on the case $F = \ell_{p^*}$.

CHAPTER 1

PRELIMINARIES

We present in this chapter some basic results known in the literature. We recall some standard notations, definitions and properties concerning the class of ideal linear operators [28].

1.1 Classical Banach spaces

1. We will denote by $\ell_p(E)$ the Banach space of absolutely p-summable sequences in E, equipped with the norm
$$\|(x_j)_j\|_p = \Big(\sum_j \|x_j\|^p\Big)^{\frac{1}{p}}. \tag{1.1}$$
2. The space formed by the bounded sequences $\ell_\infty(E)$ provided with the norm
$$\|(x_j)_j\|_\infty = \sup_{j\in\mathbb{N}} \|x_j\|. \tag{1.2}$$
3. $c_0(E)$ the closed subspace of $\ell_\infty(E)$ of the sequences which veer towards zero.
4. $\ell_p^w(E)$ the space of the sequences $(x_j)_j \subset E$ weakly p-summable, i.e. $(x_j)_j$ where $\sum\limits_j |\langle x_j, x^*\rangle|^p < +\infty$ for all $x^* \in E^*$, equipped with the norm:
$$\|(x_j)_j\|_{\ell_p^w} = \sup_{\|x^*\|\leq 1} \Big(\sum_j |\langle x_j, x^*\rangle|^p\Big)^{\frac{1}{p}}. \tag{1.3}$$

In the case $p = \infty$, $\ell_\infty^w(E) = \ell_p(E)$ and

$$\|(x_j)_j\|_{\ell_\infty^w} = \|(x_j)_j\|_\infty. \tag{1.4}$$

Remark 1.1.1. *We can define in the same way the space $\ell_p^n(E)$.*

5. Let (Ω, Σ, μ) a measured space, f a Σ-measurable function. We define according to the values of the real p the following norms

$$\|f\|_p = \begin{cases} \left(\int_\Omega |f(t)|^p d\mu(t)\right)^{\frac{1}{p}} & 1 \leq p < \infty \\ \inf\{C, |f(x)| \leq C, \quad a.e \quad sur \quad \Omega\} & p = \infty. \end{cases} \tag{1.5}$$

for $1 \leq p < \infty$, the Banach space $L_p(\mu) = L_p(\Omega, \Sigma, \mu)$ represents the space of all equivalence classes, modulo the almost equality everywhere, Σ-measurable functions such that $\|f\|_p < \infty$, Σ is the Lebesgue tribe and μ the Lebesgue measure.

1.2 Linear operators

Let E and F be two linear spaces on $\mathbb{K}$.

Definition 1.2.1. *We call an operator from E to F any map T defined from E to F by:*

$$\begin{aligned} T : E &\to F \\ x &\to y = T(x) \end{aligned}$$

The operator T is said to be linear, if for all $x, y \in E$ and $\lambda, \mu \in \mathbb{K}$:

$$T(\lambda x + \mu y) = \lambda T(x) + \mu T(y).$$

The space of all linear operators is denoted by $L(E, F)$.

Remark 1.2.1. *T calls a linear form if $F = \mathbb{K}$.*

Definition 1.2.2. *Let E, F are a normed spaces. T is continuous at point x_0, if:*

$$\forall \varepsilon > 0,\ \exists \delta > 0 : \forall x \in E,\ \|x - x_0\| \leq \delta \Longrightarrow \|T(x) - T(x_0)\| \leq \varepsilon.$$

Since the continuity of T can be characterized by the sequences, T is continuous in x_0 if for any sequence $(x_n)_n \subset E$

$$x_n \xrightarrow{\|\cdot\|} x_0 \Longrightarrow T(x_n) \xrightarrow{\|\cdot\|} T(x_0).$$

Definition 1.2.3. *A linear operator T from E in F is said to be a bounded if it is defined everywhere in E and transforms every bounded set of E into a bounded set of F.*

The linearity of T leads to the equivalence of this definition with the previous one.

Theorem 1.2.1. *[20, p. 215-216]*
Let E, F be a normed spaces and T be a linear operator from E to F. So

$$T \text{ is continuous} \Longleftrightarrow T \text{ is bounded.}$$

So we have,

$$T \text{ is bounded} \Longleftrightarrow \exists M > 0, \forall x \in E : \|T(x)\| \leq M\|x\|.$$

The lower bound of the numbers M satisfying the preceding inequality is called the norm of the operator T and is noted $\|T\|$,

$$\|T\| = \inf\{M > 0 : \|T(x)\| \leq M\|x\|\}.$$

Given a Banach spaces E and F, B_E is the closed unit ball of E and $\mathcal{L}(E, F)$ the Banach space of all continuous linear operators between E and F with the operator norm

$$\|T\| = \sup_{x \in B_E} \|Tx\| \tag{1.6}$$

The space $\mathcal{L}(E, \mathbb{K})$ of all continuous linear forms defined on E is called the topological dual of E and we denote it by E^*. For reasons that we will see, we often use the notation

$$T(x) = \langle T, x \rangle, \tag{1.7}$$

(or similar notations). The dual of E^* is called bidual of E and denoted by E^{**}, i.e. the space of continuous linear forms on E^*.

Among the main tools that we will need the concept of dual operator see [20, p 223-224].

Definition 1.2.4. *Let $T \in \mathcal{L}(E, F)$. There is a unique linear operator $T^* \in \mathcal{L}(F^*, E^*)$ is called the dual operator of T such that:*

$$\forall x \in E, \forall y^* \in F^* : \qquad \langle T(x), y^* \rangle = \langle x, T^* y^* \rangle.$$

Theorem 1.2.2. *Let* $T \in \mathcal{L}(E, F)$*, then*

$$\|T\|_{\mathcal{L}(E,F)} = \|T^*\|_{\mathcal{L}(F^*,E^*)}, \tag{1.8}$$

and let $S \in \mathcal{L}(G, E)$*, we have* $(TS)^* = S^*T^*$

Definition 1.2.5. *Let* $T \in \mathcal{L}(E, F)$ *and* $S \in \mathcal{L}(F, E)$ *that satisfies* $T \circ S = Id_F$ *and* $S \circ T = Id_E$*,* S *is called the inverse of* T *and is denoted by* T^{-1}.

1.3 Canonical injection

Consider the following canonical injection:

$$J_E : E \to E^{**},$$

which to all $x \in E$ associates $J_E(x)$ such that:

$$\langle J_E(x), x^* \rangle = \langle x, x^* \rangle, \qquad x^* \in E^*$$

this application is an isometry i.e.,

$$\|J_E(x)\|_{E^{**}} = \|x\|_E, \qquad \forall x \in E.$$

We say that the Banach space E is reflexive if

$$J_E(E) = E^{**} \qquad (J_E \quad bijection).$$

1.4 Compact operators

Definition 1.4.1. *Let* $T \in \mathcal{L}(E, F)$*. An operator* T *has finite rank, if* $Im(T)$ *is a space of finite dimension, where*

$$T = \sum_{j=1}^{n} x_j^* \otimes y_j.$$

The space of linear operators of finite rank is denoted by $\mathcal{L}_f(E, F)$*.*

For $x^* \in E^*$ *and* $y \in F$*, we define the one rank operator* $x^* \otimes y : E \longrightarrow F$ *by* $(x^* \otimes y)(x) = x^*(x)y$*. The rank one operator is called an atom.*

Definition 1.4.2. *Let $T \in \mathcal{L}(E,F)$. T is a compact operator if it sends any bounded set E_0 in E to a relatively compact set $T(E_0)$ in F. In other words, the closure $T(E_0)$ is compact.*

A Banach space F is an injective space if whenever E_0 is a subspace of Banach space E, any $T \in \mathcal{L}(E_0,F)$ has an extension $\tilde{T} \in \mathcal{L}(E,F)$

$$\begin{tikzcd}
E_0 \arrow[r, hook] \arrow[d, "T"'] & E \arrow[dl, "\tilde{T}"] \\
F &
\end{tikzcd}$$

such that $\|T\| = \|\tilde{T}\|$.

1.5 The weak topology and the weak* topology

The weak topology $\sigma(E,E^*)$: The weak topology on E is the weakest topology on E making all applications $\varphi \in E^*$ continuous. We note it $\sigma(E,E^*)$.
The weak-$*$topology $\sigma(E^*,E)$: For each $x \in E$, we consider the application $\varphi_x : E^* \longrightarrow \mathbb{K}$ defined by

$$\varphi_x(x^*) = \langle x^*, x\rangle \quad \textit{with} \quad x^* \in E^*.$$

The weak-$*$ topology on E^* is the weakest topology on E^* that makes all applications $(\varphi_x)_{x\in E}$ continuous. We note it $\sigma(E^*,E)$.

We define on E^* three topologies: the strong topology, the weak topology $\sigma(E^*,E^{**})$ and the weak-$*$ topology $\sigma(E^*,E)$.

Note that each φ_x is continuous as a linear form over E^* (with the strong topology) therefore $\varphi_x \in E^{**}$. Thus φ_x is continuous for the weak topology $\sigma(E^*;E^{**})$ and by definition of the weak-$*$ topology, we obtain that the weak-$*$ topology is weaker than the weak topology which itself is weaker than the strong topology.

Theorem 1.5.1 (The Banach-Alaoglu Theorem)**.** *[7, p. 43] The set $B_{E^*} = \{x^* \in E^* : \|x^*\| \leqslant 1\}$ is compact for the weak-$*$ topology.*

1.6 Linear operator ideals

Definition 1.6.1. *[28] A Linear operator ideal $\mathcal{I}$ is a subclass of $\mathcal{L}$ such that for every Banach spaces E and F the components*

$$\mathcal{I}(E,F) := \mathcal{L}(E,F) \cap \mathcal{I}$$

satisfy:

1- $\mathcal{I}(E,F)$ is a linear subspace of $\mathcal{L}(E,F)$ which contains the finite rank operators.

2- The ideal property: if $u \in \mathcal{L}(G,E), T \in \mathcal{I}(E,F)$ and $w \in \mathcal{L}(F,H)$, then the composition $wTu \in \mathcal{I}(G,H)$.

If $\|.\|_{\mathcal{I}} : \mathcal{I} \longrightarrow \mathbb{R}^+$ that satisfies:

$1'$- For every Banach spaces E and F the pair $(\mathcal{I}(E,F), \|\cdot\|_{\mathcal{I}})$ is a normed (Banach) space.

$2'$- $\|Id_{\mathbb{K}}\|_{\mathcal{I}} = 1$.

$3'$- If $u \in \mathcal{L}(G,E), T \in \mathcal{I}(E,F)$ and $w \in \mathcal{L}(F,H)$, then

$$\|wTu\|_{\mathcal{I}} \leq \|w\| \|T\|_{\mathcal{I}} \|u\|. \tag{1.9}$$

1.7 Examples of Linear operator ideals

1.7.1 p-summing operators

Definition 1.7.1. *Let $1 \leqslant p < \infty$ $T \in L(E,F)$. T is p-summing[28], if there is a constant $C > 0$ such that for $n \in \mathbb{N}, x_1, ..., x_n$ in E we have*

$$\Big(\sum_{j=1}^{n} \|T(x_j)\|^p\Big)^{\frac{1}{p}} \leq C \sup_{x^* \in B_{E^*}} \Big(\sum_{j=1}^{n} |x^*(x_j)|^p\Big)^{\frac{1}{p}}. \tag{1.10}$$

The collection of all p-summing operators is denoted by $\Pi_p(E,F)$ with the norm $\pi_p(.)$, $\pi_p(T)$ is the infimum of all C for which the inequality above always holds.

Basic examples on p-summing operators

Let K be a compact and let δ_k be a functional defined, for all $k \in K$, by

$$\begin{array}{rcl} \delta_k : C(K) & \to & \mathbb{K} \\ f & \to & \langle \delta_k, f\rangle = f(k) \end{array}$$

We remark that $\delta_k \in C(K)^*$. Let μ a probability measure on K and $1 \leq p < \infty$.

Let us illustrate basic examples for p-summing operators :

1. [1, p. 12] Let $\varphi \in L_p(K,\mu)$, the following multiplication operator

$$\begin{array}{rcl} M_\varphi : C(K) & \to & L_p(K,\mu) \\ f & \to & M_\varphi(f) = f\varphi, \end{array}$$

this operator is p-summing, with

$$\pi_p(M_\varphi) = \|\varphi\|_p. \tag{1.11}$$

Indeed, either $(f_k)_{k=1}^n \subset C(K)$, with a

$$\begin{array}{rcl} \|(M_\varphi(f_k))_{k=1}^n\|_p & = & \Big(\sum_{k=1}^n \|M_\varphi(f_k)\|_p^p\Big)^{\frac{1}{p}} \\ & = & \Big(\sum_{k=1}^n \|f_k\varphi\|_p^p\Big)^{\frac{1}{p}} \\ & = & \Big(\sum_{k=1}^n \int_K |\varphi(t)|^p |f_k(t)|^p d\mu(t)\Big)^{\frac{1}{p}} \\ & = & \Big(\int_K \sum_{k=1}^n |f_k(t)|^p |\varphi(t)|^p d\mu(t)\Big)^{\frac{1}{p}} \\ & \leq & \Big(\int_K \sum_{k=1}^n |\langle \delta_t, f_k\rangle|^p |\varphi(t)|^p d\mu(t)\Big)^{\frac{1}{p}} \\ & \leq & \sup_{t\in K} \Big(\sum_{k=1}^n |\langle \delta_t, f_k\rangle|^p\Big)^{\frac{1}{p}} \Big(\int_K |\varphi(t)|^p d\mu(t)\Big)^{\frac{1}{p}} \\ & \leq & \|\varphi\|_p \sup_{\delta_t \in C(K)^*} \Big(\sum_{k=1}^n |\langle \delta_t, f_k\rangle|^p\Big)^{\frac{1}{p}} \\ & \leq & \|\varphi\|_p \|(f)_{k=1}^n\|_{p,w}. \end{array}$$

So the operator in question is p-summing and we have

$$\pi_p(M_\varphi) \leq \|\varphi\|_p.$$

But, according to the inequality (1.6), we have

$$\pi_p(M_\varphi) \geq \|M_\varphi\|_p \geq \|\varphi\|_p.$$

Therefore

$$\pi_p(M_\varphi) = \|\varphi\|_p.$$

2. [15, p. 270] As a special case, the operator

$$\begin{array}{rcl} J_p : C(K) & \rightarrow & L_p(K,\mu) \\ f & \rightarrow & J_p(f) = f\varphi \end{array}$$

is p-summing and moreover

$$\pi_p(J_p) = 1. \tag{1.12}$$

Indeed, just take $\varphi = 1$ in the previous example.

3. [1, p. 13] The following inclusion operator

$$\begin{array}{rcl} i_p : L_\infty(K,\mu) & \rightarrow & L_p(K,\mu) \\ f & \rightarrow & i_p(f) = f \end{array}$$

is p-summing and

$$\pi_p(i_p) = 1. \tag{1.13}$$

Theorem 1.7.1 (Pietsch domination theorem)**.** *[12, Theorem 1]*
Let $T \in \mathcal{L}(E,F)$. Then T is p-summing if, and only if, there exist a constant C and a regular Borel probability measure μ on B_{E^*} (with the weak $*$-topology) so that

$$\|Tx\|^p \leq C^p \cdot \int_{B_{E^*}} |\langle x, x^*\rangle|^p d\mu(x^*). \tag{1.14}$$

Proof. [17, p. 98-99] Let $T \in \Pi_p(E,F)$ and $\pi_p(T) = 1$. We consider the following two subsets of $C(B_{E^*})$ (The space of continuous weak-$*$ functions on B_{E^*}):

$$S_1 = \{f \in C(B_{E^*}), \sup_{x^* \in B_{E^*}} f(x^*) < 1\},$$

and

$$S_2 = conv\{f \in C(B_{E^*}), f(x^*) = |x^*(x)|^p, \|Tx\| = 1\}.$$

Let us show that S_1 is convex. Let

$$f_1, f_2 \in S_1 \quad and \quad 0 \leq \alpha \leq 1.$$

we have

$$\begin{aligned}\sup_{x^*\in B_{E^*}} \{\alpha f_1 + (1-\alpha) f_2\}(x^*) &\leq \alpha \sup_{x^*\in B_{E^*}} f(x^*) + (1-\alpha) \sup_{x^*\in B_{E^*}} f(x^*)\\ &\leq \alpha + 1 - \alpha\\ &= 1.\end{aligned}$$

So, S_1 is convex, moreover it is open. Either, on the other hand $f \in S_2$, so there is a sequence $(x_j)_{j=1}^n \in E$ and positive scalars $\lambda_1, ..., \lambda_n$ with

$$\sum_{j=1}^n \lambda_j = 1 \quad \text{and} \quad \|Tx_j\| = 1, \quad \text{for all } j = 1, ..., n$$

such as

$$f(x^*) = \sum_{j=1}^n \lambda_j |x^*(x_j)|^p \qquad \text{(by virtue of convexity)}$$

Therefore

$$\begin{aligned}\sup_{x^*\in B_{E^*}} f(x^*) &= \sup_{x^*\in B_{E^*}} \sum_{j=1}^n \lambda_j |x^*(x_j)|^p\\ &= \sup_{x^*\in B_{E^*}} \sum_{j=1}^n |x^*(\lambda_j^{\frac{1}{p}} x_j)|^p\\ &\geq 1 \cdot \sum_{j=1}^n \lambda_j \|T\lambda_j^{\frac{1}{p}} x_j\|^p\\ From\ (1.10) &= \sum_{j=1}^n \lambda_j \|Tx_j\|^p\\ &= 1.\end{aligned}$$

Therefore the function f does not belong to S_1, so that the two subsets S_1 and S_2 are disjoint. By the application of the Hahn-Banach Theorem [7, Colloraly 1.2], it comes that there exists $\lambda > 0$ and a measure of Radon μ on B_{E^*} such that:

$$\textstyle\int_{B_{E^*}} f(x^*) d\mu(x^*) \leq \lambda \quad , \forall f \in S_1 \quad and \quad \int_{B_{E^*}} f(x^*) d\mu(x^*) \geq \lambda \quad , \forall f \in S_2.$$

As on the one hand, S_1 contains all negative functions, the measure μ must be positive, which allows us to ensure that it is a measure of probability.

On the other hand, S_1 contains the open unit ball of $C(B_{E^*})$, so

$$\int_{B_{E^*}} f(x^*)d\mu(x^*) \leq \sup_{x^*\in B_{E^*}} f(x^*), \qquad \forall f \in S_1.$$

Consequently $\lambda \geq 1$.

From the above, we can say that if $x \in E$ and $\|Tx\| = 1$

$$\int_{B_{E^*}} |x(x^*)|^p d\mu(x^*) \geq 1 = \|Tx\|^p.$$

which leads to (1.14).

Conversely let $(x_j)_{j=1}^n \subset E$ so

$$\|Tx_j\| \leq C\Big(\int_{B_{E^*}} |x^*(x_j)|^p d\mu(x^*)\Big)^{\frac{1}{p}}, \qquad \forall j = 1, ..., n.$$

Which leads to

$$\|Tx_j\|^p \leq C^p\Big(\int_{B_{E^*}} |x^*(x_j)|^p d\mu(x^*)\Big), \qquad \forall j = 1, ..., n.$$

By summing the members of the sequence, we get

$$\sum_{j=1}^n \|Tx_j\|^p \leq C^p \sum_{j=1}^n \Big(\int_{B_{E^*}} |x^*(x_j)|^p d\mu(x^*)\Big),$$

and since the sum is finite

$$\begin{aligned}
\sum_{j=1}^n \|Tx_j\|^p &\leq C^p \sum_{k=1}^n \Big(\int_{B_{E^*}} |x^*(x_j)|^p d\mu(x^*)\Big) \\
&\leq C^p \int_{B_{E^*}} \Big(\sum_{j=1}^n |x^*(x_j)|^p d\mu(x^*)\Big) \\
&\leq C^p \sup_{x^*\in B_{E^*}} \sum_{j=1}^n |x^*(x_j)|^p \int_{B_{E^*}} d\mu(x^*) \\
&\leq C^p \sup_{x^*\in B_{E^*}} \sum_{j=1}^n |x^*(x_j)|^p \mu(B_{E^*}) \\
&\leq C^p \sup_{x^*\in B_{E^*}} \sum_{j=1}^n |x^*(x_j)|^p.
\end{aligned}$$

Therefore

$$\begin{aligned}\Big(\sum_{j=1}^{n}\|x_j\|^p\Big)^{\frac{1}{p}} &\leq C\Big(\sup_{x^*\in B_{E^*}}\sum_{j=1}^{n}|x^*(x_j)|^p\Big)^{\frac{1}{p}}\\ &= C\sup_{x^*\in B_{E^*}}\Big(\sum_{j=1}^{n}|x^*(x_j)|^p\Big)^{\frac{1}{p}},\end{aligned}$$

which implies that T is p-summing. □

Let the isometric injection

$$i : E \longrightarrow C(B_{E^*})$$

and the identical application

$$j_p : C(B_{E^*}) \longrightarrow L_p(B_{E^*},\mu)$$

Theorem 1.7.2 (Factorization Theorem). *[12] The following two assertions are equivalent:*
1- $T\in\Pi_p(E,F)$
2- There is a probability measure μ on B_{E^*} and a bounded application

$$w : \overline{j_p\circ i(E)} \longrightarrow F$$

such that

$$u = w\circ j_p\circ i,$$

in this case w is chosen such that $\pi(T)=\|w\|$. The diagram of this factorization is illustrated by the next figure

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[d, "i"'] & F \\
i(x) \arrow[r, "j_p"] \arrow[d, hook] & \overline{j_p\circ i(E)} \arrow[u, "w"'] \arrow[d, hook] \\
C(B_{E^*}) \arrow[r, "j_p"] & L_p(B_{E^*},\mu)
\end{tikzcd}$$

Figure 1.1: Factorization diagram of a p-summing operator

Proof. Let $T \in \Pi_p(E, F)$, we must demonstrate the existence of the application w defined above. Let $x, y \in E$ such that

$$(j_p i)x = (j_p i)y = f \in (j_p i)(E)$$

Therefore

$$\begin{array}{rcl} \|Tx - Ty\| & = & \|T(x-y)\| \\ Theoreme\ 1.7.1 & \leq & \pi_p(T)(\int_{B_{E^*}} |x^*(x-y)|^p d\mu(x^*))^{\frac{1}{p}} \\ & \leq & \pi_p(T)(\int_{B_{E^*}} |(j_p i)(x-y)|^p d\mu)^{\frac{1}{p}} \\ & \leq & \pi_p(T)\|(j_p i)(x) - (j_p i)(y)\|_p \\ & = & 0. \end{array}$$

Therfore

$$Tx = Ty.$$

So we can define the application w de $\overline{(j_p i)(E)}$ in F by:

$$wf = Tx$$

we have

$$T = w j_p i \quad and \quad \|wf\|_F \leq \pi_p(T)\|f\|_p. \tag{1.15}$$

It comes from the last inequality that the map f is bounded.

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "i"'] & F \\ i(x) \arrow[r, "j_p"] \arrow[d, hook] & G \arrow[u, "w"'] \arrow[d, hook] \\ C(B_{E^*}) \arrow[r, "j_p"] & L_p(\mu) \end{tikzcd}$$

Figure 1.2: Factorization diagram of a p-summing operator

□

Conversely, suppose that there exists a measure μ of probability on B_{E^*} and a map w such that

$$u = wj_p i,$$

as we have shown that

$$j_p \in \Pi_p(C(B_{E^*}), L_p(B_{E^*}, \mu)) \quad \textit{and} \quad \pi_p(j_p) = 1.$$

It comes from the ideal property of p-summing operators that $\Pi_p(E, F)$ and

$$\pi_p(T) \leq \|w\|. \tag{1.16}$$

According to the inequality (1.15) and (1.16), it follows that

$$\|w\| = \pi_p(T). \tag{1.17}$$

Theorem 1.7.3. *[12, Page 39]*
If $1 \leq p \leq q < \infty$, then $\Pi_p(E, F) \subset \Pi_q(E, F)$. Moreover

$$\pi_q(T) \leq \pi_p(T), \tag{1.18}$$

T is in $\Pi_p(E, F)$.

1.7.2 p-nuclear operators

Definition 1.7.2. *Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E, F)$. T is p-nuclear if has the form*

$$T = \sum_j x_j^* \otimes y_j$$

where $(x_j^*)_j \subset E^*$ and $(y_j)_j \subset F$ satisfy

$$\|(x_j^*)_j\|_p < \infty \quad \textit{and} \quad \sup_{y^* \in B_{F^*}} \left(\sum_j |\langle y_j, y^* \rangle|^{p^*} \right)^{1/p^*} < \infty.$$

Here

$$\mathcal{N}_p((x_j^*)_j, (y_j)_j) = \|(x_j^*)_j\|_p \sup_{y^* \in B_{F^*}} \left(\sum_j |\langle y_j, y^* \rangle|^{p^*} \right)^{1/p^*}. \tag{1.19}$$

Moreover, $\nu_p(T) = \inf \mathcal{N}_p((x_j^*)_j, (y_j)_j)$, the infimum being taken over all such representations as above. The collection of all linear p-nuclear operators from E to F is denoted by $\mathcal{N}_p(E, F)$.

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[d, "b"'] & F \\
\ell_\infty \arrow[r, "M_\lambda"] & \ell_p \arrow[u, "a"']
\end{tikzcd}$$

Figure 1.3: Factorization diagram of p-nuclear operator

Theorem 1.7.4. *Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E, F)$. T is p-nuclear if, and only if, there are two operators $a \in \mathcal{L}(\ell_p, F), b \in \mathcal{L}(E, \ell_\infty)$ and a sequence $\lambda \in \ell_p$ such that the following diagram commutes:*

where $M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)$ is the diagonal operator defined as follows: $M_\lambda(\xi_j) = (\xi_j \lambda_j)_j, (\xi_j)_j \in \ell_\infty$. Then $\|M_\lambda\| = \|\lambda\|_p = \pi_p(M_\lambda)$.

We set

$$\nu_p(T) = \inf \|a\| \|M_\lambda\| \|b\|, \tag{1.20}$$

the infimum being extended over all factorizations as above.

Interchanging the roles of the sequences $(x_j)_j$ and $(y_j)_j$ one obtains in the same way another Banach space $\mathcal{N}^p(E, F)$ of operators [27], the norm being given by $\nu^p(T) = \inf \mathcal{N}^p((x_j^*)_j, (y_j)_j)$, where

$$\mathcal{N}^p((x_j^*)_j, (y_j)_j) = \sup_{\|x\| \leq 1} \left(\sum_j |\langle x, x_j^* \rangle|^{p^*} \right)^{1/p^*} \|(y_j)_j\|_p.$$

Similarly, every $T \in \mathcal{N}^p(E, F)$ can be factored with a, M_λ, b as above,

$$\begin{tikzcd}
E \arrow[r, "T"] \arrow[d, "b"'] & F \\
\ell_{p^*} \arrow[r, "M_\lambda"] & \ell_1 \arrow[u, "a"']
\end{tikzcd} \tag{1.21}$$

Figure 1.4: Factorization diagram of p-nuclear operator

with

$$\nu^p(T) = \inf \|a\| \|M_\lambda\| \|b\|. \tag{1.22}$$

Remark 1.7.1. *For $p = 1$, we have $\mathcal{N}^1(E, F) = \mathcal{N}_1(E, F)$.*

Theorem 1.7.5. *[12] If $1 \leq p \leq q < \infty$, then $\mathcal{N}_p(E,F) \subset \mathcal{N}_q(E,F)$. Moreover*

$$\nu_q(T) \leq \nu_p(T), \tag{1.23}$$

T is in $\mathcal{N}_p(E,F)$.

Proposition 1.7.1. *[24] If $T \in \mathcal{N}_p(E,F)$, then its transpose $T^* \in \mathcal{N}^p(F^*,E^*)$ and it satisfies*

$$\nu^p(T^*) \leq \nu_p(T). \tag{1.24}$$

Furthermore, assume F is reflexive. Then, if $T^* \in \mathcal{N}^p(F^*,E^*)$ we have $T \in \mathcal{N}_p(E,F)$, with

$$\nu^p(T^*) = \nu_p(T). \tag{1.25}$$

Isometries of p-Nuclear Operators

Isometries of p-nuclear linear operators was studied by Khalil and Yousef in [22],

Theorem 1.7.6. *Let $(A_j^*)_j$ be a sequence of an isometric onto operators on E^*, and B be an isometric onto operator on F. Let $T \in \mathcal{N}_p(E,F)$, such that*

$$J(T) = J(\sum_j f_j \otimes y_j) = \sum_j \alpha_j A_j^* x_{\varphi(j)}^* \otimes By_j,$$

where $(\alpha_j)_j$ in $\mathbb{R}, |\alpha_j| = 1$ and φ is a permutation on $\mathbb{N}$. Then J is an isometric operator of $\mathcal{N}_p(E,F)$.

Theorem 1.7.7. *Let A be an isometric onto operator of E into E and B be an isometric onto operator of F into F. Assume $T \in \mathcal{N}_p(E,F)$, such that*

$$J(T) = BTA.$$

Then $J(T)$ is an isometric onto operator of $\mathcal{N}_p(E,F)$.

Any operator $T : E \longrightarrow \ell_{p^*}$ can be written in the following form:

$$Tx = \sum_j a_j(x)\delta_j$$

For some $z_j^* \in E^*$ put $a_j(x) = \langle z_j^*, x\rangle$, so $T = \sum_j z_j^* \otimes \delta_j$, For $x_j^* = \frac{z_j^*}{\|z_j^*\|}$ and $\lambda_j = \|z_j^*\|$, we can write:

$$T = \sum_j \lambda_j x_j^* \otimes \delta_j$$

Therefore, if $(\lambda_j)_j \in \ell_{p^*}$ then $T \in \mathcal{N}_p(E,\ell_{p^*})$.

Theorem 1.7.8. *[22, Theorem 6] Let $T \in \mathcal{N}_p(E, \ell_{p^*})$.*

$$\nu_p(T) = \Big(\sum_j |\lambda_j|^p\Big)^{\frac{1}{p}}. \tag{1.26}$$

Theorem 1.7.9. *[22, Theorem 8] Let $2 \leq p < \infty$ and J be an isometric onto operator of $\mathcal{N}_p(E, \ell_{p^*})$. Then J preserves the rank.*

Corollary 1.7.1. *[22, Colloraly 1] Let J be an isometric onto operator of $\mathcal{N}_p(E, \ell_{p^*})$. Then J preserves atoms.*

1.7.3 Strongly p-summing operators

Definition 1.7.3. *[33] Let $T \in \mathcal{L}(E, F)$. T is strongly p-summing operators, if there exists $C > 0$ such that for all $n \in \mathbb{N}, x_1, ..., x_n \in E$ and $y_1^*, ..., y_n^* \in F^*$.*

$$\sum_j^n |\langle T(x_j), y_j^* \rangle| \leq C \Big(\sum_j^n \|x_j\|^p\Big)^{\frac{1}{p}} \sup_{\varphi \in B_{F^{**}}} \Big(\sum_j |\varphi(y^*)|^{p^*}\Big)^{\frac{1}{p^*}} \tag{1.27}$$

The collection of all strongly p-summing operators is denoted by $\mathcal{D}_p(E, F)$ and $d_p(T)$ the infimum of all C satisfying the above inequality.

Theorem 1.7.10. *[10, 21]*

1. *$T \in \Pi_p(E, F)$ if, and only if, the adjoint operator $T^* \in \mathcal{D}_{p^*}(F^*, E^*)$, with*
$$\pi_p(T) = d_{p^*}(T^*). \tag{1.28}$$
2. *$T \in \mathcal{D}_p(E, F)$ if, and only if, the adjoint operator $T^* \in \Pi_{p^*}(F^*, E^*)$, with*
$$\pi_{p^*}(T^*) = d_p(T). \tag{1.29}$$

Theorem 1.7.11. *Let $T \in \mathcal{L}(E, F)$. T is strongly p-summing if, and only if, there is a constant $C > 0$ and a regular Borel probability measure μ on $B_{F^{**}}$ (with the weak star topology) so that for all $x \in E$ and for all $y^* \in F^*$, the inequality*

$$\langle Tx, y^* \rangle \leq C\|x\| \Big(\int_{B_{F^{**}}} |\langle y^*, \psi \rangle| d\mu(\psi)\Big), \tag{1.30}$$

holds.

Theorem 1.7.12. *If $1 \leq p \leq q \leq \infty$, then $\mathcal{D}_q(E, F) \subseteq \mathcal{D}_p(E, F)$. Moreover*

$$d_q(T) \leqslant d_p(T), \tag{1.31}$$

T is in $\mathcal{D}_p(E, F)$.

CHAPTER 2

LIPSCHITZ OPERATORS IDEALS

This chapter is devoted to a reminder of some elementary notions, definitions and results concerning Lipschitz operators that will be used in Chapter 3 and Chapter 4.

2.1 Basic notions and terminologies

Definition 2.1.1. *A metric or distance on a non empty set X is a function*

$$d : X \times X \longrightarrow \mathbb{R}_+$$

such that:

1. $\forall x, x' \in X, \quad d(x, x') = 0$ *if, and only if,* $x = x'$ *(Positivity).*
2. $\forall x, x' \in X, \quad d(x, x') = d(x', x)$ *(Symmetry).*
3. $\forall x, x', z \in X, \quad d(x, x') \leq d(x, z) + d(z, x')$ *(Triangle inequality).*

We call a metric space, the set X equipped with the distance d.

Definition 2.1.2. *The Lipschitz function is natural morphism between metric spaces (X, d) and (Y, ρ). Let*

$$T : (X, d) \longrightarrow (Y, \rho)$$

be a map. T is called a Lipschitz if there is a positive constant K such that:

$$\forall x, x' \in X, \quad \rho(T(x), T(x')) \leq K d(x, x').$$

For a Lipschitz map $T : (X, d) \longrightarrow (Y, \rho)$ its Lipschitz constant is given by

$$Lip(T) = \sup\left\{\frac{\rho(T(x), T(x'))}{d(x, x')} : x, x' \in X, x \neq x'\right\} \tag{2.1}$$

A pointed metric space X is a metric space with a base point in X , that is, a designated special point, which we will always denote by 0. We denote by $Lip_0(X, Y)$ the set of all Lipschitz maps from X to Y, with $T(0) = 0$. The space $Lip_0(X, F)$ is a Banach space of all Lipschitz maps from a pointed metric space X into a Banach space F, under the Lipschitz norm given by

$$Lip(T) = \sup\left\{\frac{\|T(x) - T(x')\|}{d(x, x')} : x, x' \in X, x \neq x'\right\}. \tag{2.2}$$

For $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we use the shorthand $X^\sharp := Lip_0(X, \mathbb{K}) = Lip_0(X)$. $X^\sharp$ is the Lipschitz dual of X i.e., is the space of all real valued Lipschitz functions under the norm $Lip(\cdot)$.

Proposition 2.1.1. *[32, Proposition 1.2.3] Let $\tilde{X}, \tilde{Y}$ be the completions of X, Y successively. If $T \in Lip(X, Y)$, Then T has an extension $\tilde{T} \in Lip(\tilde{X}, \tilde{Y})$ such that*

$$Lip(T) = Lip(\tilde{T}). \tag{2.3}$$

Proposition 2.1.2. *[32, Proposition 1.2.2] Let $f \in Lip(X, Y)$, and $g \in Lip(Y, Z)$. Then $g \circ f \in Lip(X, Z)$ and*

$$Lip(g \circ f) \leq Lip(f) \cdot Lip(g). \tag{2.4}$$

2.1.1 Space of p-summable sequences

Let $(\lambda_j)_j \subset \mathbb{R}$ and $(x_j)_j, (x'_j)_j$ be points in X :

1. We will denote by $\ell_p(\mathbb{N} \times \mathbb{R} \times X \times X)$ or $\ell_p(\mathbb{R} \times X \times X)$ the p-sequence set, we denote by the strong p-norm

$$\|((\lambda_j, x_j, x'_j))_j\|_p^{strong} = \left[\sum_{j=1} |\lambda_j|^p d_X(x_j, x'_j)^p\right]^{\frac{1}{p}}. \tag{2.5}$$

2. The weak Lipschitz p-sequence set, denoted $\ell_p^{L,w}(\mathbb{N} \times \mathbb{R} \times X \times X)$ or $\ell_p^{L,w}(\mathbb{R} \times X \times X)$ we denote its weak Lipschitz p-norm

$$w_p^L((\lambda_j, x_j, x'_j)_j) = \sup_{f \in B_{X^\sharp}} \left(\sum_j |\lambda_j|^p f(x_j) - f(x'_j)|^p\right)^{\frac{1}{p}}. \tag{2.6}$$

3. We denote ∞-norm by

$$\|(\lambda_j, x_j, \acute{x}_j)_j\|_p = \sup_{j\in\mathbb{N}} |\lambda_j| d_X(x_j, \acute{x}_j). \tag{2.7}$$

4. The weak Lipschitz ∞-norm

$$w_p^{\infty}((\lambda_j, x_j, x'_j)_j) = \sup_{f\in B_{X^\sharp}} \sup_{j\in\mathbb{N}} |\lambda_j| \Big| f(x_j) - f(x'_j) \Big|. \tag{2.8}$$

Its obvious that

$$\|(\lambda_j, x_j, x'_j)_j\|_p^{\infty} = w_p^{\infty}((\lambda_j, x_j, x'_j)_j). \tag{2.9}$$

Definition 2.1.3. *[4] Let $(f_j)_j$ be a sequence in $X^\sharp$. We say that $(f_j)_j$ is Lipschitz w^*-p-summable if there is a constant C such that for all $j \in \mathbb{N}$ and for all $x, \acute{x}$ we have*

$$\|(f_j(x) - f_j(x'))_{j\in\mathbb{N}}\|_p \leq C d(x, x'). \tag{2.10}$$

The smallest such constant C will be denoted by $w_p^{L,w^*}((f_j)_j)$ and $\ell_p^{L,w^*}(X^\sharp)$ is denoted the set of all Lipschitz w^*-p-summable sequences in $X^\sharp$. Clearly,

$$w_p^{L,w^*}((f_j)_j) = \sup_{x\neq \acute{x}_{x,x'\in X}} \frac{\|(f_j(x) - f_j(x'))_{j\in\mathbb{N}}\|_p}{d(x, x')} \tag{2.11}$$

Lemma 2.1.1. *[4, Lemma 2.4] The canonical correspondence*

$$T \longmapsto (\langle T(.), e_j^* \rangle)_j$$

provides an isometric isomorphism of $Lip_0(X, \ell_p)$ into $\ell_p^{L,w^*}(X^\sharp)$.

Lemma 2.1.2. *[4, Lemma 2.5]*

$$[\ell_p^{L,w^*}(X^\sharp), w_p^{L,w^*}] = [\ell_p^{w}(X^\sharp), w_p(.)]. \tag{2.12}$$

2.2 Arens-Eells space

Definition 2.2.1. *A molecule on metric space X is a real-valued function $m : X \longrightarrow \mathbb{R}$, with finite support that satisfies*

$$\sum_{x\in X} m(x) = 0. \tag{2.13}$$

The real linear space of all molecules on X is denoted by $\mathcal{M}(X)$.

For $x, x' \in X$, the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where χ_A is the characteristic function of the set A. For $m \in \mathcal{M}(X)$, we can write

$$m = \sum_{j}^{n} \mu_j m_{x_j x'_j} \tag{2.14}$$

where μ_j are scalars. It's well known that $Lip_0(X)$ has a predual, namely the space of Arens-Eells of X, denoted $\text{Æ}(X)$ [3], which is the completion of the space of molecules with the norm

$$\|m\|_{\text{Æ}(X)} = \inf\left\{ \sum_{j=1}^{n} |\mu_j| d(x_j, x'_j) : m = \sum_{j=1}^{n} \mu_j m_{x_j x'_j} \right\}, \tag{2.15}$$

where the infimum is taken over all representations of the molecule m.

Some remarkable properties concerning the Banach space $\text{Æ}(X)$ are given in the following:

Theorem 2.2.1. *[32, Theorem 2.2.2]*
Let X be a pointed metric space. Then $\text{Æ}(X) \cong X^\sharp$. On bounded subsets of $X^\sharp$ its weak topology agrees with the topology of pointwise convergence.*

Theorem 2.2.2. *[32, Theorem 2.2.4]*
Let T be a Lipschitz map between a pointed metric space X and Banach space F. Then T has factorization $T = T_L \circ \delta_X$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "\delta_X"'] & & F \\
& \text{Æ}(X), \arrow[ur, "T_L"'] &
\end{tikzcd}$$

Figure 2.1: Factorization diagram of Lipschitz operator

where $\delta_X(x) = m_{x0}$ is an isometric embending from X into $\text{Æ}(X)$ and T_L is unique linear operator from $\text{Æ}(X)$ into F (see [32, p. 38–43]). Furthermore

$$\|T_L\| = Lip(T). \tag{2.16}$$

The correspondence $T \longleftrightarrow T_L$ establishes an isomorphism isometric between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}(Æ(X), E)$.

Theorem 2.2.3. *[19, Lemma 3.1]*

Let T be a Lipschitz map between a metric spaces X into Y, which preserves the base point. Then there is a unique bounded linear map $\hat{T} : Æ(X) \longrightarrow Æ(Y)$ such that $\hat{T}\delta_X = \delta_Y T$ that is, the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "\delta_X"'] & Y \arrow[d, "\delta_Y"] \\
Æ(X) \arrow[r, "\hat{T}"] & Æ(Y)
\end{tikzcd}$$

Furthermore,

$$\|\hat{T}\| = Lip(T). \tag{2.17}$$

2.3 Adjoint of Lipschitz mapping

Definition 2.3.1. *[31] Let $T \in Lip_0(X, Y)$, a Lipschitz adjoint (or dual) between a pointed metric spaces X and Y of T is defined as the continuous linear operator*

$$\begin{aligned} T^\sharp : Y^\sharp &\longrightarrow X^\sharp \\ f &\longmapsto T^\sharp(f) = f \circ T, \end{aligned}$$

where $f \in Y^\sharp$. The norm of $T^\sharp$ is given by $\|T^\sharp\| = Lip(T)$. If $Y = F$ is a Banach space, the restriction of $T^\sharp$ to F^* is called the Lipschitz transpose operator of T and is denoted by T^t.

The correspondence $T \longleftrightarrow T^t$ establishes an isomorphism isometric between the vector spaces $Lip_0(X, E)$ and $\mathcal{L}((E^*, w^*), (X^\sharp, w^*))$.

Theorem 2.3.1. *[9, Theorem 2] Let $T \in Lip_0(X, Y)$. The Lipschitz adjoint of T it can be factored such that the following diagrams are commutative:*

$$\begin{tikzcd}
Y^\sharp \arrow[r, "T^\sharp"] \arrow[d, "R_2"'] & X^\sharp \\
Æ(Y)^* \arrow[r, "\Phi^*(T)"] & Æ(X)^* \arrow[u, "S_1"']
\end{tikzcd}$$

or, equivalently,

$$\begin{tikzcd}
Æ(Y)^* \arrow[r, "\Phi^*(T)"] \arrow[d, "S_2"'] & Æ(X)^* \\
Y^\sharp \arrow[r, "T^\sharp"] & X^\sharp \arrow[u, "R_1"']
\end{tikzcd}$$

where S_1, R_1, *and* S_2, R_2 *be the linear isometrics between the spaces* $X^\sharp, Æ(X)^*$ *and* $Y^\sharp, Æ(Y)^*$ *respectively.*

2.4 Lipschitz operator ideals

This section dedicated the concept of Lipschitz operator ideal, with a linear case that has already been introduced and studied by others.

Definition 2.4.1. *[2] A Lipschitz operator ideal* $\mathcal{I}_{Lip}$ *is a subclass of* Lip_0 *such that for every pointed metric space* X *and every Banach space* F *the components*

$$\mathcal{I}_{Lip}(X,F) := Lip_0(X,F) \cap \mathcal{I}_{Lip}$$

satisfy:

1- $\mathcal{I}_{Lip}(X,F)$ *is a linear subspace of* $Lip_0(X,F)$.

2- $vg \in \mathcal{I}_{Lip}(X,F)$ *for* $v \in E$ *and* $g \in X^\sharp$.

3- The ideal property: if $S \in Lip_0(Y,X), T \in \mathcal{I}_{Lip}(X,E)$ *and* $w \in \mathcal{L}(E,F)$, *then the composition* $wTS \in \mathcal{I}_{Lip}(Y,F)$.

A Lipschitz operator ideal $\mathcal{I}_{Lip}$ *is a normed (Banach) Lipschitz operator ideal if there is:*

$$\|\cdot\|_{\mathcal{I}_{Lip}} : \mathcal{I}_{Lip} \longrightarrow [0,\infty[$$

that satisfies:

1′- For every pointed metric space X and every Banach space F, the pair $(\mathcal{I}_{Lip}(X,F), \|\cdot\|_{\mathcal{I}_{Lip}})$ is a normed (Banach) space and for all $T \in \mathcal{I}_{Lip}(X,F)$

$$Lip(T) \leq \|T\|_{\mathcal{I}_{Lip}}.$$

2′- $\|Id_{\mathbb{K}} : \mathbb{K} \longrightarrow \mathbb{K}, Id_{\mathbb{K}}(\lambda) = \lambda\|_{\mathcal{I}_{Lip}} = 1.$

3′- If $S \in Lip_0(Y,X), T \in \mathcal{I}_{Lip}(X,E)$ and $w \in \mathcal{L}(E,F)$, then

$$\|wTS\|_{\mathcal{I}_{Lip}} \leq Lip_0(S)\|T\|_{\mathcal{I}_{Lip}}\|T\|. \tag{2.18}$$

2.4.1 Methods to produce Lipschitz operator ideals

Definition 2.4.2. *[2, Definition 3.1] Given an operator ideal $\mathcal{I}$, a Lipschitz mapping $T \in Lip_0(X,F)$ belongs to the composition Lipschitz operator ideal $\mathcal{I} \circ Lip_0$, denoted $T \in \mathcal{I} \circ Lip_0$, if there are a Banach space E, a Lipschitz operator $S \in Lip_0(X,E)$ and an operator $u \in \mathcal{I}(E,F)$ such that $T = u \circ S$. If $(\mathcal{I}, \|\dot{}\|_{\mathcal{I}})$ is a normed operator ideal we write $\|T\|_{\mathcal{I}} = \|T\|_{\mathcal{I} \circ Lip_0}$, where the infimum is taken over all u, S as above.*

Proposition 2.4.1. *[2, Proposition 3.2] Let $\mathcal{I}$ be an operator ideal. The following are equivalent for $T \in Lip_0(X,F)$:*

1- $T \in \mathcal{I} \circ Lip_0(X,F)$.

2- $T_L \in \mathcal{I}(\AE(X), F)$.
If $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is a normed operator ideal, then

$$\|T\|_{\mathcal{I} \circ Lip_0} = \|T_L\|_{\mathcal{I}}. \tag{2.19}$$

Corollary 2.4.1. *[2, Corollary 3.3] If $\mathcal{I}$ is a (normed, closed, Banach) operator ideal then, $\mathcal{I} \circ Lip_0$ is a (respectively normed, closed, Banach) Lipschitz operator ideal.*

The dual of a Lipschitz operator ideal $\mathcal{I}^{Lip_0-dual}$ is defined as follows ([2, Definition 3.8]):

$$\mathcal{I}^{Lip_0-dual} = \Big\{T \in Lip_0(X,F) : T^t \in \mathcal{I}(F^*, X^\sharp)\Big\},$$

where $T^t : F^* \longrightarrow X^\sharp$ is the transpose of T. If $\mathcal{I}$ is a quasi-normed operator ideal, define

$$\|T\|_{\mathcal{I}^{Lip_0-dual}} = \|T^t\|_{\mathcal{I}}. \tag{2.20}$$

Theorem 2.4.1. *[2, Theorem 3.9] If $\mathcal{I}$ is an operator ideal then,*

$$\mathcal{I}^{Lip_0-dual} = \mathcal{I}^{dual} \circ Lip_0$$

Moreover, if $(\mathcal{I}, \|\cdot\|_{\mathcal{I}})$ is normed then,

$$\|\cdot\|_{\mathcal{I}^{Lip_0-dual}} = \|\cdot\|_{\mathcal{I}^{dual} \circ Lip_0}. \tag{2.21}$$

2.5 Examples of Lipschitz operator ideals

2.5.1 Lipschitz p-summing operators

The concept of Lipschitz p-summing operators was introduced in 2009 by Farmer and Johnson in their article [13]. This notion is a non-linear version of the p-summing operators.

Definition 2.5.1. *[13] Let $T \in Lip_0(X, F), T$ p-summing operator, if there is a constant $C > 0$ such that for $n \in \mathbb{N}, x_1, ..., x_n$ and $x'_1, ..., x'_n \in X$ we have*

$$\Big(\sum_{j=1}^{n} \|T(x_j) - T(x'_j)\|^p\Big)^{\frac{1}{p}} \leq C \sup_{f \in B_{X^\sharp}} \Big(\sum_{j=1}^{n} |f(x_j) - f(x'_j)|^p\Big)^{\frac{1}{p}}. \tag{2.22}$$

The collection of all Lipschitz p-summing operators is denoted by $\Pi_p^L(X, Y)$ with the norm $\pi_p^L(.)$, is the infimum of all C for which the inequality above always holds.

Theorem 2.5.1 (Pietsch Domination-Factorization theorem)**.** *[13, Theorem 1] Let $T \in Lip_0(X, Y)$. Then the following are equivalent for the operator T and $C > 0$:*

1. *T is Lipschitz p-summing.*
2. ***(Pietsch domination)** There is a probability measure μ on $B_{X^\sharp}$ such that*
$$\|Tx - Tx'\|^p \leq C^p \cdot \int_{B_{X^\sharp}} |f(x) - f(x')|^p d\mu(f). \tag{2.23}$$
3. ***(Pietsch Factorization)** For some (or any) isometric embedding J of Y into a 1-injective space Z, there is a factorization with μ is a probability and $Lip(A)Lip(B) \leq C$:*

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "B"'] & Y \arrow[r, "J"] & Z \\
L_\infty(\mu) \arrow[rr, "I_{\infty,p}"] & & L_p(\mu) \arrow[u, "A"']
\end{tikzcd}$$

Figure 2.2: Factorization diagram of Lipschitz p-summing operator

Proof. 2 $\Rightarrow$ 3) Since each space X integrates into a space $C(K)$, we have the following commutative diagram by taking $K = B_{X^\sharp}$: with $B_1 \circ I_{\infty,p} \circ$

$$\begin{tikzcd}
C\left(B_{X^\#}\right) \arrow[r, "J_\infty"] & L_\infty\left(\mu\right) \arrow[rr, "I_{\infty,p}"] & & I_{\infty,p}AX \arrow[r, hook] \arrow[d, "B_1"] & L_p \arrow[dl, "B"] \\
& X \arrow[ul, "i_X"] \arrow[u, "A"] \arrow[r, "T"] & Y \arrow[r, "J"] & Z &
\end{tikzcd}$$

Figure 2.3: Factorization diagram of $J \circ T$

$A = J \circ T$. We are able to extend B_1 to B because Z is a 1-injective with $B_1 \circ I_{\infty,p} \circ A = J \circ T$. We are able to extend B_1 to B because Z is a 1-injective with $Lip(B_1) = Lip(B)$. So $B_1 = B_{|I_{\infty,p}AX}$. Since i_X and J_∞ are Lipschitz and the composition of the maps of Lipschitz is Lipschitz by Proposition 2.1.2. Then A is also Lipschitz, since J is a isometry and by condition 2), we have for all $x, y \in X$:

$$\begin{aligned}
\|B_1 I_{\infty,p}Ax - B_1 I_{\infty,p}Ay\|_Z^p &= \|JTx - JTy\|_Z^p = \|Tx - Ty\|^p \\
&\leq C^p \cdot \int_{B_{X^\sharp}} |f(x) - f(y)|^p d\mu(f).
\end{aligned}$$

Since the function f are Lipschitz, $f \in B_{X^\sharp}$ therefore $Lip(f) \leq 1$ and $I_{\infty,p} \circ A$ is injective, then by [25, Remarque 2.2.2] we have :

$$\begin{aligned}
\|B_1 I_{\infty,p}Ax - B_1 I_{\infty,p}Ay\|_Z^p &\leq C^p \cdot Lip(f)^p \cdot \|x - y\|_X^p \\
&\leq C^p \cdot \|x - y\|_X^p \\
&= C^p \cdot \|I_{\infty,p}Ax - I_{\infty,p}Ay\|_{L_p(\mu)}^p.
\end{aligned}$$

Thus B_1 is Lipschitz and $Lip(B_1) \leq 1$. Also :

$$Lip(A) = Lip(J_\infty i_X) \leq Lip(J_\infty) Lip(i_X) \leq 1.$$

Since B_1 is an extension, i.e., B_1 extends to $B_1 = B_{|\infty,p}$ with $Lip(B_1) = Lip(B)$ we have:

$$Lip(A) \cdot Lip(B) \leq C.$$

$3 \Rightarrow 1$) Since $I_{\infty,p}$ is Lipschitz p-sommant with $\pi_p^L(I_{\infty,p}) = 1$ by [25, Remarque 1.2.4]

and J is an isometric, then by condition 3 and ideal property we have:

$$\begin{array}{rcl}\pi_p^L(T) & = & \pi_p^L(JT) = \pi_p^L(B_1 \circ I_{\infty,p} \circ A) \\ & \leq & Lip(B_1) \cdot \pi_p^L(I_{\infty,p}) \cdot Lip(A) \\ & = & Lip(B_1) \cdot \pi_p(I_{\infty,p}) \cdot Lip(A) \\ & = & Lip(B_1) \cdot Lip(A) \\ & = & Lip(B) \cdot Lip(A) \leq C.\end{array}$$

$1 \Rightarrow 2$) Suppose $\pi_p^L(T) = 1$. Let Q be convex cone in $C(B_{X^\sharp})$ composed of all positive linear combinations of the form $\|Tx - Ty\| - C^p \cdot |f(x) - f(y)|^p$, like x and y range over X. Now condition 1) Q is disjoint by positive cone $P = \{F \in C(B_{X^\sharp}) : F(f) > 0, \forall f \in B_{X^\sharp}\}$. P is clearly open and convex of $C(B_{X^\sharp})$. Indeed, P is open because $P = \cup_F F^{-1}(0, \infty)$ where $F \in C(B_{X^\sharp})$. P is convex it is a cone and so and so $Q \cap P = \varnothing$, other $g_M \in Q$ for some finished set $M \subset X$ and $g_M(f) > 0$ for all $f \in B_{X^\sharp}$ where

$$g_M = \sum_{x,y \in M} \|Tx - Ty\|^p - C^p \cdot |f(x) - f(y)|^p.$$

In fact, on the contrary, if $g_M \in Q$ for some finished set $M \subset X$ and $g_M(f) > 0$ for all $f \in B_{X^\sharp}$, then

$$\sum_{x,y \in M} \|Tx - Ty\|^p - C^p \cdot |f(x) - f(y)|^p > 0,$$

so that

$$\sum_{x,y \in M} \|Tx - Ty\|^p > C^p \cdot \sum_{x,y \in M} |f(x) - f(y)|^p.$$

where

$$\sum_{x,y \in M} \|Tx - Ty\|^p > C^p \cdot \sup_{f \in B_{X^\sharp}} \sum_{x,y \in M} |f(x) - f(y)|^p$$

Contrary T is Lipschitz p-sommant. Then, $P \cap Q = \varnothing$. Hence, by Theorem separation and Riesz representation theorem, there is finite and signed a measure of Baire μ on $B_{X^\sharp}$ and c a real number so that for all $G \in Q$ and $F \in P$,

$$\int_{B_{X^\sharp}} G d\mu \leq c < \int_{B_{X^\sharp}} F d\mu.$$

Because $0 \in Q$ then $c \geq 0$. Then, because all positive constant functions belong to P, then $c < 0$ so that $c = 0$. Since $\int_{B_{X^\sharp}} d\mu$ is positive on the positive cone, the signed measure μ is positive that we can assume by scaling is a measure of probability. From where

$$\int_{B_{X^\sharp}} G d\mu \leq 0 < \int_{B_{X^\sharp}} F d\mu$$

so that

$$\int_{B_{X^\sharp}} \|Tx - Ty\|^p - C^p \cdot |f(x) - f(y)|^p d\mu(f) \leq 0.$$

Thus

$$\|Tx - Ty\|^p \leq C^p \cdot \int_{B_{X^\sharp}} |f(x) - f(y)|^p d\mu(f).$$

$\square$

Theorem 2.5.2. *[12, Page 39]*
If $1 \leq p \leq q < \infty$, *then* $\Pi_p^L(X, Y) \subset \Pi_q^L(X, Y)$. *Moreover*

$$\pi_q^L(T) \leq \pi_p^L(T),$$

T *is in* $\Pi_p^L(X, Y)$.

Theorem 2.5.3. *[13, Theorem 2]*
Let $1 \leq p < \infty$ *and* $T \in \mathcal{L}(E, F)$. *Then*

$$\pi_p^L(T) = \pi_p(T).$$

2.5.2 Strongly Lipschitz p-nuclear operators

The notion of Lipschitz p-nuclear operators and strongly Lipschitz p-nuclear operators was introduced by Chen and Zheng in [8], such that:

Definition 2.5.2. $T \in Lip_0(X, F), T$ *is a Lipschitz* p*-nuclear operator if there exist:* $B \in Lip(X, \ell_\infty), M_\lambda \in \mathcal{L}(\ell_\infty, \ell_p)(\mathcal{L}(\ell_\infty, c_0), p = \infty) : M_\lambda((\xi_n)) = (\xi_n \lambda_n)_n, (\xi_n)_n \in \ell_\infty$, *with* $\|M_\lambda\| = \|\lambda\|_p$, *and* $A \in Lip(\ell_p, F)$, *such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "B"'] & F \\
\ell_\infty \arrow[r, "M_\lambda"] & \ell_p(c_0, p = \infty) \arrow[u, "A"']
\end{tikzcd}$$

Figure 2.4: Factorization diagram of a Lipschitz p-nuclear operator

We denote by $\mathcal{N}_p^L(X,F)$ *the space of the Lipschitz* p*-nuclear operators from* X *to* F *and*

$$\nu_p^L(T) = \inf Lip(A)\|M_\lambda\|Lip(B). \tag{2.24}$$

Theorem 2.5.4. *Let* T *be a bounded linear operator from a separable Banach space* X *into a dual space* Y*. Then*

$$\nu_p^L(T) = \nu_p(T). \tag{2.25}$$

Definition 2.5.3. *Let* $T \in Lip_0(X,F), T$ *is a strongly Lipschitz* p*-nuclear operator has the form:*

$$T = \sum_j f_j \otimes y_j,$$

where $(f_j)_j$ *in* $X^\sharp$ *and* $(y_j)_j$ *in* F*. The collection of all strongly Lipschitz* p*-nuclear operators from* X *to* F *is denoted by* $\mathcal{SN}_p^L(X,F)$*, and the strongly Lipschitz* p*-nuclear norm was defined as following*

$$s\nu_p^L(T) = \inf\Big[\Big(\sum_j (Lip(f_j))^p\Big)^{\frac{1}{p}} \sup_{y^* \in B_{F^*}} \Big(\sum_j |\langle y_j, y^*\rangle|^{p^*}\Big)^{\frac{1}{p^*}}\Big]. \tag{2.26}$$

Chen and Zheng in [8, Theorem 2.2] prove that:

Theorem 2.5.5. *Let* $T \in Lip_0(X,F), T$ *is strongly Lipschitz* p*-nuclear operator if, and only if,* T *has a factorization* $T = aM_\lambda B$ *such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "B"'] & F \\
\ell_\infty \arrow[r, "M_\lambda"] & \ell_p(c_0, p=\infty) \arrow[u, "a"']
\end{tikzcd}$$

Figure 2.5: Factorization diagram of a strongly Lipschitz p-nuclear operator

where $B \in Lip_0(X,\ell_\infty), M_\lambda \in \mathcal{L}(\ell_\infty,\ell_p)(\mathcal{L}(\ell_\infty,c_0), p=\infty)$ *is the diagonal operator defined as follows:* $M_\lambda(\xi_j) = (\xi_j\lambda_j)_j, (\xi_j)_j \in \ell_\infty$*, with* $\|M_\lambda\| = \|\lambda\|_p$*, and* $a \in \mathcal{L}(\ell_p, F)$*. Moreover* $s\nu_p^L(T) = \inf\|a\|\|M_\lambda\|Lip(B)$*, where the infimum being extended over all factorizations as above.*

Proof. Suppose that T is strongly Lipschitz p-nuclear then there exist a sequences $(f_j)_j \subset X^\sharp$ et $(y_j)_j \subset Y$ such that $T = \sum\limits_j f_j \otimes y_j$. Let

$$B : X \longrightarrow \ell_\infty, \qquad x \mapsto \left(\frac{f_j(x)}{Lip(f_j)}\right)_j.$$

$$M_\lambda : \ell_\infty \longrightarrow \ell_p, (c_0, p = \infty) \quad (t_j)_j \mapsto (Lip(f_j)t_j)_j, \quad \lambda = (Lip(f_j))_j.$$

$$a : \ell_p(c_0, p = \infty) \longrightarrow F, \qquad (s_j)_j \mapsto \textstyle\sum_j s_j y_j.$$

Then B is Lipschitz from X into ℓ_∞ then

$$B(0) = 0, \qquad \mathrm{Lip}(B) \leq 1 \qquad \text{and} \qquad \|M_\lambda\| = \left(\sum_j \mathrm{Lip}(f_j)^p\right)^{\frac{1}{p}} (1 \leq p < \infty).$$

For $p = 1$,

$$\|a((s_j)_j)\| = \sup_{y^* \in B_{F^*}} | < y^*, \sum_j s_j y_j > | \leq \left(\sum_j |s_j|\right)\left(\sup_j \|y_j\|\right).$$

Thus

$$\|a\| \leq \sup_j \|y_j\| \qquad \text{for } p = 1.$$

For $1 < p < \infty$,

$$\|a((s_j)_j)\| = \sup_{y^* \in B_{F^*}} | < y^*, \sum_j s_j y_j > | \leq \sup_{y^* \in B_{F^*}} \left(\sum_j |s_j|^p\right)^{\frac{1}{p}} \left(\sum_j | < y^*, y_j > |^{p^*}\right)^{\frac{1}{p^*}}.$$

Thus

$$\|a\| \leq \sup_{y^* \in B_{F^*}} \left(\sum_j | < y^*, y_j > |^{p^*}\right)^{\frac{1}{p^*}} \qquad \text{for } \ 1 < p < \infty$$

For $p = \infty$,

$$\|M_\lambda\| = \sup_j \mathrm{Lip}(f_j), \ \|a\| \leq \sup_{y^* \in B_{F^*}} \sum_j | < y^*, y_j > |.$$

Thus,

$$T = aM_\lambda B \qquad \text{and} \qquad \inf \|a\|\|M_\lambda\|\mathrm{Lip}(B) \leq s\nu_p^L(T).$$

Conversely, if $M_\lambda = \sum\limits_j \delta_j e_j \otimes e_j$ and $\lambda = (\delta_j)_j \in \ell_p(c_0, \ p = \infty)$, so for $x \in X$,

$$T(x) = aM_\lambda B(x) = \sum_j \delta_j < B(x), e_j > a(e_j).$$

Let $f_j = \delta_j < B(.), e_j >$. Then, $f_j \in X^\sharp$ and $T = \sum\limits_j f_j \otimes a(e_j)$.

For $p = 1$,

$$\sum_j \mathrm{Lip}(f_j) \leq \mathrm{Lip}(B)\|M_\lambda\| \quad \text{and} \quad \sup_j \|a(e_j)\| \leq \|a\|.$$

For $1 < p < \infty$,

$$\left(\sum_j \mathrm{Lip}(f_j)^p\right)^{\frac{1}{p}} \leq \mathrm{Lip}(B)\|M_\lambda\|$$

and

$$\begin{aligned}
\left(\sup_{y^* \in B_{F^*}} \sum_j |<y^*, a(e_j)>|^{p^*}\right)^{\frac{1}{p^*}} &= \sup_{y^* \in B_{F^*}} \left(\sum_j |<a^* y^*, e_j>|^{p^*}\right)^{\frac{1}{p^*}} \\
&= \sup_{y^* \in B_{F^*}} \|a^* y^*\| \\
&= \|a\|.
\end{aligned}$$

For $p = \infty$

$$\|M_\lambda\| = \sup_j |\delta_j|, \qquad \sup_j \mathrm{Lip}(f_j) \leq \mathrm{Lip}(B) \sup_j |\delta_j|$$

and

$$\lim_j \mathrm{Lip}(f_j) = 0.$$

Plus,

$$\begin{aligned}
&= \sup_{y^* \in B_{F^*}} \|a^* y^*\| \\
&= \|a\|.
\end{aligned}$$

Thus, for $1 \leq p \leq \infty$.

$$\mathcal{N}_p^L((f_j)_j, (y_j)_j) \leq \|a\| \|M_\lambda\| \mathrm{Lip}(B).$$

Which implies

$$s\nu_p^L(T) \leq \inf \|a\| \|M_\lambda\| \mathrm{Lip}(B).$$

□

Theorem 2.5.6. *If $1 \leq p \leq q < \infty$, then $\mathcal{SN}_p^L(X, F) \subset \mathcal{SN}_q^L(X, F)$. Moreover*

$$s\nu_q^L(T) \leq s\nu_p^L(T).$$

Remark 2.5.1. *It follows from Theorem 2.5.5 that strongly Lipschitz p-nuclear operators are Lipschitz p-nuclear.*

Theorem 2.5.7. *Let T be a bounded linear operator from a separable Banach space X into a dual space Y. Then*

$$s\nu_p^L(T) = \nu_p(T). \tag{2.27}$$

2.5.3 Strongly Lipschitz p-summing operators

Yahi and al. [33] introduced the Lipschitz version of strongly p-summing operators whose linear analogue has introduced by Cohen [10], and characterized those operators whose adjoints are absolutely p^*-summing linear operators

Definition 2.5.4. *[33] Let $1 < p \leqslant \infty$ $T \in Lip_0(X, F), T$ is strongly Lipschitz p-summing operators, if there is a Banach space E and an operator $S \in \mathcal{D}_p(E, F)$ [i.e $S^* \in \Pi_{p^*}(F^*, E^*)$] such that*

$$|\langle y^*, T(x) - T(x')\rangle| \leq d(x, x')\|S^*(y^*)\| \tag{2.28}$$

for all $x, x' \in X$ and $y^* \in F^*$. The collection of all strongly Lipschitz p-summing operators is denoted by $\mathcal{D}^L_{st,p}(X, F)$ and $d^L_{st,p}(T)$ the infimum of all $d_p(S)$, for S satisfying above the inequality.

For $p = 1, \mathcal{D}^L_{st,p}(X, F) = Lip_0(X, F)$. If T is a linear operator between Banach spaces E and F then $T \in \mathcal{D}^L_{st,p}(X, F)$ if, and only if, $T \in \mathcal{D}_p(E, F)$.

Theorem 2.5.8. *If $1 > p \leq q \leq \infty$, then $\mathcal{D}^L_{st,q}(X, F) \subseteq \mathcal{D}^L_{st,p}(X, F)$. Moreover*

$$d^L_{st,p}(T) \leqslant d^L_{st,q}(T).$$

Proposition 2.5.1. *[33, Proposition 3.2] Let $T \in Lip_0(X, F)$. Then $T \in \mathcal{D}^L_{st,p}(X, F)$ if, and only if, $T_L \in \mathcal{D}_p(Æ(X), F)$, with*

$$d^L_{st,p}(T) = d_p(T_L).$$

Theorem 2.5.9. *Let $1 < p \leq \infty$ and $T \in Lip_0(X, F)$. The following statements are equivalent:*
1- $T \in \mathcal{D}^L_{st,p}(X, F)$;
2- there exist a constant $C > 0$ and a probability measure μ on $B_{F^{**}}$ such that for all $x, x' \in X, y^* \in F^*$ we have

$$|\langle y^*, T(x) - T(x')\rangle| \leq C d(x, x')\Big(\int_{B_{F^{**}}} |\varphi(y^*)|^{p^*} d\mu(\varphi)\Big)^{\frac{1}{p^*}}; \tag{2.29}$$

3- there exist a constant $C > 0$ such that for all $(x_j)^n_j, (x'_j)^n_j \subset X$, all $(y^*_j)^n_j \subset E^*$ and all $(b^*_j)^n_j \subset \mathbb{R}$

$$\sum_{j=1}^{n} |b_j||\langle y^*, T(x_j) - T(x'_j)\rangle| \leq C\Big(\sum{}_j{}^n |b_j|^p d(x_j, x'_j)^p\Big)^{\frac{1}{p}} \sup_{\varphi \in F^{**}} \Big(\sum_{j=1}^{n} |\varphi(y^*_j)|^{p^*}\Big)^{\frac{1}{p^*}} \tag{2.30}$$

Furthermore, the infimum of the constants $C > 0$ in (2.29) and (2.30) is $d^L_{st,p}(T)$.

Proof. For the proof see [33]. □

2.5.4 Lipschitz compact and Lipschitz weakly compact operators

The notion of Lipschitz compact (weakly compact, finite-rank, approximable) was introduced by Vargas and al. in [18].

Definition 2.5.5. *[18] Let $T \in Lip_0(X, F)$. T is Lipschitz compact (Lipschitz weakly compact) if the set*

$$\left\{ \frac{f(x) - f(y)}{d(x, y)} : x, y \in X, x \neq y \right\}$$

is relatively compact (respectively, relatively weakly compact) in F. Denote by $Lip_{0\mathcal{K}}(X, F)(Lip_{0\mathcal{W}}(X, F))$ the collection of all Lipschitz compact operators (Lipschitz weakly compact operators) from X to F, respectively.

Proposition 2.5.2. *[18, Proposition 2.4] Let $T \in Lip_0(X, F)$. The following statements are equivalent:*

1. *T is Lipschitz compact.*
2. *T_L is compact from Æ(X) into F.*

Proposition 2.5.3. *[18, Proposition 3.5](Schauder's theorem) Let $T \in Lip_0(X, F)$. The following statements are equivalent:*

1. *T is Lipschitz compact.*
2. *T^t is compact from F^* into $X^\sharp$.*

CHAPTER 3

STRONGLY LIPSCHITZ UP-NUCLEAR OPERATORS

In this chapter, we start by giving a new concept, the Lipschitz version of N^p-nuclear operator which is introduced by Persson in [26]. We show that this class of operators is a Lipschitz operator ideal [2]. We give the factorization Theorem of this class, and we study the relationships between a strongly Lipschitz operator T and its transpose T^t.

The results of this chapter have been published in Moroccan Journal of Pure and Applied Analysis [5].

3.1 Strongly Lipschitz up-nuclear operators

Definition 3.1.1. *Let* $1 \leq p \leq \infty$, *and* $T \in Lip_0(X, F)$. T *is Lipschitz up-nuclear operator, if* T *can be written in the following form*

$$T = \sum_j f_j \otimes y_j$$

such that $(f_j)_j \subset X^\sharp$ *where* $X^\sharp$ *is the space of all real-valued Lipschitz functions under the (semi)-norm* $Lip(.)$ *and* $(y_j)_j \subset F$ *satisfy*

$$\sup_{\|m\| \leq 1} \left(\sum_j |\langle f_j, m \rangle|^{p^*} \right)^{1/p^*} < \infty \quad and \quad \|(y_j)_j\|_p < \infty,$$

where $m \in \AE(X)$. Here

$$\mathcal{N}^{p^L}((f_j)_j, (y_j)_j) = \sup_{\|m\| \le 1} \Big(\sum_j |\langle f_j, m \rangle|^{p^*} \Big)^{1/p^*} \|(y_j)_j\|_p. \tag{3.1}$$

Moreover, $\nu^{p^L}(T) := \inf \mathcal{N}^{p^L}((f_j)_j, (y_j)_j)$, the infimum being taken over all such representations as above. The collection of all strongly Lipschitz up-nuclear operators from X to F is denoted by $\mathcal{N}^{p^L}(X, F)$.

Proposition 3.1.1. *Let $1 \le p \le \infty$, and $T \in Lip_0(X, F)$. Then $T \in \mathcal{N}^{p^L}(X, F)$ if, and only if, $T_L \in \mathcal{N}^p(\AE(X), F)$, with*

$$\nu^{p^L}(T) = \nu^p(T_L). \tag{3.2}$$

Proof. It's well known that $\mathcal{N}^p(\AE(X), F)$ is a normed operator ideal (see [29]) and $T \in Lip_0(X, F)$, then by [2, Proposition 3.2] we get the equivalence is above with the necessary norm. □

Theorem 3.1.1. *$(\mathcal{N}^{p^L}(X, F), \nu^{p^L})$ is a Banach Lipschitz ideal.*

Proof. Since $\mathcal{N}^p(\AE(X), F)$ is a normed operator ideal, $T \in \mathcal{N}^p(\AE(X), F) \circ Lip_0(X, \AE(X))$ and by [2, Corollary 3.3], $\mathcal{N}^{p^L}(X, F)$ is a Banach strongly Lipschitz up-nuclear operator ideal. □

3.2 Factorization Theorem

In this section, we characterize operators in $\mathcal{N}^{p^L}(X, F)$.

Theorem 3.2.1. *Let $1 \le p \le \infty$ and $T \in Lip_0(X, F)$. Then $T \in \mathcal{N}^{p^L}(X, F)$ if, and only if, T has a factorization $T = aM_\lambda B$ such that the following diagram commutes:*

$$\begin{tikzcd}
X \arrow[r, "T"] \arrow[d, "B"'] & F \\
\ell_{p^*} \arrow[r, "M_\lambda"] & \ell_1 \arrow[u, "a"']
\end{tikzcd}$$

Figure 3.1: Factorization diagram of Lipschitz up-nuclear operator

where $B \in Lip_0(X, \ell_{p^*})$ *with* $B(0) = 0, M_\lambda \in \mathcal{L}(\ell_{p^*}, \ell_1)$ *a diagonal operator and* $a \in \mathcal{L}(\ell_1, F)$*. Moreover,* $\nu^{p^L}(T) := \inf\|a\|\|M_\lambda\|Lip(B)$*, where the infimum is taken over all the above factorizations.*

Proof. We know if $T \in Lip_0(X, F)$ there exists a unique linear map $T_L : \AE(X) \longrightarrow F$ such that $T = T_L \circ \delta_X$, more they have the same characteristics. Since $T_L \in \mathcal{N}^p(\AE(X), F)$, so we have a factorization as following

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "\delta_X"] \arrow[ddr, "B"'] & & F \\
& \AE(X) \arrow[ur, "T_L"'] \arrow[d, "b"] & \\
& \ell_{p^*} \arrow[r, "M_\lambda"] & \ell_1 \arrow[uu, "a"']
\end{tikzcd}$$

we can see that $T = aM_\lambda B$, where $B = b \circ \delta_X$.

Conversely, a similar proof as in [8, Theorem 2.2], with

$$\nu^{p^L}(T) = \inf\|a\|\|M_\lambda\|Lip(B). \tag{3.3}$$

□

Remark 3.2.1. *For* $p = 1$*, we have* $\mathcal{N}^{1^L}(X, F) = \mathcal{N}_1^L(X, F)$*.*

Using Theorem 3.2.1 we obtain the following results:

Proposition 3.2.1. *Let* $1 \leq p \leq q \leq \infty$*, then* $\mathcal{N}^{p^L}(X, F) \subseteq \mathcal{N}^{q^L}(X, F)$*, with*

$$\nu^{q^L}(T) \leq \nu^{p^L}(T). \tag{3.4}$$

Proof. We know that a multiplication operator can be factored as (see [30])

$$M_\lambda : \ell_{p^*} \xrightarrow{M_\alpha} \ell_{q^*} \xrightarrow{M_\beta} \ell_1,$$

So,

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[d, "B"] & & F \\
\ell_{p^*} \arrow[rr, "M_\lambda"] \arrow[dr, "M_\alpha"'] & & \ell_1 \arrow[u, "a"'] \\
& \ell_{q^*} \arrow[ur, "M_\beta"'] &
\end{tikzcd}$$

where M_α and M_β are multiplication operators which are given by $\alpha_n = |\lambda_n|^{1-\frac{p^*}{q^*}}$ and $\beta_n = (sign\lambda_n)|\lambda_n|^{\frac{p^*}{q^*}}(n \in \mathbb{N})$.

$$\begin{aligned}
\nu^{p^L}(T) &= \inf Lip(B)\|M_\lambda\|\|a\| \\
&= \inf Lip(B)\|M_\beta M_\alpha\|\|a\| \\
&= \inf Lip(B)\|M_\alpha\|\|M_\beta\|\|a\| \\
&\geq \inf Lip(\tilde{B})\|M_\beta\|\|a\| \\
&\geq \nu^{q^L}(T). \qquad \square
\end{aligned}$$

□

Corollary 3.2.1. *Let* $1 \leq p \leq \infty$ *and* $T \in Lip_0(X, F)$. *We have*

1. $J_E \circ T \in \mathcal{N}^{p^L}(X, F^{**}) \Leftrightarrow (T^t)^* \in \mathcal{N}^p((X^\sharp)^*, F^{**})$.

2. $T \in \mathcal{N}^{p^L}(X, F) \Rightarrow (T^t)^* \in \mathcal{N}^p((X^\sharp)^*, F^{**})$.

3.3 Applications

This section is devoted to some applications such as duality, relationships with known spaces.

Proposition 3.3.1. *If* $T \in \mathcal{N}_p^L(X, F)$, *then its transpose* $T^t \in \mathcal{N}^p(F^*, X^\sharp)$ *and it satisfies*

$$\nu^p(T^t) \leq s\nu_p^L(T).$$

Furthermore, assume F *is reflexive. Then, if* $T^t \in \mathcal{N}^p(F^*, X^\sharp)$ *we have* $T \in \mathcal{N}_p^L(X, F)$, *with*

$$\nu^p(T^t) = s\nu_p^L(T). \tag{3.5}$$

Proof. By [2, Proposition 2.7, Proposition 3.2], $T \in \mathcal{N}_p^L(X, F)$ if, and only if, $T_L \in \mathcal{N}_p(\text{Æ}(X), F)$. Then, by [24, Proposition 1], we have $(T_L)^* \in \mathcal{N}^p(F^*, \text{Æ}(X)^*)$, wich gives us $T^t \in \mathcal{N}^p(F^*, X^\sharp)$ because

$$T^t : F^* \xrightarrow{(T_L)^*} \text{Æ}(X)^* \xrightarrow{\delta_X^t} X^\sharp.$$

Furthermore, as $T^t = \delta_X^t \circ (T_L)^*$ so

$$\begin{aligned} \nu^p(T^t) = \nu^p(\delta_X^t \circ (T_L)^*) &\leq \nu^p((T_L)^*) \\ &\leq \nu_p(T_L) \\ &= s\nu_p^L(T) \qquad\qquad . \end{aligned}$$

Hence,

$$\nu^p(T^t) \leq s\nu_p^L(T). \tag{3.6}$$

Conversely, if $T^t \in \mathcal{N}^p(F^*, X^\sharp)$, then $(T^t)^* \in \mathcal{N}_p((X^\sharp)^*, F^{**})$, therefor $T \in \mathcal{N}_p^L(X, F)$ because

$$\begin{tikzcd} & F \arrow[dr, "J_F"] & \\ X \arrow[ur, "T"] \arrow[r, "K_X"] \arrow[dr, "a \circ K_X"'] & (X^\sharp)^* \arrow[r, "(T^t)^*"] \arrow[d, "a"'] & F^{**} \\ & \ell_\infty \arrow[r, "M_\lambda"] & \ell_p \arrow[u, "b"'] \end{tikzcd}$$

where K_X is the evaluation map $K_X(x)(f) = f(x), x \in X, f \in X^\sharp$ and J_E is the canonical injection from F into F^{**}, so,

$$s\nu_p^L(T) \leq \nu^p(T^t). \tag{3.7}$$

From (3.6) and (3.7) we get

$$s\nu_p^L(T) = \nu^p(T^t).$$

□

Theorem 3.3.1. *If* $T \in \mathcal{N}^{p^L}(X, F)$, *then its transpose* $T^t \in \mathcal{N}_p(F^*, X^\sharp)$ *and it satisfies*

$$\nu_p(T^t) \leq \nu^{p^L}(T). \tag{3.8}$$

Furthermore, assume F *is reflexive. Then, if* $T^t \in \mathcal{N}_p(F^*, X^\sharp)$ *we have* $T \in \mathcal{N}^{p^L}(X, F)$, *with*

$$\nu_p(T^t) = \nu^{p^L}(T). \tag{3.9}$$

Proof. By Proposition 3.1.1, $T \in \mathcal{N}^{p^L}(X, F)$ if, and only if, $T_L \in \mathcal{N}^p(Æ(X), F)$. Then, $(T_L)^* \in \mathcal{N}_p(F^*, Æ(X)^*)$, that is to say $T^t \in \mathcal{N}_p(F^*, X^\sharp)$ because

$$T^t : F^* \xrightarrow{(T_L)^*} Æ(X)^* \xrightarrow{\delta_X^t} X^\sharp,$$

it is clear that $T^t = \delta_X^t \circ (T_L)^*$ so

$$\begin{aligned} \nu_p(T^t) = \nu_p(\delta_X^t \circ (T_L)^*) &\leq \nu_p((T_L)^*) \\ &\leq \nu^p(T_L) \\ &= \nu^{p^L}(T). \end{aligned}$$

Hence

$$\nu_p(T^t) \leq \nu^{p^L}(T). \tag{3.10}$$

Conversely, we use the same method used in [**?**, Proposition 1], if $T^t \in \mathcal{N}_p(F^*, X^\sharp)$ then for any $\varepsilon > 0$, it can be written as

$$T^t y^* = \sum_j \langle y^*, y_j^{**} \rangle f_j,$$

for each $y^* \in F^*$ and $m \in Æ(X)$, with

$$\sup_{\|m\| \leq 1} \left(\sum_j |\langle f_j, m \rangle|^{p^*} \right)^{1/p^*} \|(y_j^{**})_j\|_p \leq \nu_p(T^t) + \varepsilon. \tag{3.11}$$

Since F is reflexive

$$\sup_{\|m\| \leq 1} \left(\sum_j |\langle f_j, m \rangle|^{p^*} \right)^{1/p^*} \|(y_j)_j\|_p \leq \nu_p(T^t) + \varepsilon. \tag{3.12}$$

Hence we have

$$T = \sum_j f_j \otimes y_j,$$

and (3.12) shows

$$\nu^{p^L}(T) \leq \nu_p(T^t). \tag{3.13}$$

From (3.10) and (3.13) we get

$$\nu^{p^L}(T) = \nu_p(T^t). \tag{3.14}$$

□

Remark 3.3.1. *Let $T \in \mathcal{N}_p^L(X, F)^{dual}$, then by [2, Theorem 3.9] we get $T \in \mathcal{N}_p(X, F)^{dual} \circ Lip_0$ and by [24, Proposition 1] $T \in \mathcal{N}^p(X, F) \circ Lip_0$ so $T \in \mathcal{N}^{p^L}(X, F)$.*

Proposition 3.3.2. *Let* $1 \leq p \leq \infty$ *and* $T \in Lip_0(X, F)$*. If* $T \in \mathcal{N}^{p^L}(X, F)$*, then* $T^t \in \Pi_p(F^*, X^\sharp)$*, with*

$$\pi_p(T^t) \leq \nu^{p^L}(T). \tag{3.15}$$

Proof. By Theorem 3.3.1, $T^t \in \mathcal{N}_p(F^*, X^\sharp)$ if $T \in \mathcal{N}^{p^L}(X, F)$. Then by [12, Proposition 5.5, Corollary 5.24], we get $T^t \in \Pi_p(F^*, X^\sharp)$, with

$$\pi_p(T^t) \leq \nu_p(T^t) \leq \nu^{p^L}(T).$$

□

Proposition 3.3.3. *Let* $1 \leq p \leq \infty$ *and* $T \in Lip_0(X, F)$*. If* $T \in \mathcal{N}^{p^L}(X, F)$*, then* $T \in \mathcal{D}^L_{st,p^*}(X, F)$*, with*

$$d^L_{st,p^*}(T) \leq \nu^{p^L}(T). \tag{3.16}$$

Proof. We have by Proposition 3.3.2, $T \in \mathcal{N}^{p^L}(X, F)$, then $T^t \in \Pi_p(F^*, X^\sharp)$. Hence by [33, Theorem 4.1], $T \in \mathcal{D}^L_{st,p^*}(X, F)$, and

$$d^L_{st,p^*}(T) = \pi_p(T^t) \leq \nu^{p^L}(T).$$

□

We present the previous proposition in another way, using the ideal property.

Proposition 3.3.4. *Let* $1 \leq p \leq \infty$ *and* $T \in Lip_0(X, F)$*. If* $T \in \mathcal{N}^{p^L}(X, F)$*, then there exist a Banach space* G*,* $u \in \mathcal{D}_{p^*}(G, F)$ *and* $S \in Lip_0(X, G)$ *such that* $T = u \circ S$*.*

Proof. It's direct from Proposition 3.3.3 and [33, Corollary 3.6]. □

CHAPTER 4

SOME RESULTS ABOUT LIPSCHITZ P-NUCLEAR OPERATORS

In this chapter we extend the notion of the isometries of p-nuclear operators to Lipschitz case whose linear analogue has studied by Yousef and Khalil in [22]. We studied the Lipschitz version the isometric onto operators of $\mathcal{N}_p(E.F)$. We continued in same direction with $F = \ell_{p^*}$, where we have given results, among others, concerning rank and atoms.

The results of this chapter have been published in Moroccan Journal of pure and Applied Analysis [6].

4.1 Onto isometries

In this section, we study the Lipschitz version of the isometric onto operators of $\mathcal{N}_p(E, F)$ [22, Sect. 3].

Theorem 4.1.1. *Let $(S_j^\sharp)_j$ be a sequence of an isometric onto operators on $X^\sharp$, and R be an isometric onto operator on F. Let T be a strongly Lipschitz p-nuclear operator from X into F such that*

$$J(T) = J(\sum_j f_j \otimes y_j) = \sum_j \alpha_j S_j^\sharp f_{\varphi(j)} \otimes R y_j,$$

where $(\alpha_j)_j$ in $\mathbb{R}, |\alpha_j| = 1$ and φ is a permutation on $\mathbb{N}$. Then J is an isometric operator of $\mathcal{SN}_p^L(X, F)$.

Proof. Assume that $T \in \mathcal{SN}_p^L(X,F)$, then T has the following form

$$T = \sum_j f_j \otimes y_j$$

where $(f_j)_j$ in $X^\sharp$ and $(y_j)_j$ in F, we have

$$\begin{aligned} J(T) &= \sum_j \alpha_j S_j^\sharp f_{\varphi(j)} \otimes R y_j \\ &= \sum_j \tilde{f}_j \otimes \tilde{y}_j. \end{aligned}$$

Now,

$$s\nu_p^L(T) = \inf \Big(\sum_j (Lip(f_j))^p \Big)^{\frac{1}{p}} \sup_{\|y^*\| \le 1} \Big(\sum_j |\langle y_j, y^* \rangle|^{p^*} \Big)^{\frac{1}{p^*}}.$$

Since $S_j^\sharp$ is an isometry, $|\alpha_j| = 1$ and φ is a permutation,

$$\begin{aligned} \sum_j (Lip(\alpha_j S_j^\sharp f_{\varphi(j)}))^p &= \sum_j |\alpha_j|^p (Lip(S^\sharp f_{\varphi(j)}))^p \\ &= \sum_j (Lip(f_{\varphi(j)}))^p \\ &= \sum_j (Lip(f_j))^p \end{aligned}$$

and also R is an isometric onto operator on F, so

$$\begin{aligned} \sup_{\|y^*\| \le 1} \Big(\sum_j |\langle R y_j, y^* \rangle|^{p^*} \Big)^{\frac{1}{p^*}} &= \sup_{\|y^*\| \le 1} \Big(\sum_j |\langle y_j, R^* y^* \rangle|^{p^*} \Big)^{\frac{1}{p^*}} \\ &= \sup_{\|R^* y^*\| \le 1} \Big(\sum_j |\langle y_j, R^* y^* \rangle|^{p^*} \Big)^{\frac{1}{p^*}}. \\ &= \sup_{\|\tilde{y}^*\| \le 1} \Big(\sum_j |\langle y_j, \tilde{y}^* \rangle|^{p^*} \Big)^{\frac{1}{p^*}}. \end{aligned}$$

Thus, $J(T)$ is strongly Lipschitz p-nuclear operator of $\mathcal{SN}_p^L(X,F)$ and

$$s\nu_p^L(J(T)) = s\nu_p^L(T).$$

$\square$

Theorem 4.1.2. *Let S be an isometric onto operator of X into X and R be an isometric onto operator of F into F. Assume T be a strongly Lipschitz p-nuclear operator from X into F such that*

$$J(T) = RTS.$$

Then $J(T)$ is an isometric onto operator of $\mathcal{SN}_p^L(X,F)$.

Proof. Let $T \in \mathcal{SN}_p^L(X,F)$, then T has the following representation

$$T = \sum_j f_j \otimes y_j$$

so

$$\begin{aligned} J(T)x &= RTSx \\ &= R(\sum_j (f_j(Sx))y_j \\ &= \sum_j (S^\sharp(f_j x))Ry_j \end{aligned}$$

Thus,

$$J(T) = \sum_j S^\sharp f_j \otimes Ry_j.$$

Since S is an isometric onto operator of X, then $S^\sharp$ is an isometric onto operator of $X^\sharp$, and R is an isometric onto operator of F. Now, we show that J preserves norm,

$$\begin{aligned} s\nu_p^L(J(T)) &= \inf \left(\sum_j (Lip(S^\sharp f_j))^p \right)^{\frac{1}{p}} \sup_{\|y^*\| \leq 1} \left(\sum_j |\langle y_j, R^* y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\ &= \inf \left(\sum_j (Lip(f_j))^p \right)^{\frac{1}{p}} \sup_{\|R^* y^*\| \leq 1} \left(\sum_j |\langle y_j, R^* y^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\ &= \inf \left(\sum_j (Lip(f_j))^p \right)^{\frac{1}{p}} \sup_{\|\tilde{y}^*\| \leq 1} \left(\sum_j |\langle y_j, \tilde{y}^* \rangle|^{p^*} \right)^{\frac{1}{p^*}} \\ &= s\nu_p^L(T). \end{aligned}$$

Thus,

$$s\nu_p^L(J(T)) = s\nu_p^L(T).$$

To show that J is onto, let $T = \sum_j f_j \otimes y_j$ be a strongly Lipschitz p-nuclear operator, and $\tilde{T} = \sum_j f_j S^{-1} \otimes R^{-1} y_j$, so

$$J(\tilde{T}) = \sum_j f_j \otimes y_j = T,$$

this implies that J is onto.

Hence, J is an isometric onto operator of $\mathcal{SN}_p^L(X,F)$.

□

4.2 Lipschitz operators, where $F = \ell_{p^*}$

In this section, we characterize the Lipschitz operators that takes values in ℓ_{p^*} whose linear analogue has found in [22, Sect. 4].

Theorem 4.2.1. *Let $2 \leq p < \infty$ and T be a Lipschitz operator from X into ℓ_{p^*}. Then T is a strongly Lipschitz p-nuclear operator from X into ℓ_{p^*}, with*

$$s\nu_p^L(T) = \Big(\sum_j |\lambda_j|^p\Big)^{\frac{1}{p}}, \tag{4.1}$$

where $(\lambda_j)_j \in \ell_{p^*}$.

Proof. Given the fact that each Lipschitz operator can be factored as [32, Theorem 2.2.4]

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "\delta_X"'] & & \ell_{p^*} \\
& \AE(X) \arrow[ur, "T_L"'] &
\end{tikzcd}$$

Figure 4.1: Factorization diagram of Lipschitz operator

where $\delta_X(x) = m_{x0}$ is an isometric embending from X into $\AE(X)$ and T_L is unique linear operator from $\AE(X)$ to F (for more details see [32, p. 38–43]). Since T_L is p-nuclear operator [22, Sect.4] and $T = T_L \circ \delta_X$, then by [2, Proposition 3.2] T is a strongly Lipschitz p-nuclear operator with $s\nu_p^L(T) = \nu_p(T_L)$, by means of [22, Theorem 6] we get

$$s\nu_p^L(T) = \Big(\sum_j |\lambda_j|^p\Big)^{\frac{1}{p}}.$$

□

Using Theorem 4.2.1, we obtain the following results:

Corollary 4.2.1. *Let $2 \leq p < \infty$ and T be a strongly Lipschitz p-nuclear operator from X into ℓ_{p^*}. Then T has representation for which the infimum is attained.*

We will give in the next corollary the same result as [8, Theorem 2.1] with E only a Banach space.

Corollary 4.2.2. *Let $2 \leq p < \infty$ and T be a linear operator from E into ℓ_{p^*}. Then*

$$s\nu_p^L(T) = \nu_p(T). \tag{4.2}$$

Proof. From [22, Theorem 6] and by Theorem 4.2.1 we get

$$s\nu_p^L(T) = \nu_p(T).$$

□

Corollary 4.2.3. *The space $\mathcal{SN}_p^L(X, \ell_{p^*})$ and $\ell_{p^*}^{L,w^*}(X^\sharp)$ are isometrically isomorphic.*

Proof. There is an isomorphism isometry between $Lip_0(X, \ell_{p^*})$ and $\ell_{p^*}^{L,w^*}(X^\sharp)$ [18, Lemma 2.4], and $Lip_0(X, \ell_{p^*}) = \mathcal{SN}_p^L(X, \ell_{p^*})$ Theorem 4.2.1. Hence $\mathcal{SN}_p^L(X, \ell_{p^*})$ and $\ell_{p^*}^{L,w^*}(X^\sharp)$ are isometrically isomorphic. □

Corollary 4.2.4. *Let J be an isometric onto operator of $\mathcal{SN}_p^L(X, \ell_{p^*})$. Then J is Lipschitz compact operator.*

Proof. It's known that $Lip_{0K}(X, \ell_{p^*}) \subset Lip_0(X, \ell_{p^*})$ [18] and Then by Theorem 4.2.1
$Lip_{0K}(X, \ell_{p^*}) = \mathcal{SN}_p^L(X, \ell_{p^*})$. □

The next theorem we show that an isometric onto operators of a Lipschitz operator ideal preserves the rank.

Theorem 4.2.2. *Let $2 \leq p < \infty$ and J be an isometric onto operator of $\mathcal{SN}_p^L(X, \ell_{p^*})$. Then J preserves the rank.*

Proof. It's known that an isometry on ℓ_p preserve the support. We will use this in our proof. If $T \in \mathcal{SN}_p^L(X, \ell_{p^*})$, then by Theorem 4.2.1

$$T = \sum_j \lambda_j f_j \otimes \delta_j.$$

Therefore $J(T) \in \mathcal{SN}_p^L(X, \ell_{p^*})$,

$$J(T) = \sum_j \xi_j g_j \otimes \delta_j,$$

where $\|(\lambda_j)\|_p = \|(\xi_j)\|_p$ and $Lip(f_j) = Lip(g_j)$. This produces

$$\hat{J} : \ell_p \longrightarrow \ell_p,$$

where $\hat{J}(\lambda_j) = (\xi_j)_j$. Clearly $\hat{J}$ is an isometric onto operator on ℓ_p. So it preserve the support. That is $supp(\lambda_j) \cap supp(\xi_j) = \emptyset$, then $supp(\hat{J}(\lambda_j)) \cap supp(\hat{J}(\xi_j)) = \emptyset$. Indeed

$$\begin{aligned} \|\hat{J}(\lambda_j) + \hat{J}(\xi_j)\|_p^p &= \|\hat{J}(\lambda_j + \xi_j)\|_p^p \\ &= \|\lambda_j + \xi_j\|_p^p \\ &= \|(\lambda_j)\|_p^p + \|(\xi_j)\|_p^p \\ &= \|(\hat{J}(\lambda_j)\|_p^p + \|(\hat{J}(\xi_j)\|_p^p. \end{aligned}$$

Since $rank(J(T)) = |supp(\hat{J}(\xi_j))| = |supp(\lambda_j)| = rank(T)$. So it's preserves rank. ☐

For $f \in X^\sharp$ and $y \in F$ (Banach space), we define the one rank operator $f \otimes y : X \longrightarrow F$ by $(f \otimes y)(x) = f(x)y$. The rank one operator is called an atom.

The next corollary is an immediate consequence of Theorem 4.2.2.

Corollary 4.2.5. *Let J be an isometric onto operator of $\mathcal{SN}_p^L(X, \ell_{p^*})$. Then J preserves atoms.*

Now, we give in the next theorem the main result of this section. We adapt the proof in [21, Theorem 3.3] to Lipschitz situation.

Theorem 4.2.3. *Let $J : \mathcal{SN}_p^L(X, \ell_{p^*}) \longrightarrow \mathcal{SN}_p^L(X, \ell_{p^*})$. If J preserves rank and that preserves the basic atoms, then the following are equivalent:*

1. *J is an isometric onto operator.*
2. *There exist two isometric onto operators:*
   $S : X^\sharp \to X^\sharp$ and $W : \ell_{p^*} \to \ell_{p^*}$ and a sequence $(a_j)_j, |a_j| = 1$ for all j, with
   $$J(T) = \sum_j a_j S f_j \otimes W \delta_j.$$

Proof. 1⇒2. Let J be an isometric onto operator.

Step 1: Let $M = \left\{ f \otimes \delta_1 : f \in X^\sharp \right\}$. Then

$$J(M) = \left\{ g \otimes \delta_j : \text{for fixed } j, \forall g \in X^\sharp \right\}.$$

Assume $J(f_1 \otimes \delta_1) = \hat{f}_1 \otimes \delta_{j_1}$ and $J(f_2 \otimes \delta_1) = \hat{f}_2 \otimes \delta_{j_2}$ where $\hat{f}_1, \hat{f}_2 \in X^\sharp$, and $\delta_{j_1} \neq \delta_{j_2}$. Then for some $g \in X^\sharp$ and $j \in \mathbb{N}$:

$$\begin{aligned}
\hat{f}_1 \otimes \delta_{j_1} + \hat{f}_2 \otimes \delta_{j_2} &= J(f_1 \otimes \delta_1) + J(f_1 \otimes \delta_2) \\
&= J(f_1 \otimes \delta_1 + f_1 \otimes \delta_2) \\
&= J(f_1 \otimes (\delta_1 + \delta_2)) \\
&= g \otimes \delta_j
\end{aligned}$$

This implies $\hat{f}_1 \otimes \delta_{j_1} + \hat{f}_2 \otimes \delta_{j_2} = g \otimes \delta_j$ with $\delta_{j_1} \neq \delta_{j_2}$. But $f_1 \otimes \delta_1 + f_2 \otimes \delta_1 = (f_1 + f_2) \otimes \delta_1$ is a basic atom, and J preserves basic atoms which is a contradiction because $\hat{f}_1 \otimes \delta_{j_1} + \hat{f}_2 \otimes \delta_{j_2}$ is not a basic atom, since $\delta_{j_1} \neq \delta_{j_2}$. So $J(f \otimes \delta_1) = g \otimes \delta_j$ for fixed $j \in \mathbb{N}$. Similarly for $\delta_2, \ldots$

Step 2: Let $J(f \otimes \delta_1) = g \otimes \delta_j$. Define $W : \ell_{p^*} \to \ell_{p^*}, W\delta_1 = \delta_j$, W it permutes the basis (δ_j), so W is an onto and $W\delta_1 = \delta_{\varphi(j)}$, where φ is a permutation on the set of natural numbers $\mathbb{N}$. Since J is an isometric, then W is an isometric. Similarly for $\delta_2, \ldots$

Step 3: Assume $f \otimes \delta_1 + f \otimes \delta_2 = f \otimes (\delta_1 + \delta_2)$ be a basic atom (1-rank operator). Let $J(f \otimes \delta_1) = g \otimes \delta_{\varphi(1)}$ and $J(f \otimes \delta_2) = h \otimes \delta_{\varphi(2)}$, such that $g \neq h$. Then $J(f \otimes (\delta_1 + \delta_2)) = g \otimes \delta_{\varphi(1)} + h \otimes \delta_{\varphi(2)}$, Since J is an isometry, then $Lip(f) = Lip(g) = Lip(h)$, but this implies that either g, h are independent or dependent i.e $g = \pm h$. If g, h are independent, then $J(f \otimes (\delta_1 + \delta_2)) = g \otimes \delta_{\varphi(1)} + h \otimes \delta_{\varphi(2)}$ is two rank operator which is a contradiction, because J preserves rank. Hence $J(f \otimes \delta_1) = a_1 g \otimes \delta_{\varphi(1)}$ and $J(f \otimes \delta_2) = a_2 g \otimes \delta_{\varphi(2)}$, with $|a_j| = 1$. Similarly we prove

$$\Big\{J(f \otimes \delta_j) : j \in \mathbb{N}\Big\} = \Big\{a_j g \otimes \delta_{\varphi(j)} : |a_j| = 1, j \in \mathbb{N}\Big\}.$$

Step 4: Define $S : X^\sharp \to X^\sharp, S(f) = g$, where $J(f \otimes \delta_j) = g \otimes a_j \delta_{\varphi(j)}$ with $|a_j| = 1$. It's easy to check that S is well-defined isometric linear operator. Let $g \in X^\sharp : J(f \otimes \delta_{\varphi^{-1}(j)}) = a_j g \otimes \delta_j$, where $\varphi : \mathbb{N} \longrightarrow \mathbb{N}$ is one-to-one and onto map on the set of natural numbers $\mathbb{N}$. Since J is onto, then for some $g = S(f)$. This implies S is onto.

We show now that $J(T) = \sum_j a_j S f_j \otimes W\delta_j$. We have

$$J(\sum_j f_j \otimes \delta_j) = \sum_j J(f_j \otimes \delta_j) = \sum_j a_j g_j \otimes \delta_{\varphi(j)} = \sum_j a_j S f_j \otimes W\delta_j.$$

2⇒1. Let $T \in \mathcal{SN}_p^L(X, \ell_{p^*})$. Since S, W is an isometry we have

$$J(T) = \sum_j a_j S f_j \otimes W\delta_j = \sum_j a_j g_j \otimes \delta_{\varphi(j)} = \tilde{T},$$

where $Lip(f_j) = Lip(g_j)$, with

$$s\nu_p^L(J(T)) = s\nu_p^L(\tilde{T}) = \Big(\sum_j |\lambda_j|^p\Big)^{\frac{1}{p}} = s\nu_p^L(T).$$

Thus J is an isometry.

Now we show that J is onto, let $T = \sum_j f_j \otimes \delta_j$ be a strongly Lipschitz p-nuclear operator, where $f_j = a_j g_j$, with $|a_j| = 1$, let $\tilde{T} = \sum_j S^{-1} g_j \otimes \delta_{\varphi^{-1}(j)}$, so

$$J(\tilde{T}) = \sum_j a_j g_j \otimes \delta_j = \sum_j f_j \otimes \delta_j = T,$$

this implies that J is onto.

□

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