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THEME

Missile Guidance System Design Using Optimal Control

Defense jury:

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ملخص:

الهدف الرئيسي من العمل المدرج في هذه المذكرة هو إيجاد قانون التحكم الأفضل من نوع LQR يساعد على الحصول على الحد الأدنى لمؤشر الأداء وهو عبارة عن دالة خطية تربيعية، وذلك من أجل جعل نظام توجيه الصاروخ المدروس و المنمذج بثلاث درجات حرية $3DOF$ جد دقيق وفعَّال. أولاً، تم عرض الحالة الفنية للصواريخ المتعلقة بأنواع منصات الإطلاق، أنظمة الدفع والتوجيه بالإضافة إلى طرق التحكم المثلى المستخدمة بشكل شائع في الصواريخ الموجهة. بعد ذلك، تمت نمذجة الصاروخ باعتبار ست ثم ثلاث درجات حرية بمعادلات الحركة الإنسحابية و الدورانية المعبرعنها في المعلم المثبت بمركز ثقل الصاروخ وكذا المعلم العطالي الأرضي. فيما بعد، تم تقديم وشرح طريقة التحكم الأمثل من نوع LQR واستخدامها للحصول على قانون التوجيه الأفضل لضمان تتبع دقيق وفعَّال وضرب للهدف المتوخى. أخيرًا، تم عرض نتائج المحاكاة ومناقشتها بهدف دراسة ديناميكية صاروخ منمذج بست درجات حرية بدون تحكم، بالإضافة إلى إظهار كفاءة قانون التحكم الأمثل المستنبط سلفًا والمستخدم في نظام توجيه صاروخ منمذج بثلاث درجات حرية بغية تتبع وضرب هدف سريع الحركة و لديه القدرة على تنفيذ مناورات.

كلمات مفتاحية : نمذجة الصاروخ، نموذج ذو ثلاث درجات حرية، نموذج ذو ست درجات حرية، نظام التوجيه، التحكم الأمثل من نوع LQR، هدف متحرك وله القدرة على المناورة، نتائج المحاكاة، أداء وفعالية قانون التوجيه.

Abstract :

The aim of this memoir work is to develop a Linear Quadratic Regulator (*LQR*) optimal control law that minimizes a performance index 'or cost functional' in order to perform an effective and accurate intercept guidance system of a three Degrees Of Freedom (*3DOF*) guided missile. First of all, a state of the art review, that concerns mainly missiles types and launch modes, propulsion and guidance systems, and commonly used optimal control methods used in guided missiles, has been established. Then, the missile *6DOF* and *3DOF* motion modeling has been performed by determining both velocity components and rotation rates dynamic expressions in body-fixed and inertial frames. Next, the *LQR* optimal control method has been presented and used to get the most favorable intercept guidance law in order to ensure an accurate and effective missile-target tracking and hitting. Finally, simulation results have been presented and discussed to study the dynamics of an open loop *6DOF* missile model and show the aptitude of the performed optimal control law of a *3DOF* missile guidance system to track and hit a fast moving and maneuvering target.

Key words: *Missile modeling, 6DOF model, 3DOF model, Guidance system, LQR optimal control, Moving and maneuvering target, Simulation results, Optimal guidance law performance.*

Résumé :

L'objectif principal du travail présenté dans ce mémoire est de développer une loi de commande optimale de type linéaire quadratique (*LQR*) qui minimise un indice de performance ou un critère de coût afin de mettre en œuvre un système de guidage efficace et précis d'un missile intercepteur modélisé par trois Degrés De Liberté (*3DDL*). Tout d'abord, une revue de l'état de l'art, portant principalement sur les types de missiles et les modes de lancement, les systèmes de propulsion et de guidage, ainsi que les méthodes de commande optimale couramment utilisées dans les missiles guidés, a été réalisée. Ensuite, une modélisation de la dynamique à *6DDL* et à *3DDL* du missile a été effectuée permettant de déterminer les expressions des composantes des vitesses de translation et de rotation dans un repère lié au corps du missile et dans un repère inertiel. Ensuite, une méthode de commande optimale de type *LQR* a été présentée et utilisée pour obtenir une loi de guidage la plus favorable afin d'assurer un suivi précis et efficace du missile pour bien intercepter et frapper sa cible. Enfin, des résultats de simulation ont été présentés et discutés pour étudier la dynamique d'un modèle de missile en boucle ouverte à *6DDL* et montrer l'aptitude des performances de la loi de commande optimale établie pour un système de guidage de missile à *3DDL* en termes de poursuite et d'interception d'une cible en mouvement rapide et effectuant des manœuvres.

Mots clés: *Modélisation d'un missile, Modèle à 6DDL, modèle à 3DDL, Système de guidage, Commande optimale de type LQR, Cible mobile et manœuvrant, Résultats de simulation, Performance de la loi de guidage optimale.*

Dedication

We humbly dedicate this memoir to our ***beloved families****, whose unwavering support and encouragement have been the bedrock of our academic journey. Your boundless love and unwavering belief in us have provided the strength and motivation to overcome challenges and pursue excellence.*

We also extend our heartfelt dedication to our esteemed supervisor, ***Prof. MOKRANI Lakhdar****, whose exceptional guidance and expertise have profoundly shaped our understanding of the subject matter and nurtured our intellectual growth. Your unwavering dedication to education and unwavering commitment to our development have been invaluable for us.*

Furthermore, we dedicate this memoir to ***all those who ardently strive*** *for a better and more sustainable future in the dynamic field of automation and systems. May the findings presented within these pages ignite a spark of inspiration and propel further students in tackling other challenges.*

To everyone who *has been an integral part of this arduous journey, we extend our deepest gratitude for your unyielding presence, encouragement, and unwavering support. Your beliefin us has been an enduring source of motivation, and we are forever grateful for your presence in our lives.*

Raouf & Salah

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First of all, we would like to thank God for giving us the faith, the will and the patience to be able to carry out this work.

We would also like to thank ***Prof. MOKRANI Lakhdar*** *for supervising us to achieve our memoir. May he find here the expression of our deep and respectful gratitude for having guided our work with patience and diligence throughout the preparation of this memoir.*

We also express our deep gratitude to **Dr. Keltoum Benlarbi***, for having co-supervised this work. She spared no effort to help us in the preparation of our memoir and the SIMULINK part.*

We would like to thank all the examination staff for having agreed to be part of our defense jury, thus showing their interest and taking the time to read and evaluate our work, namely the president of the jury ***Dr. BENMOUIZA Khalil*** *and the member of the jury* ***Dr. Rahmani Belkacem****.*

We would also like to thank all the teachers and all the staff of the Electrical Engineering Department of AMMAR TELIDJI – Laghouat University for their help throughout the period of our studies.

Symbols & Acronyms List

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- **Symbols**

Symbol	*Signification*
X_b	Body *x* axis
Y_b	Body *y* axis
Z_b	Body *z* axis
$\vec{\boldsymbol{F}}_{\boldsymbol{Text}}$	Vector of total external forces
$\vec{V}$	Velocity vector
$\vec{V}_M$	Velocity vector of the missile
$\vec{M}_T$	Vector of external forces moment
F_x	Force along *x* axe
F_y	Force along *y* axe
F_z	Force along *z* axe
L	Roll angle momentum
M	Pitch angle momentum
N	Yaw angle momentum
$\vec{H}$	Angular momentum vector
$\vec{\omega}$	angular velocity vector
p	Rotation rate around *x* axis
q	Rotation rate around *y* axis
r	Rotation rate around *z* axis
ψ, φ, θ	Euler angles
u	Translation velocity along *x* axis
v	Translation velocity along *y* axis
w	Translation velocity along *z* axis
T	Thrust
P_a	Ambient pressure
v_e	Exhaust velocity

A_e	Area of the motor nozzle
S	Reference area
C_L, C_D	Lift and drag coefficients respectively
Q	Dynamic pressure
C_x	Coefficient of aerodynamic forces along x axis
C_y	Coefficient of aerodynamic forces along y axis
C_z	Coefficient of aerodynamic forces along z axis
N	Proportional navigation constant
$X(t)$	State variables vector
$Y(t)$	Output vector
A	State matrix
B	Input matrix
C	Output matrix
D	Forward matrix
Qc	Controllability matrix
Qo	Observability mtrix
H	Hamiltonian operator
$\lambda(t)$	Costate variables vector
$e(t)$	Dynamic error vector
$z(t)$	Desired output trajectory
$[t_0, t_f]$	Time interval

- ***Acronyms***

Acronyms	***Signification***
LOS	Line of Sight Angle
LQR	Linear Quadratic Regulator
SISO	Single-Input-Single-Output
MIMO	Multi-Input-Multi-Output
6DOF	Six Degrees Of Freedom
3DOF	Three Degrees Of Freedom
LTV	Linear Time Variant
LTI	Linear Time Invariant
ARE	Algebraic Riccati Equation

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General Introduction

General Introduction

Guided missiles are weapons systems that are designed to navigate and direct themselves towards a specific target with the help of an onboard guidance system. Unlike unguided missiles ‘or ballistic missiles’, which follow an arcing trajectory and rely on the laws of physics to reach their intended destination, guided missiles are equipped with sophisticated technologies that enable them to actively adjust their course during flight. Therefore, the primary objective of a guided missile is to accurately deliver its payload to a particular target, which can vary depending on the mission requirements. It includes explosives which are stored in warheads.

Moreover, guided missiles are highly efficient at destroying targets due to their automatic guidance system which offers the accuracy and precision needed to hit exactly where intended. Furthermore, the quest for precision and accuracy drove the engineers behind today’s guided missiles enabling them to create advanced systems that can be armed with a range of warheads and hit targets which distance is about thousands miles with a precision of a surgical knife.

So, one of the most important components of a guidance system is the autopilot system which is responsible for control and adjust the missile's flight path to reach its intended target autonomously. This gives the access to the missile to enhance its accuracy, reliability and effectiveness. Therefore, to achieve these advantages the scientists, researchers and engineers have developed several types of controls to design an effective guidance system. One of the very interesting types is the Linear Quadratic Regulator (LQR) optimal control method which is our challenge. It is a question of establishing an optimal control law to design the guidance system for a missile expressed by three degrees of freedom ‘3DOF’ motion model.

The present memoir is divided into 4 chapters; the first chapter concerns a state of the art review of missiles types, launch modes, range, propulsion systems, warhead, guidance system and phases and finally the optimal control methods used in guided missiles.

The second chapter is dedicated to the missile 6DOf and 3DOF motion modeling by determine the linear velocity and rotation rate components in body and inertial fixed frame.

The third chapter is devoted to the presentation of the *LQR* optimal control method and its use to minimize a performance index ‘or a criterion’ in order to establish an intercept guidance law.

In the last chapter, we will present and discuss some simulation results of a *6DOF* open loop missile and a *3DOF* missile equipped with an *LQR* optimal control designed to intercept a moving and manoeuvring target.

Finally, we will end our memoir with a general conclusion which summarizes the main obtained results and presents a set of perspectives concerning a possible continuation of this work.

Chapter I

Generalities on Missiles and Guidance Systems

I.1. Introduction

From the past decades, ages and eras, beginning with the oldest kingdoms and civilizations arrived to the new concept of population distribution around the world which called 'states', man used to protect himself from enemies by using many types of weapons such as knives, swords, arrows and war cannons ...etc. Also, he had used it when he has to occupy others by force. Furthermore, nowadays the countries are following the same approach by making plans to expand to other states, which leads them to invest in a new generation of weapons. by making advanced weapons, other countries try to protect their political boundaries and their people in case of any threat from those who consider wars as an important way to expand their frontiers.

One of the most high-tech weapons in use today is guided missiles which started to become clearly developed during the world war II " 1939 – 1945 " by Germans with the V-1 rocket, while one of these principal purposes is rocket propulsion systems for aircraft and guided missiles which have significantly contributed in an unprecedented development of weapons. Therefore, it makes sense to look at the differences and similarities for each of the conventional missiles and the guided missiles, while the conventional missiles need to be launched in the correct direction or path and correct angle towards the targets, (in case of any fault in them the target will be missed). The guided missiles work in the same way, but the difference is that they correct their path automatically to track the target to destroy, which enhances their accuracy.

As we said before that guided missiles have played a critical role in modern warfare and have revolutionized the way in which wars are fought. These precise weapons have the ability to strike targets with incredible accuracy from long distance due to modern technologies involving the compatibility between the hardware and the software of the missile.

In this chapter, we will present a state of the art on missiles types, guidance system, missile components and their general control systems.

I.2. Classification of missiles

Each missile have its own characteristics that distinguishes it from others, the main factors of a missile design any depend on the threat, the operating environment and, the cost as summarized in the following sections.

I.2.1. Types of missiles

One can distinguish two kinds of missiles which are briefly described hereafter.

a) Cruise missile

A cruise missile (see figure I.1) is a type of guided missiles that is intended to hit terrestrial targets with high precision. It requires pre-set coordinates to locate its target and remains in the atmosphere throughout its flight. They missile typically maintains a relatively constant speed throughout most of its flight path. It is designed to transport a large warhead over long distances. Modern versions of cruise missiles can achieve supersonic or high subsonic speeds, feature self-navigation systems, and can fly at extremely low altitudes on a non-ballistic trajectory.

Figure I.1 Cruise missile

b) Ballistic missile

Ballistic missiles are designed to deliver one or more warheads on a predetermined target by following a ballistic trajectory. Such a trajectory refers to the path that an object follows after it has been launched but has no active propulsion during its flight.

As a result, these weapons are only guided during relatively brief periods of their flight. The trajectory is fully determined by factors such as the initial velocity, effects of gravity, air resistance, and the motion of the earth. Ballistic missiles with shorter ranges typically remain within the Earth's atmosphere. On the other hand, intercontinental ballistic missiles are launched on a sub-orbital flight trajectory, spending most of their flight outside of the atmosphere.

Figure I.2 Ballistic missile

The following table shows the difference between cruise and ballistic missiles:

Table I.1 Difference between cruise and ballistic missiles

Ballistic missile	Cruise missile
• After launch, the missile is propelled for only a short period of time.	• The missile is self-propelled for the entire duration of its flight
• Similar to a rocket engine.	• Similar to a jet engine.
• Long-range missiles exit and re-enter the Earth's atmosphere.	• The missile's flight path remains within the Earth's atmosphere.
• The missile's precision is low because it is unguided for most of its flight path and its trajectory depend on gravity and air resistance.	• The missile hits targets with high precision because it is continually propelled throughout its flight path.
• The missile can have an extensive range, ranging from 300 km to 12,000 km, since it does not require additional fuel after its initial launch trajectory.	• The missile has a limited range, typically below 500 km, as it needs to be continually propelled to achieve high precision in hitting its target.
• The missile has a significant payload carrying capacity.	• The missile has a restricted payload capacity.
• The missile has the capability to carry multiple payloads.	• The missile typically carries a single payload.
• The missile was primarily developed to transport nuclear warheads.	• The missile was primarily developed to carry conventional warheads.

I.2.2. Launch mode

Each type of missile has its own launch pad which depends on the threat to be targeted (see figure I.3).

a) Surface to surface missile

b) Surface to air missile

c) Surface to sea missile

d) Air to air missile

e) Air to surface missile

f) Sea to sea missile

g) Anti-tank missile

Figure I.3 Missiles with different launch modes

I.2.3. Range

The type of missile to be used relates to the range to be reached, and this type of ammunition and explosives loaded on the warhead. By the way, there are several kinds of missiles based on the desired range including:

a) Short range missiles,
b) Medium range missiles,
c) Intermediate range ballistic missiles,
d) Intercontinental ballistic missiles.

I.2.4. Propulsion system

It's the system that powers a rocket, lifts it off the ground, and propels it through the air. Rockets use a combination of systems and components to create the necessary thrust to lift off the ground and move through the air or space. The basic concept of a rocket is to generate a large amount of hot gas through a chemical reaction that can be expelled out of the back of the rocket at high speeds. This creates a force in the opposite direction, propelling the rocket forward one can find the following types missile propellants.

a) Solid propulsion system

Solid propellants are mode by mixing fuel and oxidizer together, and pack them into a solid cylinder that contains a hole serving as a combustion chamber (see figure I.4). When the mixture is ignited, combustion occurs on the surface of the propellant. This generates a flame front that burns into the mixture, producing vast quantities of exhaust gas at high temperature and pressure.

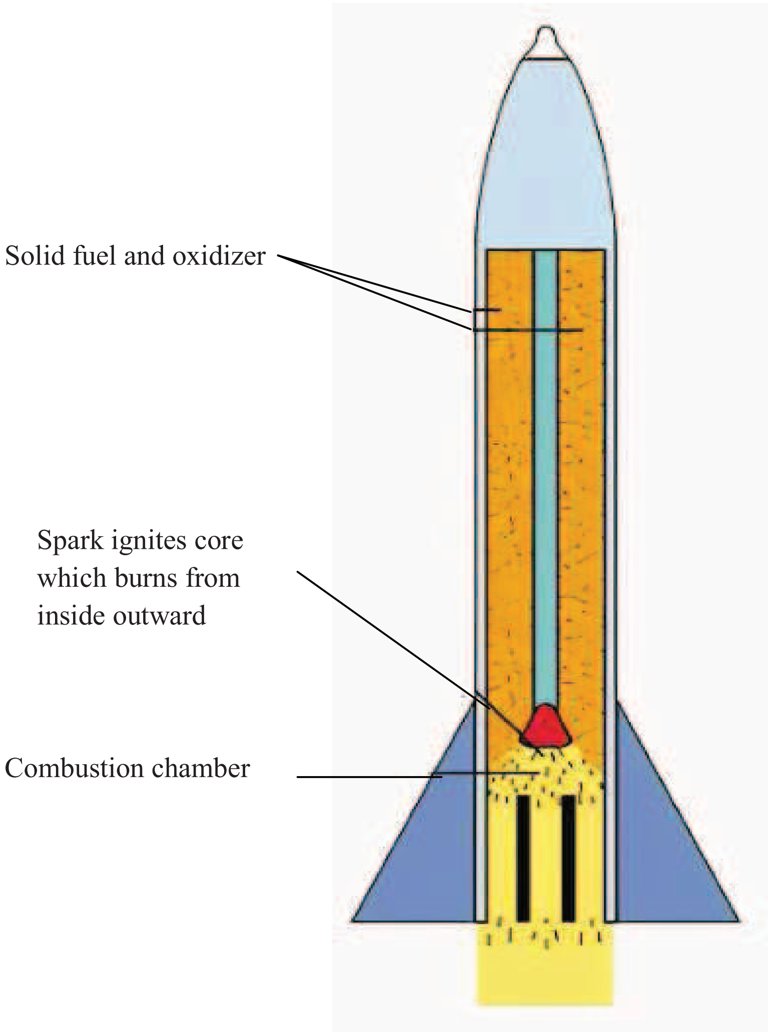


Figure I.4 Solid propulsion missile

b) Liquid propulsion system

In liquid propulsion systems, the propellant is stored in external tanks separated from the combustion chamber. These engines typically use both liquid fuel and liquid oxidizer, which are moved from their respective tanks into the engine by pumps. The pumps raise the pressure of the propellants above the operational pressure of the engine, and then the propellants are injected into the engine in a way that ensures rapid mixing and atomization to generate the necessary thrust (see figure I.5).

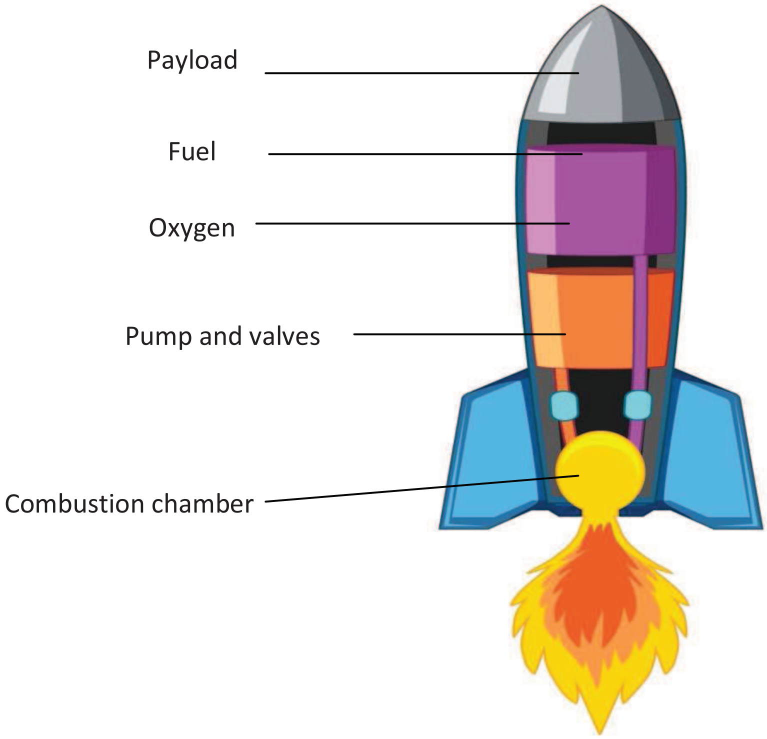


Figure I.5 Liquid propellant missile

c) Hybrid propulsion

A hybrid propulsion system is a type of vehicle propulsion system that utilizes two or more sources of propulsion (see figure I.6), such as diesel, batteries, and renewable energy sources. The concept of hybrid propulsion is not a recent innovation and has been adopted worldwide. It offers several benefits, such as improved fuel efficiency, reduced emissions, and increased range. They achieve these benefits by utilizing multiple sources of energy to power the vehicle, with each source being used in the most efficient way possible.

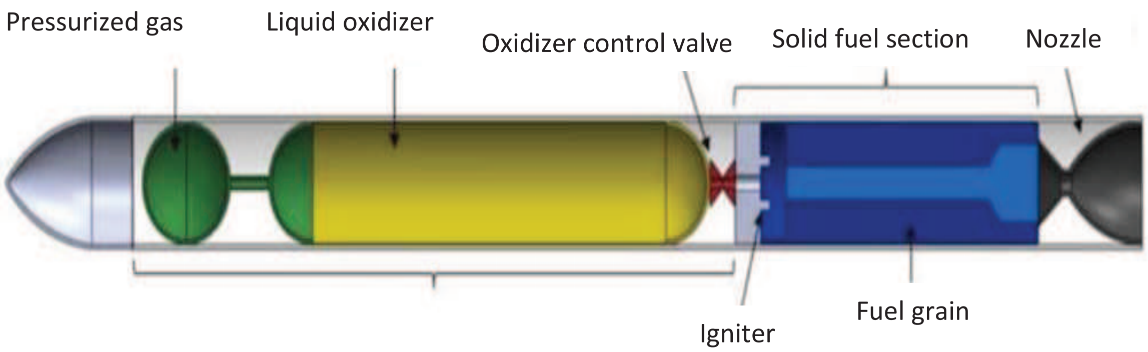


Figure I.6 Example of a hybrid propulsion missile

d) Ramjet propulsion

A ramjet engine operates differently from traditional air-breathing jet engines in fact it doesn't have a rotary compressor. Instead, it relies on the forward motion of the engine to compress incoming air (see figure I.7). However, since it requires forward motion to operate, it cannot generate thrust at zero airspeed or low speeds, and therefore cannot be used for takeoff or landing. Despite its limitations, the ramjet engine's is characterized by its simplicity, with only a few moving parts has and Its basic components that include an air intake, combustor, and exhaust nozzle.

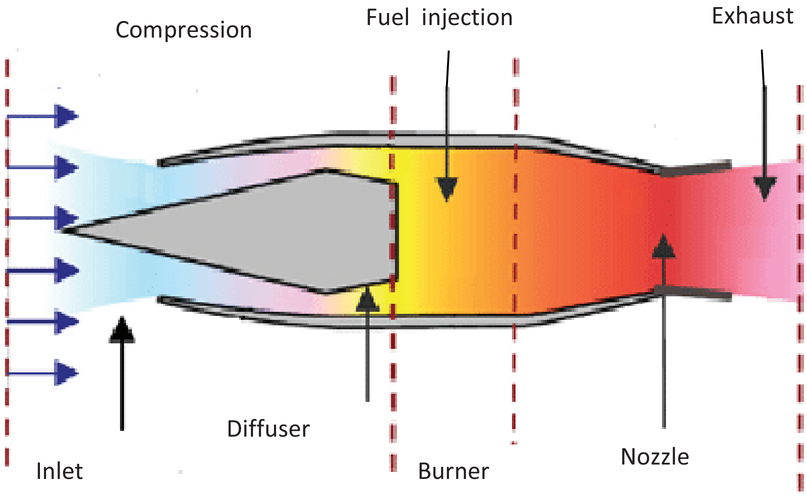


Figure I.7 Ramjet propulsion missile

e) Scramjet propulsion

A scramjet engine is a type of air-breathing jet engine that is an evolution of the ramjet. Unlike a ramjet, a scramjet engine maintains supersonic airflow throughout the engine, meaning that the air entering the engine is travelling faster than the speed of sound. Scramjets rely on the combustion of fuel and oxidizer to produce thrust, just like conventional jet engines (see figure I.7). The main difference is that scramjets obtain the oxidizer by ingesting atmospheric oxygen, rather than carrying it onboard like rockets. This limits the use of scramjets to atmospheric suborbital propulsion, as the oxygen content in the air is only sufficient for combustion within the atmosphere. Scramjet engines are a promising technology for high-speed flight, with potential applications for space access, hypersonic missiles, and other high-speed vehicles.

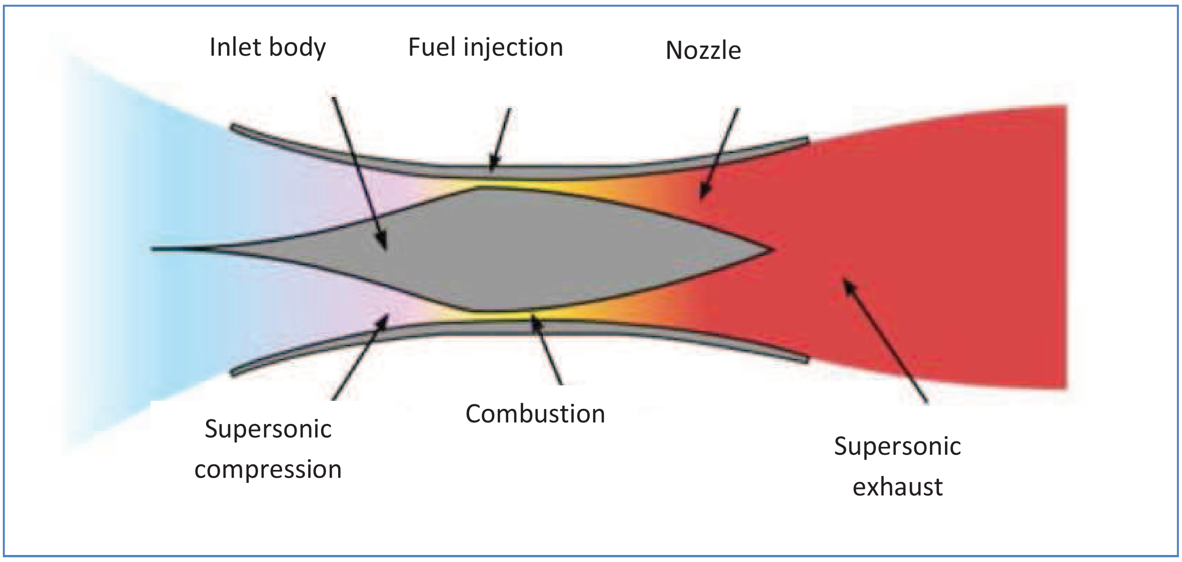


Figure I.8 Scramjet propulsion missile

f) Cryogenic propulsion system

Cryogenic engines are rocket propulsion systems known for their high efficiency and are particularly suitable for upper stages of a rocket due to their high specific impulse, which enhances payload capacity. A cryogenic rocket engine comprises various components, including igniters, a combustion chamber (thrust chamber), fuel cryogenic pumps, fuel injector, oxidizer cryogenic pumps, a gas turbine, cryogenic valves, regulators, fuel tanks, and a rocket engine nozzle(see figure I.8).

During the launch of a space vehicle, the cryogenic stage is the final stage where fuel, liquid oxygen, and liquid hydrogen stored using cryogenics are released from their respective tanks. They are then fed by individual booster pumps to the main turbo pump to ensure a high flow rate of propellants into the combustion chamber. This system ensures optimal efficiency and performance, making cryogenic engines a preferred choice for certain types of space missions.

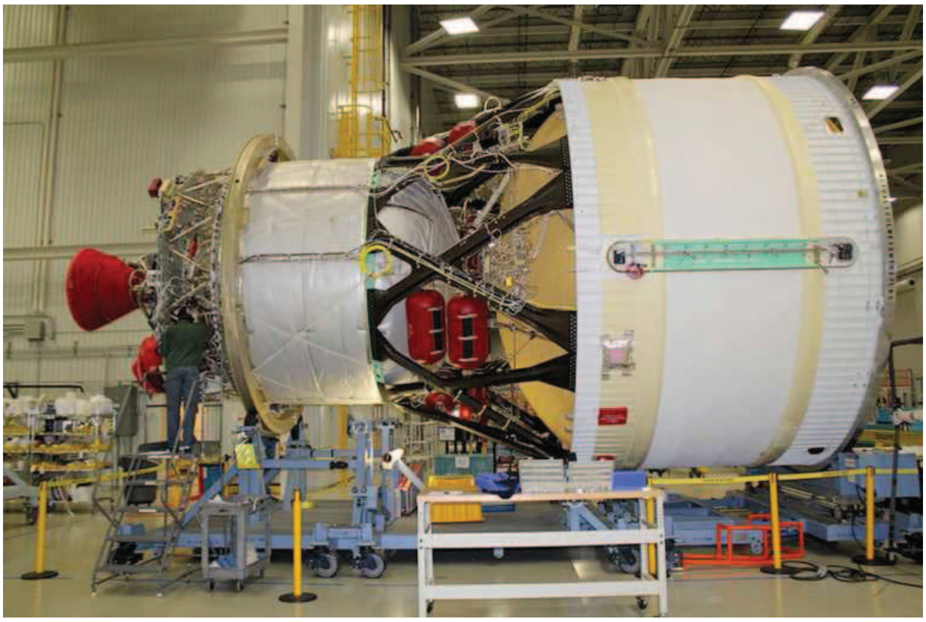


Figure I.9 Cryogenic propulsion missile

I.2.5. Warhead

The warhead is the essential component of a weapon that is responsible for achieving the desired end result, which is effective damage to the target. It is the part of the weapon that contains explosive, chemical, or biological material that is intended to cause destruction or incapacitation. The design and composition of a warhead depend on the type of weapon and the specific mission requirements. Warheads are an integral part of many types of weapons, including missiles, bombs, torpedoes and mines. Their effectiveness and reliability play a crucial role in the success of a military operation there are two kinds of warheads.

a) Conventional

Conventional warheads are propelled munitions that are highly explosive and primarily used against military targets, including armor, infantry, aircraft, buildings, naval vessels, helicopters, and other targets. They are designed to cause significant damage to these targets, but they do not contain nuclear, chemical, or biological agents. Conventional warheads are typically used in military operations where the use of nuclear weapons is not necessary or appropriate. The impact and effectiveness of the warhead depend on factors such as the type of missile, its trajectory, and the size and composition of the warhead.

b) Strategic

The use of atomic or thermonuclear devices is intended to strike an enemy's military, economic, or political power at its source, as only these weapons have the explosive power required to quickly and easily destroy the entire war-making capability of a large nation. These weapons are considered to be strategic in nature and are designed to achieve maximum destructive impact on the target.

The delivery systems for these warheads are complex and designed to ensure that they can reach their targets with accuracy and precision. These systems include intercontinental ballistic missiles, submarine-launched ballistic missiles, and strategic bombers. The use of such weapons is considered to be a last resort due to their devastating impact on the target and the surrounding area.

I.3. Guidance system

A guidance system is a collection of devices used for navigation of various crafts, such as ships, aircraft, missiles, rockets and satellites (see figure I.10). Typically, it refers to a system that navigates without direct or continuous human intervention. Conversely, systems that require a high degree of human interaction are referred to as navigation systems. One of the earliest examples of a true guidance system is the one used in the German V-1 during World War II, which consisted of a simple gyroscope for maintaining the heading, an airspeed sensor for estimating flight time, an altimeter for maintaining altitude, and other systems. A guidance system generally comprises three major sub-sections: Inputs, processing and outputs. The input section includes sensors, course data, radio and satellite links, and other sources of information. The processing section, made up of one or more CPUs, integrates this data and determines what actions are necessary to maintain or achieve the proper heading. This information is then fed to the outputs, which can directly affect the system's course. The outputs may control speed by interacting with devices such as fuel pumps, or they may more directly alter course by actuating rudders or other devices.

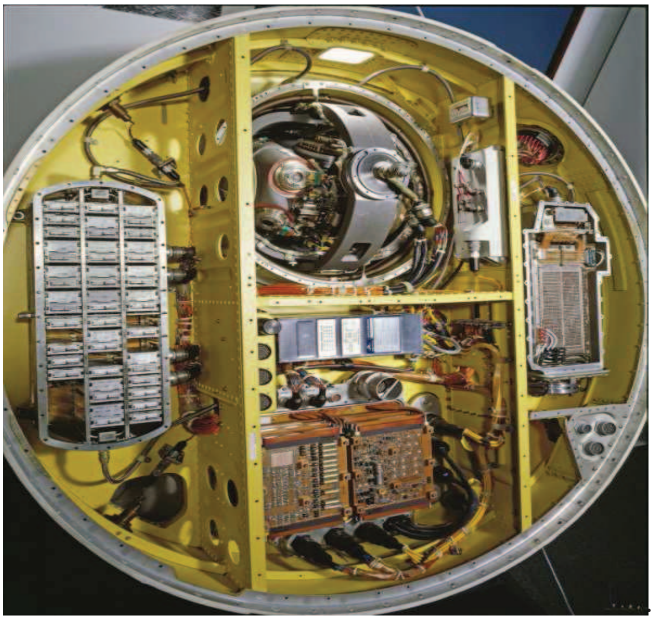


Figure I.10 Missile Guidance system in real

I.4. Phases of guidance

Missile guidance is typically divided into three phases: boost, midcourse, and terminal. Each phase corresponds to a different part of the missile's flight path. The boost phase, also known as the launch or initial phase, occurs at the beginning of the missile's flight and involves the use of rocket engines to propel the missile towards its target. The midcourse phase occurs after the boost phase and involves the missile navigating towards the target while in flight. The terminal phase occurs towards the end of the missile's flight and involves the missile homing on the target and executing its final attack.

a) Boost phase

During the boost phase, the missile is propelled to its flight speed with the assistance of the booster component. This phase begins when the missile leaves the launcher and continues until the booster has consumed its fuel. In missiles with separate boosters, the booster is jettisoned once it has burned out. The primary goal of this phase is to position the missile in space so that it can either detect the target or receive guidance signals from an external source. Some missiles lock the guidance system and aerodynamic surfaces in position during the boost phase, while others are guided throughout this phase.

b) Midcourse Phase

The midcourse phase of missile guidance typically lasts the longest in terms of distance and time. During this phase, the missile may need to make course corrections to ensure that it stays on track toward its target. There are several methods that can be used to supply information to the missile during this phase. Often, the midcourse guidance system is used to get the missile close to the target, where the final guidance system can take over. However, in some cases, the midcourse guidance system is also used during the final phase of guidance.

c) Terminal Phase

During the last phase of missile guidance, high accuracy and quick response to guidance signals are crucial. The performance of the missile becomes a critical factor during this stage, as it must execute final maneuvers for interception within a rapidly decreasing available flight time. The missile's maneuverability is influenced by both its velocity and airframe design, making it essential for the terminal guidance system to be compatible with the missile's performance capabilities. As the target's acceleration increases, the method of terminal guidance becomes more critical. While some missiles may use a single guidance system for all three phases of guidance, others, particularly long-range missiles, may use a different guidance system for each phase.

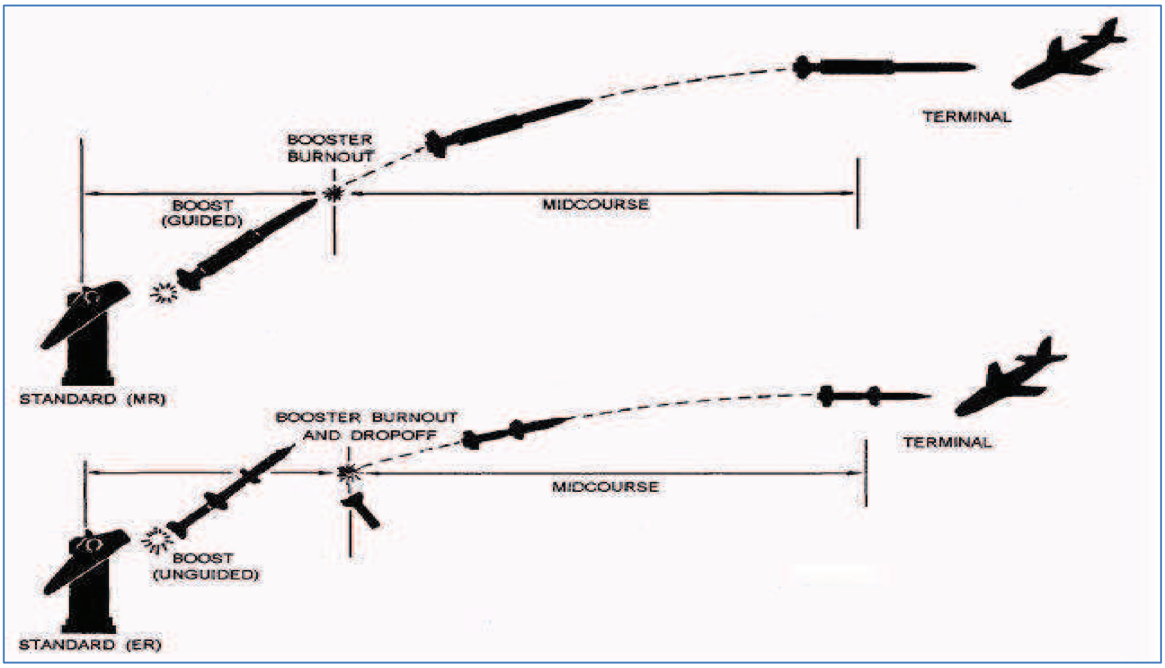


Figure I.11 Phases of missile guidance

I.5. Types of guidance systems

There are several sophisticated types of missile guidance systems. In this section, these devices will be described briefly.

I.5.1. Control guidance missile

A control guidance missile is a type of missile that is guided by direct electromagnetic radiation contact with friendly control points. This means that the missile's flight path is directed and adjusted in real-time by a control point through the use of radar or radio links. The control point sends guidance information to the missile, allowing it to make corrections to its course and successfully reach its intended target. Control guidance missiles are typically used in situations where the target is either stationary or moving at a known and constant velocity. This type of guidance system is commonly used in military applications, such as air defense systems and anti-ship missiles.

a) Radar control guidance

Radar control guidance is a type of control guidance system used to guide missiles towards their targets. It works by using radar signals to transmit guidance information from a control point to the missile. The control point uses radar to track the missile and the target, and sends guidance commands to the missile to make course corrections as necessary. In radar control guidance, the missile is equipped with a radar seeker that receives the guidance commands sent from the control point. The seeker is designed to track the radar signals from the control point and use them to adjust the missile's flight path. As the missile approaches its target, the radar seeker switches from receiving guidance commands to searching for the target using radar signals. Once the target is acquired, the missile's seeker guides it to impact with the target. Radar control guidance is commonly used in military applications, such as air defense systems and anti-ship missiles, where it provides a high degree of accuracy and allows the missile to be guided towards its target even if the target is moving.

b) Command guidance

Command guidance is a type of guidance system used to control the flight path of missiles. In this case, a ground-based operator or a control center sends commands to the missile through a radio or data link, directing it to its intended target (see figure I.12).

The operator or control center continuously monitors the missile's position, speed, and other flight parameters through the link and adjusts the missile's trajectory accordingly. The guidance commands sent to the missile include instructions to adjust its speed, altitude, and course to ensure it stays on track towards the target .Command guidance is typically used in situations where the missile must navigate through complex environments or where the target is highly mobile. It is also used in situations where the missile needs to be guided through multiple stages, such as in the launch of a satellite. However, command guidance requires a continuous link between the missile and the operator or control center, which can be vulnerable to electronic warfare or other for interference.

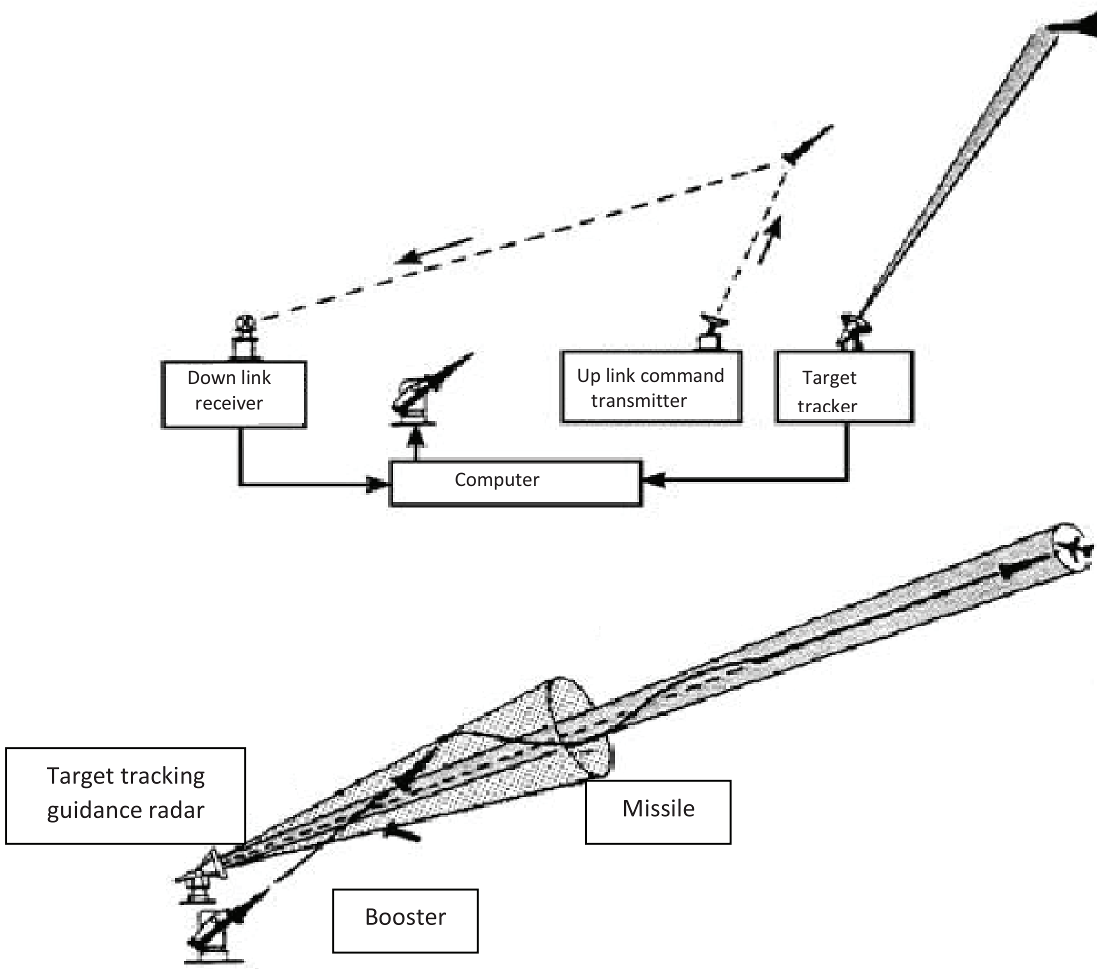


Figure I.12 Command guidance system

c) Beam-rider method

The beam-rider method is a type of guidance system used to control the flight path of missiles towards their targets. It works by using a beam of electromagnetic radiation, such as a laser or a radio signal, to guide the missile towards its intended target. In the beam-rider method, a guidance transmitter located at the launch site or on a vehicle illuminates the target with a beam of electromagnetic radiation (see figure I.14). The missile's seeker detects the beam and uses it to steer towards the target. The seeker is typically located on the missile's nose and contains sensors that detect the angle of the beam and adjust the missile's trajectory accordingly. As the missile approaches the target, the beam-rider system provides continuous guidance to ensure that the missile stays on course. The beam-rider method is highly accurate and can guide the missile towards fast-moving targets, making it well-suited for anti-aircraft and anti-missile applications. However, the beam-rider method has limitations. It requires a clear line of sight between the guidance transmitter and the missile, which can be obstructed by terrain or other obstacles. Additionally, the guidance transmitter can be vulnerable to electronic warfare or other forms of interference, which can disrupt the beam and cause the missile to lose its guidance.

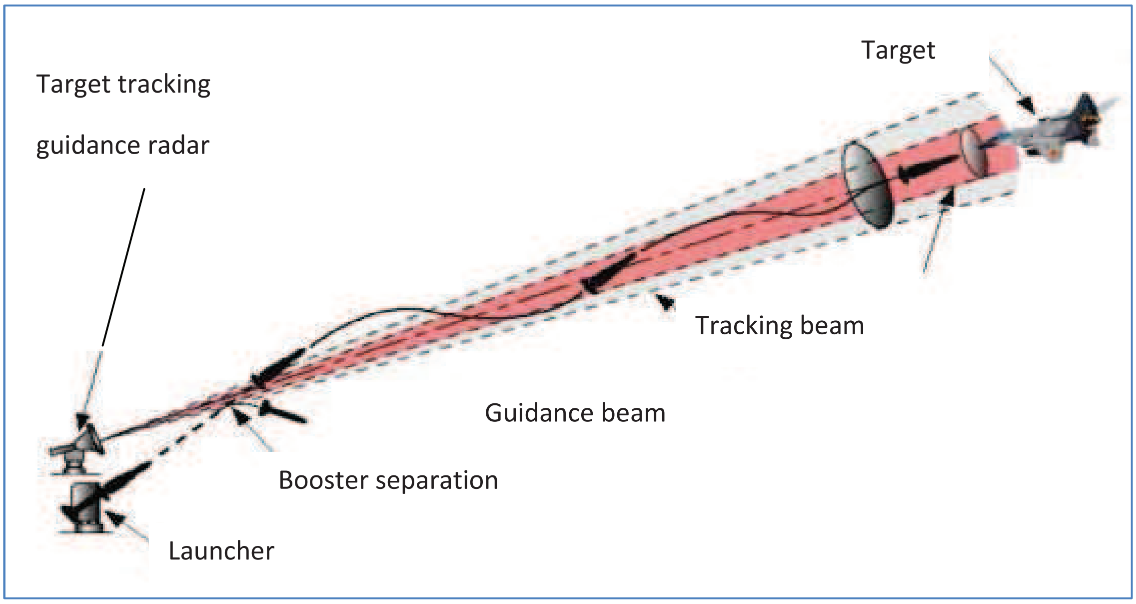


Figure I.13 Beam-rider guidance

I.5.2. Homing guidance missile

A homing guidance missile is a type of missile that uses a guidance system to track and home in on its target. The guidance system can use a variety of technologies, such as radar, infrared sensors, and GPS, to locate the target and guide the missile towards it. Once the missile is launched, the guidance system tracks the target and provides course corrections to keep the missile on track. The missile can also maneuver to avoid obstacles or to increase its chances of hitting the target. Homing guidance missiles are commonly used in military applications, such as air-to-air and surface-to-air missiles, anti-ship missiles, and guided bombs. They are highly effective weapons, as they can accurately target and destroy specific targets with a high degree of precision.

a) Active homing

Active homing is a type of guidance system used in missiles and other weapons that involves the use of active radar or sonar to locate and track a target (see figure I.15).

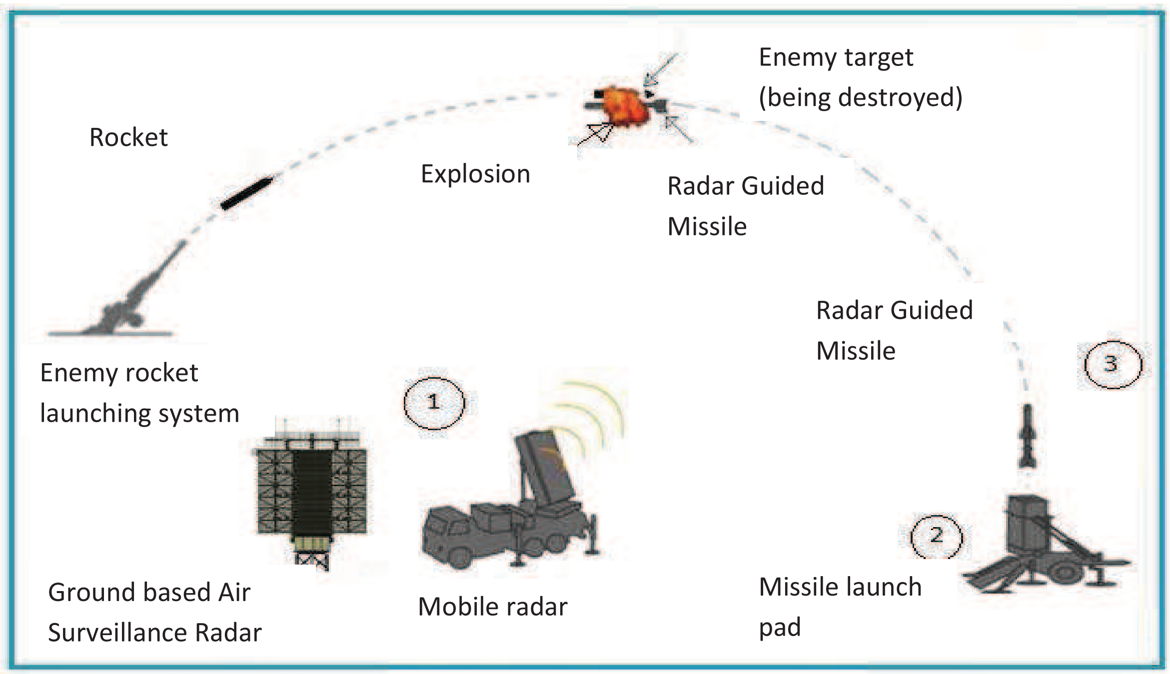


Figure I.14 Active radar homing system

b) Semi-active homing

Semi-active homing is a type of guidance system used in missiles and other weapons that involve the use of a system that is guided by an external source, such as radar or laser designated target (see figure I.17). In semi-active homing, the guidance system on the missile receives reflected energy from an external source, such as a radar or laser, which is usually located on the aircraft or vehicle that launched the missile or on a separate platform. The reflected energy provides information about the target's location, speed, and direction of travel, allowing the missile to track and home in on the target. One advantage of semi-active homing is that it can reduce the size and complexity of the missile's guidance system, making the missile smaller, lighter, and less expensive. However, semi-active homing missiles may be more vulnerable to countermeasures by the target, as the target can attempt to disrupt the external source or deploy countermeasures to confuse the missile's guidance system. This guidance system is commonly used in air-to-air and surface-to-air missiles, where the target is typically an aircraft or missile.

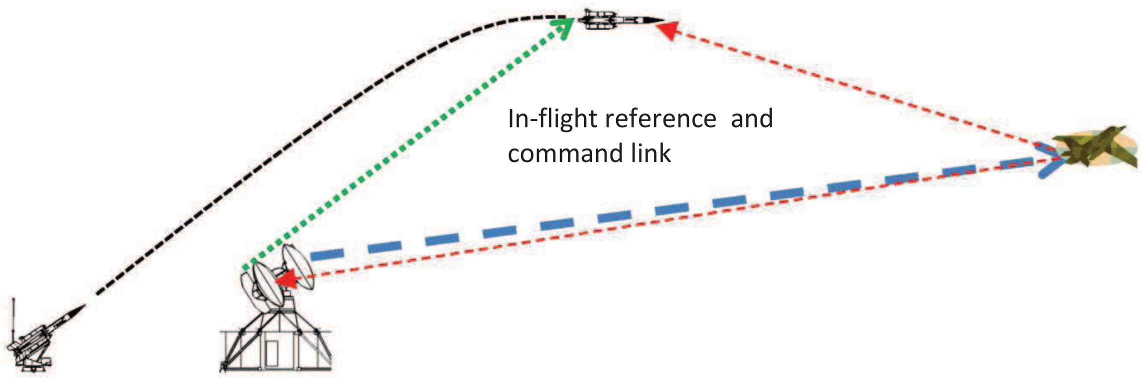


Figure I.15 Semi-active radar homing system

c) Passive homing

Passive homing is a type of guidance system used in missiles and other weapons that involves the use of sensors to detect and track a target without emitting any energy signals. In passive homing, the missile's guidance system detects and tracks the target using sensors that can detect the target's emissions, such as its radar or infrared signature (see figure I.18). This allows the missile to home in on the target without revealing its own presence to the target or any other nearby systems. Passive homing can be particularly effective in situations where the target is emitting energy signals that can be detected, but the missile does not want to reveal its own presence or location. This guidance system is commonly used in air-to-air and surface-to-air missiles, where the target is typically an aircraft or a missile. One disadvantage of passive homing is that it can be more difficult to detect and track a target without emitting any energy signals. This can make the missile less accurate or effective, especially if the target is maneuvering or deploying countermeasures to evade detection. Passive homing can also be combined with other guidance systems, such as semi-active homing or active homing, to increase the missile's effectiveness in different situations.

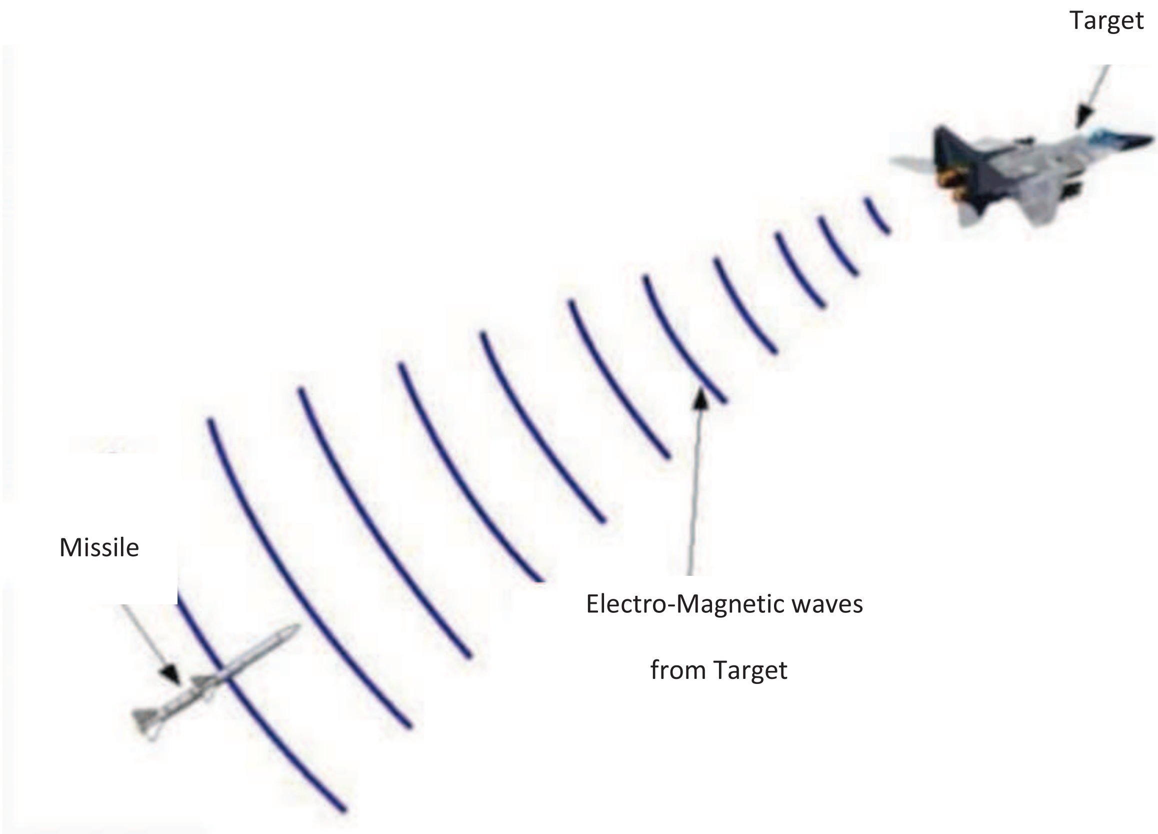


Figure I.16 Passive radar homing system

I.5.3. Self-contained guidance systems

A self-contained guidance system is a type of guidance system used in missiles and other weapons that operates independently of external sources and does not require any external guidance or control. It typically uses a combination of sensors, such as accelerometers, gyroscopes, and GPS, to determine the missile's position, velocity, and orientation. This information is then used to calculate the optimal trajectory for the missile to reach its target. Self-contained guidance systems can be particularly effective in situations where the target is located far away from the launch platform and cannot be easily illuminated or tracked by external sources. This guidance system is commonly used in long-range guided weapons, such as cruise missiles and intercontinental ballistic missiles where the target is typically a fixed location or installation.

a) Pre-set guidance system

A preset guidance system refers to a system that provides predetermined guidance to a missile based on specific parameters or conditions. The guidance system is programmed with a pre-determined flight plan, which includes the missile's trajectory, speed, altitude, and other factors that determine the missile's path towards its target. Preset guidance systems are typically used in certain types of missiles, such as cruise missiles and ballistic missiles. In cruise missiles, the preset guidance system is programmed with the missile's flight path to the target, and the missile is then launched and follows the predetermined path to its destination. In ballistic missiles, the preset guidance system is programmed with the missile's initial trajectory, and the missile follows this trajectory until it reaches the peak of its flight, at which point it switches to another guidance system to direct it towards its target. Preset guidance systems in missiles are designed to ensure that the missile follows a specific flight path and reaches its target with a high degree of accuracy. They are often used in situations where the target is a fixed location or where there are no obstacles that could cause the missile to deviate from its intended path.

b) Celestial navigation guidance

Celestial navigation guidance system is a type of guidance system used in missiles that utilizes celestial objects such as stars, planets, and the moon to determine the missile's position and orientation in space (see figure I.19). This type of guidance system was commonly used in earlier generations of missiles, and is still used in some specialized missile systems. The celestial navigation guidance system works by using sensors on the missile to detect the positions of celestial objects, and then comparing those positions with a preprogrammed map of the celestial sphere. By comparing the actual positions of the stars or planets with the positions predicted by the map, the missile's on-board computer can determine the missile's position and orientation in space. Once the missile's position is known, the guidance system can then direct the missile towards its target by calculating the required changes in course and orientation. While celestial navigation guidance systems are generally less accurate than other types of guidance systems, they have the advantage of being less vulnerable to electronic countermeasures, as they do not rely on electronic signals for navigation. Overall, celestial navigation guidance systems are an important part of missile technology, and continue to be used in certain specialized missile systems where they offer unique advantages..

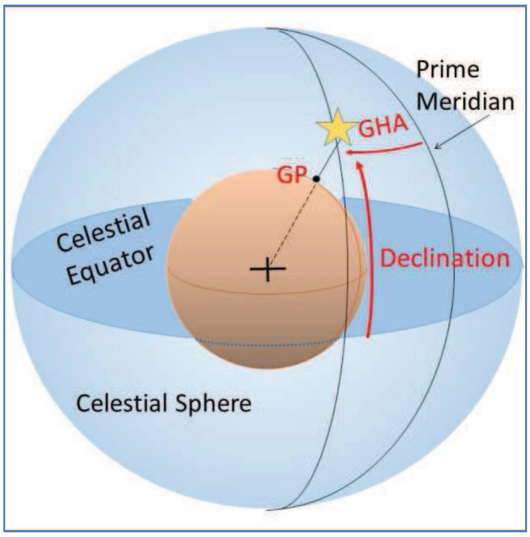


Figure I.17 Celestial navigation system

c) Inertial guidance system

An inertial guidance system is a type of guidance system used in missiles that determines the missile's position, velocity, and orientation by measuring its acceleration and rotation rates. This type of guidance system is widely used in modern missile systems (see figure I.21), including intercontinental ballistic missiles, cruise missiles, and air-to-surface missiles. The inertial guidance system works by using sensors, such as accelerometers and gyroscopes, to detect the missile's movements and changes in orientation. Based on these measurements, the guidance system can calculate the missile's position, velocity, and orientation in three-dimensional space. The system uses this information to guide the missile towards its target by making constant adjustments to the missile's course and orientation. One of the key advantages of an inertial guidance system is its ability to function independently of external signals, such as *GPS* or other navigation aids. This makes it less vulnerable to electronic jamming or other types of interference that could disrupt other types of guidance systems. In addition, inertial guidance systems can provide accurate guidance for long-range missiles that may travel beyond the range of *GPS* signals. Overall, the inertial guidance system is an important component of modern missile technology, and has played a critical role in the development of advanced missile systems.

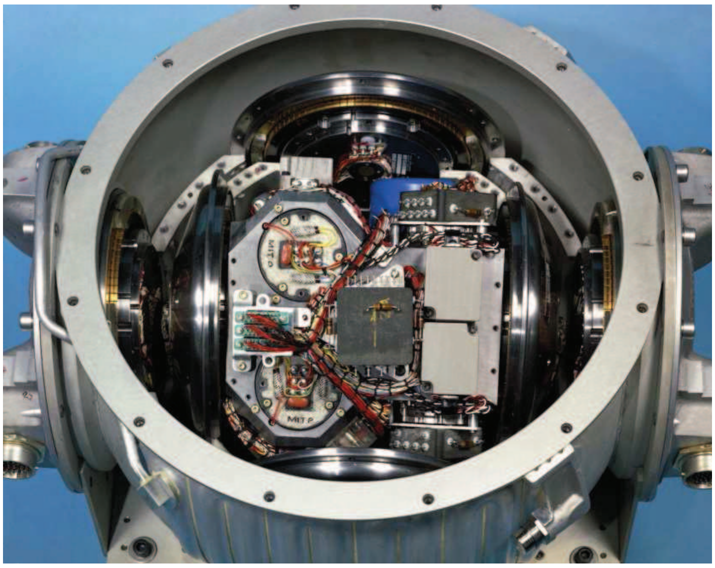


Figure I.18 Inertial guidance system

I.6.Optimal control methods used in guided missiles

There are several methods that have been used to control missiles during the pursuit of the target, each of them guaranteed the optimum path and the optimum endgame point with attendant to other factors such as cost, time and speed ...etc., In the following section we will take a look on some of these methods.

I.6.1 Linear Quadratic Regulator method

LQR (Linear Quadratic Regulator) is a method used in control theory to design optimal control systems for linear time-invariant systems. It aims to minimize a quadratic cost function, which includes the system state and control inputs. The *LQR* method provides a way to design feedback controllers that can stabilize unstable systems and optimize system performance. It is a widely used technique in aerospace, automotive, and other industries for controlling various systems, including flight control systems, robotic arms, and chemical processes. In *LQR*, the system dynamics are represented by a state-space model, and the objective is to find the optimal feedback control law that minimizes the cost function. The control law is designed as a linear function of the state variables, and the optimal gains are calculated by solving the algebraic *Riccati* equation. The *LQR* method has many advantages, including its simplicity, efficiency, and robustness. It is also a versatile method that can be applied to a wide range of systems and can handle both single-input-single-output (*SISO*) and multi-input-multi-output (*MIMO*) systems.

As a result "*LQR*" methods are very important in missiles as they enable the design of optimal control systems that can effectively stabilize and control the missile during flight. In missile guidance and control, it is used to design feedback controllers that can optimize the missile's performance while minimizing the control effort. This is critical for achieving accurate and reliable missile guidance and control, especially for long-range and high-speed missiles. It can be used to design controllers for various types of missile systems, including ballistic missiles, cruise missiles, and air-to-air missiles, which allows for the design of controllers that can handle disturbances, uncertainties, and variations in the missile's dynamics, making the missile more robust and reliable. Additionally, this method can be used for designing autopilot systems for missiles, which are critical for ensuring the missile follows a predetermined trajectory and hits its target accurately. The autopilot system uses the *LQR* method to calculate the optimal control inputs that can keep the missile on course and make any necessary adjustments in real-time.

I.6.2 *KALMAN* filter

*Kalma*n filtering is a mathematical method used for estimating the state of a dynamic system based on a sequence of measurements. It is a recursive algorithm that updates the estimated state of the system at each time step by combining the current measurement with the previous state estimate and its associated uncertainty which is particularly useful in situations where the measurements are noisy or incomplete. Moreover, the *Kalman* filter is widely used in control systems, navigation systems, robotics, and other applications where the state of a system needs to be estimated based on noisy measurements. It is especially useful in situations where the state of the system is not directly observable or where the measurements are affected by random errors or other sources of uncertainty. It works by modeling the system as a set of linear equations with random disturbances. It then uses a series of mathematical equations to estimate the system state, including a prediction step and an update step. The prediction step uses the system model to estimate the state of the system at the next time step, while the update step combines the current measurement with the predicted state estimate to generate a more accurate estimate of the current state. The *Kalman* filter is known for its efficiency, accuracy, and ability to handle noisy and incomplete data. It is a powerful tool for estimation and control problems and has been used in a wide range of applications, including aerospace and robotics.

The *Kalman* filter is an important of the method (Linear Quadratic Gauss), tool in missile guidance and control systems as it enables accurate estimation of the missile's state and improves the accuracy and effectiveness of the missile. In missile guidance and control, this method is used to estimate the missile's position, velocity, and orientation based on measurements from various sensors, such as GPS, inertial measurement units, and radar. It combines the measurements with a dynamic model of the missile's behavior to generate an estimate of the missile's current state. The estimated state of the missile is then used to calculate the optimal control inputs that can guide the missile to its target. The accuracy of the estimated state is critical for ensuring that the missile hits its target accurately and avoids any obstacles or countermeasures.

Moreover, this technique is particularly useful in missile guidance and control because it can handle noisy and incomplete measurements, which are common in high-speed and long-range missiles, it can also handle uncertainties in the missile's dynamics and can adapt to changes in the missile's behavior. Additionally, the *Kalman* filter can be used in conjunction with other control techniques, such as *LQR*, to design optimal guidance and control systems for missiles. It provides accurate state estimation, which is essential for the design of effective guidance and control strategies. Overall, the *Kalman* filter is a critical tool in missile guidance and control systems, which are essential for ensuring the accuracy and effectiveness of missiles in combat situations

I.6.3 H-infinity method

The H-infinity method, also known to as the H-infinity control theory or the optimal control theory, is a mathematical approach to designing robust control systems. The main objective of the H-infinity method is to design a controller that can handle uncertainties and disturbances in a system and provide optimal performance. In H-infinity control, the system is modeled as a linear time-invariant system with uncertain parameters, and the controller is designed to minimize the effect of uncertainties on the system's output, which is widely used in control engineering applications, such as aerospace, automotive, and manufacturing industries, where the control systems must operate reliably and with high precision in the presence of uncertainties and disturbances. Overall, the goal of this method is to design a controller that minimizes the maximum gain of the system transfer function over a certain frequency range, while still meeting certain performance and stability criteria. Hence, this technique is a powerful tool for designing control systems that can handle uncertainty and disturbances, making it a popular choice for aerospace and other high-performance applications.

The H-infinity method is particularly important in missile guidance and control systems because it provides a framework for designing controllers that are optimal and abuts against uncertainties and disturbances during flight such as aerodynamic and atmospheric variations, sensor noise, and target maneuvers, It can be used to design controllers that are able to reject disturbances and track desired trajectories with high accuracy, even in the presence of uncertainties. In addition, the H-infinity method can be used to design robust missile guidance and control systems that can handle a wide range of disturbances and uncertainties without requiring extensive tuning or redesign. This can be particularly important in military applications, where missiles may be used in a variety of different scenarios and environments. Overall, the H-infinity method is an important tool in the design of robust missile guidance and control systems, allowing for optimal performance in the face of uncertainties and disturbances.

I.6.4 Minimal time of optimal control

Solve an optimal control problem with a minimal final time. Set up and solve the optimal control input to be determined is the commanded missile lateral acceleration. Continuous control will be assumed in deriving the guidance laws, and it is desired to minimize the expected mean square of the miss distance subject to a penalty function on the total control energy. Therefore, the performance index to be minimized is assumed to be given by the terminal miss distance at the intercept time plus a weighting function on the control effort which is the commanded control. This equation states that the optimal control consists in minimizing the terminal mean-square miss distance plus the weighted integral-square missile acceleration normal to the line of sight (*LOS*).

I.7. Conclusion

Throughout this chapter, we have studied deeply and presented the fundamentals of missiles, including their launch mode, range, propulsion systems, warheads, and guidance systems. Specifically, our focus on guidance systems led us to discuss their three distinct phases (boost, midcourse, and terminal) in detail. We have then mentioned the three main types of guidance systems (control, homing, and self-contained), and the features that distinguish them from each other. Finally, we provided a general overview of optimal control methods that are employed in modern warfare and highlighted their significance in enhancing missile effectiveness.

In conclusion, guided missiles represent a technological achievement that has revolutionized the way wars are fought. They offer a level of precision and accuracy that was unimaginable, and can strike targets with high speed and effectiveness.

Chapter II

Missile Motion and Guidance System Modeling

II.1. Introduction

Missile modeling is a process that involves creating mathematical representation and simulation to study and analyze the behavior of missiles. It plays a crucial role in the design, development, and evaluation of missile systems. By constructing accurate and detailed models, engineers and researchers can gain insights into various aspects of missile performance, including aerodynamics, propulsion, guidance, and control. Therefore, aerodynamic modeling focuses on understanding how missiles interact with the surrounding air, taking into account factors such as shape, size and surface characteristics. This involves studying the forces and moments acting on the missile as it moves through different flight regimes.

Moreover, propulsion modeling involves analyzing the performance of the missile's engine, including thrust generation, fuel consumption, and acceleration characteristics. By accurately modeling the propulsion system, engineers can assess the missile's range, speed, and maneuverability.

Additionally, missile modeling allows us the evaluation of different operating conditions and parameters. It enables engineers to assess the missile's performance under different environmental conditions, target scenarios and engagement strategies. Through simulations and analysis, engineers can optimize design choices, improve system performance, and enhance mission effectiveness of missiles.

Overall, missile modeling is an important tool that helps engineers and researchers to understand, analyze and optimize the performance of missiles. Also, it provides valuable insights into aerodynamics, propulsion, guidance and control, enabling the development of more capable and efficient missile systems.

In this chapter we will focus on the derivation of the equations governing the overall motion of a missile, by considering its six degrees of freedom (*6DOF*). Subsequently, we will develop the simplified three degrees of freedom (*3DOF*) equations. Additionally, we will explore the concepts of lift and drag forces, talking about the seeker/tracker subsystem and provide an overview of commonly used navigation systems.

II.2. Six degrees of freedom missile motion modeling

In this part we will derive the missile equations of motion (see figure II.1). First of all we have to make some points clearer as possible by considering the following assumptions [1]:

a) The mass of the missile is constant.
b) The airframe motion is described by a translation of the missile gravity center and by rotations about this point.
c) The airframe is assumed to have a plane of symmetry coinciding with the vertical plane of reference.
d) The vertical reference plane is the plane defined by the missile X_b and Z_b axes as shown in Figure (II.1). The Y_b axes is perpendicular to this plane of symmetry and the term of inertia tensor I_{XY} and I_{YZ} vanish.

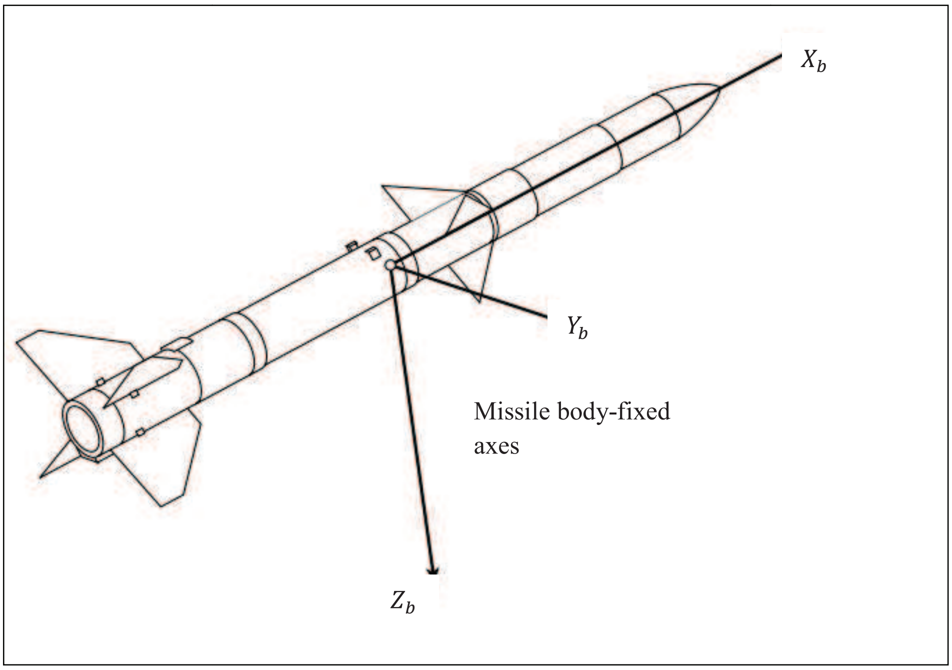


Figure II.1 Orientation of missile body fixed-axes [2]

e) It is assumed that the aerodynamic forces and moments exerted on the vehicle remain constant regardless of the missile's roll position in relation to the free-stream velocity vector. As a result, this assumption significantly simplifies the equations of motion by removing the aerodynamic cross-coupling terms between roll, pitch and yaw motions.

f) The missile equations of motion are written in the body-axes coordinate frame;
g) A spherical Earth rotating at a constant angular velocity is assumed;
h) The vehicle aerodynamics is nonlinear;
i) The undisturbed atmosphere rotates with Earth;
j) An inverse-square gravitational law is used for the spherical Earth model;
k) The gradients of low frequency winds are small enough to be neglected.

Now, we have to develop the generalized missile equations of motion. So, let us assume that the missile has six degrees of freedom (*6DOF*) that are the three translation speeds (u, v and w) and the three rotation velocities (p, q and r) about and along the missile (X_b, Y_b and Z_b) axes. Mathematically, the translation and rotation of a missile can be expressed by the Newton's laws. In inertial 'or absolute' reference frame, Newton's second laws are given by [3]:

$$\sum \vec{F}_{Text} = m\vec{a} \tag{II.1}$$

$$\sum \vec{\tau} = \frac{d}{dt}\left(\vec{r} \times m\vec{V}\right) \tag{II.2}$$

Where $\vec{\boldsymbol{F}}_{\boldsymbol{T.ext}}$ is the external forces acting on the missile, τ is the total torque applied on the missile.

$\vec{V}$ is the linear velocity vector of the missile which is define by its components (u, v and w) along the missile (X_b, Y_b and Z_b) body axes respectively. Therefore, the missile linear velocity vector $\vec{V}$ can be expressed by:

$$\vec{V} = u\vec{\imath} + v\vec{\jmath} + w\vec{k} \tag{II.3}$$

Where ($\vec{\imath}, \vec{\jmath}$ and $\vec{k}$) are the unit vectors along the respective missile body axes.

In the same way, the angular velocity vector $\vec{\omega}$ of the missile can be decomposed into its components (p, q and r) around the (X_b, Y_b and Z_b) axes, respectively, in the following manner:

$$\vec{\omega} = p\vec{\imath} + q\vec{\jmath} + r\vec{k} \tag{II.4}$$

Where ($p, q,$ and r) are called roll, pitch and yaw rates. All components are illustrated in figure (II.2).

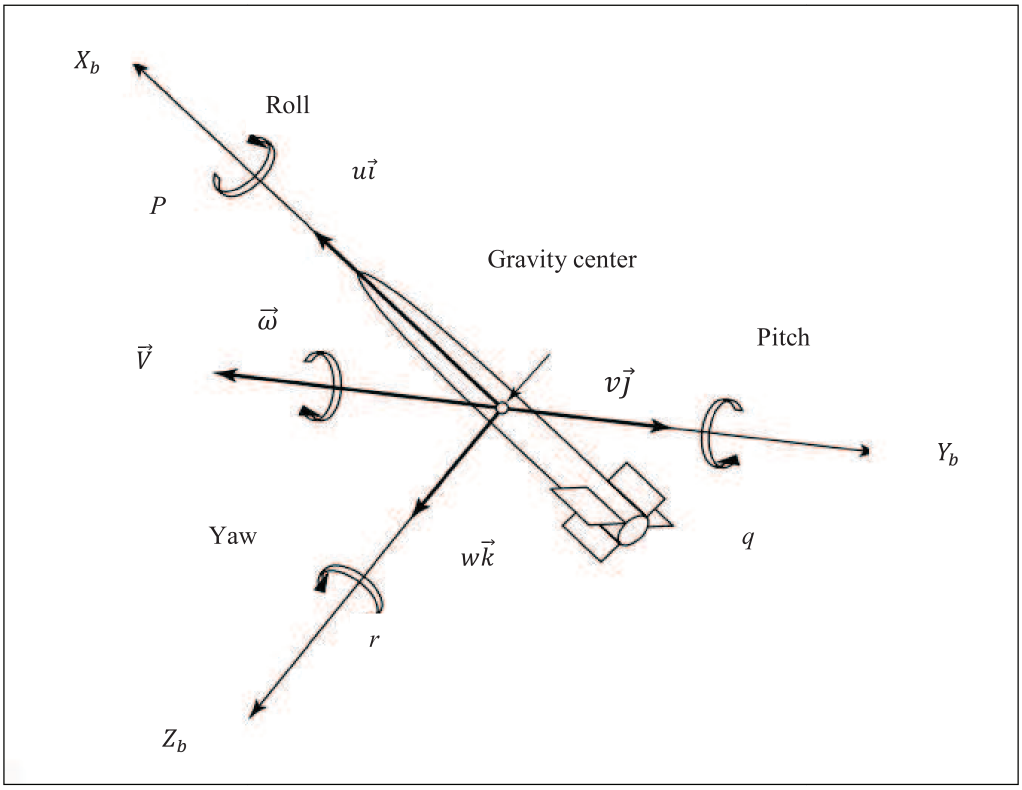


Figure II.2 Representation of the missile's six degrees of freedom [4]

Now, in case of a non-inertial reference frame, the Newton's second law can be expressed by the following equations:

$$\sum \vec{F}_{Text} = \frac{d(m\vec{V})}{dt} \qquad (II.5)$$

$$\sum \vec{M}_T = \frac{d\vec{H}}{dt} \qquad (II.6)$$

Where $\vec{H}$ is the angular momentum.

Therefore, one can write the equation (II.5) in the following manner:

$$\sum \vec{F}_{Text} = m\frac{d\vec{V}}{dt} = m\vec{a} \qquad (II.7)$$

Note that equation (II.5) can be written in scalar from as follows:

$$F_x = \frac{d(mu)}{dt}, \qquad F_y = \frac{d(mv)}{dt}, \qquad F_z = \frac{d(mw)}{dt} \qquad (II.8)$$

These components represent the aerodynamic, propulsive and gravitational forces acting on the missile. Where F_x, F_y and F_z are the total components of external forces along the missile's (X_b, Y_b and Z_b) axes.

In the same way, the moment equation (II.6) can be rewritten as follows:

$$L = \frac{dH_x}{dt}, \qquad M = \frac{dH_y}{dt}, \qquad N = \frac{dH_z}{dt} \qquad (II.9)$$

Where H_x, H_y and H_z are the components of the angular momentum moment along the X_b, Y_b and Z_b axes respectively. Also, L, M and N are the roll, pitch and yaw moment respectively.

Now, let us summarize the various forces, moments and axes used in developing the missile *6DOF* equations of motion:

a) Total external forces:

$$\vec{F}_{Text} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} \qquad (II.10)$$

b) Velocity

$$\vec{V} = u\vec{\imath} + v\vec{\jmath} + \omega\vec{k} \qquad (II.11)$$

c) Moment of external Forces

$$\sum \vec{M}_T = L\vec{\imath} + M\vec{\jmath} + N\vec{k} \qquad (II.12)$$

d) Angular momentum

$$\vec{H} = H_x\vec{\imath} + H_y\vec{\jmath} + H_z\vec{k} \qquad (II.13)$$

e) Angular Velocity

$$\vec{\omega} = p\vec{\imath} + q\vec{\jmath} + r\vec{k} \qquad (II.14)$$

The next step is to develop the relation that allows us to express missile's linear velocity vector $\vec{V}$ in the earth fixed X_e , Y_e and Z_e axes system. This is given by the following relation:

$$\left(\frac{d\vec{V}}{dt}\right)_E = \left(\frac{d\vec{V}}{dt}\right)_{body-fixed} + \vec{\omega} \times \vec{V} \qquad (II.15)$$

Where $\vec{\omega}$ is the total angular velocity vector of the missile with respect to the earth.

By these relation, the force equation (II.5) can be written as follows:

$$\vec{F}_{Text} = m\left(\frac{d\vec{V}}{dt}\right)_{body-fixed} + m\,(\vec{\omega}\times\vec{V}) \qquad (II.16)$$

Noting that the first term on the right-hand side of the equation (II.15) can be broke up into the components below:

$$\left(\frac{d\vec{V}}{dt}\right)_{body-fixed} = \left(\frac{du}{dt}\right)\vec{\imath} + \left(\frac{dv}{dt}\right)\vec{\jmath} + \left(\frac{dw}{dt}\right)\vec{k} \qquad (II.17)$$

Where $\left(\frac{du}{dt}\right)$, $\left(\frac{dv}{dt}\right)$ and $\left(\frac{dw}{dt}\right)$ are the forward, right wing and downward accelerations respectively.

The cross vector is calculated as below:

$$\vec{\omega}\times\vec{V} = \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ p & q & r \\ u & v & w \end{bmatrix} = (wq - vr)\vec{\imath} + (ur - \omega p)\vec{\jmath} + (vp - uq)\vec{k} \qquad (II.18)$$

Thus, from equations (II.10), (II.17) and (II.18) one can write the missile's linear equations of motion by considering $(X_b\ Z_b)$ plane of symmetry. So we have:

$$\sum F_x = m(\dot{u} + wq - vr) \qquad (II.19)$$

$$\sum F_y = m(\dot{v} + ur - wp) \qquad (II.20)$$

$$\sum F_z = m(\dot{w} + vp - uq) \qquad (II.21)$$

Until now, we have developed the right-hand side of equation (II.5). In the same way, we can develop also the right-hand side of equation (II.6).

In the case, the expression of the angular momentum $\vec{H}$ is given by [5]:

$$\vec{H} = \int \vec{r}\times(\vec{\omega}\times\vec{r})dm \qquad (II.22)$$

Where $\vec{r}$ is the position of the mass element measured from the gravity center and $\vec{\omega}$ is the angular velocity vector .It can be expressed as follows:

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k} \qquad (II.23)$$

Now, the next step is to calculate the cross product:

$$\vec{\omega} \times \vec{r} = \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ p & q & r \\ x & y & z \end{bmatrix} = (zq - yr)\vec{\imath} + (xr - zp)\vec{\jmath} + (yp - xq)\vec{k} \qquad (II.24)$$

Moving to the other cross product:

$$\vec{r} \times (\vec{\omega} \times \vec{r}) = \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & z \\ (zq - yr) & (xr - zp) & (yp - xq) \end{bmatrix}$$
$$= [(y^2 + z^2)p - xyq - xzr]\vec{\imath} + [(z^2 + x^2)q - yzr - xyp]\vec{\jmath}$$
$$+ [(x^2 + y^2)r - xzp - yzq]\vec{k} \qquad (II.25)$$

By considering $(X_b\,, Y_b)$ as a plane of symmetry of the missile, some the inertia tensor elements vanish ($I_{xy} = I_{yz} = 0$). This leads to express the momentum moment components as follows:

$$H_x = p\int (y^2 + z^2)dm - r\int xzdm = pI_x - rI_{xz} \qquad (II.26)$$

$$H_y = q\int (x^2 + z^2)dm = qI_y \qquad (II.27)$$

$$H_z = r\int (x^2 + y^2)dm - p\int xzdm$$
$$= rI_z - pI_{xz} \qquad (II.28)$$

And by developing the expression of $\vec{H}$ into its components since it changes in magnitude and direction, one can write the equation (II.6) as follows:

$$\sum \vec{M}_T = \left(\frac{d\vec{H}}{dt}\right) + \vec{\omega} \times \vec{H} \qquad (II.29)$$

Next, the momentum components first derivative is given as follows:

$$\left(\frac{dH_x}{dt}\right) = \left(\frac{dp}{dt}\right)I_x - \left(\frac{dr}{dt}\right)I_{xz} \qquad (II.30)$$

$$\left(\frac{dH_y}{dt}\right) = \left(\frac{dq}{dt}\right) I_y \qquad (II.31)$$

$$\left(\frac{dH_z}{dt}\right) = \left(\frac{dr}{dt}\right) I_z - \left(\frac{dp}{dt}\right) I_{xz} \qquad (II.32)$$

Now, the vector cross product of equation (II.29) is calculated hereafter:

$$\vec{\omega} \times \vec{H} = \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ p & q & r \\ H_x & H_y & H_z \end{bmatrix} = \left(qH_z - rH_y\right)\vec{i} + (rH_x - pH_z)\vec{j} + \left(pH_y - qH_x\right)\vec{k} \qquad (II.33)$$

From, (II.12), (II.30 to II.33) and substituting from (II.26, 27 and 28), the angular momentum equations will be written as follow:

$$L = \dot{p}I_x + \left(I_z - I_y\right)qr - (\dot{r} + pq)I_{xz} \qquad (II.34)$$

$$M = \dot{q}I_y + (I_x - I_z)pr + (p^2 - r^2)I_{xz} \qquad (II.35)$$

$$N = \dot{r}I_z + \left(I_y - I_x\right)pq - (\dot{p} + qr)I_{xz} \qquad (II.36)$$

Where $\dot{p}$, q and r are the roll, pitch and yaw accelerations respectively.

Therefore, equations (II.19) to (II.21) and (II.34) to (II.36) represents the complete *6DOF* missile translation and rotation motion model. Note that these equations which describe the behavior of a missile are non-linear. It should be noted also that for cruciform missiles with rotational symmetry, I_y is equal to I_z and I_{xz} is equal to zero. Hence, one can write.

$$L = \dot{p}I_x + qr\left(I_z - I_y\right) \qquad (II.37)$$

$$M = \dot{q}I_y + (I_x - I_z)pr \qquad (II.38)$$

$$N = \dot{r}I_z + \left(I_y - I_x\right)pq \qquad (II.39)$$

From these equations, the rotation accelerations can be expressed as follows:

$$\dot{p} = qr\left[\frac{(I_y - I_z)}{I_x}\right] + \left(\frac{L}{I_x}\right) \qquad (II.40)$$

$$\dot{q} = pr\left[\frac{(I_z - I_x)}{I_y}\right] + \left(\frac{L}{I_y}\right) \qquad (II.41)$$

$$\dot{r} = pq\left[\frac{(I_x - I_y)}{I_z}\right] + \left(\frac{L}{I_z}\right) \quad (II.42)$$

Elsewhere, the equations of angular velocities can be written in terms of Euler-angels (ψ, φ, θ) and the rates $(p, q, \text{and } r)$ as follows (see figure II.3):

$$\left(\frac{d\psi}{dt}\right) = \frac{q sin(\varphi) + r cos(\varphi)}{cos(\theta)} \quad (II.43)$$

$$\left(\frac{d\varphi}{dt}\right) = p + \left(\frac{d\psi}{dt}\right)\sin(\theta) \quad (II.44)$$

$$\left(\frac{d\theta}{dt}\right) = q cos(\varphi) - r\sin(\varphi) \quad (II.45)$$

Consequently, one can obtain the values of these angels (ψ, φ, θ) by the integration as below:

$$\psi = \psi_0 + \int_0^t \left(\frac{d\psi}{dt}\right) dt \quad (II.46)$$

$$\varphi = \varphi_0 + \int_0^t \left(\frac{d\varphi}{dt}\right) dt \quad (II.47)$$

$$\theta = \theta_0 + \int_0^t \left(\frac{d\theta}{dt}\right) dt \quad (II.48)$$

Euler angles are illustrated in the following figure:

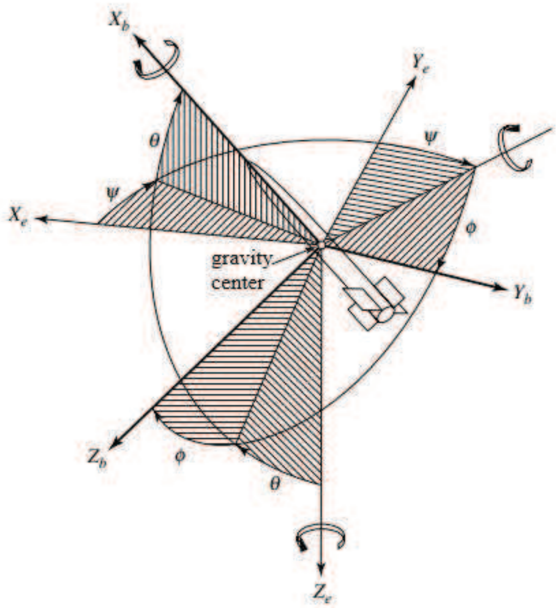


Figure II.3 Euler angles and body fixed axes [6]

The components of the missile velocity in the Earth-fixed coordinate system (X_e, Y_e and Z_e) in terms of (u, v, w) and (ψ, φ, θ) are given by the following transformation:

$$\frac{dX_e}{dt} = \big(cos(\theta)\cos(\psi)\big)u + \big(\cos(\psi)\sin(\varphi)\sin(\theta) - \sin(\psi)\,co(\,\varphi)\big)v +$$

$$(\cos(\psi)\cos(\varphi)\sin(\theta) + \sin(\psi)\sin(\varphi))w \qquad (II.49)$$

$$\frac{dY_e}{dt} = \big(\cos(\theta)\sin(\psi)\big)u + \big(\sin(\psi)\sin(\varphi)\sin(\theta) + \cos(\psi)\cos(\varphi)\big)v \qquad (II.50)$$

$$\frac{dZ_e}{dt} = -\big(sin(\theta)\big)u + \big(sin(\varphi)\cos(\theta)\big)v + \big(cos(\theta)\cos(\varphi)\big)w \qquad (II.51)$$

In the matrix form, equations (II.49) to (II.51) can be written as:

$$\frac{d}{dt}\begin{bmatrix} X_e \\ Y_e \\ Z_e \end{bmatrix} = C_e^b \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad (II.52)$$

Where C_e^b is the transformation matrix between body-fixed and earth-fixed coordinates (see figure II.3).

In the same manner, one can express $X_e, Y_e\ and\ Z_e$ as follows:

$$X_e = X_{e0} + \int_0^t \left(\frac{dX_e}{dt}\right) dt \qquad (II.53)$$

$$Y_e = Y_{e0} + \int_0^t \left(\frac{dY_e}{dt}\right) dt \qquad (II.54)$$

$$Z_e = Z_{e0} + \int_0^t \left(\frac{dZ_e}{dt}\right) dt \qquad (II.55)$$

Note that the missile altitude is $h = -Z_e$, since the *Z*-axes is downward.

Hence, we have determined the expressions of both right-hand sides of equations (II.5) and (II.6). Now, let us develop the expressions of the left-hand sides (external forces and moments). Hereafter, we take a look on the external forces acting on the missile body.

a) Thrust

The primary force propelling a missile forward is known as thrust. It is generated by the missile's propulsion system. This force is created through the expulsion of a reaction mass, either in the form of solid or liquid propellant. The total thrust T is the combination of two components: momentum thrust and pressure thrust. These components can be expressed as follows [7]:

$$T = m_p v_e + (p_e - P_a) A_e \qquad (II.56)$$

Where m_p is the mass expelled in unit time (the propellant mass flow rate), p_e is the exhaust pressure, P_a is the ambient pressure, v_e is the exhaust velocity, and A_e is the exit area of the motor nozzle.

b) Drag and lift

Drag and lift are categorized as aerodynamic forces. Drag is a force that acts along the velocity vector and hinders the movement of the missile. As a result, it slows the missile speed, so reducing its acceleration capability. Furthermore, lift is directed perpendicularly up with respect to drag. It is the dominant force governing the flight of a missile. The drag and lift forces equations are given below [8]:

$$Lift = C_L Q S \qquad (II.57)$$

$$Drag = C_D Q S \qquad (II.58)$$

Where S represents the reference area, C_L and C_D are the lift and drag coefficients respectively. Q represents the dynamic pressure, which is dependent on the atmospheric pressure (*press*) and the Mach number as follows [9]:

$$Q = 0.7\, PRESS\, (Mach)^2 \qquad (\text{II}.59)$$

c) Gravitational forces

The gravitational forces are given below:

$$\begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} = g \begin{bmatrix} -\sin\theta \\ \cos\theta \sin\varphi \\ \cos\theta \cos\varphi \end{bmatrix} \qquad (II.60)$$

Where the gravitational forces are written in terms of Euler angels $\psi, \varphi,$ and θ which are corresponding the following order of rotation about the Z_e axes through angle θ, rotation about the new position of the Y_e axes through angle φ, putting the X_e axes into coincidence with the x axis; rotation about the *x* axes through angle ψ (see Figure II.3).

Let us move now to develop the left-hand side of forces and moments equations (II.5) and (II.6). The general equation of motion for a broad range of missiles, known as the standard body-axes *6DOF* equation, can be represented as follows [10]:

$$\begin{cases} u = rv - qw + X + G_x + T_x \\ v = -ru + pw + Y + G_y + T_y \\ w = qu - pv + Z + G_z + T_z \\ \dot{p} = -L_{pq}pq - L_{qr}qr + L + L_T \\ \dot{q} = -M_{rp}rp - M_{r^2p^2}(r^2 - p^2) + M + M_T \\ \dot{r} = -N_{pq}pq - N_{qr}qr + N + N_T \end{cases} \quad (II.61)$$

In this equations $u, v,$ and w represents the components of velocity along the $x, y,$ and z axes, respectively; $p, q,$ and r denote the roll, pitch, and yaw rates, respectively; G_x, G_y , and G_z represents the gravity components; $X, Y,$ and Z model accelerations produced by the aerodynamic forces; $L, M,$ and N model angular accelerations produced by the aerodynamic moments; T_x , T_y , and T_z model propulsion system forces; and $L_T, M_T,$ and N_T model the moments produced by the propulsion system. Note that all the variables appearing on the right-hand side of equations (II.61) are expressed in acceleration unit. Moreover, the coefficients $L_{pq}, L_{qr}, M_{rp}, N_{pq}, M_{r^2p^2}$ and N_{qr} are derived through the simplification of the more general and comprehensive moment equations ($m_x, m_y,$ and m_z) as below:

$$\begin{cases} m_x = I_{xx}\dot{p} - (I_{yy} - I_{zz})qr + I_{yz}(r^2 - q^2) - I_{xz}(pq - \dot{r}) + I_{xy}(rp - \dot{q}) \\ m_y = I_{yy}\dot{q} - (I_{zz} - I_{xx})rp + I_{xz}(r^2 - p^2) - I_{xy}(qr - \dot{p}) + I_{yz}(pq - \dot{r}) \\ m_z = I_{zz}\dot{r} - (I_{xx} - I_{yy})pq + I_{xy}(q^2 - p^2) - I_{yz}(rp - \dot{q}) + I_{xz}(qr - \dot{p}) \end{cases} \quad (II.62)$$

Where I_{xy}, I_{xz} and I_{yz} are the mass product of inertia about the x and y, x and z, y and z axes, respectively. Also, I_{xx}, I_{yy} and I_{zz} denote the mass moments of inertia about the $x, y,$ and z axes. It is preferred for the $x, y,$ and z axes to coincide with the principal axes of inertia, ensuring that the product of inertia terms become zero and for symmetry about the xz plan both of I_{yz} and I_{xy} are equal to zero, In this case, equations (II.62) will be written as follows:

$$\begin{cases} m_x = I_{xx}\dot{p} - (I_{yy} - I_{zz})qr - I_{xz}(pq - \dot{r}) \\ m_y = I_{yy}\dot{q} - (I_{zz} - I_{xx})rp - I_{xz}(r^2 - p^2) \\ m_z = I_{zz}\dot{r} - (I_{xx} - I_{yy})pq + I_{xz}(qr - \dot{p}) \end{cases} \quad (II.63)$$

In the case of a cruciform configuration, which exhibits symmetry about both the $xy\ and\ xz$ planes, the value of I_{xz} also becomes zero. Consequently, the equation (II.63) is simplified to:

$$\begin{cases} m_x = I_{xx}\dot{p} - \left(I_{yy} - I_{zz}\right)qr \\ m_y = I_{yy}\dot{q} - (I_{zz} - I_{xx})rp \\ m_z = I_{zz}\dot{r} - \left(I_{xx} - I_{yy}\right)pq \end{cases} \qquad (II.64)$$

By the use of the formulations for the lift and drag equations (II.57) and (II.58), the coefficients $C_x, C_y\ and\ C_z$ representing the aerodynamic forces are modelled as non-dimensional quantities that are scaled to units of force, thus:

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \frac{QS}{m}\begin{bmatrix} C_x \\ C_y \\ C_z \end{bmatrix} \qquad (II.65)$$

Note that *m* represent the mass of missile.

The set of equations (II.63) can be expressed as the moment equations from (II.61), where the aerodynamic moments exerted on the body are represented as:

$$\begin{cases} L = \frac{QSl}{I_{xx}\,I_{zz}-{I_{xz}}^2} C_l I_{zz} + C_n I_{xz} \\ M = \frac{QSl}{I_{yy}} C_m \\ N = \frac{QSl}{I_{xx}\,I_{zz}-I_{xz}^2} C_n I_{xx} + C_l I_{xz} \end{cases} \qquad (II.66)$$

In this representation, $C_l,\ C_m\ and\ C_n$ represent the non-dimensional coefficients for aerodynamic moments related to rolling, pitching, and yawing, respectively. The length l is a reference dimension, typically corresponding to the diameter of the missile. The cross-axis inertia symmetry term couples the roll–yaw moment equations. And the coefficients $L_{pq},\ L_{qr},\ M_{rp},\ N_{pq},$ $M_{r^2p^2}$ and N_{qr} of the equation (II.61) are written as follows:

$$\begin{cases} L_{pq} = \frac{I_{xz}\,I_{xx}-I_{yy}I_{zz}}{I_{xx}\,I_{zz}-{I_{xz}}^2},\ I_{qr} = \frac{I_{zz}\left(I_{zz}-I_{yy}\right)-{I_{xz}}^2}{I_{xx}\,I_{zz}-{I_{xz}}^2},\ M_{pr} = \frac{I_{xx}-I_{zz}}{I_{yy}} \\ M_{r^2p^2} = \frac{I_{xy}}{I_{yy}}, \quad N_{qr} = \frac{I_{xz}\,I_{ZZ}-I_{xx}I_{yy}}{I_{xx}\,I_{zz}-{I_{xz}}^2},\ N_{pq} = \frac{I_{xx}\,I_{xx}-I_{yy}-{I_{xz}}^2}{I_{xx}\,I_{zz}-{I_{xz}}^2} \end{cases} \qquad (II.67)$$

Note that equations (II.61), (II.66) and (II.67) presented above are formulated for missile configurations that exhibit symmetry about the $x - z$ plane, resulting in $I_{yz} = 0\ and\ I_{xy} = 0$, both mass and geometrical symmetry are assumed here, while it is conceivable that small mass asymmetries could be present in an actual missile configuration.

Moreover, if the mass distribution is such that $I_{yy} = I_{zz}$ (applicable to missiles with circular body cross sections), the above expressions can be simplified. Also, in the case of a cruciform configuration that exhibits symmetry about both the xy and xz planes, I_{xz} is also equal to zero. Consequently, equations (II.61), (II.66) and (II.67) can be further simplified.

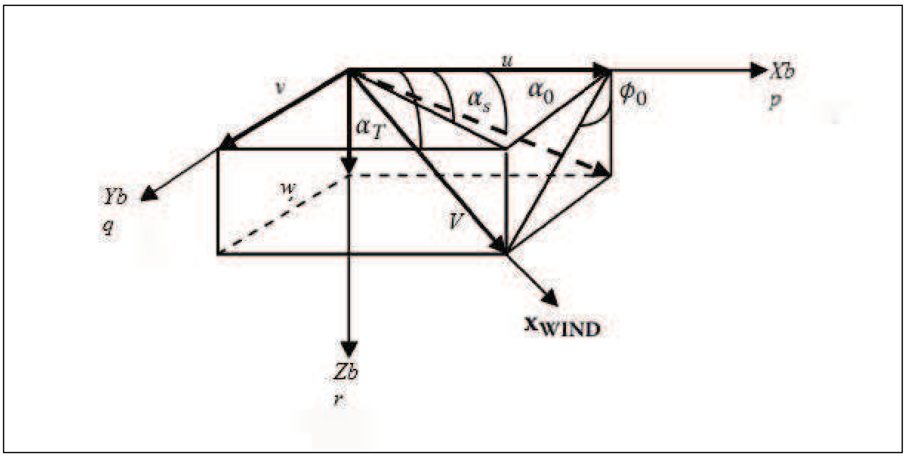


Figure II.4 Coordinate systems used in missile dynamics [11]

When the expressions that define the angels α_0, α_s, and φ_0 are given by (see figure II.4):

$$\begin{cases} \alpha_0 = \tan^{-1}(w/u) \\ \alpha_s = \tan^{-1}(v/u) \\ \varphi_0 = \tan^{-1}(v/w) \end{cases} \qquad (II.68)$$

Note that the total angel α_T is defined as the angle between the X_b axes and the magnitude of the missile velocity vector as show in figure II.4. It is given by the following relationship:

$$\tan^2(\alpha_T) = \frac{{v_y}^2 + {v_z}^2}{{v_x}^2} = \tan^2(\alpha_0) + \tan^2(\alpha_s) \qquad (II.69)$$

By assuming insignificant changes of speed (i.e. $\dot{v}_x \approx 0$,), the accelerations $\dot{v}_y$ and $\dot{v}_z$ can be presented as $\dot{v}_y \approx v_x\alpha_s$ and $\dot{v}_z \approx v_x\alpha_0$. Also for small angles of attack and sideslip $\alpha_0 \approx \frac{v_z}{v_x}$, $\alpha_s \approx \frac{v_y}{v_x}$ and $v_x = V$, the force equations (II.61) can be simplified to the following equations:

$$\begin{cases} Vq\alpha_0 - r\alpha_0 = X + G_x + T_x \\ V\dot{\alpha_s} + r - p\alpha_0 = Y + G_y + T_y \\ V(\dot{\alpha_0} - q + p\alpha_s) = Z + G_z + T_z \end{cases} \quad (II.70)$$

The aerodynamic coefficients C_x, C_y and C_z which determine the aerodynamic forces, as well as C_l, C_m and C_n, which relate to aerodynamic moments, are commonly represented as functions of various factors. These factors include the pitch-plane angle of attack α_0, the sideslip angle α_s in the yaw-plane, the aerodynamic roll angle φ_0 (see figure II.4), the Mach number, the body rates $p, q,$ and r, the deflections of aerodynamic control surfaces in pitch, yaw and roll ($\delta P, \delta Y, \delta R$), changes in the center of gravity, and whether the main propulsion system is active or inactive.

Based on equation (II.61), the components A_x, A_y and A_z represent the body axis accelerations are written as follows:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} X + G_x + T_x \\ Y + G_y + T_y \\ Z + G_z + T_z \end{bmatrix} \quad (II.71)$$

According to (II.65), the equation (II.71) can be transformed into:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \frac{QS}{m} \begin{bmatrix} C_x(\alpha_0, \alpha_s, \delta P, \delta Y, \delta R) \\ C_y(\alpha_0, \alpha_s, \delta P, \delta Y, \delta R) \\ C_z(\alpha_0, \alpha_s, \delta P, \delta Y, \delta R) \end{bmatrix} + \begin{bmatrix} G_x \\ G_y \\ G_z \end{bmatrix} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} \quad (II.72)$$

The functions $C_x, C_y \; and \; C_z$ are linearized and presented as follows:

$$\begin{cases} C_x = C_{x0} + C_{x\alpha_0}\alpha_0 + C_{x\alpha_s}\alpha_s + C_{xV}V + C_{x\delta P}\delta P + C_{x\delta Y}\delta Y + C_{x\delta R}\delta R \\ C_y = C_{y0} + C_{y\alpha_0}\alpha_0 + C_{y\alpha_s}\alpha_s + C_{yV}V + C_{y\delta P}\delta P + C_{y\delta Y}\delta Y + C_{y\delta R}\delta R \\ C_z = C_{z0} + C_{z\alpha_0}\alpha_0 + C_{z\alpha_s}\alpha_s + C_{zV}V + C_{z\delta P}\delta P + C_{z\delta Y}\delta Y + C_{z\delta R}\delta R \end{cases} \quad (II.73)$$

A similar approximation can be made for $C_l, C_m \; and \; C_n$:

$$\begin{cases} C_l = C_{l0} + C_{l\alpha_0}\alpha_0 + C_{l\alpha_s}\alpha_s + C_{lV}V + C_{l\delta P}\delta P + C_{l\delta Y}\delta Y + C_{l\delta R}\delta R \\ C_m = C_{m0} + C_{m\alpha_0}\alpha_0 + C_{m\alpha_s}\alpha_s + C_{mV}V + C_{m\delta P}\delta P + C_{m\delta Y}\delta Y + C_{m\delta R}\delta R \\ C_n = C_{n0} + C_{n\alpha_0}\alpha_0 + C_{n\alpha_s}\alpha_s + C_{nV}V + C_{n\delta P}\delta P + C_{n\delta Y}\delta Y + C_{n\delta R}\delta R \end{cases} \quad (II.74)$$

II.3. The three degrees of freedom missile motion modeling

In missile motion modeling, *3DOF* (Three Degrees of Freedom) refers to a simplified approach that captures the essential dynamics of a missile's motion by considering three primary degrees of freedom. In this section, let us assume that the missile is in the same plan with the target, which is moving in constant speed and don't making maneuvers. It means that the missile and the target are moving in 2D plan which make both of making two translations along x 'forward or longitudinal movement' and z 'upward/downward movement' axes, in addition to one rotation about y 'right-wing' axes.

Moreover, the maneuverability of the target is limited, in other way it can only maneuver on a single axe by making a rotation about y axes. Therefore, the motion equations of a missile are simpler than the *6DOF* equations due to the elimination of some terms that are not acting on the missile. The *3DOF* model of the missile is given then by:

$$\begin{cases} u = rv - qw + X + G_x + T_x \\ w = qu - pv + Z + G_z + T_z \\ \dot{q} = -M_{rp}rp - M_{r^2p^2}(r^2 - p^2) + M + M_T \end{cases} \qquad (II.75)$$

II.4. Guidance techniques

Guidance techniques refer to the methods and strategies employed to steer and control missiles or other guided systems towards their intended targets. These techniques are designed to optimize the trajectory, improve accuracy, and increase the probability of successfully engaging the desired objective. Here are some commonly used guidance techniques [12]:

a) Pure Collision: It is a direct trajectory pursued by an interceptor or weapon, aiming to directly collide with the target.

b) Pursuit: It is a guidance strategy employed in missile systems to track and intercept moving targets. It involves continuously adjusting the missile's trajectory to maintain a relative velocity and heading that closely matches the target's velocity and heading.

The pursuit guidance strategy aims to minimize the relative motion between the missile and the target by adjusting the missile's trajectory to match the target's velocity and heading. It ensures that the missile stays on a pursuit path, increasing the chances of successfully intercepting and engaging the moving target.

c) Lead Pursuit: An interceptor, such as a missile, flies a lead pursuit course where its velocity vector is directed at an angle from the target. This angle ensures that projectiles launched from any position along the course will strike the target if it falls within the weapon's effective range.

d) Deviated Pursuit: The interceptor missile monitors the target and generates commands using a guidance law similar to pure pursuit. However, in this case, the missile's heading leads the line-of-sight by a constant angle. When the fixed lead angle is set to zero, deviated pursuit transforms into pure pursuit. While no missile is intentionally designed to follow a deviated pursuit course, instances of deviated pursuit can occur due to random errors or unintended biases in the guidance system.

e) Proportional navigation: It is a strategy where the lead angle is altered proportionally to the angular rate of the line of sight towards the target. The missile gauges the rotation of the line of sight and adjusts its turning rate accordingly. The fundamental aim of the classical proportional navigation guidance law is to minimize the heading error in order to intercept the target. Mathematically, proportionnel navigation can be expressed as [13]:

$$\alpha_n = NV_c \left(\frac{d\lambda}{dt}\right) \tag{II.76}$$

Where N is the constant of proportionality between the turn rate and line-of-sight rate, it is called the navigation constant. Furthermore, the navigation constant has a substantial impact on the trajectory flown by the missile. This constant is upheld by equating the missile's lateral acceleration α_n to the product of the line-of-sight rate $\left(\frac{d\lambda}{dt}\right)$ and the closing velocity V_c.

II.5. The seeker/tracker

The general role of the seeker in a guided missile is to detect, track, and locate the target. The seeker is typically equipped with sensors, such as radar, infrared, which enable it to gather information about the target's position, velocity, and other relevant parameters. By continuously monitoring the target and providing precise guidance information, the seeker plays a critical role in directing the missile towards its intended target, increasing the probability of a successful engagement. Moreover, the main functions of the seeker called also homing eye are summarized hereafter [14]:

a) The seeker and radar receiver are used to measure the line of sight LOS rate $\left(\frac{d\lambda}{dt}\right)$ (see figure II.5) and the closing velocity V_c which is the relative speed between missile and target.
b) Keep the antenna or receiver pointed at the target.
c) Track the target continuously after acquisition.

The line of sight (LOS), is defined as the angle λ between a line from the center of the seeker antenna to the target, and some arbitrary non-rotating (inertial) reference line. Therefore, $\lambda(t)$ at time t is the total change in the angular position of the LOS relative to the initial LOS. Note that θ_m is the angular position of the missile body centreline which is measured relative to the initial LOS, θ_h is the angular position of the antenna centerline. It is called the gimbal angle. Thus, the LOS angle expression is given by (see figure II.5):

$$\lambda = \theta_m + \theta_h + \varepsilon \qquad (II.77)$$

Where ε is the boresight error between antenna center line and line of sight to the target. It takes into account both the missile's attitude relative to inertial space and the angular position of the antenna relative to the missile's centerline.

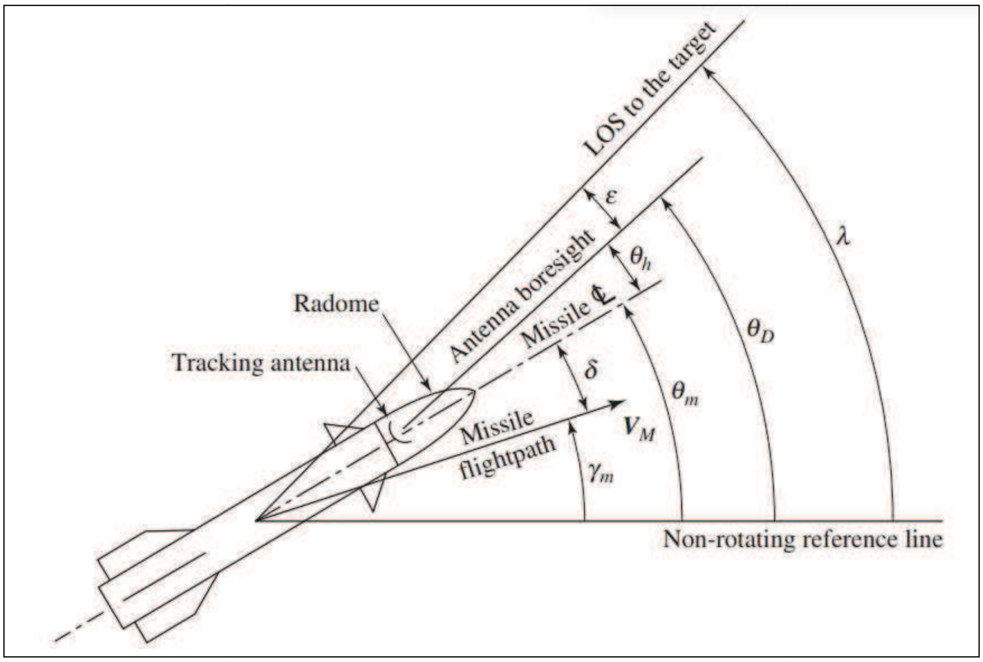


Figure II.5 Missile seeker showing angular geometry [15]

II.6. Conclusion

Missile motion modeling is a complex and evolving field because it requires a comprehensive knowledge about physics and mathematics. Therefore, it can range from simplified approaches, such as *3DOF* models to more complex and realistic models, such as *6DOF* models that incorporate all both translational and rotational movements. The choice of modeling approach depends on the level of accuracy required and the specific objectives of the simulation or analysis. Moreover, the accurate missile motion modeling enables to study various aspects of missile behavior, such as flight characteristics, stability, control, guidance, and interception capabilities. It aids in evaluating the effectiveness of seeking-tracking technique and different guidance strategies, assessing the impact of environmental factors, and optimizing missile design parameters.

By employing advanced mathematical techniques, computational algorithms, and real-world data, missile motion modeling provides valuable insights into the missile's performance, allowing for the refinement of designs, the development of effective guidance strategies, and the assessment of mission success probabilities.

In conclusion, missile motion modeling is a critical aspect of understanding and analyzing the behavior and performance of missiles. It involves developing mathematical models and simulation techniques to represent the motion, trajectory and dynamics of missiles during flight.

Chapter III

LQR Optimal Control Applied to Missile Guidance

III.1. Introduction

In automation, the aim of control is to regulate the behavior of systems such as processes, machines, robots ...etc without human intervention. The performance of a control system depends on many factors such as stability, accuracy, responsiveness and robustness against disturbances that negatively affect on these performances.

There are common types of control problems that can be summarized in two main problems regulation and tracking. In this work we will present the optimal control method and show how it deals with these problems.

First of all, we have to define a criterion 'or cost functional' to optimize, then we should ensure the optimality of the criteria which is the objective of control law. A set of constraints control inputs and states can be added to generate the flexibility of the control.

In our case, we will consider a guidance problem that allows the missile to track its target. For this purpose, we choose a Linear Quadratic Regulator (LQR) optimal control method to generate the optimal commands that maintain the proper course of the missile to hit its target accurately. It should be a balance between the performance and the energy consumed by the missile as possible.

In this chapter, we will derive the expressions of the LQR optimal control method. First of all some properties and concepts of multi-input multi-output systems are recalled. Then, Hamilton approach is used to establish the LQR optimal control law. Finally, an optimal control law that optimizes a performance criterion of an intercept guidance system is performed.

III.2. Dynamic of a *MIMO* systems

The dynamic state space equation of *MIMO* systems with 'n' dimension state variables can be expressed as follow:

$$\dot{X}(t) = F(X(t), u(t), t) \qquad (III.1)$$

$$Y(t) = H(X(t), u(t), t) \qquad (III.2)$$

Where $X(t)$ is the vector of state variables $X \in \mathbb{R}^{\mathrm{n}}$, $u(t)$ is the input vector $u \in \mathbb{R}^{\mathrm{m}}$, $Y(t)$ is the output vector $Y \in \mathbb{R}^{\mathrm{r}}$.

A linear system is represented by a set of first-order differential equations that describe the system's behavior in terms of its internal state variables and inputs. Moreover, it can be time variant or invariant.

III.2.1. Linear time variant systems

A linear time variant '*LTV*' system is a class of systems in which the system parameters or coefficients change by time, while still preserving linearity. This means that the system's response varies as the system parameters change over time.

This kind of systems can be represented by the following state space:

$$\dot{X}(t) = A(t)X(t) + B(t)u(t) \qquad (III.3)$$

$$Y(t) = C(t)X(t) + D(t)u(t) \qquad (III.4)$$

A, B, C and D are matrices that describe the system's dynamics and the relation between the inputs and outputs, where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{r\times n}$ and $D \in \mathbb{R}^{r\times m}$.

III.2.2. Linear Time-Invariant systems

A Linear Time Invariant *LTI* system follows the principle of superposition accordingly to which any delay provided in input must be reflected in the output.

The state space representation of *LTI* systems is given as follows:

$$\dot{X}(t) = AX(t) + Bu(t) \qquad (III.5)$$

$$Y(t) = CX(t) + Du(t) \qquad (III.6)$$

Note that in the case non linear systems, the linearization of $(III.1)$ and $(III.2)$ is given by:

$$A = \frac{\partial F}{\partial x}, \qquad \mathrm{B} = \frac{\partial F}{\partial u}, \qquad \mathrm{C} = \frac{\partial H}{\partial x}, \qquad \mathrm{D} = \frac{\partial F}{\partial u} \qquad (III.7)$$

The state space representation of *LTI* systems can be presented by the following block diagram:

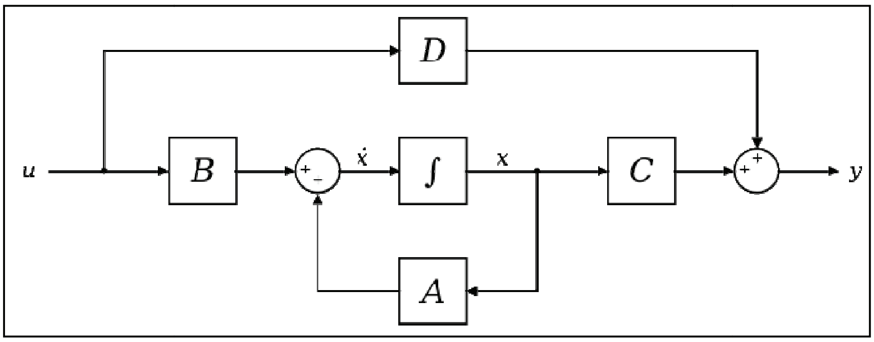


Figure III.1 Block diagram of *LTI* systems state-space representation [1]

III.3. Some properties of *MIMO* systems

The *LTI* systems properties that may be checked before dealing with performing a feedback state space control law such as optimal control once state controllability and observability.

III.3.1. Controllability

A system is said to be a completely state controllable if it is possible to transfer the system state from any initial state $x(t_0)$ to any desired state $x(t)$ in a specified finite time by a control vector $u(t)$.

To verify the controllability of *LTI* systems, one can use the following *Kalman's* test stated hereafter [2]:

The state equation $(III.5)$ is completely controllable if and only if the rank of the composite matrix Q_c is equal to *n* where *n* is the dimension of the state matrix A: $Rank(Q_c) = Rank(A)$, where:

$Q_c = [B \ \vdots \ AB \ \vdots \ \cdots \ \vdots \ A^{n-1}B]$

III.3.2. Observability

A system is said to be completely observable, if every state $x(t_0)$ can be completely identified by measurements of the outputs $Y(t)$ over a finite time interval $(to \leq t \leq t1)$.

To verify the observability of *LTI* systems, one can use the following *Kalman's* test:

The output equation $(III.6)$ is completely observable if and only if the rank of the observability matrix Q_O is *n*: $Rank(Q_O) = Rank(A)$, where:

$Q_O = [C^t \ \vdots \ A^t C^t \ \vdots \ \cdots \ \vdots \ (A^t)^{n-1} \ C^t]$

III.4. Hamilton approach

Let us consider a dynamical system of *n* first order differential equations:

$$\dot{X}(t) = f(X(t), u(t), t) \qquad (III.8)$$

The solution of this differential equations $x(t)$ for a given control vector $u(t)$ and an initial state variable $x(t_0)$ is called the state space trajectory. It can be found if and only if the initial conditions $x(t_0) = x_0$ and the control vector $u(t)$ are already defined. In optimal control, the problem is to determine $u(t)$ which minimizes or maximizes a certain objective function in the interval of time $[t_0, t_f]$.

Hopefully, the goal is to optimize a performance index $V(x(t), u(t), t)$ given by:

$$min_{u(t)}\, J = S(x_f, t_f) + \int_{t_0}^{t_f} V(x(t), u(t), t)dt \qquad (III.9)$$

For this purpose, one can use the Hamiltonian equation which is defined as follows:

$$H = V(x(t), u(t), t) + \lambda^t(t) f(x(t), u(t), t) \qquad (III.10)$$

Where $\lambda(t)$ is the costate vector.

With this equation, one can find an optimal control $u^*(t)$ that determines the optimal trajectory $x^*(t)$ that minimizes the Hamiltonian equation by the following optimality conditions [3]:

$$\left(\frac{\partial H}{\partial x}\right)_* = -\dot{\lambda}^*(t) \qquad (III.11)$$

$$\left(\frac{\partial H}{\partial u}\right)_* = 0 \qquad (III.12)$$

$$\left(\frac{\partial H}{\partial \lambda}\right)_* = \dot{x}^*(t) \qquad (III.13)$$

$$\left\{H + \left(\frac{\delta S}{\delta t}\right)_{*t_f}\right\}\delta_{t_f} + \left\{\left(\frac{\delta s}{\delta x}\right) - \lambda(t)\right\}^t_{*t_f} \delta_{x_f} = 0 \qquad (III.14)$$

III.5. *LQR* optimal control method

In this section we will develop two different cases of using the *LQR* method, especially in the tracking problem.

III.5.1. case of linear time-variant systems

Consider the state space equation $(III.3)$ and $(III.4)$, suppose that 'D' matrix is null and consider the following equation:

$$e(t) = z(t) - Y(t) \qquad (III.15)$$

Where $e(t)$ is the error matrix and $z(t)$ is the output desired trajectory.

Consider also the following quadratic criteria below:

$$J(u(t)) = \frac{1}{2}\{e^t(t_f)F(t_f)e(t_f)\} + \frac{1}{2}\int_{t_0}^{t_f}\{e^t(t)Q(t)e(t) + u^t(t)R(t)u(t)\}dt \qquad (III.16)$$

Where $Q, R\ and\ F$ are weighting positive defined matrices ($Q > 0, R > 0$ and $F > 0$).

To construct the solution, the first step is to express the Hamiltonian as follows:

$$H = \frac{1}{2} * \{e^t(t)Q(t)e(t)\} + \frac{1}{2} * \{u^t(t)R(t)u(t)\} + \lambda^t(t)\{A(t)x(t) + B(t)u(t)\} \qquad (III.17)$$

By replace equation (III.15) in $(III.16)$ one can write:

$$H = \frac{1}{2} * \left\{\big(z(t) - C(t)x(t)\big)^t Q(t)(z(t) - C(t)x(t))\right\} + \frac{1}{2} * \{u^t(t)R(t)u(t)\} + \lambda^t(t)\{A(t)x(t) + B(t)u(t)\} \qquad (III.18)$$

Accordingly to optimality conditions of *Hamilton* we have:

$$\left(\frac{\partial H}{\partial u}\right)_* = 0 \Rightarrow R(t)u^*(t) + B^t(t)\lambda^*(t) \Rightarrow u^*(t) = -R^{-1}(t)B^t(t)\lambda^*(t) \qquad (III.19)$$

$$\left(\frac{\partial H}{\partial \lambda}\right)_* = \dot{x}^*(t) \Rightarrow \dot{x}^*(t) = A(t)x^*(t) - B(t)R^{-1}(t)B^t(t)\lambda^*(t) \qquad (III.20)$$

$$-\left(\frac{\partial H}{\partial x}\right)_* = \dot{\lambda}^*(t) \Rightarrow \dot{\lambda}^*(t) = -C^t(t)Q(t)C(t)x^*(t) + C^t(t)Q(t)z(t) - A^t(t)\lambda^*(t) \qquad (III.21)$$

Let us write these two latter equation in a matrix form:

$$\begin{bmatrix} \dot{x}^*(t) \\ \dot{\lambda}^*(t) \end{bmatrix} = \begin{bmatrix} A(t) & -B(t)R^{-1}(t)B^t(t) \\ -C^t(t)Q(t)C(t) & -A(t) \end{bmatrix} * \begin{bmatrix} x^*(t) \\ \lambda^*(t) \end{bmatrix} + \begin{bmatrix} 0 \\ C^t(t)Q(t)z(t) \end{bmatrix} \quad (III.22)$$

The final condition can be expressed as follows:

$$\lambda^*(t_f) = \frac{1}{2}\left\{\frac{\partial}{\partial x(t_f)}\; e^t(t_f)F(t_f)e(t_f)\right\} \quad (III.23)$$

This implies:

$$\lambda^*(t_f) = C^t(t_f)F(t_f)C(t_f)x(t_f) - C^t(t_f)F(t_f)z(t_f) \quad (III.24)$$

Let us put:

$$\lambda^*(t) = P(t)x^*(t) - G(t) \quad (III.25)$$

By deriving these equations, we get:

$$\dot{\lambda}^*(t) = \dot{P}(t)x^*(t) + P(t)\dot{x}^*(t) - \dot{G}(t) \quad (III.26)$$

This letter equation is used with equations (III.20) and (III.21) to get:

$$\dot{P}(t) = -P(t)A(t) - A^t(t)P(t) + P(t)B(t)R^{-1}(t)B^t(t)P(t) - C^t(t)Q(t)C(t) \quad (III.27)$$

$$\dot{G}(t) = \{-A^t(t)G(t) - P(t)B(t)R^{-1}(t)B^t(t)\}G(t) - C^t(t)Q(t)z(t) \quad (III.28)$$

From the final condition equation, on can write:

$$\lambda^*(t_f) = P(t_f)x^*(t_f) - G(t_f) \quad (III.29)$$

Or:

$$\lambda^*(t_f) = C^t(t_f)F(t_f)C(t_f)x^*(t_f) - C^t(t_f)F(t_f)z(t_f) \quad (III.30)$$

By identification, one can obtain the final condition on $P(t)$ and $G(t)$. So, we have:

$$P(t_f) = C^t(t_f)F(t_f)C(t_f) \quad (III.31)$$

$$G(t_f) = C^t(t_f)F(t_f)z(t_f) \quad (III.32)$$

So, the optimal control can be expressed as follows:

$$u^*(t) = -R^{-1}(t)B^t(t)\lambda^*(t) = -R^{-1}(t)B^t(t)P(t)x^*(t) + R^{-1}(t)B^t(t)G(t) \quad (III.33)$$

This implies:

$$u^*(t) = -K(t)x^*(t) + R^{-1}(t)B^t(t)G(t) \qquad (III.34)$$

III.5.2. Case of linear time-invariant system with infinite horizon

In this case, the criteria in given by [4]:

$$J(u(t)) = \lim_{t_f\to\infty}\{\frac{1}{2}\int_{t_0}^{t_f}(x^t(t)Qx(t) + u^t(t)Ru(t))dt\} \qquad (III.35)$$

Where Q and R are weighting matrices Q is the performance matrix and R is the energy matrix. Since we have an infinite horizon problem and an LTI system, one can assume that $P(t) \to P$ and $(t) \to \bar{G}(t)$, P and $\bar{G}(t)$ are obtained by the given equations:

Hence, we can write:

$$PA + A^tP + PBR^{-1}B^tP - C^tQC = 0 \qquad (III.36)$$

This is called the Algebraic Riccati Equation (*ARE*).

And we have also:

$$\dot{\bar{G}}(t) = (-A^t + PBR^{-1}B^t)\bar{G}(t) - C^tQz(t) \qquad (III.37)$$

After solving these two differential equations, we get the control input vector:

$$u^*(t) = -R^{-1}B^t(Px^*(t) - \bar{G}(t)) \qquad (III.38)$$

III.6. Application of optimal control on missiles

In this lecture we will show the general guidance system which illustrate as follow:

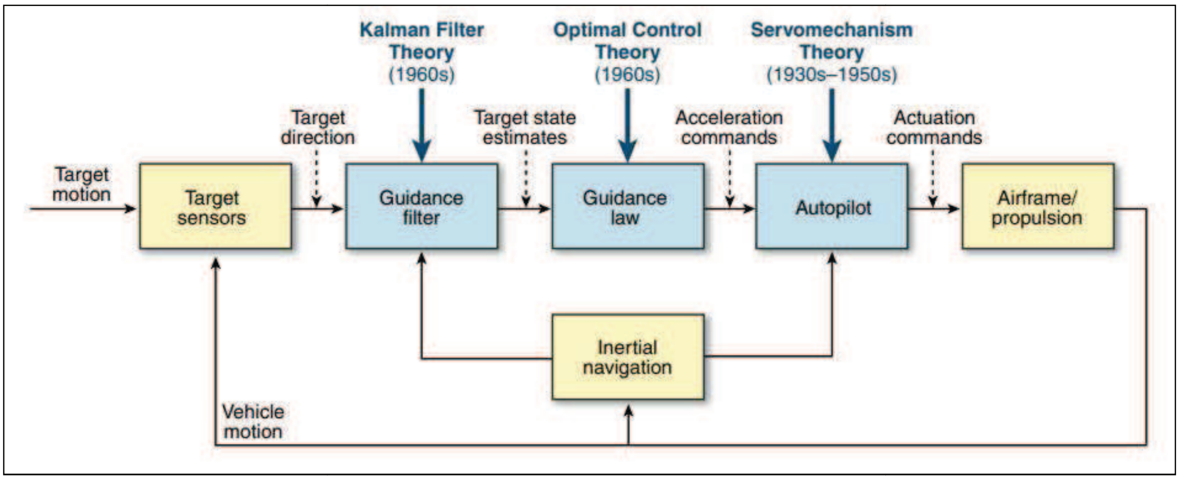


Figure III.2 General guidance system scheme [5]

This figure shows how a guidance system actually works. Therefore, our work requires establishing the guidance law for a linear quadratic criterion by using the optimal control theory which is in our case the *LQR* method to minimize the miss distance. Also, the control 'acceleration commands' or the demanded acceleration 'a_{zd}' which pass it to the autopilot system to determine the right actuation commands 'or fin commands' that ensures the equivalent deflection of wings by the fin actuator. Moreover, one can note that the miss distance have been expressed by linear quadratic criteria that we will define it later.

In guided missiles, there is several performance index 'or cost functional' to consider in the optimal control. In this section, we will present some common criteria to minimize used in this application.

a) Minimum time: The objective is to efficiently move the missile from an arbitrary initial state $x(t_0) = x_0$ to a predefined target in the shortest time possible. In the case of the minimum-time problem, the performance measure is expressed as follows:

$$J(u) = t_f - t_0 = \int_{t_0}^{t_f} dt \qquad (III.39)$$

Where t_f is the first time that $x(t)$ intersects the target.

b) Minimum fuel: The task involves moving the missile from an arbitrary initial state $x(t_0) = x_0$ to a specified target within a given time while minimizing a weighted sum of the absolute values of the control vector components. This is the criteria expression in the case of the minimum-fuel problem:

$$J(u) = \int_{t_0}^{t_f} \left[\sum_{i=1}^{m} c_i \, |u_i(t)| \right] dt \qquad (III.40)$$

Where $c_i > 0$.

c) Minimum energy: Here, the objective is to transfer the missile from an arbitrary initial state $x(t_0) = x_0$ to a specified target within a given time while minimizing a weighted combination of the squared values of the control vector components. Here is the index performance of the minimum-energy problem:

$$J(u) = \int_{t_0}^{t_f} [u^t(t) R u(t)] dt \qquad (III.41)$$

The norm of the control vector is determined by the positive definite weighting matrix 'R'.

d) ***Terminal Control:*** The objective is to minimize the deviations of the final system state values from some desired values. Consequently, here is the focus of the terminal control problem:

$$J(u) = \left[x(t_f) - d(t_f)\right]^t Q\left[x(t_f) - d(t_f)\right] \qquad (III.42)$$

Where $d(t_f)$ is the desired final value of the state variables vector and 'Q' is a positive weighting matrix.

e) ***Tracking Control:*** This involves the minimization of deviations, which can be weighted if desired, between the system's state values and desired values over the entire operational interval. This objective is applied specifically to the tracking-control problem. Here is the corresponding performance index:

$$J(u) = \int_{t_0}^{t_f} [x(t) - d(t)]^t Q[x(t) - d(t)]dt \qquad (III.43)$$

In case of bounded controls this performance index will be written as follows:

$$J(u) = \int_{t_0}^{t_f} ([x(t) - d(t)]^t Q[x(t) - d(t)] + u^t(t)Ru(t))dt \qquad (III.44)$$

On the other hand, for unbounded (unconstrained) controls, the desired state value $d(t)$ is defined throughout the interval $[t_0, t_f]$ while the weighing matrices 'Q' and 'R' are possibly time varying.

f) ***Regulating Control:*** This represents a specific scenario within tracking control, where the desired state values are set to zero. In the case of regulating control, where the desired state values remain zero throughout the interval $[t_0, t_f]$, the performance measure can be defined as follows:

$$J(u) = \int_{t_0}^{t_f} [x^t(t)Qx(t) + u^t(t)Ru(t)]dt \qquad (III.45)$$

A comprehensive mathematical representation for the performance measure of a control system objective, encompassing all the aforementioned cases, can be expressed as follows:

$$J(u) = \left[S(x_f, t_f)\right] + \int_{t_0}^{t_f} V[x(t), u(t), t]dt \qquad (III.46)$$

The performance measure for a missile control problem may also take the following form:

$$J(u) = \left[x(t_f) - d(t_f)\right]^t F\left[x(t_f) - d(t_f)\right] + \int_{t_0}^{t_f} dt \qquad (III.47)$$

While $d(t_f)$ represents the specified target point.

In the above objective function, the first quadratic term indicates the weighted deviations of the final states of the missile from the target (such as the miss distance), and the second integral indicates the time of flight. The elements of the positive weighting matrix 'F' can be selected so as to reflect the relative importance between the two terms ($F = 0$ gives a strict minimal-time optimal control problem).

III.7. Optimal control for a missile intercept guidance system

In this section, we will apply the *LQR* control to get an optimal guidance law. This optimal control law is performed for minimizing the terminal miss distance. So, let us start from the following assumptions:

- The engagement takes place in a plane (see figure III.3).
- The target acceleration is set to zero (this assumption implies constant target velocity).
- The control vector is the missile's inertial acceleration. In general, the thrust of a missile is typically optimized to maximize its velocity during the initial stages of flight. This approach aims to minimize the window of time available for the target to execute evasive maneuvers.

In the derivation of guidance laws, it will be assumed that a continuous control is employed. Thus, the performance index to be minimized is assumed to be given by:

$$J = y_d^2(T_f) + \gamma \int_0^{T_f} u^2(t)\,dt \qquad (III.48)$$

Where 'y_d' is terminal miss distance at the intercept time 'T_f', γ ($\gamma \geq 0$) is the weighting factor on the control effort, and $u(t)$ is the control vector.

This equation states that the optimal control consists in minimizing the terminal mean-square miss distance plus the weighted integral-square missile acceleration normal to the line of sight (*LOS*). In general, the missile commanded acceleration normal to the *LOS* is constrained by $|u| \leq u_{max}$.

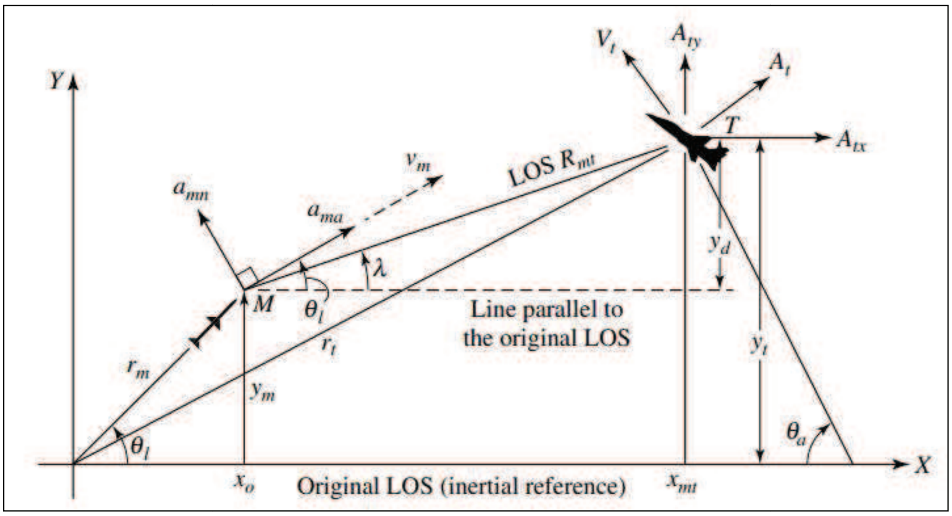


Figure III.3 Intercept geometry [6]

Let us define 'θ_l' which is the missile lead angle, 'θ_{he}' is the missile heading error, 'θ_a' is the target aspect angle, 'λ' is line of sight (*LOS*), 'V' is the missile velocity and 'V_T' is the target velocity.

Now, let's consider that r_M, V and a_M are the interceptor missile's position, velocity, and acceleration vectors relative to an inertial reference frame. Furthermore, let r_T, V_T and a_T be the target's corresponding position, velocity, and acceleration vectors relative also to the inertial reference frame. Moreover, assuming that the time to go 't_{go}' is known and can be computed separately. Additionally, the closing velocity 'V_c' is defined as the relative velocity measured along the *LOS* (see figure III.3) which is given by the following equation [7]:

$$V_c = V\cos(\theta_l + \theta_{he} - \lambda) + V_T\cos(\theta_a + \lambda) \qquad (III.49)$$

One can note that for a given aspect angle 'θ_a', the collision course missile lead angle expression is written as follow:

$$\theta_{lc} = \sin^{-1}\left[\left(\frac{V_T}{V}\right)\sin(\theta_a)\right] \qquad (III.50)$$

Furthermore, form figure (III.3) the *LOS* angle 'λ' is given by:

$$\lambda = tan^{-1}\left(\frac{y_d}{x_{TM}}\right) \qquad (III.51)$$

Where 'x_{TM}' is the missile-to-target range measured along the original *LOS*. For small angle, the *LOS* angle 'λ' as follows:

$$\lambda \cong \frac{y_d}{x_{TM}} \qquad (III.52)$$

At this point, let us present the missile and target relationships in state space notation below:

$$\begin{aligned} \dot{x_1} &= x_3 \\ \dot{x_2} &= x_4 \\ \dot{x_3} &= a_{Tx} - a_{Mx} \\ \dot{x_4} &= a_{Tz} - a_{Mz} \end{aligned} \qquad (III.53)$$

Where $x_1 = p_{Tx} - p_{Mx}$ is the distance between missile and target along *x* axes, p_{Tx} and p_{Mx} are target and missile positions respectively, $x_2 = p_{Tz} - p_{Mz}$ is the distance 'or height' between missile and target along z axes, p_{Tz} and p_{Mz} are the positions 'or altitude' of target and missile respectively, x_3 is the missile- target relative velocity in the direction of *x* axes, and x_4 is the missile-target relative velocity in the direction of z axes.

Therefore, if we consider that the target movement is a disturbance to compensate, then the equation $(III.53)$ can be written in the following form:

$$\begin{aligned} \dot{x_1} &= x_3 \\ \dot{x_2} &= x_4 \\ \dot{x_3} &= u_1 = -a_{Mx} \\ \dot{x_4} &= u_2 = -a_{Mz} \end{aligned} \qquad (III.54)$$

The next step is to write the equations $(III.54)$ in the following canonical form:

$$\dot{X}(t) = AX + Bu = \begin{bmatrix} 0 & | & I \\ - & + & - \\ I & | & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ I \end{bmatrix} u \qquad (III.55)$$

Where *I* is the 2 × 2 identity matrix, depending on the performance index which given as follow:

$$J(u) = x^t(t)Qx(t) + \frac{1}{2}\int_{t_0}^{t_f} u^t(t)Ru(t)dt \qquad (III.56)$$

Where $Q=\begin{bmatrix} I & | & 0 \\ - & + & - \\ 0 & | & 0 \end{bmatrix}$ and $R=\begin{bmatrix} b & | & 0 \\ - & + & - \\ 0 & | & b \end{bmatrix}$.

One can note that b is an element of the positive definite matrix R. So, if b is chosen to be small, the missile willing to expend whatever acceleration is required to minimize the terminal miss distance (assuming, of course, that the missile is capable of producing and sustaining such accelerations). However, if b is chosen to be large, the magnitude of the acceleration available will be limited in achieving small miss distance.

Now, the performance index can be rewritten as follow:

$$J(u) = x_1^2(t_f) + x_2^2(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (u_1^2 + u_2^2)\, dt \qquad (III.57)$$

We have the expression the solution:

$$u^*(x,t) = -R^{-1}B^t P(t)x(t) = -\left(\frac{1}{b}\right)\begin{bmatrix} 0 \\ I \end{bmatrix}^t P(t)x(t) \qquad (III.58)$$

To determine the solution, we have to calculate the matrix *P* by solving the first order differential equation *Riccati* equation which is given as follow:

$$-\frac{dP}{dt} = P\begin{bmatrix} 0 & | & I \\ - & + & - \\ I & | & 0 \end{bmatrix} + \begin{bmatrix} 0 & | & I \\ - & + & - \\ I & | & 0 \end{bmatrix} P - P\begin{bmatrix} 0 \\ I \end{bmatrix}\left(\frac{1}{b}\right)[0\ I]P \qquad (III.59)$$

The equations $(III.58)$ and $(III.59)$ can be solved analytically yielding to the control law as follow:

$$u = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{3t_{go}}{3b + t_{go}^3} & 0 & -\frac{3t_{go}^2}{3b + t_{go}^3} & 0 \\ 0 & -\frac{3t_{go}}{3b + t_{go}^3} & 0 & -\frac{3t_{go}^2}{3b + t_{go}^3} \end{bmatrix} \qquad (III.60)$$

Where $t_{go} = t_f - t_0 = -\frac{R}{(\frac{dR}{dt})}$.

Now, if we assume that $b = 0$, the equation $(III.60)$ becomes:

$$\begin{cases} u_1(t) = -\left(\frac{3}{t_{go}^2}\right)x_1 - (3/t_{go})x_3 \\ u_2(t) = -\left(\frac{3}{t_{go}^2}\right)x_2 - (3/t_{go})x_4 \end{cases} \qquad (III.61)$$

Where $u_1(t)$ and $u_2(t)$ are the controls along the *x* axes and z axes respectively. Furthermore, if we assume that the *LOS* angle is small, then the control $u_1(t)$ vanish '$u_1(t) = 0$'. So, the final guidance law is written as follow:

$$u(t) = u_2(t) = 3\left(\frac{dR}{dt}\right)\left(\frac{d\lambda}{dt}\right) \qquad (III.62)$$

Where $\left(\frac{\mathrm{dR}}{\mathrm{dt}}\right)$ is the missile closing velocity expression V_c. In our case, the desired result will be considered as the proportional navigation guidance law with the effective navigation ratio $N' = 3$ while in practice the navigation ratios of 4 and 5 are commonly used based on the classical control theory analysis.

II.8. Conclusion

In this chapter, we have already covered the essential and necessary points that concern the *MIMO* dynamics of *LTV* and *LTI* systems, in addition to some properties of these systems such as observability and controllability. Then, we presented the *Hamilton* approach and the corresponding optimality conditions.

In the second part of this chapter, we have presented the *LQR* optimal control method in case of *LTV* and *LTI* systems with infinite horizon. Moreover, we introduced the most common performance index used in missiles control systems. Finally, we have performed an optimal guidance law for a missile intercept guidance system that may ensure an effective navigation of the missile and a good target tracking in different circumstances.

Chapter IV

Simulation Results Presentation and Discussion

IV.1. Introduction

Simulation is widely used to evaluate and analyze the performance, behavior and effectiveness of missiles during various scenarios. By creating virtual environments, it is possible to observe the dynamics of missile flight, test guidance strategies and explore the impact of external factors to choose the right response of the system in a specific scenario. For this purpose, we will use the MATLAB environment to simulate a missile dynamics and it's guidance system then discuss the simulation results.

More accurately, we will present and discuss the obtained simulation results of a missile in two different cases.

The first case concerns a six degrees of freedom (*6DOF*) open loop missile model in two different scenarios "the model and the data of the missile are prebuilt in *MATLAB-SIMULINK* environment" [1]. First, we will apply fixed inputs 'constant thrust force' during the simulation time. Then we will apply variable thrust force to the system. The applied external moments on the missile dynamics model are the same.

The second case concerns an *LQR* optimal control method of a missile guidance system applied to a three degrees of freedom (*3DOF*) model of a missile motion. We will determine the optimal gain of the navigation control to minimize the miss distance between the missile and the target. Then, we will present the simulation results of this method to show the effectiveness of the optimal guidance law in terms of tracking a moving and maneuvering target.

IV.2. *6DOF* open loop missile model

In this section we will present a *6DOF* open loop missile model. Let's take a look on the general diagram that describes this system (see figure IV.1).

This block diagram is based on *6DOF* mathematical model of the missile that has been developed in the second chapter. Therefore, this model represents the motion equations of the missile in subsystem form. It has 12 output variables, three positions and three orientations of the vehicle and their rates or velocities expressed in the body reference frame. This diagram summarizes the main parts of the missile dynamics model 'or subsystems' that we will discuss below after present the following figure [2]:

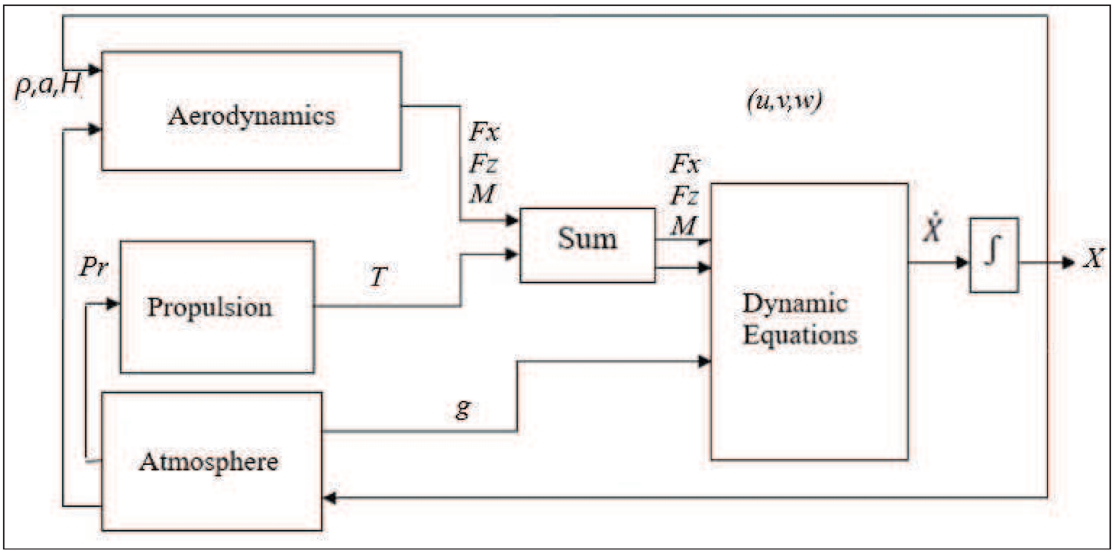


Figure IV.1 General block diagram of the missile dynamics [3]

a) ***Dynamics block:*** The primary function of the dynamics block is to generate the 12 outputs signals that effectively governs and influences the equations of motion. These signals are generated in accordance with the inputs provided by the mathematical model representing the missile aerodynamics and propulsion system. Also, it receives several inputs, including the forces acting on the body, body momentum, mass, moment of inertia, gravity, and time. These inputs are crucial for accurately determining and calculating the behavior and motion of the missile.

b) ***Aerodynamics block:*** This block performs the essential task of computing the body forces and momentums that serve as control variables acting on the missile body within the dynamics block. These computations are based on the input signals, namely velocity, atmosphere density (ρ), acceleration, height, and center of gravity. The aerodynamics block plays a critical role in determining the aerodynamic forces and moments that influence the motion and stability of the missile.

c) ***Atmosphere block:*** It performs calculations to determine several parameters as output signals based on the position of the missile as input. These parameters include atmosphere density (ρ), acceleration, gravity, height, and pressure at a specific level. The atmosphere block is responsible for modeling the atmospheric conditions surrounding the missile, providing crucial information that influences the missile's performance and behavior during flight.

d) ***Propulsion block:*** This block plays a vital role in determining the thrust force for the dynamic block based on the calculated pressure. Inputs such as the missile's gravity and moment of inertia are provided to the propulsion block to facilitate these calculations. By considering these parameters, the propulsion block ensures that the dynamic block receives accurate information about the thrust force needed to propel the missile effectively.

The following figure presents previously diagram block developed under *MATLAB-Simulink*.

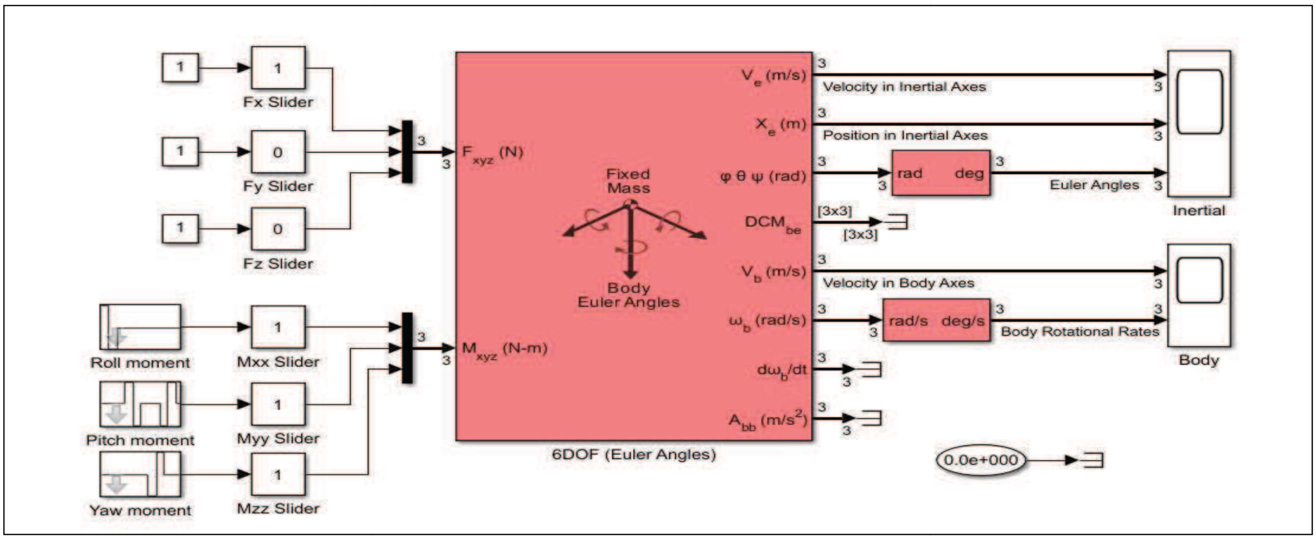


Figure IV.2 Block diagram of a *6DOF* open loop missile [4]

The studied system (the missile) receives input signals which are the three external forces components and the three external moment's components along the x, y and z axes. By applying these inputs, the system will generate output signals 'or response' such as the velocity of the missile in inertial axes V_e, the position in the inertial axes X_e, the velocity in body fixes axes V_b, body rotational rates ω_b and Euler angels (φ, θ, ψ). The thrust force applied to the missile model is assumed to have only one component F_x along x axes which means that F_y and F_z are equal to zero. Also, the roll, pitch and yaw moments $(M_x, M_y\ and\ M_z)$ values applied to the system have specific profiles in order to test and discuss the missile dynamics.

These inputs are used by the system internally to determine total forces and moments, that directly contribute to generate the response of the system such as V_e is the velocity and $X_e$is the position in the inertial axes, the V_b is the velocity, ω_b is the rotational rates and (φ, θ, ψ) are the Euler angles of the body-fixed axes.

In the next part, we will get inside the *6DOF* missile model to explain in more details the internal structure of system (see figure IV.3).

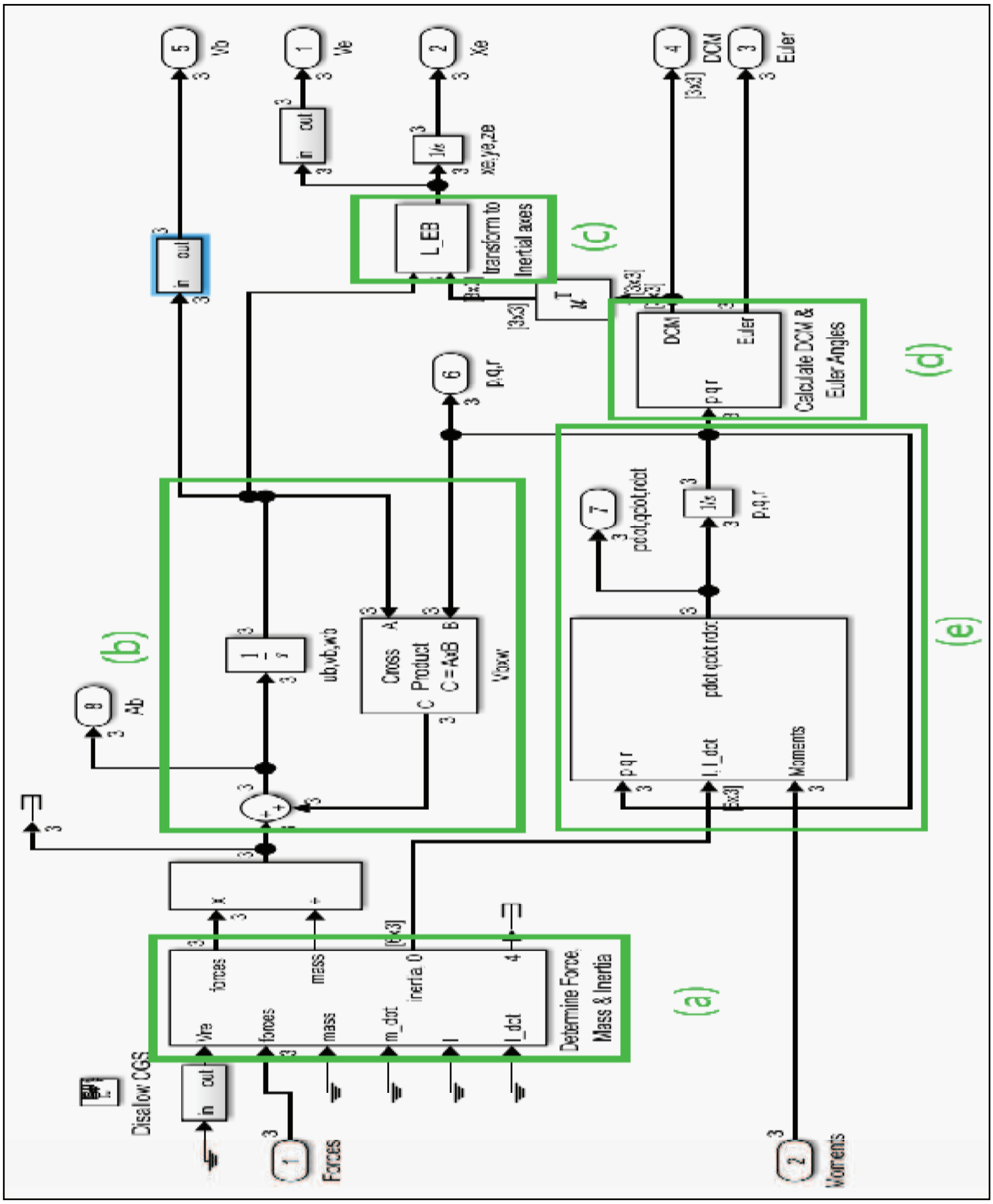


Figure IV.3 Developed block diagram of the studied *6DOF* open loop missile

The subsystem (a) receives external forces F_x, F_y and F_z, mass changes during the flight. The thrust force is developed by the consumption of the propellant which leads the mass decreasing. So, it depends on how we would like consider the mass of the missile. This subsystem will determine effective forces, mass and inertia of the missile.

The subsystem (b) receives the cross product of linear velocity in body axes u, v and w with body rotational rates p, q and r and adds them to the new calculated forces to determine the body accelerations. Then, the velocity in body axes are determined by a simple integration of these acceleration components. Now, subsystem (e) when it fed by the body rotational rates p, q and r besides to inertia and moments and it computes the angular accelerations p_{dot}, q_{dot} and r_{dot}. Note that the body rotational rates are determined also by the integration of these angular accelerations.

The subsystem (d) takes as inputs the body rotational rates and calculates Euler angles (φ, θ and ψ) and DCM. Finally, the subsystem (c) determines the velocity V_e and the position of the missile in inertial axes by transforming these variables from body axes to inertial axes. In the next section, we will present and discuss the obtained simulation results to show the dynamic behavior of the studied missile.

IV.2.1. Simulation results and discussion

In this section we will present and discuss the obtained simulation results of the two scenarios of the studied *6DOF* open loop missile model for two different scenarios as follow:

a) First scenario

Here we will apply a constant thrust force with one component F_x along x axes and excite the system by varied profiles of roll, pitch and yaw moments (see figure IV.4).

In this section, we will present and discuss the obtained simulation results that concern this scenario which is illustrated below:

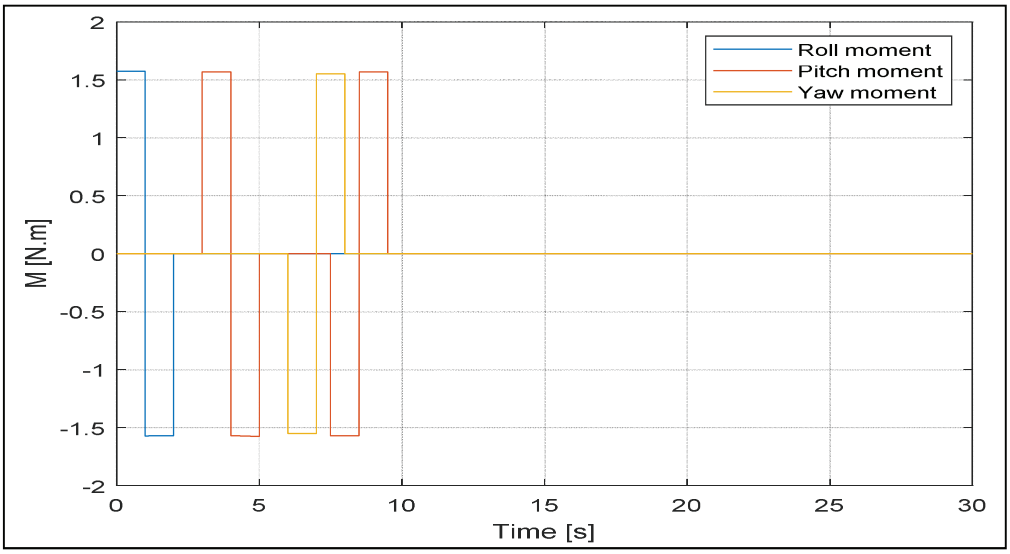


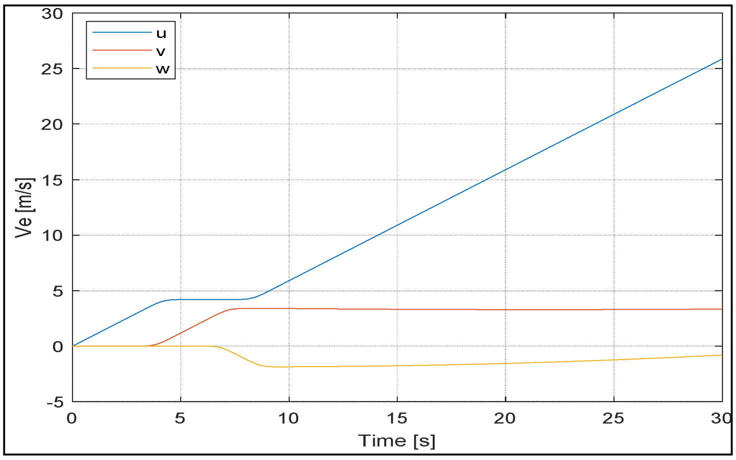
Figure IV.4 Profiles of moments acting on the studied missile

In this scenario, we not the increasing of the velocity along x axes since the thrust is assumed to have one component F_x. At the same time, by applying a roll moment on the missile body it makes a rotation around roll axes 'x axes' in other way it turns around see Euler angle 'φ' which increases the corresponding roll moment at the same time. Then, we apply pitch moment on the system. One can observe a rotation around the pitch axes y axes, see the variation of the Euler angle θ.

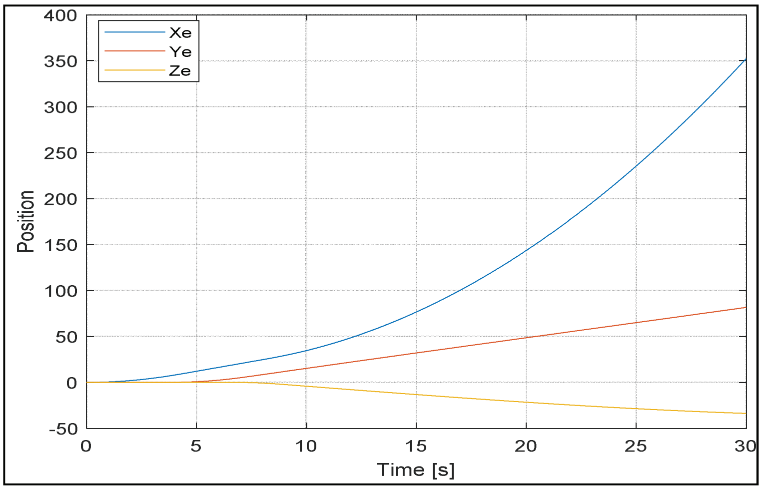
Likewise, application the velocity component 'v' along the pitch axes varies accordingly. Then, one can note a new rotation around yaw axes (see Euler angle 'ψ' variation) caused by a yaw moment step application. Which generates a variation of the yaw component velocity w.

We should mention that negative values of 'w' and the position along z axes refers to the fact that the missile is moving up and the z axes is directed backward.

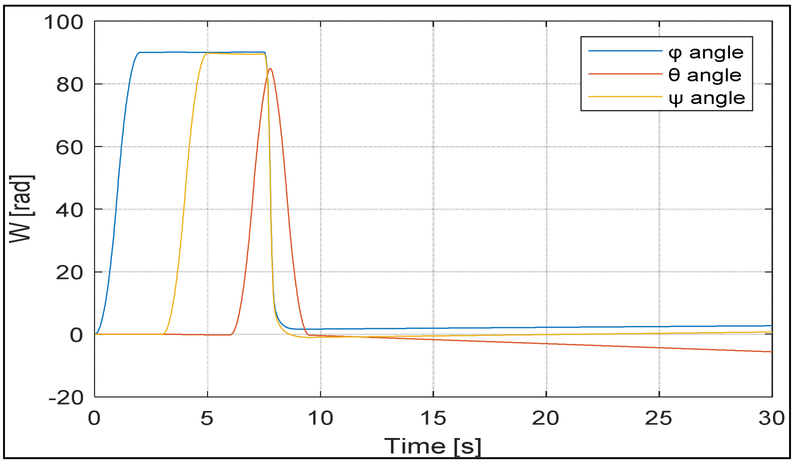
Elsewhere, the missile position varies according to its velocity, note that the major movement of missile is happening forward along roll axes 'x axes' since the applied thrust is purely along this axes. The obtained simulations are show in figure (IV.5).



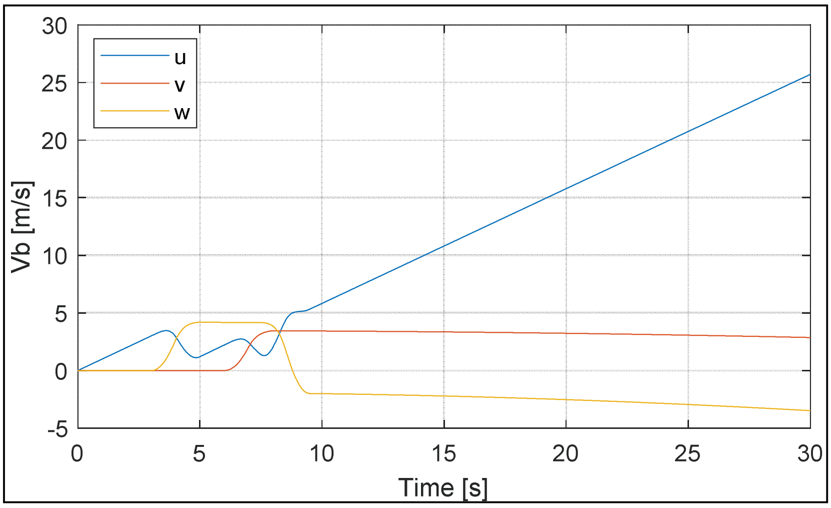
a) Velocity components along inertial frame axes



b) Missile position in the inertial axes



c) Euler angles



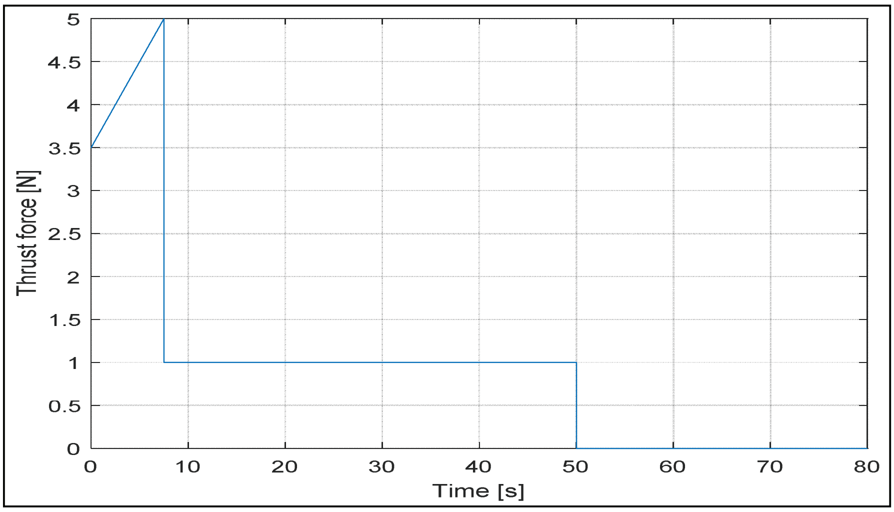
d) Velocity components along body fixed axes

Figure IV.5 Simulation results of the studied missile *6DOF* model obtained in the case of 1st scenario

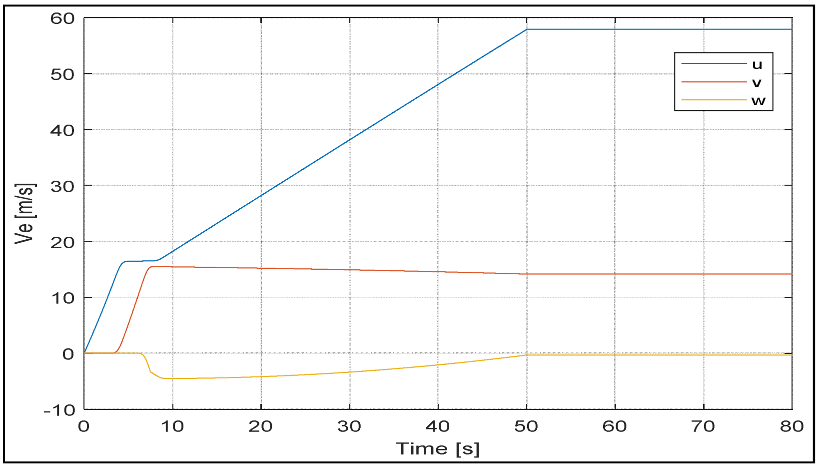
One can note also that body-fixed velocity components are influenced by the missile rotations around its own axes with show the relationship between translation and rotation movement of the missile and complicate the control operation (see figure IV.5.d).

b) Second scenario

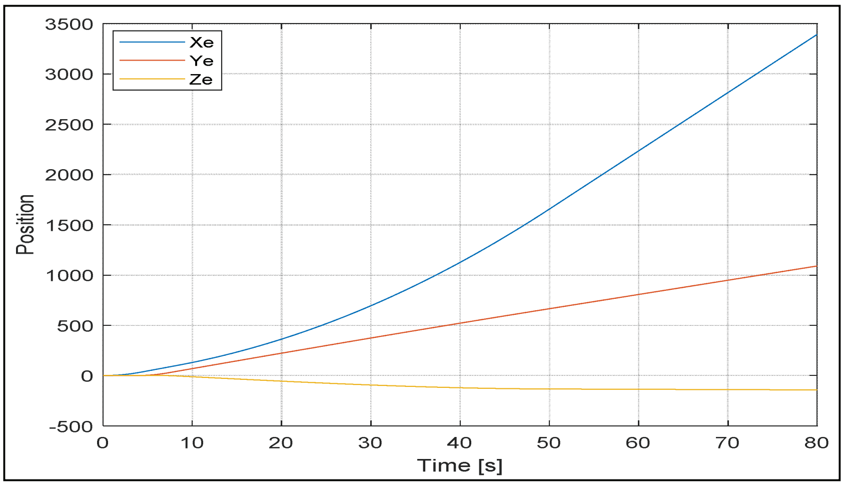
In this case a more realistic variable thrust ‘ F_x ’ is applied along the roll axes (see figure IV.6.a), also a variable the profiles of roll, pitch and yaw moment variation are similar to those of the first scenario except the roll moment that varies differently by the end of the scenario (see figure IV.6.d).



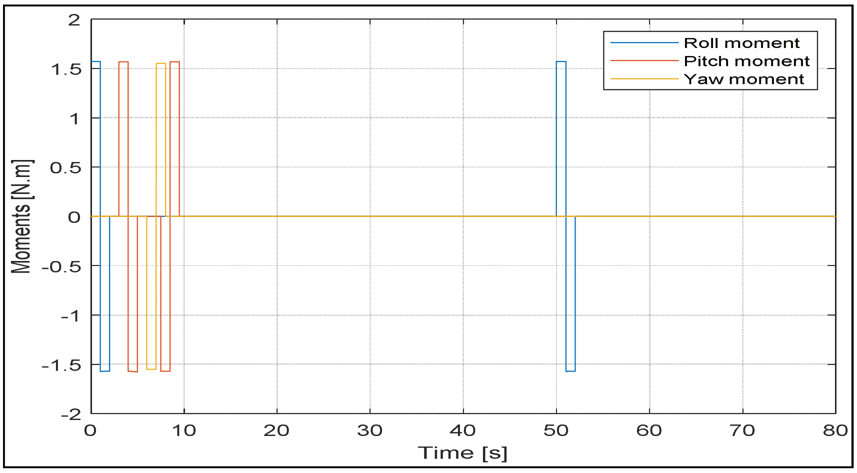
a) Thrust profile



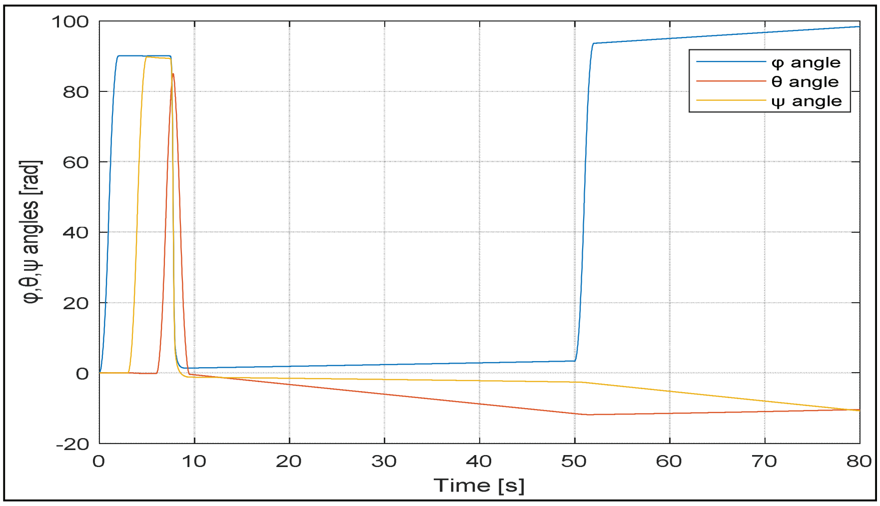
b) Velocity components along inertial axes



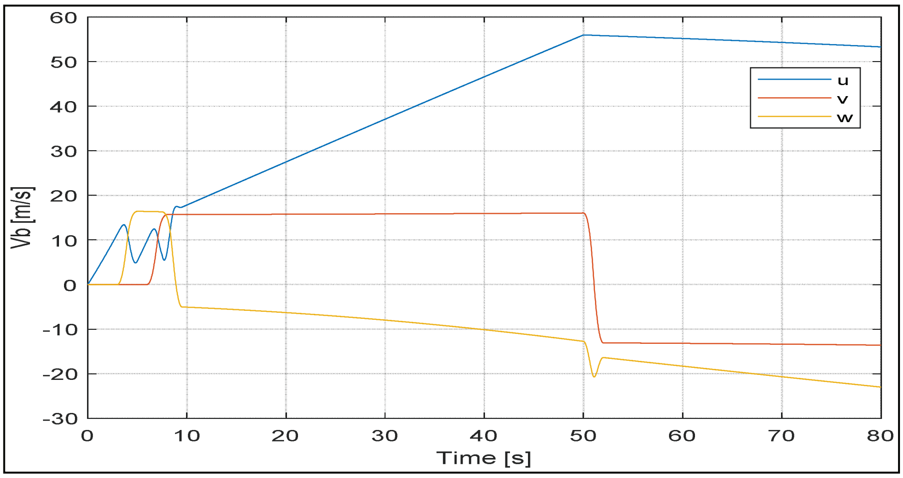
c) Missile position along the inertial axes



d) Profiles of the moments



e) Euler angles



f)Velocity components along body-fixed axes

Figure IV.6 Simulation results of the studied missile *6DOF* model obtained in the case of the 2nd scenario

Note that, in the first interval [0, 30]s both a thrust along *x* axes and roll, pitch and yaw moments are acting on the missile. Therefore, the velocity along *x* axes '*u*' is the first component that starts to increase then both velocities *v* and *w* increase during this period of time respectively, at the same time of applying roll and yaw moments. Note also that one can rotate the missile by (φ and ψ) angles around these axes in the two directions by applying positive/negative steps of moments. Moreover, the position components rise up according to the velocity vector. In the second interval [30, 50]s, the thrust is applied as there is no rotation around the three axes can be seen from (figure

IV.6.e). However, both velocities u and v also positions $X_e, Y_e\ and\ Z_e$ of the missile are still increasing.

Moreover, in the third interval [49, 51]s the engine thrust is vanished a roll moment is applied briefly. One can observe that the missile make a rotation even in the absence of thrust. Therefore, in this case velocities and position of the missile are still varying. In the last interval [51, 80]s, we eliminate all the inputs, in this case we did not observe a considerable rotation around axes, on the other hand the velocity and position of the missile are still varying thanks to gravity forces increase. It is included also that the thrust has no influence practically on Euler angels (rotations). In counterpart, moment components can be used to rotate the missile in different directions.

IV.3. *3DOF* missile guidance system design

In this section, we will present and discuss a missile guidance system block diagram carried out (prebuilt) under *MATLAB-SIMULINK* (see figure IV.7).

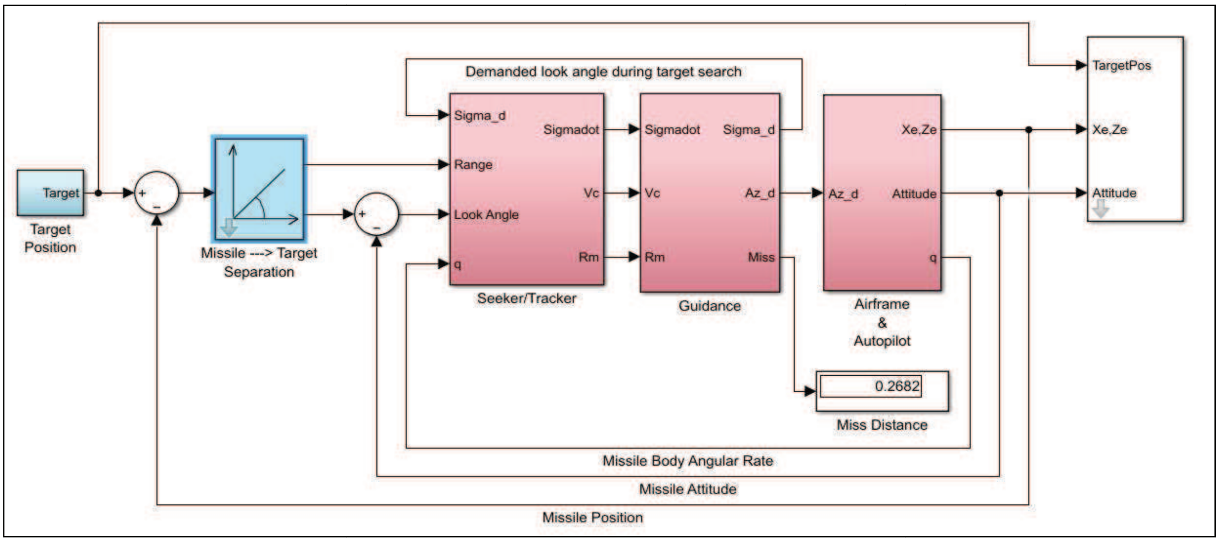


Figure IV.7 Block diagram of a missile guidance system

In this case of *3DOF*, both missile and target are moving in same plan 'ox, oz'. First of all, the guidance system should have both target and missile position as inputs. Here, the target position is defined by its initial position, its movement direction and its velocity. The missile position is determined in the same manner in the airframe and autopilot subsystem. From these positions information, the missile-target separation bloc calculate the distance 'or range' between the missile and target and the sightline angle λ.

The look angle and the range (given by the missile-target separation block), the demanded look angle by the guidance system and the missile attitude are used as inputs by the seeker-tracker block to generate the sightline rate "sigmadot", the closing velocity V_c and the range R_m. These signals are the inputs of the next subsystem which is the guidance subsystem. This latter uses the proportional navigation technique to calculate the normal acceleration demand (A_{zd}) and pass it to the airframe and autopilot subsystem. Also, it will do some calculations to determine the miss distance between the missile and the target and the demanded look angle (Sigma_d). This letter information is used by the seeker/tracker subsystem to track and lock on the target.

In the airframe and autopilot subsystem, the autopilot uses the normal acceleration demand (A_{zd}) besides some factors such as incidence angle (Alpha), velocity of the missile in Mach unit, measured acceleration along z axes (A_{zm}), measured rotation (q) and the demanded acceleration (A_{zd}) to calculate the equivalent deflection 'or angle' of the wings fins of the missile to generate this acceleration in order to adjust the missile direction towards the target. Also, this subsystem determines the position of the missile in 'ox, oz ' plan, its attitude and its body angular rate (q).

Let's now present the airframe subsystem of the missile guidance system (see figure IV.8) as follow:

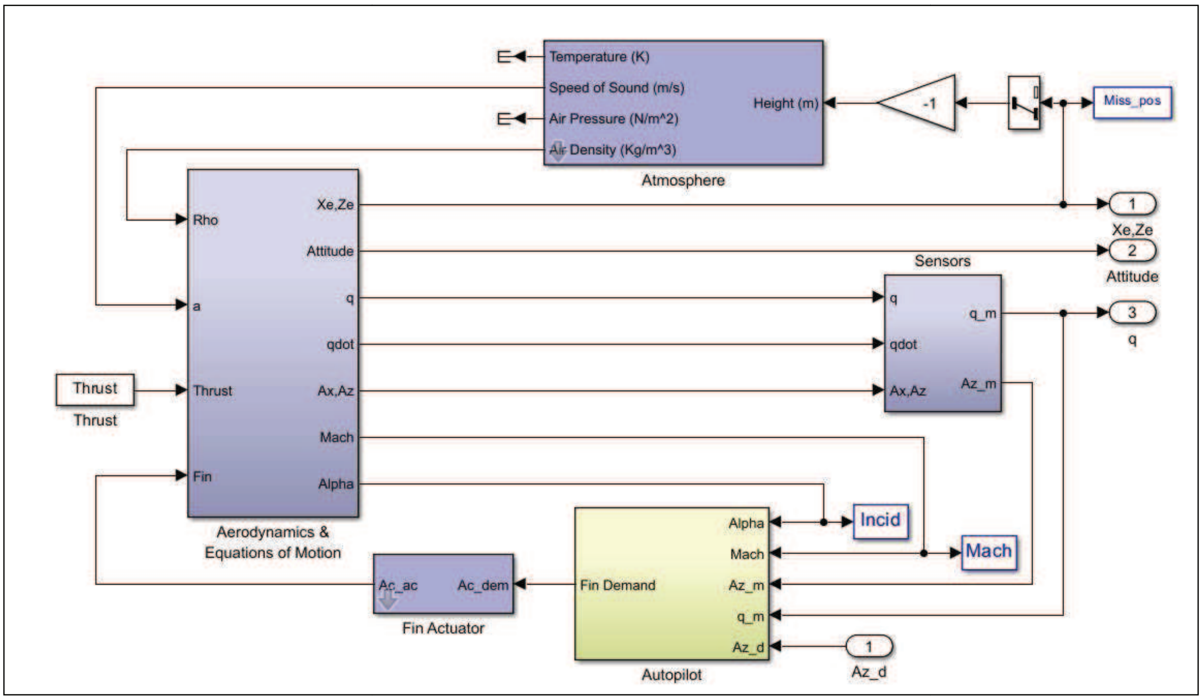


Figure IV.8 Block diagram of the airframe subsystem

The airframe model is composed of 4 main subsystems that are by the demanded normal. The Atmosphere model determines how atmospheric conditions vary with altitude that yields to calculate the actual density 'Rho' and the speed of sound to pass them to the aerodynamics block, while the 'Fin Actuator' and 'Sensors' models connect the autopilot subsystem to the Aerodynamics and Equation of Motion subsystem.

Therefore, the sensors ensure the measure of the accelerations 'A_x and A_z' along 'x and z axes', the rotation rate (q) and the angular acceleration (qdot). This subsystem gives the real 'or measured' rotation rate (q_m) and the acceleration along 'z' axes to the autopilot block. Moreover, the fin actuator is modeled by a 2nd order transfer function which is designed to adjust the wings of the missile to change its path towards the target. After Autopilot block receives also the incidence angle and the Mach number from the aerodynamic and Equation of Motion subsystem.

Additionally, 'the Aerodynamics and Equations of Motion' subsystem calculates the forces and moments acting on the missile body and integrates the motion equations to determine the missile state variable in both body fixed and inertial frames.

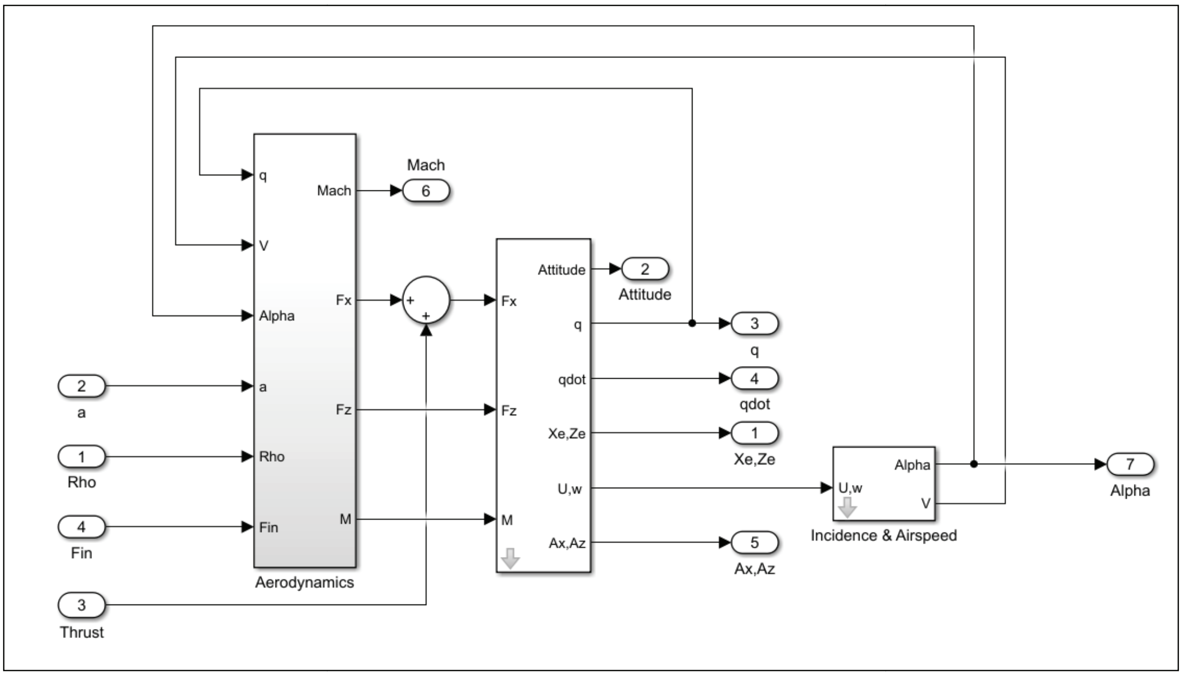


Figure IV.9 Block diagram of Aerodynamics and Equations of Motion

In aerodynamic block, the actual air density 'Rho', the speed of sound 'a', the fin that comes from fin actuator, the rotation rate (q), the missile velocity and the incidence angle 'Alpha' are received as inputs. Then, it calculates the velocity in Mach unit besides to forces 'F_x and F_z' and the moment yaw 'M' and feed them to 'Equations of Motion' block after add the thrust force to the force component 'F_x' (see figure IV.9). The aerodynamic coefficients C_x, C_z and C_m used to calculate (F_x, F_z and M) are determined by lookup tables given as data that depends on incidence angle 'Alpha' and missile velocity on Mach number.

After receive the incidence angle '*Alpha*' and the velocity in Mach unit, the lookup table blocks determine the right aerodynamic coefficients accordingly to the missile operating conditions.

Moreover, the equations of motion block expressed in the body-fixed axes will calculate the attitude of the missile, the body rotation rate 'q' as well as the yawing angular acceleration 'qdot', the position of the missile in 'ox, oz' plane 'X_e, Z_e', the acceleration 'A_z and A_x'and the linear velocity 'along x and z axes' 'u and w'. These velocity components are used by the 'Incidence and Airspeed' block to calculate the incidence angle (between the body x axes and the velocity vector) and the missile velocity magnitude.

Let us now present autopilot subsystem based on scheduled gains anti-windup PI regulator. This prebuilt scheduled gains regulator simply implemented to calculate the suitable coefficients for different operating conditions of the system 'the missile' (see figure IV.10). These gains have been calculated and stored in lookup tables and scheduled to be suitable for all operating conditions.

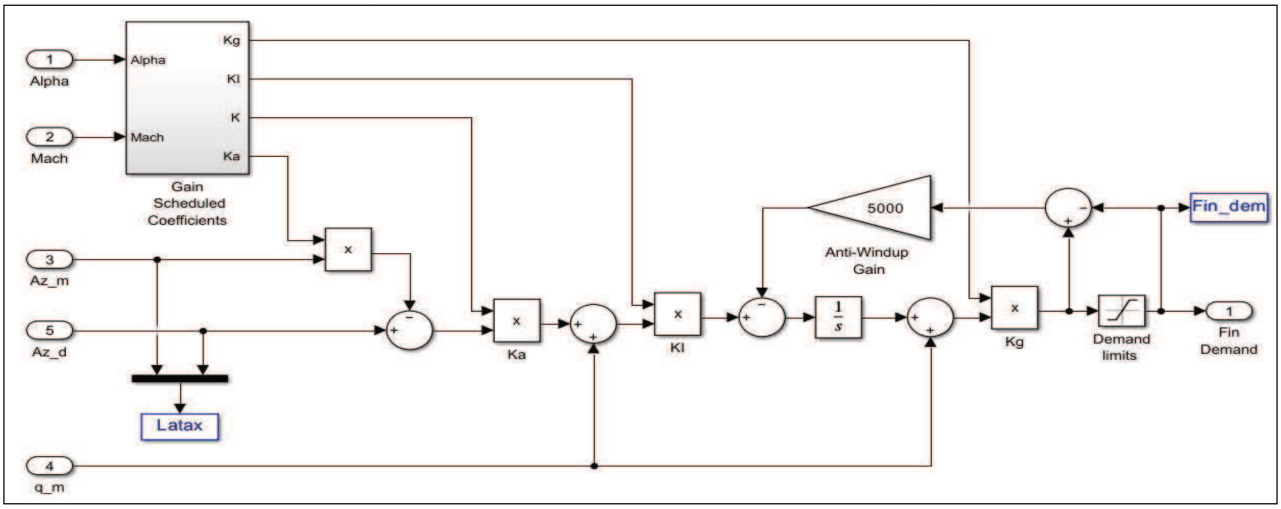


Figure IV.10 Block diagram of gain scheduled components autopilot regulator

The gain scheduled coefficients block receives the incidence angle '$Alpha$' and the velocity of the missile 'in Mach unit'. Then, depending on these parameters, it determines the correct gains to feed to the autopilot regulator. Moreover, the autopilot block receives the measured acceleration 'A_{zm}', the demanded acceleration 'A_{zd}', and the body rotation rate 'q'. Briefly, the acceleration is added to the measured body rotation rate 'q_m' and given to the autopilot PI regulator to generate the fin demand.

Now, after we explained the main parts of the guidance system, the airframe and the autopilot systems. Let us recall that we have already proved that the *LQR* optimal control law that minimizes a criterion based on the terminal miss distance gives us a proportional navigation guidance law with a gain varying from 3 to 5.

In the upcoming step, we will determine this optimum gain for the studied missile that minimizes the miss distance.

Table IV.1 show the influence of this gain on the missile miss distance and the taken time to intercept the target (t_{taken}).

Table IV.1 Influence of proportional navigation gain on the interception time and the miss distance

Gain	3	3.2	3.4	3.5	3.6	3.7	3.8	4.5	4.7	5
$Miss_{dis}$ [m]	2.519	2.684	1.185	0.2682	0.512	0.9497	1.032	4.239	7.631	31.14
t_{taken} [s]	3.4531	3.4555	3.4590	3.4609	3.4625	3.4644	3.4659	3.4747	3.4757	3.4681

The navigation gain of '3.5' have been chosen to achieve the smaller miss distance between the missile and the target (see table IV.1) and an acceptable time to reach the intended target. Therefore, in the next part of the simulation results, the chosen gain is always used and fixed in the guidance system.

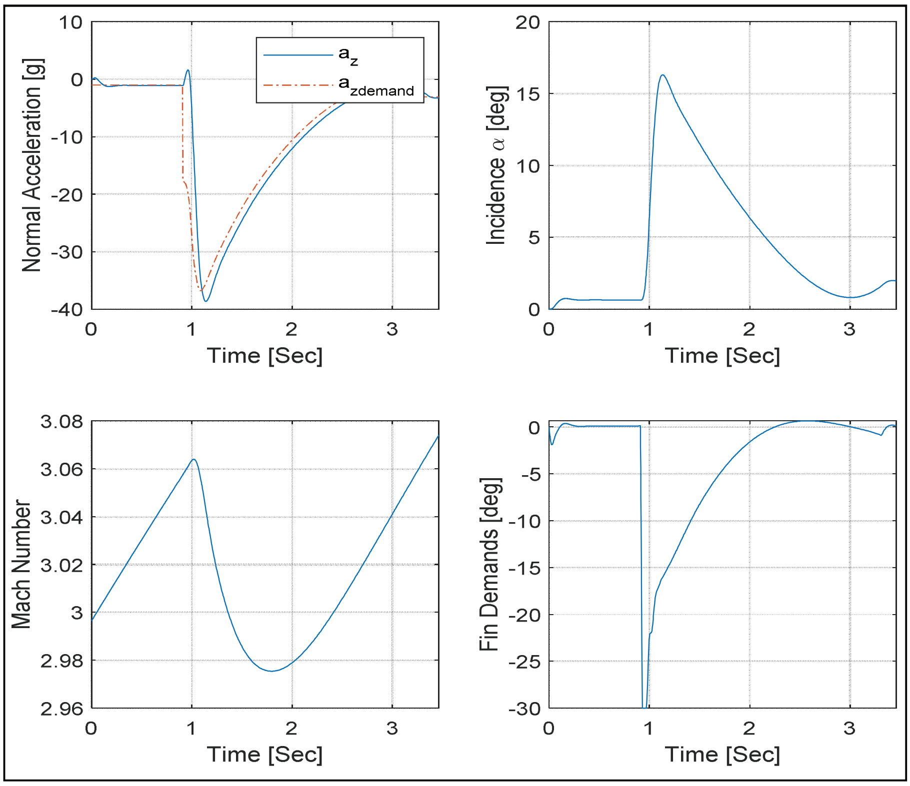
IV.3.b. Simulation results

In this section, we will present and discuss the simulation results of the following different scenarios of how the missile track and follow the target to destroy it:

- *First scenario:* The target is moving forward on a straight path. It is getting closer and closer toward the missile over time.
- *Second scenario:* The target is moving further away from the missile on a straight path**.**
- *Third scenario:* The target is moving further away from the missile and making maneuvers.
- *Fourth scenario:* The target is accelerating further away from the missile and making maneuvers.

The simulation results are obtained by using the optimum gain that we have determined previously. Now, let us present and discuss the closed loop simulation results of a *3DOF* missile guidance system in the case of the above scenarios.

a) ***First scenario:*** The obtained simulation results are illustrated in figure IV.11.



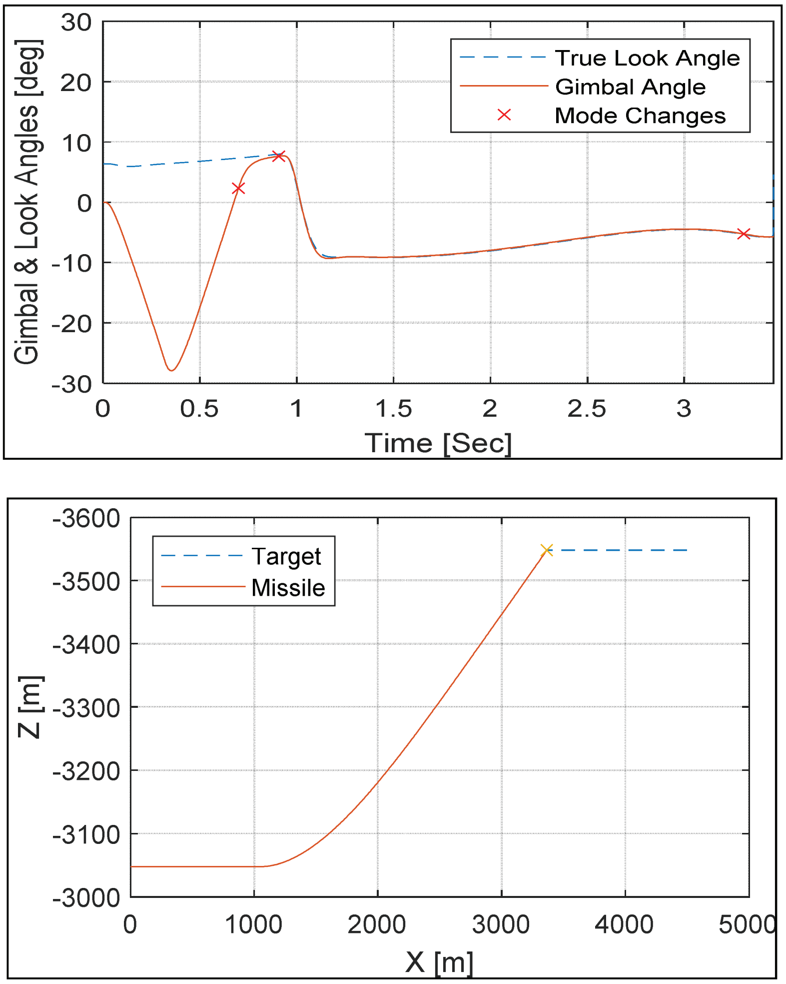


Figure IV.11 Simulation results of a *3DOF* missile guidance system 1st scenario

The following table summarizes the initial positions and velocities of the missile and target. It gives also the intercept position, time and the miss distance corresponding to the first scenario.

Table IV.2 Some data and simulation results of the 1st scenario

	V_{ini} [m/s]	X_{ini} [m]	Z_{ini} [m]	$X_{detonate}$ [m]	$Z_{detonate}$ [m]	t_{taken} [s]	$Miss_{dis}$ [m]
Missile	984	0	-3048	3364.7	-3547.94	3.4609	0.1313
Target	328	4500	-3548				

From the graphs and the table IV.2 we note that once the missile is launched, it follows a straight horizontal path for a certain time. During this time, the radar of the 'seeker/tracker' system is still searching for target. Once the target is locked in less than 1 second (see figure IV.11), the radar locks on the target and tracks it until the missile destroy it.

Consequently, the missile adjusts instantly its fins by a specific angle to achieve the demanded acceleration a_{zd} then it alters its position towards the target with extraordinary speed of up to 3 Mach (see figure IV.11). Moreover, the variation of the real acceleration 'a_z' is almost the same as the demanded one 'a_{zd}'. One can note that the missile intercept and destroy the target in about 3.5 seconds which is amazing.

To show the influence of initial conditions of both missile and target velocities on the spent time and the miss distance, these two speeds have been varied and the results are given in the tables IV.3 and IV.4.

Table IV.3 Variation of intercept time in function of the missile initial velocity (1st scenario)

V_m_{ini} [m/s]	600	800	1000	1200
t_{taken} [s]	4.6009	3.9288	3.4255	3.0345
$X_{detonate}$ [m]	2990.66	3211.36	3376.34	3504.01
$Z_{detonate}$ [m]	-3547.9	-3547.99	-3543.94	-3547.71
$Miss_{dis}$ [m]	0.2503	0.0022	0.1091	0.7396

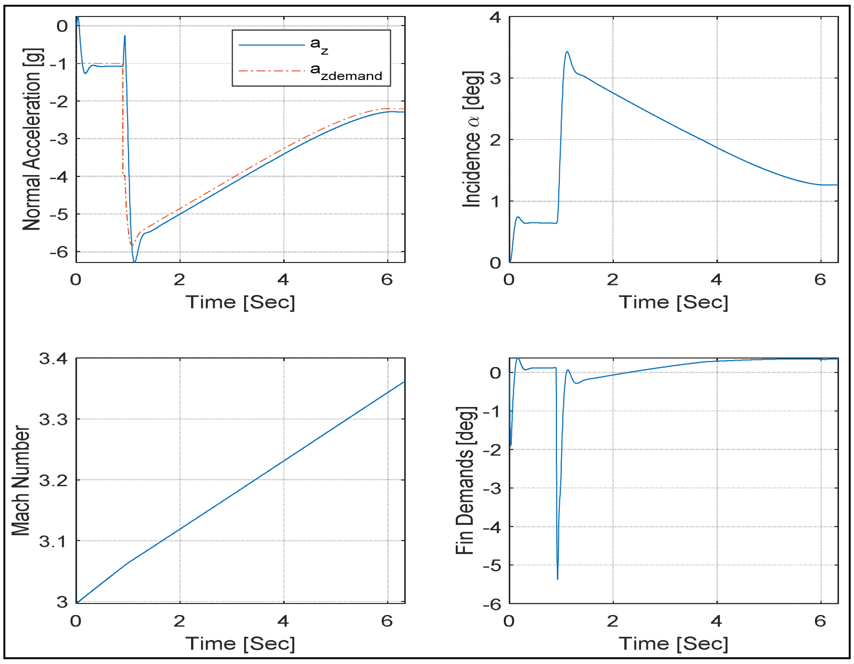
Table IV.4 Variation of time intercept in function of the target initial velocity (1st scenario)

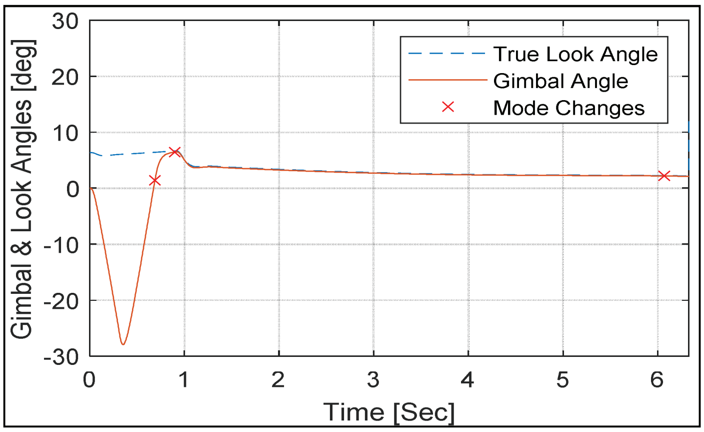
V_t_{ini} [m/s]	200	400	600
t_{taken} [s]	3.7869	3.3042	2.9378
$X_{detonate}$ [m]	3742.5	3178.23	2728.5
$Z_{detonate}$ [m]	-3547.93	-3547.99	-3546.8
$Miss_{dis}$ [m]	0.1340	0.0851	8.8882

Obviously, when the initial velocity of the missile rise up, it takes less time to intercept the target also, it gets really close to the intended target (see table IV.3).

Moreover, from table IV.4, one can note that it takes less time for a missile to reach its target, both the spent time ‘or detonation time’ and distance are with the increase of the initial velocity of the target. So, once we increment the initial velocity we obtain a fastest interception since the target moves backward in this first scenario.

b) ***Second scenario:*** The obtained simulation results of this scenario are shown in figure IV.12.





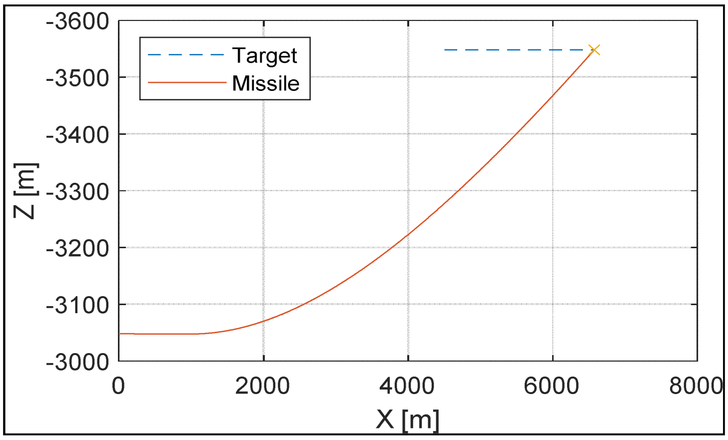


Figure IV.12 Simulation results of a *3DOF* missile guidance system 2nd scenario

In the same manner, the table IV.5 summarize missile and target initial positions and velocities and gives us the final intercept point, time and miss distance of the second scenario.

Table IV.5 Some data and simulation results of the 2nd scenario

	V_{ini} [m/s]	X_{ini} [m]	Z_{ini} [m]	$X_{detonate}$ [m]	$Z_{detonate}$ [m]	t_{taken} [s]	$Miss_{dis}$ [m]
Missile	984	0	-3048	6575.44	-3547.98	6.3277	0.0579
Target	328	4500	-3548				

Let us now discuss the obtained results of the second scenario.

From the same initial conditions, the seeker/tracker system 0.6 second to locate the target and lock it. Then the missile moves towards the target thanks to its guidance system by changing its position via adjusting the fins to perform the right acceleration and stay tracking it until interception. As a result, the applied acceleration 'a_z' is almost the same as the demanded one 'a_{zd}' by the guidance subsystem.

In this case also, the missile adjust its incidence angle and moves towards the target with an unbelievable speed 'up to 3.4 Mach' to detonate the target (see figure IV.12) in about 6.33 seconds 'see Table IV.5'.

Form table IV.6, it can be concluded that when the initial velocity of the missile rise up, the missile takes less time to destroy the intended target. In all cases, the miss distance is less than one meter which reflects the tracking high accuracy. In fact, in all cases the miss distance between the missile and target is truly small.

Table IV.6 Variation of intercept time in function of the missile initial velocity (2nd scenario)

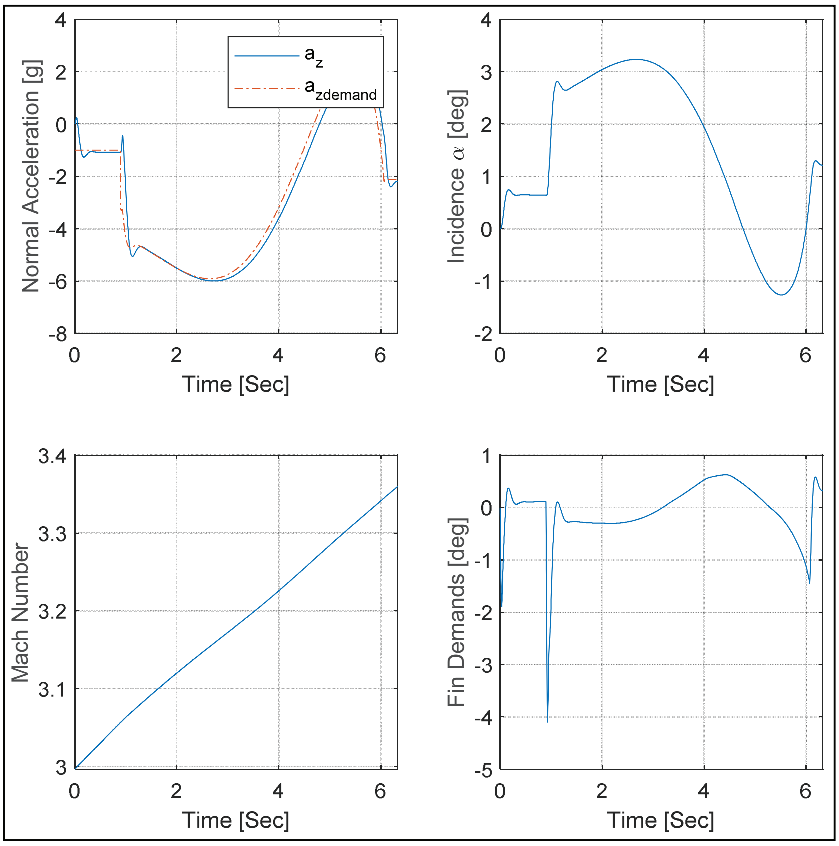
V_m_{ini} [m/s]	600	800	1000	1200
t_{taken} [s]	10.1750	7.8274	6.2199	5.1016
$X_{detonate}$ [m]	7837.35	7067.41	6540.09	6173.15
$Z_{detonate}$ [m]	-3548	-3548	-3547.98	-3547.93
$Miss_{dis}$ [m]	0.0336	0.0133	0.0602	0.1807

Furthermore, in table IV.7, the time to intercept the target is getting more and more with the target velocity. And the miss distance is getting less and less with the target initial velocity increasing.

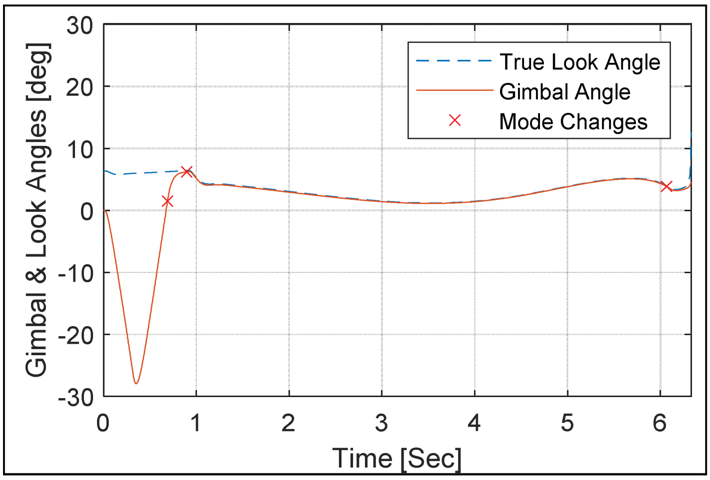
Table IV.7 Variation of intercept time in function of the target initial velocity (2nd scenario)

V_t_{ini} [m/s]	200	400	600
t_{taken} [s]	5.4404	6.9609	9.5479
$X_{detonate}$ [m]	5587.98	7284.35	10228.7
$Z_{detonate}$ [m]	-3547.96	-3547.99	-3548
$Miss_{dis}$ [m]	0.1130	0.0162	0.0416

c) ***Third scenario:*** The obtained simulation results of the 3rd scenario are presented and discussed hereafter.



Normal Acceleration [g]
Time [Sec]
a_z
a_zdemand
Incidence α [deg]
Time [Sec]
Mach Number
Time [Sec]
Fin Demands [deg]
Time [Sec]



Gimbal & Look Angles [deg]
Time [Sec]
True Look Angle
Gimbal Angle
Mode Changes

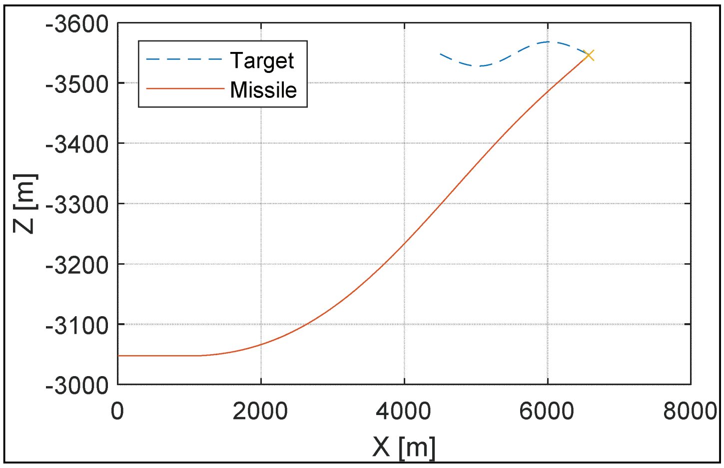


Figure IV.13 Simulation results of a *3DOF* missile guidance system 3nd scenario

Also, mention that we have table IV.8 presents the initial conditions of velocities and coordinates ‘or positions’ of both missile and target in addition to the position ‘or coordinates’ of the intercept point and the taken time to reach the target ‘or to destroy it’.

Table IV.8 Some data and simulation results of the 3nd scenario

	V_{ini} [m/s]	X_{ini} [m]	Z_{ini} [m]	$X_{detonate}$ [m]	$Z_{detonate}$ [m]	t_{taken} [s]	$Miss_{dis}$ [m]
Missile	984	0	-3048	6573.3	-3545.77	6.3293	3.8470
Target	328	4500	-3548				

In the same way, it can noted from figure IV.13 that once the missile is launched, it follows a straight path. During this time, the target is getting away from the missile with making maneuvers and the missile is moving forward. The radar of the ‘seeker/tracker’ system still searching for the intended target. Once the target is located as illustrated in figure IV.13, this happens also in less than 1 second which is incredible, the radar locks on the target and track it until the end of the mission ‘destroy of the target’.

At this moment, the missile starts immediately to deflect its fins by a specific angle to accomplish the demanded acceleration a_{zd} to change its direction towards the target with a remarkable speed (up to 3 Mach), (see figure IV.13). Moreover, it can be seen from figure IV.13 that the variation of the real acceleration 'a_z' is almost the same as the demanded one 'a_{zd}'. Obviously, one can note also that the missile intercept 'or destroy' the target in about 6.33 seconds since the target is moving forward and maneuvering.

One more, results of table IV.9 show that when the initial velocity of the missile rise up, it takes less time to destroy the intended target.

Table IV.9 Variation of intercept time in function of the missile initial velocity (3[nd] scenario)

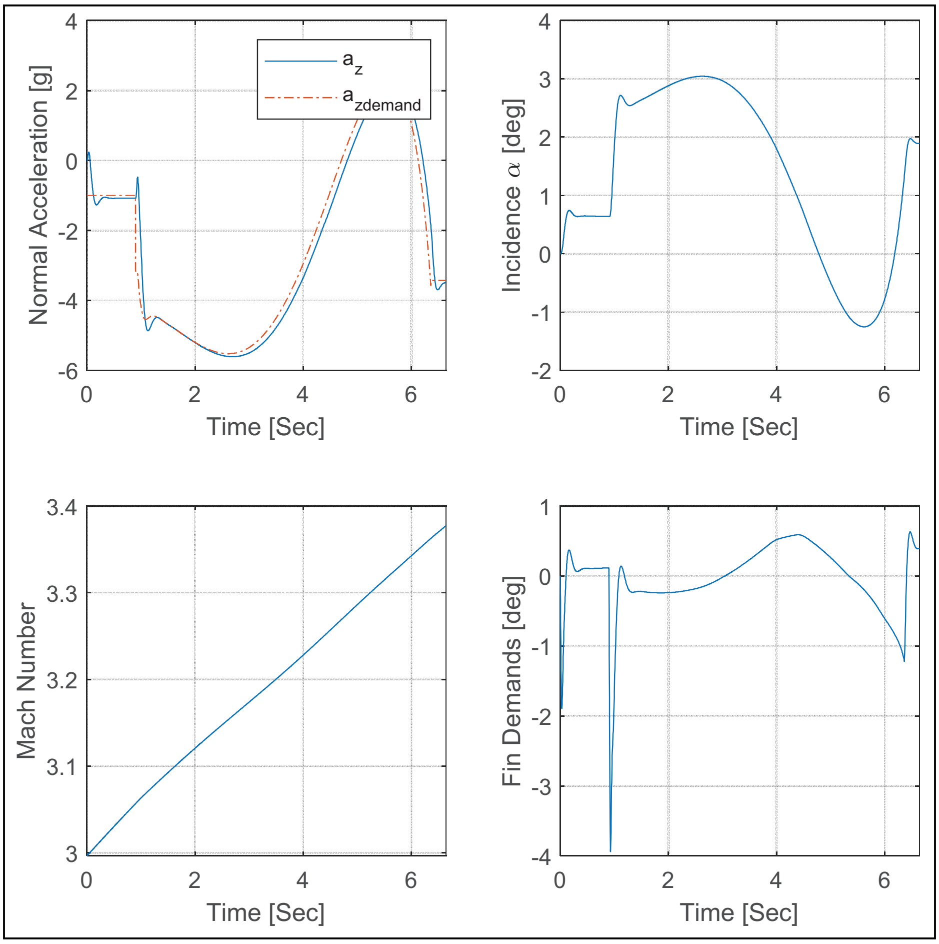
V_m_{ini} [m/s]	600	800	1000	1200
t_{taken} [s]	10.1594	7.8570	6.2193	5.0864
$X_{detonate}$ [m]	7814.69	7097.02	6535.15	6147.94
$Z_{detonate}$ [m]	-3561.06	-3528	-3548.01	-3565.96
$Miss_{dis}$ [m]	4.1900	0.0578	3.7252	1.8894

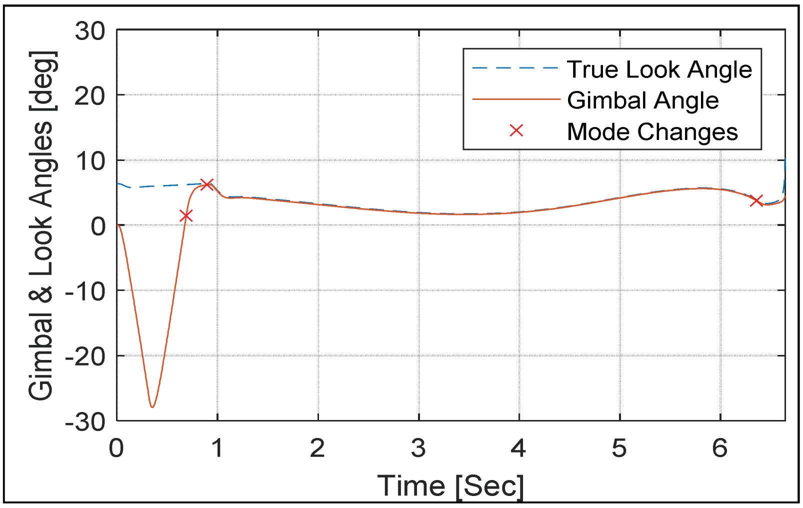
Moreover, results of table IV.10 shown that the taken time is increasing with the augmentation of target velocity. In this case, the missile reaches its target in longer distances and also it spent more time to intercept it obviously.

Table IV.10 Variation of intercept time in function of the target initial velocity (3[nd] scenario)

V_t_{ini} [m/s]	200	400	600
t_{taken} [s]	5.4286	6.9765	9.5377
$X_{detonate}$ [m]	5568.43	7300.28	10212.8
$Z_{detonate}$ [m]	-3562.29	-3534.1	-3549.54
$Miss_{dis}$ [m]	2.3491	3.2986	7.5924

a) ***Fourth scenario*****:** The obtained simulation results of the 4[th] scenario are presented and discussed hereafter.


Normal Acceleration [g]
a_z
a_zdemand
Time [Sec]
Incidence α [deg]
Time [Sec]
Mach Number
Time [Sec]
Fin Demands [deg]
Time [Sec]


Gimbal & Look Angles [deg]
True Look Angle
Gimbal Angle
Mode Changes
Time [Sec]

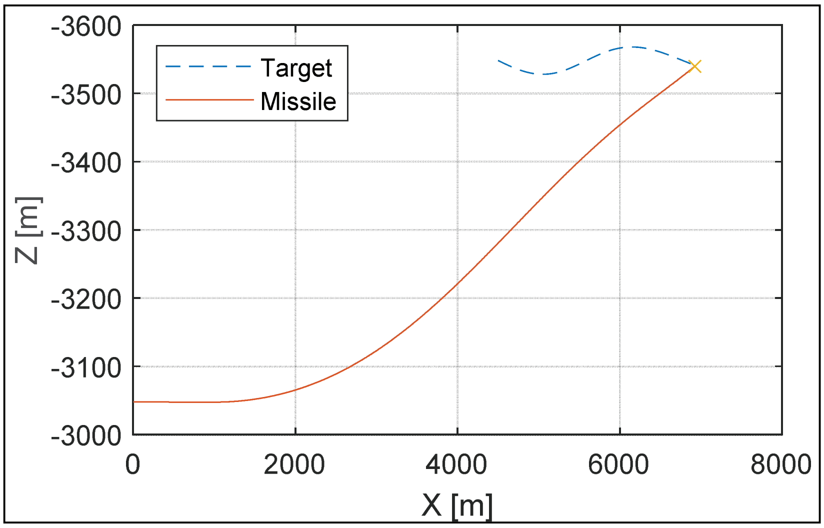


Figure IV.14 Simulation results of a *3DOF* missile guidance system 4th scenario

In the same way, from figure IV.14 one can remark that when the missile is launched, it also follows a straight path, over time, the target is getting away from the missile with making maneuvers and accelerating as possible to escape from the missile. Initially, the missile moves forward besides the radar 'seeker/tracker' system is searching for the intended target. Once it determines the target as illustrated in figure IV.14 (this happens also in less than 1 second which is a very small time), the radar locks on the target and track it until the end of the mission 'target destroying'.

Immediately, the missile guidance system is enable to adjust the fins deflection by a certain angle to accomplish the demanded acceleration (a_{zd}) then, in order to move towards the target and adapt its trajectory in this difficult circumstance with a remarkable speed which is up to 3 Mach (see figure IV.14). Consequently, the real acceleration 'a_z' is almost the same as the demanded one 'a_{zd}' (see figure IV.14). Clearly, one can note that the missile intercept its target in about 6.7 seconds which is a great time.

Table IV.11 Some data and simulation results of the 4th scenario

	V_{ini} [m/s]	X_{ini} [m]	Z_{ini} [m]	$X_{detonate}$ [m]	$Z_{detonate}$ [m]	t_{taken} [s]	$Miss_{dis}$ [m]
Missile	984	0	-3048	6921.15	-3539.47	6.6400	5.0260
Target	328	4500	-3548				

Furthermore, results of IV.12 show the influence of the initial velocity on the detonation time and distance. Therefore, both distance and spent time to reach the target are getting less, the augmentation of the missile velocity.

Table IV.12 Variation of intercept time in function of the missile initial velocity (4th scenario)

V_m_{ini} [m/s]	600	800	1000	1200
t_{taken} [s]	11.2819	8.4100	6.5153	5.2491
$X_{detonate}$ [m]	8876.52	7659.57	6869.36	6353.06
$Z_{detonate}$ [m]	-3566.93	-3530.85	-3541.92	-3564.64
$Miss_{dis}$ [m]	0.6017	2.5467	4.5564	2.1995

Once again, results of table IV.13 shown that for a fixed initial velocity of the missile and a variable acceleration of the target, when the target acceleration is boosted, the missile takes more time to destroy the intended target. Also, it takes longer distance to reach the target.

Table IV.13 Variation of intercept time in function of the target initial velocity (4th scenario)

Tgt_{acc} [m^2/s]	8	9	10.9
V_t_{ini} [m/s]	328	328	328
t_{taken} [s]	6.5500	6.5800	6.6400
$X_{detonate}$ [m]	6820.34	6853.93	6921.15
$Z_{detonate}$ [m]	-3541.16	-3540.59	-3539.47
$Miss_{dis}$ [m]	5.1874	5.2289	5.0260

IV.4. Conclusion

In this chapter, we have studied a *6DOF* open loop missile model and explain the different parts of the system, then we presented the dynamic behavior ‘or the response’ of the missile for two different scenarios and discussed via simulation results.

Besides, we presented a closed loop guidance system of a missile modeled by *3DOF* missile model. Moreover, we have designed and optimal control law of the guidance system and applied it to ensure a good tracking of a moving and maneuvering target.

It is shown through simulation results that the studied missile could track and hit a fast moving maneuvering target thanks to the optimal designed guidance law. In conclusion, missile motion modeling is a critical aspect of understanding and analyzing the behavior and performance of missiles. It involves developing mathematical models and simulation techniques to represent the motion, trajectory and dynamics of missiles during flight.

General Conclusion

General Conclusion

It is obvious that guided missiles provide enhanced accuracy and operational flexibility compared to unguided or ballistic projectiles and rockets thanks to their sophisticated guidance systems and autopilot technology to autonomously navigate and direct themselves towards specific targets. Furthermore, it has been noted these war engines can be classified into various types, including air to air, surface to air, anti ship and anti tank …etc. Each type is designed to fulfill specific mission objectives, ranging from engaging enemy aircraft to destroying armored vehicles or striking targets at long distances.

Moreover, it has been concluded also that the guidance systems enable the missiles to acquire, track, and engage targets accurately. Besides, the autopilot systems play a critical role in controlling the missile's flight path, maintaining stability, and executing effectively maneuvers, enhancing its overall effectiveness. As technology continues to advance, guided missiles are likely to undergo further improvements, such as increased range, speed, and target recognition capabilities.

In this work, we have successfully achieved the targeted objectives that concern optimal control of a missile guidance system. In fact, we have first established a state of the art review that concerns missiles types, launch modes, range, propulsion systems, warheads, guidance systems phases. Then we have presented briefly the most common optimal control methods used in guided missiles.

Next, we have performed the missile *6DOF* and *3DOF* motion modeling by determining the linear velocity components and rotation rates dynamic equations in missile fixed-body and geo-inertial frames. Moreover, we presented the *LQR* optimal control method and shown how apply it to minimize a performance index 'or cost functional' in order to ensure the most favourable intercept guidance law.

Finally, some simulation results of a *6DOF* open loop missile and a *3DOF* missile equipped with an *LQR* optimal control guidance system designed to intercept a target, have been presented and discussed. It is shown that the studied missile could reach and destroy effectively and accurately it's fast moving (with a high speed up to more than 1 Mach) and manoeuvring target in few seconds.

The effects of some key parameters (target speed and acceleration, missile speed, initial miss distance) on the missile guidance system performance have been discussed and shown the effectiveness of the studied system.

By the end, one can consider as perspectives the following points deemed to be of primary interest:

- Design of the *Kalman* filter, *H* infinity to get an optimal control law in order to guarantee more effective guidance system of missiles,
- Investigation of cruise missiles case, with variable mass due to the consumption of propellants during the flight,
- Application of *3D* guidance law in case of interception of all directions maneuvering targets in space requiring normal and side accelerations,
- Synchronization between two or more guided missiles to destroy a specific target.

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