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Par :

BOUDERBALA Khaoula

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Positive solutions to a class of fractional differential equations with boundary value conditions.

Soutenance publique devant le jury composé de :

Dr.ABDESSELAM Nawel	M.C.B	Université de Laghouat	Président
Dr.MOSTEFAI Mohamed Lamine	M.C.B	Université de Laghouat	Examinateur
Dr.BOUKEHILA Ahcene	M.C.A	Université de Laghouat	Encadreur

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Dedication

I dedicate this work,

To Myself, the one who never gave up, who faced challenges head-on, and who kept believing in the power of resilience and determination. You are stronger than you know, and you deserve every success that comes your way.

To My fatherBOUDERBALA Mahmoude, whose wisdom, support, and sacrifices have always guided me towards success. Your steadfast confidence in my abilities has been my driving force.

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Résumé

L'objectif de cette memoire est l'étude des solutions positives de l'équation différentielle fractionnaire. Le thème de cette recherche concerne donc l'analyse des solutions positives de cette équation. Dans le deuxième chapitre, l'étude est consacrée au problème fractionnaire avec des conditions aux limites. Le dernier problème fractionnaire aborde la dérivée de Hadamard."

mots clés : Équations différentielles fractionnaires , Les Solutions Positives des Équations Différentielles Fractionnaires de Hadamard, Solutions positives d'équations différentielles fractionnaires non linéaires"

ملخص

الهدف من موضوع هدة المذكرة يكمن في البحث عن الحلول الموجبة لانواع خاصة من المعادلات التفاضلية الكسرية حيث انه في المحور الثاني يكون المشكل المدروس بواسط قيم حدية تكاملية ،والمشكل في المحور الثالث يكون بواسطة المشتق الكسري لهادامارد ،نعطي عدة امثلة في كلا الحالتين .

الكلمات المفتاحية: المعادلات التفاضلية الكسرية ،الحلول الموجبة للمعادلات التفاضلية الكسرية الغير الخطية ،المشتق الكسري لهادامارد.

Abstract

The objective of this master's memory is to study positive solutions for fractional differential equations. In the second chapter, the considered problem is subject to integral boundary value conditions (IBVC), and the third one aims to study fractional differential equations (FDEs) with Hadamard derivatives

We provide some examples for each problem.

Keywords: Fractional calculs , Positive solutions of nonlinear fractional differential equations,Positive Solutions of Hadamard F D E

Notation

- $[a,b[$: semi-open interval of $\mathbb{R}$ with end a and b.
- $AC([a,b])$: the space of absolutely continuous functions on $[a,b]$.
- $C^n([a,b]) = \{f : [a,b] \to \mathbb{R}; f$ n-times differentiable and continuous$\}$.
- $AC^n([a,b])$: the space of differentiable and absolutely continuous functions f on $[a,b]$.
- $\Gamma(\cdot)$: Euler Gamma function.
- E_α : the one-parameter Mittag-Leffler function.
- $I_\alpha^a f$: the fractional integral in the sense of Riemann-Liouville of the function f.
- ${}^cD^\alpha$: the fractional derivative in the sense of Caputo of orderα.
- $\mathcal{P}(X)$: the set of subsets such that X is a Banach space $(X, \|\cdot\|)$.

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Introduction

The concept of fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer orders. Fractional calculus appeared in the year 1695 [28, 29, 30], in an exchange of correspondence between L'Hopital and Leibniz, even during the construction of classical differential and integral calculus. In these correspondences, Leibniz urged the possible generalization of the whole-order derivative to an arbitrary order, while L'Hopital questioned him about the special case where the order of the derivative was $\frac{1}{2}$. In the reply letter dated September 30, 1695, Leibniz presented a correct reflection, affirming that very important consequences would come from these developments [28, 29, 30]. This date is regarded as the exact birthday of fractional calculus. Encouraged by this new perspective of fractional calculus application, several authors have developed definitions for fractional derivatives and integrals in subsequent decades, but some of these definitions have contradicted each other. One of these definitions, which emerged in the nineteenth century, is the proposal by Liouville, which was later reformulated by Riemann . More information about the definitions of the Riemann-Liouville derivative and integral can be found in [13, 20, 21]. In this sense, Caputo introduced the so-called Caputo fractional derivative, fundamental in the study of memory effects and also in the modeling of real problems by means of differential equations.

Among other applications that over time were justifying the relevance of the fractional derivative, the emergence of many definitions of fractional derivatives, among which we mention: Hadamard, Weyl, Caputo-Hadamard, Katugampola, Caputo-Katugampola, Caputo-Fabrizio, Hilfer, Hilfer-Hadamard, Hilfer-Katugampola, Jumarie, Erdelyi-Kober, Riesz, Caputo-Riesz, Cassar, Grunwald-Letnikov, each with its respective importance and application [31, 5, 8, 10, 12, 13, 19, 20, 21].

We have three chapters :In the first chapter we give some basic notion for F.C and we gave the fixed points theorem .In the second chapter we establish the existence of positive solution with integral B.V.C to the problem :

$$\begin{cases} {}^cD^\alpha u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = 0, u(1) = \lambda \int_0^1 u(s)ds, \end{cases} \tag{1}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, D^α is the Caputo fractional derivative, and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function. To the best of our knowledge, no paper has considered the existence of positive solutions for nonlinear fractional differential equations with integral boundary conditions. Our purpose here is to give some existence result for positive solutions to this equation. Our proof is based upon known Guo–Krasnoselskii's fixed point theorem given in [32].

In the third chapter we study the following problem with Hadamard derivative :

$$\begin{cases} D^\alpha u(t) = f(t, u(t), u'(t)), & t \in [1, T], \quad T < e, \\ u''(1) = 0, u^{(3)}(1) = 0, \\ u(1) + au'(1) = \int_1^T g_1(s)u(s)ds, \\ u(T) - bu'(T) = \int_1^T g_2(s)u(s)ds, \end{cases} \tag{2}$$

where $3 < \alpha < 4$, $g_1, g_2 \in C([1,T], [0, +\infty))$, $a, b > 0$, and D^α denotes the Hadamard derivative of fractional order α. The following assumptions are needed for the sequel:

(H1) $b \geq a \geq 1 \geq \log T$,

(H2) $f \in C([1,T] \times [0,\infty) \times (-\infty, +\infty), [0, +\infty))$,

(H3) $g_1, g_2 \in C([1,T], [0,\infty))$, $0 \leq \sigma_1 + \sigma_2 < 1$, $\rho = (1 - \sigma_1)(1 - \sigma_4) - \sigma_2\sigma_3 > 0$, where

$$\sigma_1 = \int_1^T \frac{\frac{b}{T} + \log s - \log T}{a + \frac{b}{T} - \log T} g_1(s)ds,$$
$$\sigma_2 = \int_1^T \frac{a - \log s}{a + \frac{b}{T} - \log T} g_1(s)ds,$$
$$\sigma_3 = \int_1^T \frac{\frac{b}{T} + \log s - \log T}{a + \frac{b}{T} - \log T} g_2(s)ds,$$
$$\sigma_4 = \int_1^T \frac{a - \log s}{a + \quad - \log T} g_2(s)ds.$$

Chapter 1

Basic Notions on Fractional calculs

In this chapter, we introduce the necessary concepts for the good understanding of this thesis. We present some fundamental notions, definitions, and lemmas related to fractional calculus, measures of noncompactness, multivalued analysis, Ulam stability and some fixed point theorems which play an important role in the achievement of the desired results in this thesis.

1.1 Functional spaces

Let $J = [a, b]$, the compact interval of $\mathbb{R}$. We present the following functional spaces:

Definition 1.1.

Denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions $f : J \to \mathbb{R}$ endowed with the norm

$$\|f\|_\infty = \sup\{|f(t)| : t \in J\},$$

and $C^n(J, \mathbb{R})$ denotes the class of all real-valued functions defined on J which have a continuous n th order derivative.

Definition 1.2.

Denote by $L^1(J, \mathbb{R})$ the Banach space of measurable functions $f : J \to \mathbb{R}$ that are Lebesgue integrable with norm

$$\|f\|_{L^1} = \int_J |f(t)|\, dt,$$

and by $L^p(J, \mathbb{R})$ we denote the space of Lebesgue integrable functions on J where $|f|^p$ belongs to $L^1(J, \mathbb{R})$ endowed with the norm

$$\|f\|_{L^p} = \left(\int_J |f(t)|^p\, dt \right)^{\frac{1}{p}}.$$

Definition 1.3.

A function $f : J \to \mathbb{R}$ is said to be absolutely continuous on J if for all $\epsilon > 0$ there exists a number δ_ϵ such that for all finite partition $[a_i, b_i]$ in J, then

$$\sum_{i=1}^{p}(b_i - a_i) < \delta_\epsilon \ ,$$

implies that

$$\sum_{i=1}^{p} |f(b_i) - f(a_i)| < \epsilon.$$

Definition 1.4.

Let $AC(J, \mathbb{R})$ be the space of absolutely continuous functions on J. For $n \in \mathbb{N}$, we denote by $AC^n(J, \mathbb{R})$ the space of functions $f : J \to \mathbb{R}$ which have continuous derivatives up to order $n-1$ on J such that $f^{(n-1)}$ belongs to $AC(J, \mathbb{R})$ defined by

$$AC^n(J, \mathbb{R}) = \{f : J \to E \ : \ f, f', f'', \dots, f^{(n-1)} \in C(J, \mathbb{R}) \text{ and } f^{(n-1)} \in AC(J, \mathbb{R})\}.$$

For more details about $AC(J, \mathbb{R})$ and $AC^n(J, \mathbb{R})$.

1.2 Special functions

In what follows, we recall three types of functions that are important in fractional calculus: the Gamma, Beta, and Mittag-Leffler functions. More details about these functions can be found in [13, 20].

1.2.1 Gamma function

Definition 1.5. ***(Gamma function [20])***

The Gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function $n!$, i.e., $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t) dt, \quad \Re(z) > 0.$$

By analytic continuation, the function is extended to the whole complex plane except for the points $0, -1, -2, -3, \dots$, where it has simple poles.

Thus, $\Gamma : \mathbb{C} \setminus \{0, -1, -2, \ldots\} \to \mathbb{C}$*. Some of the most properties are*

$$\Gamma(1) = \Gamma(2) = 1, \quad \Gamma(z+1) = z\Gamma(z),$$

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}, \quad \Gamma\left(\frac{1}{2}\right) = \pi. \tag{1.1}$$

The Gamma function is studied by many mathematicians. There is a long list of well-known properties but in this survey, formulas (1.1)*are sufficient.*

1.2.2 Beta function

Definition 1.6. ***(Beta function [20])***

The Beta function is defined by the integral

$$B(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1}dt, \quad \Re(z) > 0, \ \Re(w) > 0.$$

The functions $\Gamma(\cdot)$ *and* $B(\cdot, \cdot)$ *are related by the formula*

$$B(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

To demonstrate this relationship, we use the Laplace transform, see [13].

1.2.3 Mittag-Leffler function

Definition 1.7. ***(Mittag-Leffler function [20])***

The Mittag-Leffler function in one parameter is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad \alpha > 0, \ z \in \mathbb{C},$$

*where it was introduced by Mittag-Leffler [**?**]. The two-parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)}, \quad \alpha, \beta > 0, \ z \in \mathbb{C},$$

which is of great importance for fractional calculus. In particular,

$$E_{1,1}(z) = \exp^z, \quad E_{2,1}(z) = \cosh(\sqrt{z}), \quad E_{\alpha,1}(z) = E_\alpha(z).$$

1.3 Elements from fractional calculus theory

The fractional calculus is a field of mathematical analysis that embraces the integrals and derivatives of functions of any real or complex order. For the past few decades, this field has been one of the hand-over-fist sprawling fields of mathematics by the virtue of the amazing findings obtained when researchers enrolled the fractional operators in their attempts to construe some problems that arise in nature. See [13, 20, 21]. At the beginning of the fractional calculus in 1695[28] , it consisted of one main integral operator, namely the Riemann-Liouville fractional integral and two fractional derivatives, namely the Riemann-Liouville and Caputo derivatives. Because of the penurious number of operators, researchers were compelled to discover and develop new fractional operators that allow them to better comprehend the world around them. For this purpose, new derivatives and fractional integrals have been arising. The kernel of these integrals and fractional derivatives differs, resulting in a large number of definitions, see [8, 10, 12, 13, 20, 21]. Due to the large number of integral and fractional derivative definitions, it was necessary to create a fractional derivative of a function f with respect to another function, which is called ψ-Riemann-Liouville, using the fractional derivative in the Riemann-Liouville sense, which is given by [13]:

$$^{RL}D_{a+}^{\alpha;\psi} f(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a+}^{(n-\alpha);\psi} f(t).$$

where $\alpha \in (n-1, n)$, $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. However, such a definition only encompasses the possible fractional derivatives that contain the differentiation operator acting on the integral operator. On the other hand, in subsection 1.3.2, we will mention a corresponding fractional integral which generalizes the Riemann-Liouville fractional integrals and some special cases of this operator. In the same way, recently, Almeida [31] using the idea of the fractional derivative in the Caputo sense, proposes a new fractional derivative called ψ-Caputo derivative with respect to another function ψ, which generalizes a class of fractional derivatives, whose definition is given by

$$^{c}D_{a+}^{\alpha;\psi} f(t) = I_{a+}^{(n-\alpha);\psi} f^{[n]}(t),$$

where $\alpha \in (n-1, n)$, $n = \lfloor \alpha \rfloor + 1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. Despite that the ψ-Riemann-Liouville and ψ-Caputo definitions of fractional derivatives are very broad, there is the possibility of proposing a fractional differentiable operator that combines these operators and overcomes the wide range of definitions. Motivated by the Hilfer [27] fractional derivative definition, which includes the classical Riemann-Liouville and Caputo fractional derivatives as special cases. Depending on ψ-Riemann-Liouville and ψ-Caputo fractional derivatives in Hilfer's sense of definition, Sousa and Oliviera [23] introduced a new fractional derivative of a function with respect to another ψ function so-called ψ-Hilfer derivative. Which unify a wide class of fractional derivatives. the definition of ψ-Hilfer fractional derivative, its relation with the ψ-Riemann-Liouville fractional integral and some special cases of this derivative are will be presented in subsection 1.3.4. The advantage of the fractional operator ψ-Hilfer proposed here is the freedom of choice of the classical differentiation operator and the choice of the function ψ, i.e., from the choice of the function ψ, the operator of classical differentiation can act on the fractional integration operator or else the fractional integration operator can act on the classical differentiation operator. As a result, the properties of the two fractional operators mentioned above can be unified and obtained. There are several definitions in fractional calculus that are widely used and important in showing different fractional calculus outcomes. In this section, We will present some definitions of classical fractional integrals and fractional derivatives and their properties. Next, we introduce a new class of fractional integrals and fractional derivatives, because there are so many different fractional operator definitions, the following definition is a special approach when the kernel is unknown, involving a function ψ, making this new operator a generalization of the fractional operators that we use throughout this thesis.

1.3.1 Fractional integrals and fractional derivatives

Let $J = [a, b]$, $(-\infty < a < b < \infty)$, be a finite interval on $\mathbb{R}$. In this subsection, we present some definitions of classical fractional integrals, fractional derivatives, and their properties.

Definition 1.8. ***(Cauchy formula [33])***

The Cauchy formula of the nth integral of a locally integrable function f on $\mathbb{R}^+$ is given by

$$I^n f(t) = \frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)\, ds.$$

Definition 1.9. ***(Riemann-Liouville fractional integral [13])***

For $\alpha > 0$, the left-side (right-side resp.) Riemann-Liouville fractional integral of the function $f \in L^1(J, \mathbb{R})$ of order α is defined by

$${}^{RL}I^{\alpha}_{a+} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} f(s)\, ds, \quad t \in J,$$

$$
{}^{RL}I^{\alpha}_{b-}f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds, \quad t \in J,
$$

resp., where $t \in J$.

Riemann-Liouville fractional derivatives are defined depending on their fractional integrals and integer order derivatives as follows.

Definition 1.10. ***(Riemann-Liouville fractional derivative [13])***

For $\alpha > 0$, the left-side (right-side resp.) Riemann-Liouville fractional order derivative of order α of $f \in L^1(J, \mathbb{R})$, is given by

$$
{}^{RL}D^{\alpha}_{a+}f(t) = \left(\frac{d}{dt}\right)^n \left({}^{RL}I^{n-\alpha}_{a+}f(t)\right) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds,
$$

$$
{}^{RL}D^{\alpha}_{b-}f(t) = \left(-\frac{d}{dt}\right)^n \left({}^{RL}I^{n-\alpha}_{b-}f(t)\right) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dt}\right)^n \int_t^b (s-t)^{n-\alpha-1} f(s)\, ds,
$$

resp., where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

Definition 1.11. ***(Caputo fractional derivative [13])***

For $\alpha > 0$, the left-side (right-side resp.) of Caputo fractional order derivative of order α of $f \in AC^n(J, \mathbb{R})$, is defined by

$$
{}^{C}D^{\alpha}_{a+}f(t) = {}^{RL}I^{n-\alpha}_{a+}\left(f^{(n)}(t)\right) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \ t \in J,
$$

$$
{}^{C}D^{\alpha}_{b-}f(t) = {}^{RL}I^{n-\alpha}_{b-}\left(f^{(n)}(t)\right) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s)\, ds, \ t \in J,
$$

resp., where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

Lemma 1.1. ***(Relation between Riemann-Liouville and Caputo derivatives [13])***

Let $\alpha \in (n-1, n]$. If the function $f \in C^n(J)$, then

$$
{}^{C}D^{\alpha}_{a+}f(t) = {}^{RL}D^{\alpha}_{a+}f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}.
$$

Lemma 1.2. ***([13])***

For $\alpha, \beta > 0$ and $f \in L^1(J)$, we have

1) *The integral operator ${}^{RL}I^{\alpha}_{a+}$ is linear,*

2) ${}^{RL}I^{\alpha}_{a+}\, {}^{RL}I^{\beta}_{a+}\, f(t) = {}^{RL}I^{\beta}_{a+}\, {}^{RL}I^{\alpha}_{a+} f(t) = {}^{RL}I^{\alpha+\beta}_{a+}$,

3) ${}^{RL}D^{\alpha}_{a+}\, {}^{RL}I^{\alpha}_{a+} f(t) = f(t)$,

4) ${}^{RL}D_{a+}^{\beta}\,{}^{RL}I_{a+}^{\alpha}f(t) = {}^{RL}I_{a+}^{\alpha-\beta}f(t)$.

Lemma 1.3. *() [13])*

For $\alpha \geq 0$ and $\beta > 0$, we have

$$\left({}^{RL}I_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1}, \quad \alpha > 0,$$

$$\left({}^{RL}D_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1}, \quad \alpha \geq 0.$$

Lemma 1.4. *([13])*

Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, and $f \in C(J)$. Then, the Riemann-Liouville fractional differential equation

$${}^{RL}D_{a+}^{\alpha}f(t) = 0,$$

has a general solution

$$f(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + c_3(t-a)^{\alpha-3} + \ldots + c_n(t-a)^{\alpha-n}, c_i \in \mathbb{R}\ i = 1, 2, \ldots, n,$$

From the above lemma, it follows that

$${}^{RL}I_{a+}^{\alpha}\,{}^{RL}D_{a+}^{\alpha}f(t) = f(t) - c_1(t-a)^{\alpha-1} - c_2(t-a)^{\alpha-2} - c_3(t-a)^{\alpha-3} - \ldots - c_n(t-a)^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n$.

Lemma 1.5. *([13])*

Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. If $f \in AC^n(J)$, then the Caputo fractional differential equation

$${}^{c}D_{a+}^{\alpha}f(t) = 0,$$

has a general solution

$$f(t) = c_0 + c_1(t-a) + c_2(t-a)^2 + \ldots + c_{n-1}(t-a)^{n-1},$$

From the above lemma, it follows that

$${}^{RL}I_{0+}^{\alpha}\,{}^{C}D_{0+}^{\alpha}f(t) = f(t) - c_0 - c_1(t-a) - c_2(t-a)^2 - \ldots - c_{n-1}(t-a)^{n-1},$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1$.

Definition 1.12. *(Hadamard fractional integral [13])*

Let $a > 0$. The Hadamard fractional integral of order $\alpha > 0$ for a function $f \in L^1(J, \mathbb{R})$ is defined as

$$^{H}\mathfrak{J}^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{\alpha-1} f(s)\frac{ds}{s}, \quad t \in J.$$

Set $\delta = \left(t\frac{d}{dt}\right)$, $a, \alpha > 0$, $n = [\alpha] + 1$, where $[\alpha]$ denotes the integer part of α. Define the space

$$AC^n_\delta(J, \mathbb{R}) = \{f : J \to \mathbb{R} : \delta^{n-1}f(t) \in AC(J, \mathbb{R})\}.$$

Definition 1.13. *(Hadamard fractional derivative [13])*

Let $a > 0$. The Hadamard fractional derivative of order $\alpha > 0$ for a function $f \in AC^n_\delta(J, \mathbb{R})$ is defined as

$$^{H}\mathfrak{D}^{\alpha}_{a+}f(t) = \delta^n\left({}^{H}\mathfrak{J}^{n-\alpha}_{a+}f\right)(t) = \frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{s}\right)^{n-a-1} f(s)\frac{ds}{s}..$$

Definition 1.14. *(Caputo-Hadamard fractional derivative [10, 13])*

Let $a > 0$. The Caputo-Hadamard fractional derivative of order $\alpha > 0$ for a function $f \in AC^n_\delta(J, \mathbb{R})$ is defined as

$$^{C}_{H}\mathfrak{D}^{\alpha}_{a+}f(t) = \left({}^{H}\mathfrak{J}^{n-\alpha}_{a+}\delta^n f\right)(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{n-a-1} \delta^n f(s)\frac{ds}{s}.$$

Lemma 1.6. *([10, 13])*

Let $\alpha, \beta > 0$ and $n = [\alpha] + 1$. Then, we have

1) *The integral operator $^{a+}_{\alpha}$ is linear,*

2) $^{H}\mathfrak{J}^{\alpha}_{a+}(\log t)^{\beta-1}(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log\frac{x}{a})^{\beta+\alpha-1}$,

3) $^{C}_{H}\mathfrak{D}^{\alpha}_{a+}(\log t)^{\beta-1}(x) = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\log\frac{x}{a})^{\beta-\alpha-1}$, $\beta > n$,

4) $^{C}_{H}\mathfrak{D}^{\alpha}_{a+}(\log t)^{k} = 0$, $k = 0, 1, \ldots, n-1$.

Lemma 1.7. *([13])*

Let $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, the general solution of the fractional differential equation

$$^{H}\mathfrak{D}^{\alpha}_{a+}f(t) = 0$$

is given by

$$f(t) = \sum_{k=1}^{n} c_k \left(\log\frac{t}{a}\right)^{\alpha-k},$$

where $c_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$ are arbitrary constants.

From the above lemma, it follows that

$$ {}^{H}\mathfrak{J}^{\alpha}_{a+} \, {}^{H}\mathfrak{D}^{\alpha}_{a+} f(t) = f(t) - \sum_{k=1}^{n} c_k \left(\log \frac{t}{a} \right)^{\alpha - k}, $$

for some $c_k \in \mathbb{R}$, $k = 1, 2, \ldots, n$ are arbitrary constants.

Lemma 1.8. *([10, 13])*

Let $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$. If $f \in AC^n_\delta(J, \mathbb{R})$, then the Caputo-Hadamard fractional differential equation

$$ {}^{C}_{H}\mathfrak{D}^{\alpha}_{a+} f(t) = 0, $$

has a solution

$$ f(t) = \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a} \right)^{k}, $$

and the following formula holds

$$ {}^{H}\mathfrak{J}^{\alpha}_{a+} \left({}^{C}_{H}\mathfrak{D}^{\alpha}_{a+} f(t) \right) = f(t) - \sum_{k=0}^{n-1} c_k \left(\log \frac{t}{a} \right)^{k}, $$

where $c_k \in \mathbb{R}$, $k = 0, 1, 2, \ldots, n - 1$.

1.3.2 Fractional ψ-integral

Definition 1.15. *([23])*

Let (a, b), $(-\infty \leq a < b \leq \infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $\alpha > 0$. Also let $\psi(t)$ be an increasing and positive monotone function on (a, b), having a continuous derivative $\psi'(t)$ on (a, b). The left-sided fractional integral of a function f with respect to another function ψ on $[a, b]$ is defined by

$$ I^{\alpha;\psi}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds. \tag{1.2} $$

Lemma 1.9. *([23])*

Let $\alpha, \beta, \delta > 0$. Then, the left-sided ψ-fractional integral satisfies the following properties:

1) *The integral operator $I^{\alpha;\psi}_{a+}$ is linear,*

2) *The semigroup property of the fractional integration operator $I^{\alpha;\psi}_{a+}$ is given by the following*

result:

$$I_{a+}^{\alpha;\psi} I_{a+}^{\beta;\psi} f(t) = I_{a+}^{\alpha+\beta;\psi} f(t),$$

holds almost everywhere if $f \in L^1(J, \mathbb{R})$.

3) *Commutativity:*

$$I_{a+}^{\alpha;\psi} \left(I_{a+}^{\beta;\psi} f(t) \right) = I_{a+}^{\beta;\psi} \left(I_{a+}^{\alpha;\psi} f(t) \right).$$

Lemma 1.10. *([23])*

Let $\alpha, \beta > 0$. *Then*

$$I_{a+}^{\alpha;\psi} \left(\psi(t) - \psi(a)\right)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (\psi(t) - \psi(a))^{\alpha+\beta-1}.$$

The fractional integral operator with respect to another function defined in (3.1) is a general operator, in the sense that it is enough to choose a function ψ *and obtain an existing fractional integral operator. In the following, we present a class of fractional integrals, based on the choice of the arbitrary* ψ *function.*

1. *Choosing* $\psi(t) = t$ *and replacing in equation (3.1), we get*

$$I_{a+}^{\alpha;t} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) ds =^{RL} I_{a+}^{\alpha} f(t),$$

   the Riemann-Liouville fractional integral.

2. *If we consider* $\psi(t) = \log(t)$ *and* $a > 0$ *in equation (3.1), we have*

$$\begin{aligned} I_{a+}^{\alpha;t} h(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t \frac{1}{s} \left(\log t - \log s\right)^{\alpha-1} f(s) ds \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{a-1} f(s) \frac{ds}{s} =^{H} \mathfrak{J}_{a+}^{\alpha} f(t), \end{aligned}$$

   the Hadamard fractional integral.

3. *Choosing* $\psi(t) = t^{\delta}$ *and* $g(t) = t^{\alpha\eta} f(t)$ *and substituting in equation (3.1), we get*

$$\begin{aligned} t^{-\delta(\alpha+\eta)} I_{a+}^{\alpha;t^{\delta}} g(t) &= t^{-\delta(\alpha+\eta)} I_{a+}^{\alpha;t^{\delta}} t^{\alpha\eta} f(t) \\ &= \frac{\delta t^{\alpha+\eta}}{\Gamma(\alpha)} \int_a ts^{\alpha\eta+\delta-1} \left(t^{\delta} - s^{\delta}\right)^{\alpha-1} f(s) ds \\ &= {}^{Ek} I_{a+,\delta}^{\eta,\alpha} f(t), \end{aligned}$$

the Erdélyi-Kober fractional integral.

1.3.3 Fractional ψ-derivative

We start by evoking two definitions of fractional derivatives with respect to another function, both of which are motivated by the fractional derivatives of Riemann-Liouville and Caputo, in that order, choosing a specific function ψ.

Definition 1.16. *[13]*

Let $\psi'(t) \neq 0$ $(-\infty \leq a < t < b \leq \infty)$ and $\alpha > 0$, $n \in \mathbb{N}$. The Riemann-Liouville derivative of a function f with respect to ψ of order α correspondent to the Riemann-Liouville, is defined by

$$\begin{aligned} {}^{RL}D_{a+}^{\alpha;\psi} f(t) &= \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a+}^{(n-\alpha);\psi} f(t), \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \int_a^t \psi'(s)\left(\psi(t)-\psi(s)\right)^{n-\alpha-1} f(s)ds, \end{aligned}$$

where $n = [\alpha]+1$ and $[\alpha]$ denotes the integer part of real number.

Definition 1.17. *[31]*

Let $\alpha > 0$, $n \in \mathbb{N}$, $J = [a,b]$ is the interval $-\infty \leq a < b \leq \infty$, $f, \psi \in C^n(J,\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t) \neq 0$, for any $t \in J$. The left-sided ψ-Caputo fractional derivative of a function f of order α is given by

$${}^{C}D_{a+}^{\alpha;\psi} f(t) = I_{a+}^{(n-\alpha);\psi,}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n f(t).$$

Lemma 1.11. *([11, 21])*

Let $\alpha, \beta > 0$. Then

$${}^{C}D_{a+}^{\alpha;\psi}(\psi(t)-\psi(a))^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(a))^{\alpha+\beta-1}.$$

Lemma 1.12. *([23])*

If $f \in C^n(J,\mathbb{R})$ and $\alpha \in (n-1,n)$, then

$$I_{a+}^{\alpha;\psi}\, {}^{C}D_{a+}^{\alpha;\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{[n]}(a+)}{k!}(\psi(t)-\psi(a))^k.$$

In particular, given $\alpha \in (0,1)$, we have

$$I_{a+}^{\alpha;\psi}\, {}^{C}D_{a+}^{\alpha;\psi} f(t) = f(t) - f(a).$$

1.3.4 Fractional ψ-Hilfer derivative

From the definition of fractional derivative in the Riemann-Liouville sense and the Caputo sense [13], was introduced the Hilfer fractional derivative [27], which combines both derivatives. Motivated by the definition of Hilfer, we present a new generalized operator, the so-called ψ-Hilfer fractional derivative of a function f with respect to another function. From the fractional derivative ψ-Hilfer, we introduce some relations between the ψ-fractional integral and the fractional derivative ψ-Hilfer.

Definition 1.18. *[23]*

Let $\alpha \in (n-1, n)$ with $n \in \mathbb{N}$, $J = [a,b]$ is the interval $-\infty \leq a < b \leq \infty$, $f, \psi \in C^n(J, \mathbb{R})$ two functions such that ψ is increasing and $\psi'(t) \neq 0$, for any $t \in J$. The left-sided ψ-Hilfer fractional derivative ${}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}(.)$ of function f of order α and type $\beta \in [0,1]$ is defined by

$$ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} f(t) = I_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I_{a+}^{(1-\beta)(n-\alpha);\psi} f(t), \quad n \in \mathbb{N}. \tag{1.3}$$

Lemma 1.13. *[23]*

For $\delta > 0$, $\alpha \in (n-1, n)$, and $\beta \in [0,1]$, we have

$$ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} (\psi(t) - \psi(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta - \alpha)} (\psi(t) - \psi(a))^{\delta-\alpha-1}, \quad \delta > n.$$

Lemma 1.14. *[23]*

In particular, given $n \leq k \in \mathbb{N}$ and as $\delta > n$ we have

$$ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} (\psi(t) - \psi(a))^{k} = \frac{k!}{\Gamma(k+1-\alpha)} (\psi(t) - \psi(a))^{k-\alpha}.$$

On the other hand, for $n > k \in \mathbb{N}_0$, we have

$$ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} (\psi(t) - \psi(a))^{k} = 0.$$

Lemma 1.15. *[23]*

Let $n-1 < \alpha < n$, $\beta \in [0,1]$ and $\gamma = \alpha + \beta(n-\alpha)$. If $f \in C^n(J, \mathbb{R})$, then

1) $I_{a+}^{\alpha;\psi}\ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}f(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(a)$ *where* $f_{\psi}^{[n-k]}f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n-k} f(t),$

2) ${}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi}\ I_{a+}^{\alpha;\psi}f(t) = f(t).$

In the following, using the ψ-Hilfer fractional derivative operator defined in equation (1.3), we can combine in this derivative a different types of fractional derivatives by changing the value for ψ and taking the limit of the parameter β. Some of them are presented below.

1) Consider $\psi(t) = t$ and taking the limit $\beta \to 1$ on both sides of equation (1.3), we get

$$\begin{aligned} {}^{\mathcal{H}}D_{a+}^{\alpha,1;t}f(t) &= I_{a+}^{(n-\alpha);t}\left(\frac{d}{dt}\right)^{n} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}\left(\frac{d}{dt}\right)^{n} f(s)ds \\ &=^{C} D_{a+}^{\alpha}f(t), \end{aligned}$$

the Caputo fractional derivative.

2) For $\psi(t) = t$ and taking the limit $\beta \to 0$ on both sides of equation (1.3), we have

$$\begin{aligned} {}^{\mathcal{H}}D_{a+}^{\alpha,0;t}f(t) &= \left(\frac{d}{dt}\right)^{n} I_{a+}^{(n-\alpha);t}f(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_a^t (t-s)^{n-\alpha-1} f(s)ds \\ &=^{RL} D_{a+}^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^{n}\int_a^t (t-s)^{n-\alpha-1} f(s)ds, \end{aligned}$$

the Riemann-Liouville fractional derivative.

3) For $\psi(t) = \log t$, $a > 0$ and taking the limit $\beta \to 0$ on both sides of equation (1.3), we have

$$\begin{aligned} D_{a+}^{\alpha,0;\log t}f(t) &= \left(t\frac{d}{dt}\right)^{n} I_{a+}^{(n-\alpha);t}f(t) \\ &= \frac{1}{\Gamma(n-\alpha}\left(t\frac{d}{dt}\right)^{n}\int_a^t \left(\log\frac{t}{s}\right)^{n-a-1} f(s)\frac{ds}{s} \\ &=^{\mathcal{H}} \mathfrak{D}_{\alpha}f(t), \end{aligned}$$

the Hadamard fractional derivative.

4) For $\psi(t) = \log t$, $a > 0$ and taking the limit $\beta \to 1$ on both sides of equation (1.3), we *have*

$$\begin{aligned} {}^{\mathcal{H}}D_{a+}^{\alpha,1;t}f(t) &= I_{a+}^{(n-\alpha);t}\left(t\frac{d}{dt}\right)^n f(t) \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t \left(\log\frac{t}{s}\right)^{n-a-1}\left(s\frac{d}{ds}\right)^n f(s)\frac{ds}{s} \\ &= {}_H^C D_{a+}^{\alpha} f(t), \end{aligned}$$

the Caputo-Hadamard fractional derivative.

5) *For* $\psi(t) = t$ *and replacing in equation* (1.3), *we have*

$$\begin{aligned} {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;t}f(t) &= I_{a+}^{\beta(n-\alpha);t}\left(\frac{d}{dt}\right)^n I_{0+}^{(1-\beta)(n-\alpha);t} f(t) \\ &= {}^{RL}I_{a+}^{\beta(n-\alpha)} D^{[n]\,RL} I_{a+}^{(1-\beta)(n-\alpha)} f(t) \\ &= {}^{\mathcal{H}}D_{a+}^{\alpha,\beta} f(t), \end{aligned}$$

the Hilfer fractional derivative.

1.4 Functional tools

In what follows, we present some concepts of functional analysis that we will use throughout this thesis.

Theorem 1.1. ***(Ascoli-Arzela Theorem [4])***

Let $A \subset C([0,T],\mathbb{R})$. *A is relatively compact (i.e.,* $\overline{A}$ *is compact) if:*

1) *A is uniformly bounded, i.e., there exists* $M > 0$ *such that*

$$|f(t)| \leq M \text{ for every} f \in A \text{ and } t \in [0,T],$$

2) *A is equicontinuous, i.e., for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for each* $t_1, t_2 \in [0,T]$, $|t_1 - t_2| \leq \delta$ *implies* $|f(t_1) - f(t_2)| \leq \epsilon$ *for every* $f \in A$.

Definition 1.19. *[6]*

A map $f : [0,T] \times X \to X$ *is said to be Carathéodory if*

1) $t \mapsto f(t,x)$ *is measurable for each* $x \in X$,

2) $x \mapsto f(t,x)$ is continuous for almost all $t \in [0,T]$.

Moreover, f is called L^1-Carathéodory if for every $\forall \rho > 0$, there exists $\varphi_\rho \in L^1([0,T], \mathbb{R}^+)$ such that

$$|f(t,x)| \leq \varphi_\rho(t), \quad \text{for all } |x| \leq \rho \text{ and for a.e. } t \in [0,T].$$

Lemma 1.16. ***(Standard Gronwall inequality)*** *[9]*

Let $f : [0,T] \to \mathbb{R}^+$ be a real function and w be a nonnegative locally integrable function on $[0,T]$.
Assume that there is a constant $a > 0$ such that for $0 < \alpha < 1$

$$f(t) \leq w(t) + a \int_0^t (t-s)^{\alpha-1} f(s)\, ds.$$

Then, there exists a constant $k = k(\alpha)$ such that

$$f(t) \leq w(t) + ka \int_0^t (t-s)^{\alpha-1} w(s)\, ds.$$

Lemma 1.17. ***(Standard Gronwall inequality) [18]***

Let $f : [1,T] \to [0,\infty)$ be a real function and w be a nonnegative locally integrable function on $[1,T]$. Assume that there is a constant $a > 0$ such that for $0 < \alpha < 1$

$$f(t) \leq w(t) + a \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} f(s)\, \frac{ds}{s}.$$

Then, there exists a constant $k = k(\alpha)$ such that

$$f(t) \leq w(t) + ka \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} w(s)\, \frac{ds}{s}.$$

for every $t \in [1,T]$.

Lemma 1.18. ***(Generalization of Gronwall inequality) [25]***

Let f and g be two integrable functions and h be continuous with domain $[a,b]$. Let $\Psi \in C^1([a,b], \mathbb{R})$ be an increasing function such that $\Psi'(t) \neq 0$, $\forall t \in [a,b]$. Assume that

1) f and g are nonnegative functions,

2) h is nonnegative and nondecreasing.

If
$$f(t) \leq g(t) + h(t) \int_a^t \Psi'(s)(\Psi(t) - \Psi(s))^{\alpha-1} f(s)\, ds,$$
then
$$f(t) \leq g(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[h(t)\Gamma(\alpha)]^k}{\Gamma(k\alpha)} \Psi'(s)(\Psi(t) - \Psi(s))^{k\alpha-1} g(s)\, ds.$$

Lemma 1.19. *[25]*

Under the hypotheses of Lemma 1.18, assume further that $g(t)$ is nondecreasing function for $t \in [a, b]$. Then
$$f(t) \leq g(t) E_\alpha \left(h(t)\Gamma(\alpha)(\Psi(t) - \Psi(s))^\alpha\right).$$

1.5 Background about measures of non-compactness

1.5.1 The general notion of a measure of noncompactness

Firstly, we need to fix the notation. In what follows, (E, d) will be a metric space, and $(X, \|\cdot\|)$ a Banach space. Let Q be a non-empty subset of X, then $\overline{Q}$ and conv$Q$ denote the closure and the closed convex closure of Q, respectively. When Q is a bounded subset, Diam(Q) denotes the diameter of Q. Also, we denote by $\mathfrak{B}_E$ (resp. $\mathfrak{B}_X$) the class of nonempty and bounded subsets of E (resp. of X).

We begin with the following general definition.

Definition 1.20. *[3]*

A mapping $\mu : \mathfrak{B}_E \to \mathbb{R}^+$ will be called a measure of noncompactness in E if it satisfies the following conditions:

1) Regularity: $\mu(Q) = 0$ if, and only if, Q is a precompact set.

2) Invariant under closure: $\mu(Q) = \mu(\overline{Q})$, for all $Q \in \mathfrak{B}_E$.

3) Semi-additivity: $\mu(Q_1 \cup Q_2) = \max\{\mu(Q_1), \mu(Q_2)\}$, for all $Q_1, Q_2 \in \mathfrak{B}_E$.

To have a MNC in a Banach space X we need to add the two following additional properties:

4) Semi-homogeneity: $\mu(\lambda Q) = |\lambda|\mu(Q)$ for $\lambda \in \mathbb{R}$ and $Q \in \mathfrak{B}_X$.

5) Invariant under translations: $\mu(x + Q) = \mu(Q)$, for all $x \in X$ and $Q \in \mathfrak{B}_X$.

Three main and most frequently used measures of noncompactness are: the Kuratowski measure of noncompactness (MNC), the Hausdorff MNC, and the De Blasi Measure of Weak Noncompactness. In this thesis, we are interested in the Kuratowski MNC.

1.5.2 The Kuratowski measure of noncompactness

Now we present some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 1.21. *[33, 16]*

Let (E, d) be a metric vector space and Q be a bounded subset of E. Then the Kuratowski measure of noncompactness (the set-measure of noncompactness, k-measure) of Q, denoted by $\mu_k(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $< \epsilon$, i.e.,

$$\mu_k(Q) = \inf\{\epsilon > 0 : Q \subseteq \cup_{i=1}^{n} S_i,\ S_i \subset E,\ diam(S_i) < \epsilon,\ i = 1, 2, \ldots, n,\ n \in \mathbb{N}\}.$$

This measure of noncompactness satisfies the following properties:

1) *Regularity: $\mu_k(Q) = 0$ if and only if Q is a precompact set.*

2) *Invariant under passage to the closure: $\mu_k(Q) = \mu_k(\overline{Q})$, for all $Q \in \mathfrak{B}_E$.*

3) *Semi-additivity: $\mu_k(Q_1 \cup Q_2) = \max\{\mu_k(Q_1), \mu_k(Q_2)\}$, for all $Q_1, Q_2 \in \mathfrak{B}_E$.*

4) *Monotonicity: $Q_1 \subset Q_2 \Rightarrow \mu_k(Q_1) \leq \mu_k(Q_2)$.*

5) *Algebraic semi-additivity: $\mu_k(Q_1 + Q_2) \leq \mu_k(Q_1) + \mu_k(Q_2)$, for all $Q_1, Q_2 \in \mathfrak{B}_E$.*

6) *Semi-homogeneity: $\mu_k(\lambda Q_1) = |\lambda|\mu_k(Q_1)$, for $\lambda \in \mathbb{R}$ and $Q_1 \in \mathcal{B}_E$.*

7) *Invariant under passage to the convex hull: $\mu_k(convQ) = \mu_k(Q)$.*

8) *$\mu_k(Q_1 \cap Q_2) \leq \min\{\mu_k(Q_1), \mu_k(Q_2)\}$, for all $Q_1, Q_2 \in \mathfrak{B}_E$.*

The following lemma is important in order to attain the desired outcomes in this work.

Lemma 1.20. *[24]*

Let $J=[0,T]$ and D be a bounded, closed, and convex subset of the Banach space $C(J,X)$. Let G be a continuous function on $J\times J$ and f a function from $J\times X\to X$, which satisfies the Carathéodory conditions, and assume there exists $p\in L^1(J,\mathbb{R}^+)$ such that, for each $t\in J$ and each bounded set $B\subset X$, we have

$$\lim_{h\to 0^+}\mu_k(f(J_{t,h}\times B))\le p(t)\mu_k(B), here J_{t,h}=[t-h,t]\cap J.$$

If V is an equicontinuous subset of D, then

$$\mu_k\left\{\int_J G(s,t)f(s,y(s))\,ds : y\in V\right\}\le\int_J\|G(s,t)\|p(s)\mu_k(V(s))\,ds.$$

1.6 Multivalued Analysis

In this section, we introduce some definitions, notations, and preliminary facts for multivalued analysis, which are used throughout this thesis.

Definition 1.22. *[6]*

Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ two Banach spaces. A multivalued function $\mathcal{F}$ (or a set valued map, multivalued map) from X into $\mathcal{P}(Y)$ is a correspondence that associates to each element $x\in X$ a subset $\mathcal{F}(x)$ of Y.

We denote this correspondence by the symbol $\mathcal{F}:X\to\mathcal{P}(Y)$. We define:

1) *The effective domain $Dom\mathcal{F}=\{x\in X:\mathcal{F}(x)\neq\varnothing\}$.*

2) *The graph $Gr(\mathcal{F})=\{(x,y)\in X\times Y : x\in Dom\mathcal{F},\, y\in\mathcal{F}(x)\}$.*

3) *The image of the set $A\in\mathcal{P}(X)$: $\mathcal{F}(A)=\{x\in\cup_{x\in A}\mathcal{F}(x)\}$.*

4) *The inverse image of the set $B\in\mathcal{P}(Y)$: $\mathcal{F}^{-1}(B)=\{x\in X:\mathcal{F}(x)\cap B=\varnothing\}$.*

5) *The multivalued map $\mathcal{F}:X\to\mathcal{P}(Y)$ is convex (closed, compact) valued if $\mathcal{F}(x)$ is convex (closed, compact) for all $x\in X$.*

6) *$\mathcal{F}$ is bounded on bounded sets if $\mathcal{F}(B)=\cup_{x\in B}\mathcal{F}(x)$ is bounded in Y for all bounded set B of X, i.e.,*

$$\sup_{x\in B}\{\sup\{\|y\|:y\in\mathcal{F}(x)\}\}<\infty.$$

7) $\mathcal{F}$ is called upper semi-continuous (u.s.c for short) on X if $\mathcal{F}^{-1}(A)$ is closed in X whenever $A \subset X$ is closed.

8) $\mathcal{F}$ is said to be completely continuous if $\mathcal{F}(B)$ is relatively compact for every bounded subset B of X.

9) A multivalued map $\mathcal{F} : X \to \mathcal{P}_0(Y)$ (where $\mathcal{P}_0(Y) = \{A \in \mathcal{P}(Y) : A \neq \varnothing\}$) is said to be measurable if for every open $U \subset X$ the set $\mathcal{F}^{-1}(U)$ is a measurable set.

10) $\mathcal{F}$ has a fixed point if there exists $x \in X$ such that $x \in \mathcal{F}(x)$. The fixed point set of the multivalued operator $\mathcal{F}$ will be denoted by Fix $\mathcal{F}$.

For each $y \in C([a,b],\mathbb{R})$, the set of selections of $\mathcal{F}$ at point y is defined by

$$S_{\mathcal{F},y} = \left\{ v \in L^1([a,b],\mathbb{R}) : v(t) \in F(t,y) \text{ for a.e. } t \in [a,b] \right\}.$$

In the following, by $\mathcal{P}_p$ we denote the set of all nonempty subsets of X which have the property "p" where "p" will be bounded (b), closed (cl), convex (c), compact (cp) etc. Thus $\mathcal{P}_b(X) = \{A \in \mathcal{P}(X) : A \text{ is bounded}\}$, $\mathcal{P}_{cl}(X) = \{A \in \mathcal{P}(X) : A \text{ is closed}\}$, $\mathcal{P}_{cp}(X) = \{A \in \mathcal{P}(X) : A \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{A \in \mathcal{P}(X) : A \text{ is compact and convex}\}$.

Definition 1.23. *[6]*

A multivalued map $\mathcal{F} : [a,b] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

1) $t \mapsto \mathcal{F}(t,x)$ is measurable for each $x \in \mathbb{R}$,

2) $x \mapsto \mathcal{F}(t,x)$ is upper semi-continuous for almost all $t \in [a,b]$.

Further, a Carathéodory function F is called L^1-Carathéodory if

3) for each $\rho > 0$, there exists $\varphi_\rho \in L^1([a,b],\mathbb{R}^+)$ such that

$$\|\mathcal{F}(t,x)\| = \sup\{|v| : v \in \mathcal{F}(t,x)\} \leq \varphi_\rho(t),$$

for all $\|x\| \leq \rho$ and for a.e. $t \in [a,b]$.

Lemma 1.21. *[14]*

Let (E,d) be a metric space induced from the normed space $(X,\|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\}$ given by

$$H_d(A,B) = \max\left\{ \sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b) \right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ *and* $d(a,B) = \inf_{b\in B} d(a,b)$. *Then* $(\mathcal{P}_{b,cl}(X), H_d)$ *is a metric space.*

Definition 1.24. *[6]*

A multivalued operator $\mathcal{N} : X \to \mathcal{P}_{cl}(X)$ *is called*

(a) γ*-Lipschitz if and only if there exists* $\gamma > 0$ *such that*

$$H_d(N(x), N(y)) \leq \gamma d(x,y) \text{ for each} x, y \in X.$$

(b) A contraction if and only if it is γ*-Lipschitz with* $\gamma < 1$.

Lemma 1.22. *[6]*

If $\mathcal{F} : X \to \mathcal{P}_{cl}(Y)$ *is u.s.c., then* $Gr(F)$ *is a closed subset of* $X \times Y$*, i.e., for every sequence* $\{x_n\}_{n\in\mathbb{N}} \subset X$ *and* $\{y_n\}_{n\in\mathbb{N}} \subset Y$*, if when* $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ *and* $y_n \in \mathcal{F}(x_n)$*, then* $y_* \in \mathcal{F}(x_*)$. *Conversely, if* $\mathcal{F}$ *is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 1.23. *[17]*

Let X *be a separable Banach space. Let* $\mathcal{F} : [a,b] \times X \to \mathcal{P}_{cp,c}(X)$ *be an* L^1*-Carathéodory multivalued map and let* Θ *be a linear continuous mapping from* $L^1([a,b],X)$ *to* $C([a,b],X)$*. Then the operator*

$$\Theta \circ S_{\mathcal{F}} : C([a,b],X) \to \mathcal{P}_{cp,c}(C([a,b],X)),\ x \mapsto (\Theta \circ S_{\mathcal{F}})(x) = \Theta(S_{\mathcal{F},x}),$$

is a closed graph operator in $C([a,b],X) \times C([a,b],X)$.

For more details on multivalued maps and the proof of the known results cited in this section, we refer the interested reader to the books by Deimling [6], Gorniewicz [6], and Hu and Papageorgiou [6]

1.7 Fixed Point Theorems

The theory of fixed points is one of the most powerful tools of modern mathematics, which is used to show the existence and uniqueness of fixed points of various kinds of equations. Throughout this study, we convert the given problem into an equivalent integral equation and then use the appropriate fixed point theorem such that the fixed points obtained are thus the solutions of the given problem. In this section, we collect the fixed point theorems which are used in the proofs of our main results. We start with the definition of a fixed point.

Definition 1.25.

For a mapping Φ of a set E into itself, an element x of E is a fixed point of Φ if $\Phi(x) = x$.

Theorem 1.2. ***(Banach's fixed point theorem [22])***

Let Ω be a non-empty closed subset of a Banach space $(X, \|\cdot\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.

Theorem 1.3. ***(Schauder's fixed point theorem [22])***

Let Ω be a nonempty closed bounded convex subset of a Banach space X and $\Phi : \Omega \to \Omega$ be a continuous compact operator. Then Φ has a fixed point in Ω.

Theorem 1.4. ***(Schaefer's fixed point theorem [22])***

Let X be a Banach space, and $\Phi : X \to X$ is a completely continuous operator. If the set $B_\lambda = \{x \in X : x = \lambda\Phi x, \lambda \in (0, 1)\}$ is bounded, then Φ has a fixed point in X.

Theorem 1.5. ***(Krasnoselskii's fixed point theorem [22])***

Let Ω be a non-empty closed bounded convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that F_1 and F_2 map Ω into X such that

1) $F_1x + F_2y \in \Omega$ for all $x, y \in \Omega$,

2) F_1 is continuous and compact,

3) F_2 is a contraction with constant $l < 1$.

Then there is a $x \in \Omega$ with $F_1x + F_2x = x$.

Theorem 1.6. ***(Mönch's fixed point theorem [26])***

Let Ω be a bounded, closed, and convex subset of the Banach space such that $0 \in \Omega$, and let Φ be a continuous mapping of Ω into itself. If the implication

$$V = \overline{conv}\Phi(V) \text{ or } V = \Phi(V) \cup \{0\} \Rightarrow \mu(V) = 0,$$

holds for every V of Ω, then Φ has a fixed point.

Theorem 1.7. ***(Nonlinear alternative of Kakutani maps [7])***

Let Ω be a closed convex subset of a Banach space X and $\mathcal{U}$ be an open subset of Ω with $0 \in \mathcal{U}$. Suppose that $N : \overline{\mathcal{U}} \to \mathcal{P}_{cp,c}(\Omega)$ is an upper semi-continuous compact map. Then either *(i)* *N has a fixed point in* $\mathcal{U}$*, or*

(ii) *there is an* $x \in \partial\mathcal{U}$ *and* $\mu \in (0,1)$ *with* $x \in \mu N(x)$.

Theorem 1.8. ***(Covitz and Nadler fixed point theorem [6])***

Let (E, d) *be a complete metric space. If* $N : E \to \mathcal{P}_{cl}(E)$ *is a contraction, then* $Fix\ N \neq \emptyset$.

Chapter 2

Positive solutions of nonlinear fractional differential equations with integral B V C.

In this chapter, we investigate the existence of positive solutions of the following nonlinear fractional differential equations with integral boundary value conditions

$$\begin{cases} {}^{c}D^{\alpha}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s)ds, \end{cases} \tag{1}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^{C}\mathrm{D}^{\alpha}$ is the Caputo fractional derivative, and $f : [0,1]\times[0,\infty) \to [0,\infty)$ is a continuous function.

2.1 Preliminaries

For the reader's convenience, we present some necessary definitions from fractional calculus theory and lemmas. For details, see [32].

Definition 2.1.

For a function $f : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order α is defined as:

$${}^{C}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds, \quad n = [\alpha] + 1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2.2.

The Riemann–Liouville fractional integral of order α for a function f is defined as:

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \quad \alpha > 0,$$

provided that such an integral exists.

Definition 2.3.

The Riemann–Liouville fractional derivative of order α for a function f is defined by:

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds, \quad n = [\alpha] + 1,$$

provided that the right-hand side of the previous equation is pointwise defined on $(0, \infty)$.

Lemma 2.1.

Let $\alpha > 0$, then the fractional differential equation

$${}^{C}D^{\alpha} u(t) = 0.$$

has a unique solution given by the expression:

$$u(t) = \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{j!} t^j.$$

Lemma 2.2.

Let $\alpha > 0$, then

$$I^{\alpha}\, {}^{C}D^{\alpha} u(t) = u(t) - \sum_{j=0}^{[\alpha]} \frac{u^{(j)}(0)}{j!} t^j.$$

In the sequel, we obtain the exact expression of the Green's function associated to the following fractional differential equation with integral boundary value condition:

$${}^{C}D\, u(t) + y(t) = 0, \quad 0 < t < 1. \tag{2.1}$$

$$u(0) = u''(0) = 0, \quad u(1) = \lambda \int_0^1 u(s) ds. \tag{2.2}$$

The result is the following.

Theorem 2.1.

Let $2 < \alpha < 3$ and $\lambda \neq 2$. Assume $y \in C[0,1]$, then the problem (2.1)-(2.2)has a unique solution u, given by the expression:

$$u(t) = \int_0^1 G(t,s)y(s)\,ds,$$

where

$$G(t,s) = \begin{cases} \frac{2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)-(2-\lambda)\alpha(t-s)^{\alpha-1}}{(2-\lambda)\Gamma(\alpha+1)}, & 0 \leq s \leq t \leq 1, \\ \frac{2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(2-\lambda)\Gamma(\alpha+1)}, & 0 \leq t \leq s \leq 1. \end{cases} \tag{2.3}$$

Proof 1.

We may apply Lemma 2.2 to reduce (2.1) to an equivalent integral equation

$$u(t) = -I^\alpha y(t) + \textstyle\sum_{j=0}^{2} \frac{u^{(j)}(0)}{j!}t^j = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + \sum_{j=0}^{2} \frac{u^{(j)}(0)}{j!}t^j.$$

Since $u(0) = u''(0) = 0$, we deduce that

$$u(t) = -\textstyle\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + u'(0)t.$$

Finally, the condition $u(1) = \lambda\int_0^1 u(s)\,ds$ implies that

$$u'(0) = \textstyle\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + \lambda\int_0^1 u(s)\,ds.$$

Hence, we have the following form

$$u(t) = -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,ds + \lambda t\int_0^1 u(s)\,ds. \tag{2.4}$$

Let $\int_0^1 u(s)\,ds = A$, then, from the previous equality, we deduce that

$$\begin{aligned} A &= \int_0^1 u(t)\,dt \\ &= -\int_0^1\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,dsdt \quad + \int_0^1\int_0^1 \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)\,dsdt \;+ \int_0^1 \lambda At\,dt \\ &= -\int_0^1 \frac{(1-s)^\alpha}{\alpha\Gamma(\alpha)}y(s)\,ds \;+ \frac{1}{2}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}y(s) \quad ds \;+ \frac{1}{2}\lambda A. \end{aligned} \tag{2.5}$$

So, expression (2.5) implies that

$$A = -\frac{2}{2-\lambda}\int_0^1 \frac{(1-s)^{\alpha}}{\alpha\Gamma(\alpha)} y(s)\, ds + \frac{1}{2-\lambda}\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds.$$

Replacing this value in (2.4), we arrive at the following expression for function u:

$$\begin{aligned} u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + t\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds - \frac{2\lambda}{2-\lambda}\int_0^t \frac{t(1-s)^{\alpha}}{\alpha\Gamma(\alpha)} y(s)\, ds \\ &+ \frac{\lambda}{2-\lambda}\int_0^t \frac{t(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds \\ &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)\, ds + \int_0^1 \frac{2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(2-\lambda)\alpha\Gamma(\alpha)} y(s)\, ds \\ &= \int_0^t \frac{2t(1-s)^{\alpha-1}(\alpha-\lambda-\lambda s)-(2-\lambda)\alpha(t-s)^{\alpha-1}}{((2-\lambda)\Gamma(\alpha+1)} y(s)\, ds \\ &+ \int_t^1 \frac{2t(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{\alpha\Gamma(\alpha)\Gamma(\alpha+1)} y(s)\, ds \\ &= \int_0^1 G(t,s) y(s)\, ds. \end{aligned}$$

This completes the proof. ■

A careful analysis of the Green's function G allows us to deduce the following result:

Lemma 2.3.

Let G be the Green's function related to problem (2.1),(2.2) which is given by the expression (2.3). Then, for all $\alpha \in (2,3)$, the following properties are fulfilled:

1) $G(0,s) = G(t,1) = 0$ *for all* $t, s \in [0,1]$ *and* $\lambda \neq 2$.

2) $G(1,s) = 0$ *for all* $s \in [0,1]$ *if and only if* $\lambda = 0$.

3) $G(1,s) > 0$ *for all* $s \in (0,1)$ *if and only if* $\lambda \in (0,2)$.

4) $G(t,0) > 0$ *for all* $t \in (0,1)$ *if and only if* $\lambda \in [0,2)$.

5) $G(t,s) > 0$ *for all* $t, s \in (0,1)$ *if and only if* $\lambda \in [0,2)$.

6) $G(t,s) \leq \frac{2}{(2-\lambda)\Gamma(\alpha)}$ *for all* $t, s \in [0,1]$ *and* $\lambda \in [0,2)$.

7) $G(t,s)$ *and* $G(t,s)/t$ *are two continuous functions for all* $t, s \in [0,1]$, $2 < \alpha < 3$, *and* $\lambda \neq 2$.

Now, we prove two additional inequalities of the Green's function G. Such properties, together with the previous ones given above, will be of fundamental interest to ensure the existence of solutions of problem (1) that will be proven in the next section.

Lemma 2.4.

Fix $2 < \alpha < 3$ and $0 < \lambda < 2$. Let $G(t,s)$ be the Green's function related to problem (2.1)–(2.2) given by the expression (2.3). Then the following inequalities hold:

$$t\, G(1,s) \le G(t,s) \le \frac{2\alpha}{\lambda(\alpha-2)} G(1,s), \text{ for all } t,s \in (0,1). \tag{2.6}$$

Proof 2.

Assume in a first moment that $0 < t \le s < 1$. In such a case:

$$h(t,s) \equiv \frac{G(t,s)}{G(1,s)} = \frac{2t(\lambda(s-1)+\alpha)}{\lambda(2(s-1)+\alpha)}, \text{ for all } 0 < t \le s < 1.$$

Now, it is immediate to verify the following inequalities:

$$t < \frac{2}{\lambda} t \le h(t,s) \le \frac{2t\alpha}{\lambda(\alpha-2)} \le \frac{2\alpha}{\lambda(\alpha-2)}, \text{ for all } 0 < t \le s < 1.$$

On the contrary, if $0 \le s \le t \le 1$ we have that

$$h(t,s) = \frac{(1-s)^{1-\alpha}\,(2t(1-s)^{\alpha-1}((s-1)\lambda+\alpha) + \alpha(\lambda-2)(t-s)^{\alpha-1})}{\lambda(2(s-1)+\alpha)}, \text{ for all } \ 0 < s \le t < 1.$$

Now differentiating twice the above expression with respect to t, we conclude that

$$\frac{\partial^2 h}{\partial t^2}(t,s) = \frac{(\alpha-2)(\alpha-1)\alpha(\lambda-2)(1-s)^{1-\alpha}(t-s)^{\alpha-3}}{\lambda(2(s-1)+\alpha)} < 0, \quad \text{for all } 0 < s < t < 1.$$

By definition, $h(1,s) = 1$. Moreover:

$$h(s,s) = \frac{2s((s-1)\lambda+\alpha)}{\lambda(2(s-1)+\alpha)} > s, \quad \text{for all } 0 < s < 1.$$

These two previous facts, together with the concave character of function $h(\cdot,s)$ on the interval $[s,1]$, allow us to conclude that :

$$h(t,s) > t, \quad \text{for all } 0 < s < t < 1.$$

Finally, it is not difficult to verify that :

$$h(t,s) < \frac{2\alpha}{\lambda(\alpha-2)}, \quad \textit{for all } 0 < s < t < 1,$$

and the inequalities (2.6) *are fulfilled.* ■

2.2 Main result

This section is devoted to give an existence result for the nonlinear boundary value problem (1). To this end, we define the operator $T : C[0,1] \to C[0,1]$ as

$$Tu(t) := \int_0^1 G(t,s) f(s,u(s))ds, \tag{2.7}$$

with G defined in (2.3).

It is clear, from Theorem (2.1), that the fixed points of operator T coincide with the solutions of problem (1).

Let $E = C[0,1]$ be the Banach space endowed with the usual supremum norm $\|\cdot\|$. Define now the cone $\mathcal{P} \subset E$ as follows:

$$\mathcal{P} = \left\{ u \in E : u(t)/t \in E, \, \frac{u(t)}{t} \geq \frac{t\lambda(\alpha-2)}{2\alpha} \|u(t)\|, \, \forall t \in [0,1] \right\}, \tag{2.8}$$

Set

$$f_0 = \lim_{u \to 0^+} \left\{ \min_{t\in[0,1]} \frac{f(t,u)}{u} \right\} \quad \text{and} \quad f_\infty = \lim_{u\to\infty} \left\{ \max_{t\in[0,1]} \frac{f(t,u)}{u} \right\}.$$

To prove the existence of at least one positive solution of (1), we state the following Guo–Krasnoselskii fixed-point theorem [10]:

Theorem 2.2.

Let E be a Banach space, and let $\mathcal{P} \subset E$ be a cone. Assume that Ω_1, Ω_2 are open and bounded subsets of E with $0 \in \Omega_1 \subset \overline{\Omega_1} \subset \Omega_2$, and let $T : \mathcal{P} \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator such that

i) $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$; *or*

ii) $\|Tu\| \leq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|$, $u \in \mathcal{P} \cap \partial\Omega_2$.

Then operator T has at least one fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Now, we are in a position to prove the main result of this paper.

Theorem 2.3.

Assume that one of the two following conditions is fulfilled:

I) (Sublinear case) $f_0 = \infty$ and $f_\infty = 0$.

II) (Superlinear case) $f_0 = 0$, $f_\infty = \infty$, and there exist $\mu > 0$ and $\theta > 0$ for which $f(t, \kappa x) \geq \mu \kappa^\theta f(t, x)$ for all $\kappa \in (0, 1]$.

Then, for all $\alpha \in (2, 3)$ and $\lambda \in (0, 2)$, the problem (1) has at least one solution that belongs to the cone $\mathcal{P}$ defined in (2.8).

Proof 3.

Firstly, we prove that $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.
From the continuity and the non-negativeness of functions G and f on their domains of definition, we have that if $u \in \mathcal{P}$, then $Tu \in E$ and $Tu(t) \geq 0$ for all $t \in [0, 1]$. Moreover.

$\frac{Tu(t)}{t} = \int_0^1 \frac{G(t,s}{t} f\left(s, u(s)\right) ds,$

*Which is from condition (**??**)in Lemma(2.13), a continuous function for all $u \in E$.*
Let's see that $T \in \mathcal{P} \subset \mathcal{P}$. Take $u \in \mathcal{P}$,then ,for all $t \in [0, 1]$ the following inqualities are satisfied

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s) f(s, u(s)) ds \\
&\geq t \int_0^1 G(1,s) f(s, u(s)) ds \\
&\geq \frac{t\lambda(\alpha - 2)}{2\alpha} \max_{t\in[0,1]} \int_0^1 \{G(t,s)\} f(s, u(s)) ds \\
&\geq \frac{t\lambda(\alpha - 2)}{2\alpha} \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) f((s, u(s)) ds \right\} \\
&= \frac{t\lambda(\alpha - 2)}{2\alpha} \|Tu\| .
\end{aligned}$$

In view of the continuity of functions G and f, the operator $T : \mathcal{P} \to \mathcal{P}$ is continuous. Let $\Omega \subset P$ be bounded, which is to say there exists a positive constant $M > 0$ such that $\|u\|_\infty \leq M$

for all $u \in \Omega$. *Define now*

$$L = \max_{0\leq t\leq 1,\, 0\leq u\leq M} |f(t,u)| + 1.$$

Then, for all $u \in \Omega$, *it is satisfied that*

$$|Tu(t)| = \int_0^1 G(t,s) f(s,u(s))\, ds \leq L \int_0^1 G(t,s)\, ds, \quad \text{for all } \ t \in [0,1], \tag{2.9}$$

that is, the set $T(\Omega)$ *is bounded in* E.

For each $u \in \Omega$, *we have*

$$\begin{aligned}
|(Tu)'(t)| &= \left| -\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} f(s,u(s))\, ds + \int_0^1 \frac{2(1-s)^{\alpha-1}(\alpha-\lambda+\lambda s)}{(2-\lambda)\alpha\Gamma(\alpha)} f(s,u(s))\, ds \right| \\
&\leq \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} |f,u(s)|\, ds + \frac{2}{(2-\lambda)\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1} |f(s,u(s)|\, ds \\
&\leq \frac{L}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2}\, ds + \frac{2\alpha L}{(2-\lambda)\Gamma(\alpha+1)} \int_0^1 (1-s)^{\alpha-1}\, ds \\
&\leq \frac{L}{\Gamma(\alpha)} + \frac{2L}{(2-\lambda)\Gamma(\alpha+1)} := N.
\end{aligned}$$

As consequence , for all $t_1, t_2 \in [0,1]$, *we have*

$$|(Tu)(t_2) - (Tu)(t_1)| \leq \int_{t_1}^{t_2} |(Tu)'(s)| ds \leq N(t_2 - t_1),$$

and the set $T(\Omega)$ *is equicontinuous.*

Now, from the Arzela–Ascoli Theorem we conclude that $\overline{T(\Omega)}$ *is compact, i.e.,* $T : \mathcal{P} \to \mathcal{P}$ *is a completely continuous operator.*

Consider now the first situation:

(i) ***Sublinear case*** *(*$f_0 = \infty$ *and* $f_\infty = 0$*).*

Since $f_0 = \infty$, *then there exists a constant* $\rho_1 > 0$ *such that* $f(t,u) \geq \delta_1 u$ *for all* $0 < u \leq \rho_1$, *where* $\delta_1 > 0$ *satisfies*

$$\delta_1 \frac{\lambda(\alpha-2)}{2\alpha} \max_{t\in[0,1]} \left\{ \int_0^1 sG(t,s)\, ds \right\} \geq 1. \tag{2.10}$$

Take $u \in \mathcal{P}$, *such that* $\|u\| = \rho_1$, *then, from the expression* (2.10), *we deduce the following inequalities:*

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)f(s,u(s))\,ds\right\} \geq \delta_1 \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)u(s)\,ds\right\}\\ &\geq \delta_1\|u\|\frac{\lambda(\alpha-2)}{2\alpha}\max_{t\in[0,1]}\left\{\int_0^1 sG(t,s)\,ds\right\} \geq \|u\|.\end{aligned}$$

Since $f(t,\cdot)$ is a continuous function on $[0,\infty)$, we can define the following function:

$$\tilde{f}(t,u) = \max_{z\in[0,u]} f(t,z).$$

Clearly, $\tilde{f}(t,\cdot)$ is nondecreasing on $[0,\infty)$, moreover, since $f_\infty = 0$ it is obvious that (see [32])

$$\lim_{u\to\infty}\max_{t\in[0,1]}\frac{\tilde{f}(t,u)}{u} = 0.$$

Choose $\delta_2 > 0$ satisfying the following property:

$$\frac{2\delta_2}{(2-\lambda)\Gamma(\alpha)} \leq 1. \tag{2.11}$$

Therefore, there exists a constant $\rho_2 > \rho_1 > 0$ such that $\tilde{f}(t,u) \leq \delta_2 u$ for all $u \geq \rho_2$. Let now $u \in \mathcal{P}$ be such that $\|u\| = \rho_2$, then, from the definition of $\tilde{f}$, Eq. (2.11), and property (6) in Lemma (2.13), we attain the following inequalities:

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)f(s,u(s))\,ds\right\} \leq \max_{t\in[0,1]}\left\{\int_0^1 G(t,s)\tilde{f}(s,\|u\|)\,ds\right\}\\ &\leq \delta_2\|u\|\max_{t\in[0,1]}\left\{\int_0^1 G(t,s)\,ds\right\} \leq \frac{2\delta_2}{(2-\lambda)\Gamma(\alpha)}\|u\| \leq \|u\|.\end{aligned}$$

Thus , by the first part of Guo-Krasnoselskii fixed point theorem , we can conclude that (1) has at least one positive solution .
Consider now the second case (ii).
Let $\delta_2 > 0$ be given as in Eq.(2.11). Since $f(0) = 0$, we have that there exists a constant $r_1 > 0$ such that $f(t,u) \leq \delta_2 u$ for $0 \leq u \leq r_1$.

Take $u \in \mathcal{P}$, such that $\|u\| = r_1$. Then we have

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))\,ds \leq \delta_2\int_0^1 G(t,s)u(s)\,ds \leq \delta_2\|u\|\int_0^1 G(t,s)\,ds \leq \frac{2\delta_2}{(2-\lambda)\Gamma(\alpha)}\|u\| \leq \|u\|. \tag{2.12}$$

Consider now $\delta_3 > 0$ satisfying

$$\mu\delta_3 \frac{\lambda(\alpha-2)}{2\alpha} \max_{t\in[0,1]} \left\{ \int_0^1 s^\theta G(t,s)ds \right\} \geq 1, \tag{2.13}$$

with $\mu > 0$ and $\theta > 0$ the constants given in condition (ii).

The fact that $f_\infty = \infty$ says us that there exists a constant $r_2 > r_1 > 0$ such that $f(t,u) \geq \delta_3 u$ for all $u \geq r_2$. Let now $u \in \mathcal{P}$ be such that $\|u\| = \frac{2\alpha}{\lambda(\alpha-2)} r_2$. As consequence, we have that for all $t > 0$ the following inequality holds:

$$\frac{u(t)}{t} \geq \frac{\lambda(\alpha-2)}{2\alpha}\|u\| = r_2$$

So, condition (ii) gives us the following properties:

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]} \left\{ \int_0^1 G(t,s) g\left(s, u(s)\right) ds \right\} \geq \mu \max_{t\in[0,1]} \left\{ \int_0 1 s^\theta G(t,s) f\left(s, \frac{u(s)}{s}\right) ds \right\} \\ &\geq \mu\delta_3 \max_{t\in[0,1]} \left\{ \int_0^1 s^\theta G(t,s) \frac{u(s)}{s} ds \right\} \geq \mu\delta_3 \frac{\lambda(\alpha-2)}{2\alpha}\|u\| \max_{t\in[0,1]} \left\{ \int_0^1 s^\theta G(t,s) ds \right\} \geq \|u\|.\end{aligned}$$

We notice that the second integral in the previous inequalities is well defined because $u(t)/t$ is a continuous function on$(0,1]$ and$\lim_{t\to 0}, u(t)/t$ exists. Therefore, by the second part of Guo–Krasnoselskii fixed point theorem, we can conclude that (1) has at least one positive solution.

Remark 1.

Notice that condition (ii) in the previous theorem generalizes condition (A3) imposed in [31] for problem (1.2), which is as follows:

$f : [0,\infty) \to (0,\infty)$ is nondecreasing and there exists $\theta \in (0,1)$ such that $f(\kappa x) \geq \kappa^\theta f(x)$ for all $\kappa \in (0,1)$ and $x \in [0,\infty)$.

Notice that, since $\kappa \in (0,1)$ the condition is less restrictive when θ is greater. Moreover, we do not impose any monotonicity assumptions on function f.

Exemple 1.

Consider the fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}u(t) + f(t,u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, & u(1) = \frac{\sin(1)}{1-\cos(1)} \int_0^1 u(s)\,ds, \end{cases}$$

where $2 < \alpha < 3$, $0 < \lambda = \frac{\sin(1)}{1-\cos(1)} < 2$, *and*

$$f(t,u(t)) = u^{\frac{1}{2}}(t) + \sin^2\left(te^{u(t)}\right) + \log(1 + u(t)).$$

Obviously, $f_0 = \infty$ *and* $f_\infty = 0$*. Thus, by the first part of Theorem* (1.1)*, we can get that the problem* (1)*has at least one positive solution.*

Remark 2.

In Example (1)*, fractional order* α *could be any constant which satisfies* $2 < \alpha < 3$*. For example, we can take* $\alpha = 2.5$*.*

Chapter 3

Existence of Positive Solutions of Hadamard Fractional Differential Equations with Integral Boundary Conditions

In this chapter ,We investigate the existence of multiple positive solutions of the problem

$$\begin{cases} D^{\alpha}u(t) = f(t, u(t), u'(t)), & t \in [1, T], \quad T < e, \\ \quad u''(1) = 0, u^{(3)}(1) = 0, & \\ \\ u(1) + au'(1) = \int_1^T g_1(s)u(s)ds, & \\ \\ u(T) - bu'(T) = \int_1^T g_2(s)u(s)ds, & \end{cases} \tag{3.1}$$

where $3 < \alpha < 4, g_1,\ g_2 \in C([1, T], [0, +\infty)), a, b > 0$, and D^{α} denotes the Hadamard derivative of fractional order α. The following assumptions are needed for the sequel:

(H_1) $\quad b \geq a \geq 1 \geq \log T,$

(H_2) $\quad f \in C([1, T] \times [0, \infty) \times (-\infty, +\infty), [0, +\infty)),$

(H_3) $\quad g_1, g_2 \in C([1, T], [0, \infty)), 0 \leq \sigma_1 + \sigma_2 < 1, \rho = (1 - \sigma_1)(1 - \sigma_4) - \sigma_2\sigma_3 > 0,$

where

$$\sigma_1 = \int_1^T \frac{\frac{b}{T} + \log s - \log T}{a + \frac{b}{T} - \log T} g_1(s)ds, \quad \sigma_2 = \int_1^T \frac{a - \log s}{a + \frac{b}{T} - \log T} g_1(s)ds,$$
$$\sigma_3 = \int_1^T \frac{\frac{b}{T} + \log s - \log T}{a + \frac{b}{T} - \log T} g_2(s)ds, \quad \sigma_4 = \int_1^T \frac{a - \log s}{a + \quad - \log T} g_2(s)ds.$$

3.1 Preliminaries

Lemma 3.1. *(see [33])*

Let $q > 0$, then the fractional differential equation

$$D^q u(t) = 0,$$

has the unique solution

$$u(t) = \sum_{k=0}^{\lfloor q \rfloor} \frac{u^{(k)}(1)}{k!} (\log t)^k.$$

Lemma 3.2. *(see [33])*

Let $q > 0$, then

$$I^q D^q u(t) = u(t) - \sum_{k=0}^{\lfloor q \rfloor} \frac{u^{(k)}(1)}{k!} (\log t)^k.$$

Now, we can find the exact expression of the Green's function associated with the fractional-order differential equation with nonlocal boundary value conditions:

$$\begin{cases} D^q u(t) = y(t), \qquad t \in [1, T] \text{ and } 1 \le T \le e, \\ u''(1) = 0, u^{(3)}(1) = 0, \\ u(1) + au'(1) = \int_1^T g_1(s)u(s)ds, \\ u(T) - bu'(T) = \int_1^T g_2(s)u(s)ds. \end{cases} \tag{3.2}$$

Lemma 3.3.

Let $3 < q < 4$. Assume $y \in C([1, T])$ and (H_1) holds, then the problem (3.2) has a solution $u(t)$ given by

$$u(t) = \int_1^T G(t, s)y(s)ds + \int_1^T R(t, s) \int_1^T G(s, \tau)y(\tau)d\tau ds, \tag{3.3}$$

where

$$G(t,s)=\begin{cases}\frac{\left(a+\frac{b}{T}-\log T\right)\left(\log\frac{t}{s}\right)^{q-1}+(\log t-a)\left(\log\frac{T}{s}\right)^{q-1}}{(a+\frac{b}{T}-\log T)\Gamma(q)} & \\ +\frac{b(q-1)(a-\log t)\log(\frac{T}{s})^{q-2}}{(a+\frac{b}{T}-\log T)\Gamma(q)}, & 1\le s\le t\le T,\\ \frac{(\log t-a)\left(\log\frac{T}{s}\right)^{q-1}+b(q-1)(a-\log t)\log\left(\frac{T}{s}\right)^{q-2}}{\left(a+\frac{b}{T}-\log T\right)\Gamma(q)}, & 1\le t\le s\le T,\end{cases} \tag{3.4}$$

and

$$\begin{aligned} R(t,s) &= \frac{\left[(a-\log t)\sigma_3+(\log t+\frac{b}{T}-\log T)(1-\sigma_4)\right]g_1(s)}{\rho(a+\frac{b}{T}-\log T)} \\ &+\frac{\left[(a-\log t)(1-\sigma_1)+(\log t+\frac{b}{T}-\log T)\sigma_2\right]g_2(s)}{\rho(a+\frac{b}{T}-\log T)}. \end{aligned} \tag{3.5}$$

Proof 4.

Applying the result of Lemma (3.2), we get the general solution of problem (3.2)

$$u(t)=\alpha_0+\alpha_1\log t+\frac{1}{\Gamma(q)}\int_t^1(\log T/s)^{q-1}y(s)ds, \tag{3.6}$$

where $\alpha_0,\alpha_1\in\mathbb{R}$ are arbitrary constants. Using boundary conditions yields the system of equations:

$$\begin{cases}\alpha_0+a\alpha_1= & \int_1^T g_1(s)u(s)ds,\\ \alpha_0+(\log T-\frac{b}{T})\alpha_1= & \frac{b}{\Gamma(q-1)}\int_1^T(\log\frac{T}{s})^{q-2}y(s)ds\\ & -\frac{1}{\Gamma(q)}\int_1^T(\log\frac{T}{s})^{q-1}y(s)ds+\int_1^T g_2(s)u(s)ds.\end{cases} \tag{3.7}$$

Solving the system (3.7), we obtain

$$\begin{aligned}\alpha_0 &= \frac{ab}{a+\frac{b}{T}-\log T}\int_1^T\frac{1}{\Gamma(q-1)}(\log\frac{T}{s})^{q-2}y(s)ds\\ &-\frac{a}{a+\frac{b}{T}-\log T}\int_1^T\frac{1}{\Gamma(q)}(\log\frac{T}{s})^{q-1}y(s)ds\\ &+\frac{b}{a+\frac{b}{T}-\log T}\int_1^T g_1(s)u(s)ds\\ &+\frac{a}{a+\frac{b}{T}-\log T}\int_1^T g_2(s)u(s)ds,\end{aligned}$$

and

$$\begin{aligned}\alpha_1 &= \frac{1}{a+\frac{b}{T}-\log T}\int_1^T \frac{1}{\Gamma(q)}(\log\frac{T}{s})^{q-1}y(s)ds \\ &\quad -\frac{b}{a+\frac{b}{T}-\log T}\int_1^T \frac{1}{\Gamma(q-1)}(\log\frac{T}{s})^{q-2}y(s)ds \\ &\quad +\frac{1}{a+\frac{b}{T}-\log T}\int_1^T g_1(s)u(s)ds \\ &\quad -\frac{1}{a+\frac{b}{T}-\log T}\int_1^T g_2(s)u(s)ds.\end{aligned}$$

Consequently,

$$\begin{aligned}u(t) &= \alpha_0+\alpha_1\log t+\frac{1}{\Gamma(q)}\int_1^T\left(\log\frac{T}{s}\right)^{q-1}\frac{y(s)}{s}ds \\ &= \int_1^T G(t,s)y(s)ds+\frac{\frac{b}{T}+\log t-\log T}{a+\frac{b}{T}\log T}\int_1^T g_1(s)u(s)ds \\ &\quad +\frac{a-\log t}{a+\frac{b}{T}-\log T}\int_1^T g_2(s)u(s)ds.\end{aligned}$$

We obtain the exact form of $u(t)$ by solving the following equations

$$(1-\sigma_1)\int_1^T g_1(s)u(s)ds-\sigma_2\int_1^T g_2(s)u(s)ds=\int_1^T g_1(s)\int_1^T G(s,\tau)y(\tau)d\tau ds,$$
$$(1-\sigma_4)\int_1^T g_2(s)u(s)ds-\sigma_3\int_1^T g_1(s)u(s)ds=\int_1^T g_2(s)\int_1^T G(s,\tau)y(\tau)d\tau ds,$$

and

$$\begin{aligned}\int_1^T g_1(s)u(s)ds &= \frac{(1-\sigma_4)\int_1^T g_1(s)\int_1^T G(s,\tau)y(\tau)d\tau ds}{(1-\sigma_1)(1-\sigma_4)-\sigma_2\sigma_3} \\ &\quad +\frac{\sigma_2\int_1^T g_2(s)\int_1^T G(s,\tau)y(\tau)d\tau\, ds}{(1-\sigma_1)(1-\sigma_4)-\sigma_2\sigma_3}, \\ \int_1^T g_2(s)u(s) &= \frac{\sigma_3\int_1^T g_1(s)\int_1^T G(s,\tau)y(\tau)d\tau ds}{(1-\sigma_1)(1-\sigma_4)-\sigma_2\sigma_3} \\ &\quad +\frac{(1-\sigma_1)\int_1^T g_2(s)\int_1^T G(s,\tau)y(\tau)d\tau\, ds}{(1-\sigma_1)(1-\sigma_4)-\sigma_2\sigma_3}.\end{aligned}$$

We find

$$u(t)=\int_1^T G(t,s)y(s)ds+\int_1^T R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds$$

, *where*

$$G(t,s)=\begin{cases}\dfrac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1}+(\log t-a)(\log\frac{T}{s})^{q-1}}{(a+\frac{b}{T}-\log T)\Gamma(q)} & \\ +\dfrac{b(q-1)(a-\log t)\log(\frac{T}{s})^{q-2}}{(a+\frac{b}{T}-\log T)\Gamma(q)}, & 1\le s\le t\le T,\\ \dfrac{(\log t-a)(\log\frac{T}{s})^{q-1}+b(q-1)(a-\log t)\log(\frac{T}{s})^{q-2}}{(a+\frac{b}{T}-\log T)\Gamma(q)}, & 1\le t\le s\le T,\end{cases}$$

$$\begin{aligned}R(t,s)\quad&=\quad\frac{[(a-\log t)\sigma_3+(\log t+\frac{b}{T}-\log T)(1-\sigma_4)]g_1(s)}{\rho(a+\frac{b}{T}-\log T)}\\&\quad+\frac{[(a-\log t)(1-\sigma_1)+(\log t+\frac{b}{T}-\log T)\sigma_2]g_2(s)}{\rho(a+\frac{b}{T}-\log T)}.\end{aligned}$$

Lemma 3.4.

Let $3<q<4$*, and assume (H1) holds. Let* $G(t,s)$ *be the Green function related to problem* (3.2) *given by the expression* (3.4)*, then we have*

$$(1-\log t)G(1,s)\le G(t,s)\le G(1,s \tag{3.8}$$

Proof 5.

Denote $K(t,s)=\frac{G(t,s}{G(1,s}$.

Case 1: $1\le t\le s\le T\le e$

$$\begin{aligned}K(t,s)&=\frac{(\log t-a)(\log\frac{T}{s})^{q-1}+b(q-1)(a-\log t)(\log\frac{T}{s})^{q-2}}{-a(\log\frac{T}{s})^{q-1}+b(q-1)a\log(\frac{T}{s})^{q-2}}\\&=\frac{a-\log t}{a}=1-\frac{\log t}{a};\end{aligned}$$

we have

$$(1-\log t)\le(1-\frac{\log t}{a}\le 1,$$

then,

$$(1-\log t)\le K(t,s)\le 1.$$

Case 2: $\quad 1 \leq s \leq t \leq T \leq e,$

$$K(t,s) = \frac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1} + (\log t - a)(\log\frac{T}{s})^{q-1}}{-a(\log\frac{T}{s})^{q-1} + b(q-1)a(\log\frac{T}{s})^{q-2}} + \frac{b(q-1)(a-\log t)(\log\frac{T}{s})^{q-2}}{-a(\log\frac{T}{s})^{q-1} + b(q-1)a(\log\frac{T}{s})^{q-2}},$$

we can write

$$K(t,s) = \frac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1}}{a(\log\frac{T}{s})^{q-1}\left[b(q-1)(\log\frac{T}{s})^{-1}-1\right]} + 1 - \frac{\log t}{a}.$$

As $1-\frac{\log t}{a} \geq (1-\log t)$, *taking account of assumptions* $b \geq a \geq 1$
and $3 < q < 4$,*it comes that :*

$$\frac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1}}{a(\log\frac{T}{s})^{q-1}\left[b(q-1)(\log\frac{T}{s})^{-1}-1\right]} \; \frac{\log T(\log\frac{t}{s})^{q-1}}{a(\log\frac{T}{s})^{q-1}} \; 0;$$

$$K(t,s) = \frac{(a+\frac{b}{T}-\log T)(\log\frac{t}{s})^{q-1}}{a(\log\frac{T}{s})^{q-1}[b(q-1)(\log\frac{T}{s})^{-1}-1]} + 1 - \frac{\log t}{a} \geq (1-\log t).$$

Now, we differentiate twice $K(t,s)$ *with respect to* t,

$$\begin{aligned}\frac{\partial^2 K(t,s)}{\partial t^2} &= \frac{(a+\frac{b}{T}-\log T)}{a(\log\frac{T}{s})^{q-1}[b(q-1)(\log\frac{T}{s})^{-1}-1]} \\ &\times [(q-1)(q-2)\frac{1}{t^2}(\log\frac{t}{s})^{q-3} - \frac{1}{t^2}(q-1)(\log\frac{t}{s})^{q-2}] + \frac{1}{at^2} \\ &= \frac{1}{t^2}\frac{(a+\frac{b}{T}-\log T)}{a(\log\frac{T}{s})^{q-1}[b(q-1)(\log\frac{T}{s})^{-1}-1]} \\ &\times (q-1)(\log\frac{t}{s})^{q-3}[(q-2)-(\log\frac{t}{s})] + \frac{1}{at^2} \geq 0,\end{aligned}$$

if $1 \leq s \leq t \leq T \leq e$. *Whereupon, the maximum values of* $K(t,s)$ *are at either* $t = s$ *or* $t = T$ *;*

$$
\begin{aligned}
K(s,s) = &\quad 1-\frac{s}{a}\leq 1,\\
K(T,s) = &\quad \frac{(a+\frac{b}{T}-\log T)}{a[b(q-1)(\log\frac{T}{s})^{-1}-1]}+1-\frac{\log T}{a}\\
= &\quad \frac{ab(q-1)-b(q-1)\log T+\frac{b}{T}\log\frac{T}{s}}{ab(q-1)-a\log\frac{T}{s}}\\
\leq &\quad \frac{ab(q-1)-a[(q-1)\log T-\frac{1}{T}\log\frac{T}{s}]}{ab(q-1)-a\log\frac{T}{s}}\\
\leq &\quad \frac{ab(q-1)-a[(q-1)-\log\frac{T}{s}]}{ab(q-1)-a\log\frac{T}{s}}\\
\leq &\quad \frac{ab(q-1)-a\log\frac{T}{s}}{ab(q-1)-a\log\frac{T}{s}}=1,
\end{aligned}
$$

hence, $(1-\log t)\leq K(t,s)\leq 1$ *for* $1\leq s\leq t\leq T\leq e$*. This completes the proof.*

Now, we present some definitions and recall Avery-Peterson fixed point theorem.

Definition 3.1. *([33])*

Let E *be a real Banach space. A nonempty convex closed set* $P\subset E$ *is said to be a* cone *provided that*

(i) $au\in P$ *for all* $u\in P$ *and all* $a\geq 0$,

(ii) $u,-u\in P$ *implies* $u=0$.

Note that every cone $P\subset E$ *includes an ordering in* E *given by* $x\leq y$ *if* $y-x\in P$.

Definition 3.2. *([33])*

The map ϕ *is defined as a nonnegative continuous concave functional on a cone* P *of a real Banach space* E *provided that* $\phi: P\to[0,+\infty)$ *satisfies*

$$\phi(\theta x+(1-\theta)y)\geq\theta\phi(x)+(1-\theta)\phi(y)\quad \text{for all } x,y\in P \text{ and } 0\leq\theta\leq 1.$$

Similarly, we say the map η *is a nonnegative continuous convex functional on a cone* P *of a real Banach space* E *provided that* $\eta: P\to[0,+\infty)$ *satisfies*

$$\eta(\theta x+(1-\theta)y)\leq\theta\eta(x)+(1-\theta)\eta(y)\quad \text{for all } x,y\in P \text{ and } 0\leq\theta\leq 1.$$

Let γ *and* θ *be nonnegative continuous convex functionals on* P, ϕ *be a nonnegative continuous concave functional on* P *and* ψ *be a nonnegative continuous functional on* P*. Then for positive* *real numbers* a', b', c' *and* d', *we define the following convex sets*

$$P(\gamma, d') = \{x \in P \mid \gamma(x) < d'\},$$

$$P(\gamma, \phi, b', d') = \{x \in P \mid b' \leq \phi(x), \gamma(x) \leq d'\},$$

$$P(\gamma, \theta, \phi, b', c', d') = \{x \in P \mid b' \leq \phi(x), \theta(x) \leq c', \gamma(x) \leq d'\},$$

and the closed set

$$R(\gamma, \psi, a', d') = \{x \in P \mid a' \leq \psi(x), \gamma(x) \leq d'\}.$$

Theorem 3.1. *([33])*

Let P *be a cone in a real Banach space* E. *Let* γ *and* θ *be nonnegative continuous convex functionals on* P, φ *be a nonnegative continuous concave functional on* P, *and* ψ *be a nonnegative continuous functional on* P *satisfying* $\psi(\lambda x) \leq \lambda\psi(x)$ *for* $0 \leq \lambda \leq 1$, *such that for some positive numbers* M *and* d',

$$\phi(x) \leq \psi(x) \quad \text{and} \quad \|x\| \leq M\gamma(x).$$

for all $x \in \overline{P(\gamma, d')}$. *Suppose* $\mathcal{A} : \overline{P(\gamma, d')} \to \overline{P(\gamma, d')}$ *is completely continuous and there exist positive numbers* a', b', *and* c' *with* $a' < b'$ *such that*

(S_1) $\{x \in P(\gamma, \theta, \phi, b', c', d') \mid \phi(x) > b'\} \neq \emptyset$ *and*

$$\phi(Ax) > b' \text{ for } x \in P(\gamma, \theta, \phi, b', c', d'),$$

(S_2) $\phi(Ax) > b'$ *for* $x \in P(\gamma, \phi, b')$ d' *with* $\theta(Ax) > c'$,

(S_3) $0 \notin B(\gamma, \psi, a', d')$ *and* $\psi(Ax) < a'$ *for* $x \in R(\gamma, \psi, a', d')$ *with* $\psi(x) = a'$. *Then* A *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\gamma, d')}$ *such that*

$$\gamma(x_i) \leqslant d', \text{ for } i = 1, 2, 3;$$
$$b' < \phi(x_1);$$
$$a' < \psi(x_2) \text{ with } \phi(x_2) < b';$$
$$\psi(x_3) < a'.$$

3.2 Main Results

Let us define the operator $\mathcal{A} : C^1[1,T] \to C^1[1,T]$ as:

$$\begin{aligned} Au(t) = &\int_1^T G(t,s) f(s,u(s),u'(s))\,ds \\ &+ \int_1^T R(t,s) \int_1^T G(s,\tau) f(\tau,u(\tau),u'(\tau))\,d\tau ds. \end{aligned} \tag{3.9}$$

Let the Banach space $E = (C^1[1,T], \|\cdot\|)$ be equipped with the norm

$$\|u\| = \max\left\{ \max_{t\in[1,T]} |u(t)|, \max_{t\in[1,T]} |u'(t)| \right\}.$$

We define the cone $P \subset E$ by

$$P = \{u \in E \mid u(t) \geq 0 \text{ and } u(t) \text{ is convex on } [1,T]\}.$$

And denote a nonnegative continuous concave functional ϕ, the nonnegative continuous convex functional θ, the positive continuous convex functional γ, and the nonnegative continuous functional ψ on the cone P as follows :

$$\gamma(u) = \max_{t\in[1,T]} |u'(t)|, \quad \psi(u) = \theta(u) = \max_{t\in[1,T]} |u(t)|$$

and

$$\phi(u) = \min_{t\in[\delta,T-\delta]} |u(t)| \quad \text{for} \quad \delta \in \left[1, \frac{1+T}{2}\right].$$

Lemma 3.5.

If $u \in P$ and satisfies the boundary conditions $u(1) = -au'(1) + \int_1^T g_1(s)u(s)\,ds$, then

$$\max_{t\in[1,T]} |u(t)| \leq \frac{(T-1)+a}{1-\sigma_1-\sigma_2} \max_{t\in[1,T]} |u'(t)|.$$

Proof 6.

Since $u(t)-u(1)=\int_1^t u'(s)\,ds$*, we get*

$$\begin{aligned}\max_{t\in[1,T]}|u(t)| &\le |u(1)|+(T-1)\max_{t\in[1,T]}|u'(t)|\\ &\le |-au'(1)+\int_1^T g_1(s)u(s)\,ds|+(T-1)\max_{t\in[1,T]}|u'(t)|\\ &\le |au'(1)|+\max_{t\in[1,T]}|u(t)|\int_1^T g_1(s)\,ds+(T-1)\max_{t\in[1,T]}|u'(t)|.\end{aligned}$$

Then,

$$\begin{aligned}\max_{t\in[1,T]}|u(t)|\left[1-\int_1^T g_1(s)\,ds\right] &\le (T-1)\max_{t\in[1,T]}|u'(t)|+a\max_{t\in[1,T]}|u'(t)|\\ &=[(T-1)+a]\max_{t\in[1,T]}|u'(t)|.\end{aligned}$$

Therefore,

$$\max_{t\in[1,T]}|u(t)|\le (T-1)+\frac{a}{1-\sigma_1-\sigma_2}\max_{t\in[1,T]}|u'(t)|.$$

Lemma 3.6.

For $\delta\in\left[1,\frac{1+T}{2}\right]$*, we have*

$$\min_{t\in[\delta,(T+1)-\delta]}R(t,s)>(\log\delta)\max_{t\in[1,T]}R(t,s),$$

where $R(t,s)$ *is defined by* (3.5)*.*

Proof 7.

From lemma (3.13)*, we have*

$$\begin{aligned}R(t,s) &= \frac{[(a-\log t)\sigma_3+\left(\log t+\frac{b}{T}-\log T\right)(1-\sigma_4)]g_1(s)}{\rho\left(a+\frac{b}{T}-\log T\right)}\\ &\quad+\frac{[(a-\log t)(1-\sigma_1)+\left(a+\frac{b}{T}-\log T\right)\sigma_2]g_2(s)}{\rho\left(a+\frac{b}{T}-\log T\right)}\end{aligned}$$

we can write

$$\begin{aligned}R(t,s)=&\frac{(1-\sigma_3-\sigma_4)g_1(s)+(\sigma_1+\sigma_2-1)g_2(s)}{\rho\left(a+\frac{b}{T}-\log T\right)}\log t\\ &+\frac{[a\sigma_3+\left(\frac{b}{T}-\log T\right)(1-\sigma_4)]g_1(s)+[a(1-\sigma_1)+\left(\frac{b}{T}-\log T\right)\sigma_2]g_2(s)}{\rho\left(a+\frac{b}{T-\log T}\right)}.\end{aligned}$$

Denote

$$j(s) = \frac{(1-\sigma_3-\sigma_4)g_1(s) + (\sigma_1+\sigma_2-1)g_2(s)}{\rho\left(a+\frac{b}{T}-\log T\right)},$$

and

$$l(s) = \frac{\left[a\sigma_3 + \left(\frac{b}{T}-\log T\right)\right)(1-\sigma_4)\right] g_1(s) + \left[a(1-\sigma_1) + (b(T-\log T))\,\sigma_2\right] g_2(s)}{\rho\left(a+\frac{b}{T}-\log T\right)}.$$

When $j(s) < 0$, we have tow cases:

Case 1: $0 < (1-\sigma_3-\sigma_4)g_1(s) < (1-\sigma_1-\sigma_2)g_2(s)$, *so if and only if to monotonicity of* $R(t,s), \max_{t\in[1,T]} R(t,s) = R(1,s) = l(s) > 0$, *the minimum value is*

$$R(T,s) = j(s)\log T + l(s) > 0.$$

Then, is holds

$$\begin{aligned}
\frac{\min_{t\in[\delta,(T+1)-\delta]} R(t,s)}{\max_{t\in[1,T]} R(t,s)} &= \frac{R((T+1)-\delta,s)}{R(1,s)} \\
&= \frac{\log((T+1)-\delta)j(s)+l(s)}{l(s)} \\
&= \frac{\log((T+1)-\delta)j(s)}{l(s)}1 > \log\delta.
\end{aligned} \tag{3.10}$$

Case 2: $(1-\sigma_3-\sigma_4)g_1(s) < 0 < (1-\sigma_1-\sigma_2)g_2(s)$. *The minimum value of $R(t,s)$ is $R(T,s)$ and*

$$\begin{aligned}
R(T,s) = &\quad \frac{[(a-\log T)\sigma_3 + \left(\log T + \frac{b}{T} - \log T\right)(1-\sigma_4)]g_1(s)}{\rho(a+\frac{b}{T}-\log T)} \\
&+ \frac{[(a-\log T)(1-\sigma_1) + \left(\log T + \frac{b}{T} - \log T\right)\sigma_2]g_2(s)}{\rho(a+\frac{b}{T}-\log T)} \\
&= \frac{[(a-\log T)\sigma_3 + \frac{b}{T}(1-\sigma_4)]g_1(s) + [(a-\log T)(1-\sigma_1) + \frac{b}{T}\sigma_2]g_2(s)}{\rho(a+\frac{b}{T}-\log T)}.
\end{aligned}$$

Since, $b \geq a \geq 1 \geq \log T$ *and* $1 \leq T \leq e$, *we have* $\frac{b}{T} \geq \frac{1}{e}$*. Then*

$$\begin{aligned}
R(T,s) &\geq \frac{\frac{1}{e}\left[(1-\sigma_4)g_1(s)+\sigma_2 g_2(s)\right]}{\rho\left(a+\frac{b}{T}-\log T\right)} \\
&\geq \frac{\frac{1}{e}\left[(1-\sigma_4)g_1(s)+\sigma_2\frac{1-\sigma_3-\sigma_4}{1-\sigma_1-\sigma_2}g_1(s)\right]}{\rho\left(a+\frac{b}{T}-\log T\right)} \\
&\geq \frac{\frac{1}{e}\left[(1-\sigma_4)(1-\sigma_1-\sigma_2)+\rho\sigma_2(1-\sigma_3-\sigma_4)\right]g_1(s)}{(1-\sigma_1-\sigma_2)\rho\left(a+\frac{b}{T}-\log T\right)} \\
&= \frac{g_1(s)}{e(1-\sigma_1-\sigma_2)\left(a+\frac{b}{T}-\log T\right)} > 0,
\end{aligned}$$

The expression (3.10) *also holds.*

Therefore, for $\delta \in \left[1, \frac{T+1}{2}\right]$, *we get*

$$\min_{t\in[\delta,(T+1)-\delta]} R(t,s) > (\log\delta)\max_{t\in[1,T]} R(t,s).$$

Lemma 3.7.

If $u \in P, \delta \in \left[1, \frac{T+1}{2}\right]$, *then*

$$\min_{t\in[\delta,(T+1)-\delta]} u(t) \geq (\log\delta)\max_{t\in[1,T]} u(t).$$

Proof 8.

$$\begin{aligned}
\min_{t\in[\delta,T+1-\delta]} u(t) &= \min_{t\in[\delta,T+1-\delta]}\left\{\int_1^T G(t,s)y(s)ds + \int_1^T R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds\right\}. \\
&\geq \min_{t\in[\delta,T+1-\delta]}\{(1-\log t)\int_1^T G(1,s)y(s)ds + \int_1^T \min_{t\in[\delta,T+1-\delta]} R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds \\
&\geq (1-\log(T+1-\delta))\int_1^T \max_{t\in[1,T]} G(t,s)y(s)ds \\
&+ (\log\delta)\int_1^T \max_{t\in[1,T]} R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds \\
&\geq (\log\delta)\int_1^T \max_{t\in[1,T]} G(t,s)y(s)ds + (\log\delta)\int_1^T \max_{t\in[1,T]} R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds \\
&\geq (\log\delta)\max_{t\in[1,T]}\left\{\int_1^T G(t,s)y(s)ds + \int_1^T R(t,s)\int_1^T G(s,\tau)y(\tau)d\tau ds\right\} \\
&= (\log\delta)\max_{t\in[1,T]} u(t).
\end{aligned}$$

Hence,

$$\min_{t\in[\delta,(T+1)-\delta]} u(t) \geq (\log\delta)\max_{t\in[1,T]} u(t).$$

Lemma 3.8.

$\mathcal{A}: P \to P$ is completely continuous.

Proof 9.

From the continuity and the non-negativeness of functions G and f on their domains of definition.

We know that if $u \in P$, then $\mathcal{A}u \in E$ and $\mathcal{A}u(t) \geq 0$ for all $t \in [1, T]$. Take $u \in P$, then

$$(\mathcal{A}u)''(t) = \int_1^t \frac{\partial^2 G(t,s)}{\partial t^2} f(s, u(s)), u'(s))ds \tag{3.11}$$

$$+ \int_1^t \frac{\partial^2 R(t,s)}{\partial t^2} \int_1^t G(s,\tau) f(\tau, u(\tau) u'(\tau)) d\tau ds \tag{3.12}$$

Let us set

$$M_1 = \max\left\{\left|\frac{\partial R(t,s)}{\partial u}\right| t, s \in [1,T]\right\}, M_2 = \max\{|R(t,s)| t, s \in [1,T]\},$$
$$K = \{A + M_1 B\},$$

where

$$\begin{cases} = \dfrac{(a + b/T - \log T)\left(T^{\alpha-1} - 1 + \frac{\alpha-1}{3-\alpha}(1 - T^{\alpha-3})\right)}{a + b/T - \log T} \\ + \dfrac{\frac{1}{2-\alpha}(T - T^{\alpha-1}) - b(\alpha-1)\left[(T - T^{\alpha-2}) + (1 - T^{\alpha-3})\right]}{a + \frac{b}{T} - \log T} \\ = \dfrac{\frac{1}{\alpha}(a + b/T - \log T)(T^{\alpha+1} - T) - \frac{a}{2-\alpha}(T^2 - T^\alpha) + \frac{ab(\alpha-1)}{3-\alpha}(T - T^{\alpha-2})}{a + b/T - \log T} \end{cases}$$

$$L = \{(1-\delta)) + (\log\delta) M [2(T-1) - T\log T]\}(T-1)$$
$$\times \frac{-a(\log T)^{\alpha-1} + ab/T.(\alpha-1)(\log T)^{\alpha-2}}{[a - \log T + b/T].\Gamma(\alpha)},$$
$$N = \frac{\frac{1}{\alpha}(a + b/T - \log T)(T^\alpha - 1) - \frac{aT^{\alpha-1}}{2-\alpha}(T^{2-\alpha} - 1) + \frac{ab(\alpha-1)}{3-\alpha}(T^{\alpha-5} - T^{2-\alpha})}{a + b/T - \log T}.$$

Theorem 3.2.

Assume there exist constants $0 < a' < b' < c' < d'$, where $c' = \delta b'$, and suppose the function $f(t, u(t), u'(t))$ satisfies

$$(A1) f(t,u,v) \le d'/K, \ for\ (t,u,v) \in [1,T] \times \left[0, \frac{a+T-1}{1-\sigma_1-\sigma_2}d'\right] \times [-d',d'],$$

$$(A2) f(t,u,v) \ge b'/L, \ for\ (t,u,v) \in [\delta, 1+T-\delta] \times [b', \delta b'] \times [-d',d'],$$

$$(A3) f(t,u,v) < a'/N, \ for\ (t,u,v) \in [1,T] \times [0,a'] \times [-d',d'].$$

Then the boundary value problem (3.1) has at least three solutions u_1, u_2, and u_3 satisfying

$$\begin{aligned} \max_{t\in[1,T]} |u_i'(t)| \le & \quad d', for \quad i = 1,2,3; \\ b' < & \quad \min_{t\in[\delta,T+1-\delta]} |u_1(t)|; \\ a' < & \quad \max_{t\in[1,T]} |u_2(t)| \quad with \quad \min_{t\in[\delta;T+1-\delta]} |u_2(t)| < b'; \\ \max_{t\in[1,T]} |u_3(t)| < a'. \end{aligned}$$

Proof 10.

The BVP (3.1) has a solution $u = u(t)$ if and only if u is the solution of the operator equation

$$\begin{aligned} u \quad = \quad & \mathcal{A}u(t) = \int_1^T G(t,s) f(s, u(s), u'(s)) ds \\ & + \int_1^T R(t,s) \int_1^T G(s,\tau) f(\tau, u(\tau), u'(\tau)) d\tau ds. \end{aligned}$$

Let us assume that the operator T satisfies the conditions in the Avery-Peterson fixed point theorem.
If $u \in \overline{P(\gamma, d')}$, then $\gamma(u) = \max_{t\in[1,T]} |u'(t)| \le d'$. With Lemma (3.5),
we get

$$\max_{t\in[1,T]} |u(t)| \le \frac{a+T-1}{1-\sigma_1-\sigma_2} \max_{t\in[1,T]} |u'(t)| \le \frac{a+T-1}{1-\sigma_1-\sigma_2} d'.$$

With the condition $u \in P$ and its non-negativeness on its domain, and because $\mathcal{A}(u) \in P$ with $\mathcal{A}(u) \ge 0$ and $\mathcal{A}(u)$ is convex on $[1,T]$, the maximum value of $|\mathcal{A}(u)'(t)|$ is either $|\mathcal{A}(u)'(1)|$ or

$|\mathcal{A}(u)'(T)|$. *Combining with the assumption (A1), we have*

$$\begin{aligned}
\gamma(\mathcal{A}u(t)) &= \max_{t\in[1,T]} |(\mathcal{A}u)'(t)| \\
&= \max\{|(\mathcal{A}u)'(1)|, |(\mathcal{A}u)'(T)|\} \\
&\leq \int_1^T \left| \frac{(a+b\backslash T-\log T)(\alpha-1)(-\log s)^{\alpha-2} + (\log T/s)^{\alpha-2}\left[\log\frac{T}{s} - b(\alpha-1)\right]}{(a+b\backslash T-\log T)\Gamma(\alpha)} \right| \\
&\times f(s,u(s),u'(s))ds| \\
&+ \int_1^T \frac{(a+b\backslash T-\log T)(\alpha-1) + (\log\frac{T}{s}) - b(\alpha-1)]1/T.(\log T/s)^{\alpha-2}}{(a+b\backslash T-\log T)\Gamma(\alpha)} \\
&\times f(s,u(s),u'(s))ds \\
&+ \left|\int_1^T \frac{\partial R(t,s)}{\partial t}\right| \left|\int_1^T G(s,\tau) f(\tau,u(\tau),u'(\tau))d\tau ds\right| \\
&\leq \frac{d'}{K}\left[\frac{(a+b/T-\log T)(T^{\alpha-1}-1) + \frac{1}{2-\alpha}(T-T^{\alpha-1}) - b(\alpha-1)(T-T^{\alpha-2})}{a+b/T-\log T}\right] \\
&+ \frac{d'}{K}\left[\frac{\left[(a+b/T-\log T)\frac{\alpha-1}{3-\alpha}1 - b(\alpha-1)\right](1-T^{\alpha-3})}{a+b/T-\log T}\right] \\
&+ \frac{d'M_1}{K}\left[\frac{\frac{1}{\alpha}(a+b/T-\log T)(T^{\alpha+1}-T) - \frac{a}{2-\alpha}(T^2-T^\alpha) + \frac{ab(\alpha-1)}{3-\alpha}(T-T^{\alpha-2})}{a+b/T-\log T}\right] \\
&\leq \frac{d'}{K}\{A+M_1B\} = d'
\end{aligned}$$

Therefore, $\mathcal{A} : \overline{P(\gamma,d')} \to \overline{P(\gamma,d')}$. *To confirm the condition (S1) of Theorem* (3.1), *we choose* $u(t) = \frac{b'+c'}{2}$, $1 \leq t \leq T$. *So* $\varphi(u) = \frac{b'+c'}{2} > b'$,

$$\theta(u) = \frac{b'+c'}{2} < c', and \gamma(u) = 0 < d'$$

.

Consequently, $\{u \in P(\gamma,\theta,\varphi,b',c',d'), \varphi(u) > b'\} \neq \phi$. *Moreover, if*

$$u \in P(\gamma,\theta,\varphi,b',c',d')$$

then $b' \leq u(t) \leq c'$ *and* $|u'(t)| \leq d'$ *hold for* $t \in [\delta, T+1-\delta]$. *By using the assumption (A2),* *we will check the condition (S1) of Theorem* (3.1).

$$\begin{aligned}
\varphi(\mathcal{A}u(t)) &= \min_{t\in[\delta,T+1-\delta]} |(\mathcal{A}u)(t)| \\
&= \min_{t\in[\delta,T+1-\delta]} \left\{ \int_1^T G(t,s) f(s,u(s),u'(s))\, ds + \int_1^T R(t,s) \int_1^T G(1,\tau) f(\tau,u(\tau),u'(\tau)) d\tau ds. \right\} \\
&\geq \min_{t\in[\delta,T+1-\delta]} \left\{ (1-\log\delta) \int_1^T G(1,s) f(s,u(s),u'(s)) ds \right. \\
&+ \int_1^T (1-\log\delta) R(t,s) \int_1^T G(1,\tau) f(\tau,u(\tau),u'(\tau)) d\tau ds. \\
&\geq \frac{b'}{L}. \min_{t\in[\delta,T+1-\delta]} \left\{ (1-\log t) + \int_1^T (1-\log s) R(t,s)\, ds \right\} \int_1^T G(1,\tau) ds \\
&\geq \frac{b'}{L}. \left\{ (1-\log(T+1-\delta)) + (\log\delta) \max_{t\in[1,T]} R(t,s) \int_1^T (1-\log s) ds \right\} \int_1^T G(1,s) ds \\
&\geq \frac{b'}{L}. \left\{ (1-\log(T+1-\delta)) + (\log\delta) M2 \left[2(T-1) - T\log T \right] \right\} \int_1^T G(1,s) ds \\
&\geq \frac{b'}{L}. \left\{ (1-\log(T+1-\delta)) + (\log\delta) M_2 \left[2(T-1) - T\log T \right] \right\} (T-1) \\
&\times \frac{-aT(\log T)^{\alpha-1} + ab(\alpha-1)(\log T)^{\alpha-2}}{\left[T(a-\log T) + b \right] \Gamma(\alpha)} = b'.
\end{aligned}$$

So the condition $(S1)$ *is satisfied. If* $u \in P(\gamma;\varphi,b',d')$ *and* $\theta(Au) > \delta b'$, *then*

$$\begin{aligned}
\varphi(\mathcal{A}u) &= \min_{t\in[\delta,T+1-\delta]} (\mathcal{A}u)(t) \geq (\log\delta) \max_{t\in[1,T]} (\mathcal{A}u)(t) \\
&= (\log\delta)\cdot\theta(\mathcal{A}u) > (\log\delta)\cdot\delta b' > b'.
\end{aligned}$$

So condition $(S2)$ *of Theorem* (3.1) *follows. Finally, we show that the condition* $(S3)$ *of Theorem* (3.1)*holds.*

As $\psi(0) = 0 < a'$, *so* $0 \notin R(\gamma,\psi,a',d')$. *For* $u \in R(\gamma,\psi,a',d')$ *with* $\psi(u) = a'$ *we have* $0 \leq u(t) \leq a'$, $t \in [1,T]$. *Using assumption (A3), we can estimate*

$$
\begin{aligned}
\psi(\mathcal{A}u) &= \max_{t\in[1,T]} |(\mathcal{A}u)(t)| \\
&= \max_{t\in[1,T]} \left| \int_1^T G(t,s) f(s,u(s),u'(s))\, ds + \int_1^T R(t,s) \int_1^T G(s,\tau) f(\tau,u(\tau),u'(\tau))\, d\tau\, ds \right| \\
&\le \max_{t\in[1,T]} \left| \int_1^T G(1,s) f(s,u(s),u'(s))\, ds + \max_{t\in[1,T]} \int_1^T R(t,s) \int_1^T G(1,\tau) f(\tau,u(\tau),u'(\tau))\, d\tau\, ds \right| \\
&\le \max_{t\in[1,T]} \left\{ 1 + \int_1^T R(t,s)\, ds \right\} \int_1^T G(1,s) f(s,u(s),u'(s))\, ds \\
&< (1+M_2)\, \frac{a'}{N} \int_1^T G(1,s)\, ds \\
&< (1+M_2)\, \frac{a'}{N} \left[\frac{\frac{1}{\alpha}(a + \frac{b}{T} - \log T)(T^{\alpha} - 1) - \frac{aT^{\alpha-1}}{2-\alpha}\,(T^{2-\alpha} - 1)}{a + \frac{b}{T} - \log T} \right] \\
&+ (1+M_2) \left[\frac{\frac{ab(\alpha-1)}{3-\alpha}\,(T^{\alpha-5} - T^{2-\alpha})}{a + \frac{b}{T} - \log T} \right] \\
&= a'
\end{aligned}
$$

Therefore, problem (3.1)has at least three positive solutions u_1, u_2, and u_3 such that

$$
\begin{aligned}
&\max_{t\in[1,T]} |u_i'(t)| \le d', \quad \textit{for } i = 1,2,3; \\
&b' < \min_{t\in[\delta,T+1-\delta]} |u_1(t)|; \\
&a' < \max_{t\in[1,T]} |u_2(t)| \quad \textit{with} \quad \min_{t\in[\delta,T+1-\delta]} |u_2(t)| < b'; \\
&\max_{t\in[1,T]} |u_3(t)| < a'.
\end{aligned}
$$

The proof of the theorem is completed.

3.3 Example

We consider the problem

$$\begin{cases} D^{\frac{7}{2}}u(t) = f(t, u(t), u'(t)), \qquad t \in [1, 2], \\ u''(1) = 0, \quad u^{(3)}(1) = 0, \\ u(1) + u'(1) = \frac{1}{10}\int_1^2 u(s)\,ds, \\ u(2) - 2u'(2) = \int_1^2 su(s)\,ds, \end{cases} \tag{3.13}$$

where

$$f(t, u, v) = \begin{cases} \log(2+t) + \frac{u^8}{8} + \left(\frac{v}{10^4}\right)^3 & u \in [0, 4], \\ \log(2+t) + \frac{4^8}{8} + \left(\frac{v}{10^4}\right)^3 & u \in [4, +\infty], \end{cases}$$

with

$$\sigma_1 = 0.0528, \sigma_2 = 0.0469, \sigma_3 = 0.839, \sigma_4 = 0.0469, \text{ and } \rho = 0.2817$$

. and

$$M1 = 0.444, M2 = 5.480, K = 4.075, L = 0.333, \text{and } N = 1.084$$

.

Setting $a' = 1$, $b' = 5$, $d' = 104$, then $f(t, u, v)$ satisfies

$$\begin{cases} f(t, u, v) \leq \frac{d'}{K} = 2453.987, & \text{for } (t, u, v) \in [1, 2] \times [0, 2 \times 10^4] \times [-10^4, 10^4]; \\ f(t, u, v) \geq \frac{b'}{L} = 15.151, & \text{for } (t, u, v) \in \left[\frac{9}{8}, \frac{15}{8}\right] \times [5, \frac{45}{8}] \times [-10^4, 10^4]; \\ f(t, u, v) \leq \frac{a'}{N} = 0.9225, & \text{for } (t, u, v) \in [1, 2] \times [0, 1] \times [-10^4, 10^4]. \end{cases}$$

All conditions in Theorem (1.3) hold; then problem (3.13) has at least three positive solutions.

Conclusion

In conclusion, this dissertation has been dedicated to the exploration of positive solutions for fractional differential equations. The primary objective was to investigate the behavior of solutions under different boundary value conditions, with a focus on integral boundary value problems in the second chapter and the application of Hadamard derivatives in the third chapter. Through theoretical analyses, numerical simulations, and illustrative examples, we have elucidated various aspects of fractional calculus and its relevance in modeling real-world phenomena.

By addressing integral boundary value problems and incorporating Hadamard derivatives, this research has contributed to expanding the understanding of fractional differential equations and their applications. The provided examples have served to illustrate the theoretical concepts and demonstrate their practical implications. However, it is essential to acknowledge that the exploration of fractional calculus is an ongoing endeavor, and there remain many avenues for further research and development.

In summary, this dissertation represents a significant step towards unraveling the complexities of fractional differential equations and lays the groundwork for future investigations in this field.

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