People's Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research
Amar Telidji University of Laghouat
Faculty of Sciences, Department of Mathematics

Master's Thesis

Presented for the attainment of the **Master Degree**

In : Mathematics

Option : Functional Analysis and Applications

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Title

On the pencil of a matrix and its applications

A thesis discussed on: 25/09/2024, in front of the jury composed of :

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ACKNOWLEDGEMENTS

At this modest work, I would like to thank Miss. **Nor El-Houda Beghersa** for agreeing to supervise me, for his valuable advice and guidance. I will always have fond memories of his deeply human qualities.

I extend my deepest thanks to Mr. **Merrad Boulerbah** for agreeing to chair the jury, and to Mr. **Abdelaziz Rahmoune** for agreeing to examine this thesis.

My sincere thanks to all the faculty members, especially the Department of Mathematics.

I also thank everyone who contributed, either directly or indirectly, to the completion of this thesis.

Sabrina Adjallat

RÉSUMÉ

La présente thèse constitue une contribution importante à la théorie des matrices et à ses applications. Elle se compose d'une introduction et de quatre chapitres. Dans le premier chapitre, nous passons en revue certaines définitions de base concernant les pencils, où nous introduisons le concept de polynômes invariants.

Les deuxième et troisième chapitres sont consacrés à l'application des pencils pour résoudre des systèmes différentiels implicites selon deux approches développées par les mathématiciens **F.R. Gantmacher** et **S.L. Campbell**.

Dans le dernier chapitre, nous appliquons le faiseau de matrices $zA + B$ pour trouver une solution à tout système dégénéré discret. Nous fournissons également un exemple pour illustrer la méthode proposée. Nous terminons notre thèse par une conclusion générale et une bibliographie comportant 22 titres.

Mots clés: Faiseau de matrices, Systèmes implicites, Algorithmes de réduction.

ABSTRACT

The present thesis is an important contribution of matrix theory and its applications. It consists of an introduction and (04) chapters. In the first one, we review certain basic definitions about pencils where we introduce the concept of invariant polynomials.

The second and the third chapters are dedicated to the application of of pencils in order to solve implicit differential systems by two approaches of the mathematicians: **F.R. Gantmacher**, **S.L Campbell**.

In the last chapter, we apply the pencil of matrices $zA + B$ to find a solution to any discrete degenerate systems also, we provide an example to illustrate the given method then, we ended our thesis by general conclusion and a bibliography listing 22 titles.

Key words: Pencil of matrices, Implicit systems, Reduction algorithms. .

INTRODUCTION

In control theory and other scientific domains including stability theory [1], we usually deal with the case of explicite differential systems of two types: **Continuous** or **discrete** in the form

$$x'(t) = Tx(t) + F(t); \quad t \geq 0. \tag{1}$$

or

$$x_{n+1} = Tx_n + F_n; \quad n = 0, 1, ..., \tag{2}$$

where T is a matrix or linear operator.

Since 1970, many mathematician took a wide attention to certain systems described by some implicit equations:

$$Ax'(t) = Bx(t) + f(t), \quad t \geq 0, \tag{3}$$

with the initial condition $x(t_0) = x_0$, where A in general is a singular matrix or non-invertible operator and $f(.)$ is a contineous function.

Historically, the fondation of this theory can be traced back to the work of Weierstrass in 1867, which focuced on the regular pencil of matrices $\lambda A - B$ (where $det(\lambda A - B) \neq 0$). However, the most general case includes the contributions of Kronecker (1890). In 1965, the mathematician F.R. Gantmacher provided a comprehensive treatment of this theory represented in [10].

In recent years, the system (3) appears in numerous domains, just to name a few:

- **Control Systems**: H. Rosenbrock (1970) [16], G.C. Verghese (1978) [19], D. Cobb (1984) [6], F. Lewis (1986) [11], M. Malabre (1989) [14].

- **Economics**: Leontiff model (see D.G Luenberger [12]).
- **Demography**: Leslie model (see S.L Campbell (1980) [5]).
- **Circuits**: R.L.C (see R.W. Newcomb (1981) [15]).

For example, let consider the following R.L.C circuit [17]:

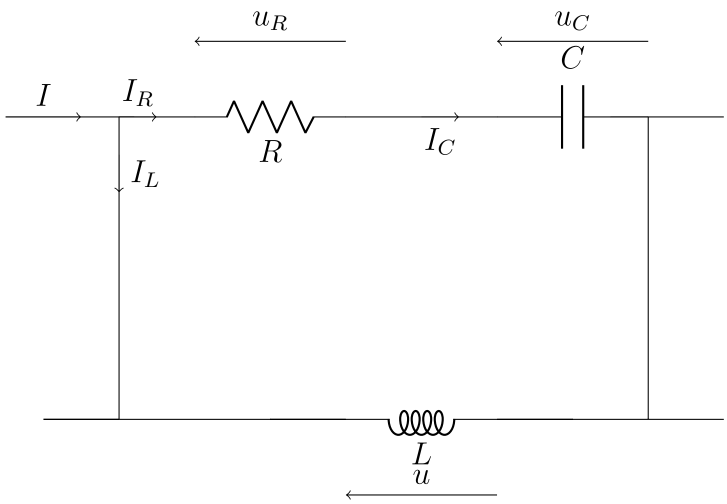

Figure 1: Electrical Circuit

Here, at time $t_0 = 0$, we give the initial values $I_R(0)$, $I_L(0)$, and $U_C(0)$. Our problem is to determine the current intensity I_R through the resistor R, the current intensity I_L through the inductor L, and the voltage U_C across the capacitor C at any given time t.

According to Kirchhoff's laws, we provide the following system of equations:

$$\begin{cases} I = I_R + I_L; \\ L\dfrac{dI_L}{dt} = U; \\ RI_R + U_C = U. \end{cases} \tag{4}$$

The sought state is denoted by: $x(t) = \begin{pmatrix} I_R(t) \\ I_L(t) \\ U_C(t) \end{pmatrix}$ and its derivative by: $x'(t) = \begin{pmatrix} I'_R(t) \\ I'_L(t) \\ U'_C(t) \end{pmatrix}$.

We can easily verified that the system (3) is satisfied when

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & L & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & -1 & 0 \\ 0 & 0 & 0 \\ -R & 0 & -1 \end{pmatrix}, \quad f(t) = \begin{pmatrix} I(t) \\ U_L(t) \\ U(t) \end{pmatrix},$$

also, we note that the matrix A is not necessary invertible.

We will find more pratical examples of various fields including physics and chemistry (quantum mechanics, statistical mechanics) in [3, 9, 18, 20].

CHAPTER 1

SOME PRELIMINARIES ON THE THEORY OF MATRICES

In the classical matrix theory, the study of pencil of matrices based on some properties of their canonical forms. In this section, we will provide some definitions which have an important role in our thesis.

1.1 Pencil of matrices

1.1.1 Fundamental concepts

Definition 1.1. *We call a polynomial matrix, any square matrix denoted by $A(\lambda)$ represented as polynomial in λ with matrix coefficients such that:*

$$A(\lambda) = \|a_{ik}(\lambda)\|_1^n = \|a_{ik}^0\lambda^n + a_{ik}^1\lambda^{n-1} + \cdots + a_{ik}^n\|_1^n = A_0\lambda^n + A_1\lambda^{n-1} + \ldots + A_n; \tag{1.1}$$

where $A_j = \|a_{ik}^j\|_0^n$ for $j = \overline{0, n}$, and $A_0 \neq 0$.

Example 1.1. *The following polynomial matrix is of degree 3 and order 2:*

$$A(\lambda) = \begin{pmatrix} \lambda^3 + \lambda & 2\lambda^3 + \lambda^2 \\ -\lambda^3 - 2\lambda^2 + 1 & 3\lambda^3 + \lambda \end{pmatrix}.$$

Definition 1.2. *Let's consider a matrix $A = \|a_{ik}(\lambda)\|_1^n$ where it charactéristic matrix is denoted by $\lambda I - A$. We call a charactéristic polynomial of A, the scalar polynomial in λ given by $det(A) = |\lambda I - A|$.*

Definition 1.3. *If $A(\lambda)$ is a polynomial matrix of rank r and $D_j(\lambda)$ is the greatest common divisor of all the minors of order j, such that $j = \overline{1, r}$, then the series:*

$$D_r(\lambda), D_{r-1}(\lambda), \ldots, D_1(\lambda), D_0(\lambda) \equiv 1, \tag{1.2}$$

form the corresponding quotients given by:

$$i_1(\lambda) = \frac{D_r(\lambda)}{D_{r-1}(\lambda)}, i_2(\lambda) = \frac{D_{r-1}(\lambda)}{D_{r-2}(\lambda)}, \ldots, i_r(\lambda) = \frac{D_1(\lambda)}{D_0(\lambda)}, \tag{1.3}$$

are called the invariant polynomials of $A(\lambda)$.

Remark 1.1. *If $i_j(\lambda)$ accepts a finite decomposition into irreductible factors over the field $\mathbb{k}$, for all $j = \overline{1, r}$ where:*

$$\begin{aligned} i_1(\lambda) &= [\varphi_1(\lambda)]^{c_1}[\varphi_2(\lambda)]^{c_2} \ldots [\varphi_s(\lambda)]^{c_s}, \\ i_2(\lambda) &= [\varphi_1(\lambda)]^{d_1}[\varphi_2(\lambda)]^{d_2} \ldots [\varphi_s(\lambda)]^{d_s}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ i_r(\lambda) &= [\varphi_1(\lambda)]^{l_1}[\varphi_2(\lambda)]^{l_2} \ldots [\varphi_s(\lambda)]^{l_s}, \end{aligned} \tag{1.4}$$

with $c_k \geq d_k \geq \ldots \geq l_k$ and $k = 1, 2, \ldots, s$, then all the powers among $[\varphi_1(\lambda)]^{c_1}$, $[\varphi_2(\lambda)]^{d_2}$,..., $[\varphi_s(\lambda)]^{l_s}$ in the equation (1.4) which are distinct from 1, are called the elementary divisors of the matrix A.

Definition 1.4. *We define a pencil of matrices $\lambda A - B$ as a polynomial matrix whose coefficients are polynomials of degree less than or equal to 1, where A and B are two arbitrary matrices of the same order.*

Definition 1.5. *A pencil of matrices $\lambda A - B$ is said to be regular if:*

- *A and B are square matrices of the same order n.*
- *The determinant $|\lambda A - B|$ does not vanish identically.*

- *If the rank $r < m = n$, then the given pencil is said to be* **singular square pencil** *(S.S.P).*
- *If $r < m \neq n$, it is named* **singular rectangular pencil** *(S.R.P).*

CHAPTER 2

GANTMACHER APPROACH

In this chapter, we will present Kronecker-Weierstrass Theory for regular or singular pencil of matrices also, we will introduce certain algorithms of reduction for their canonical Forms.

A canonical form of regular pencils was established by Weierstrass in 1867 based on his theory of elementary divisors. Since 1890, the mathematician Kronecker, through his research provided a fundamental result for singular pencils concerning the existence of bases in which the pencil $\lambda A - B$ admits a quasi-diagonal canonical form that is more general than that of Weierstrass.

2.1 Some results on regular and singular pencil of matrices

Theorem 2.1. *[10](**Weierstrass**). Evry regular pencil $\lambda A + B$ can be reduced to a quasi-diagonal (equivalent) form:*

$$\lambda A - B \sim \begin{pmatrix} N^{(u_1)} & & & & \\ & N^{(u_2)} & & & \\ & & \ddots & & \\ & & & N^{(u_s)} & \\ & & & & J - \lambda I \end{pmatrix}; \tag{2.1}$$

where the first s diagonal blocks correspond to the infinite elementary divisors $\mu^{(n_1)}, \mu^{(n_2)}, \dots, \mu^{(n_s)}$ of the pencil $\lambda A - B$, and the normal form of the last diagonal block $J - \lambda I$ is uniquely determined by the finite *elementary divisors of the given pencil, where:*

$$N^{(u_k)} = I^{(u_k)} + \lambda H^{(u_k)} = \begin{pmatrix} 1 & \dots & \dots & 0 \\ \vdots & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{pmatrix}_{u_k \times u_k} + \lambda \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}_{u_k \times u_k} = \begin{pmatrix} 1 & \lambda & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \lambda \\ 0 & \dots & \dots & 1 \end{pmatrix}, k = \overline{1, s}.$$

The last block $J - \lambda I$ corresponds to the Jordan normal form for the matrix $T = (\alpha A - B)^{-1}$, where $|\alpha A - B| \neq 0$.

2.1.1 Reduction Algorithms

Case of regular pencils of matrices:

- **First step.** Consider the regular matrix pencil $\lambda A - B$ of order n.

  1. Compute all the minors Δ_j of order j for $\lambda A - B$; $j = 1, 2, ..., n$. There are $C_n^j \times C_n^j$ minors of order j.
  2. Calculate the greatest common divisor (G.C.D) $D_j(\lambda)$ of all the minors Δ_j.
  3. Ensuring to have 1 as the coefficient of the highest degree in the sequence of polynomials in $D_j(\lambda)$. Each polynomials is divisible by the preceding one.
  4. By factoring the polynomials into irreductible factors over the field of numbers, we obtain finite elementary divisors (F.E.D) of the pencil $\lambda A - B$.

- **Second step.** Consider $\lambda A - \mu B$ as a homogeneous matrix pencil in λ and μ, calculate $\Delta(\lambda, \mu) = |\lambda A - \mu B)|$.

  1. By computing the $D_j(\lambda, \mu) = G.C.D$ of all minors of order$j$ where $j = \overline{1; n}$. There are $C_n^j \times C_n^j$ of order j.
  2. Find the invariant polynomials $i_j(\lambda, \mu)$, for $j = \overline{1; l}$:

  $$i_1(\lambda, \mu) = \frac{D_n(\lambda, \mu)}{D_{n-1}(\lambda, \mu)}; \quad i_2(\lambda, \mu) = \frac{D_{n-1}(\lambda, \mu)}{D_{n-2}(\lambda, \mu)}; \dots; \quad i_l(\lambda, \mu) = \frac{D_1(\lambda, \mu)}{D_0(\lambda, \mu)}.$$

  3. Reduce them into an irreductible form to obtain **infinite elementary divisors**[1] (I.E.D) of the pencil $\lambda A - B$.

[1]The factors of the form μ^q are called **infinite elementary divisors** (I.E.D) of the pencil $\lambda A - B$, exist only if A is not invertible ($det(A) = 0$). .

Now, we will provide certain examples to illustrate the previous theorem.

Example 2.1. *Consider the matrices:*

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix} \; ; \; B = \begin{pmatrix} -2 & -1 & -3 \\ -3 & -2 & -5 \\ -3 & -2 & -6 \end{pmatrix} .$$

The pencil $\lambda A - B$ can be expressed as follows:

$$\begin{pmatrix} \lambda+2 & \lambda+1 & 2\lambda+3 \\ \lambda+3 & \lambda+2 & 2\lambda+5 \\ \lambda+3 & \lambda+2 & 3\lambda+6 \end{pmatrix} .$$

Therefore, $\Delta_3(\lambda) = |\lambda A - B| = \lambda + 1 \neq 0$.

1. *First, we search for the finite elementary divisors of the pencil $\lambda A - B$. We have:*

   - *$\Delta_3(\lambda)$ is the only minor of order 3. Therefore, $D_3(\lambda) = |\lambda A - B| = \lambda + 1$.*
   - *Next, we calculate all minors of order 2 and their greatest common divisor is $D_2(\lambda) = 1$.*
   - *Similarly, we calculate all minors of order 1 and their greatest common is $D_1(\lambda) = 1$.*
   - *Finally, we compute the invariant polynomials using the relation (1.3), we find:*

   $$i_1(\lambda) = \lambda + 1, \quad i_2(\lambda) = 1, \quad i_3(\lambda) = 1, \quad \textit{since} \quad D_0 \equiv 1.$$

   Thus, $\lambda + 1$ is the only finite elementary divisor.

2. *Now, we are searching for the infinite elementary divisors (I.E.D) of the pencil $\lambda A - B$ by introducing the pencil $\lambda A - \mu B$:*

   $$\lambda A - \mu B = \begin{pmatrix} \lambda+2\mu & \lambda+\mu & 2\lambda+3\mu \\ \lambda+3\mu & \lambda+2\mu & 2\lambda+5\mu \\ \lambda+3\mu & \lambda+2\mu & 3\lambda+6\mu \end{pmatrix} .$$

   We have $|\lambda A - \mu B| = \mu^3 + \lambda\mu^2 = \mu^2(\lambda + \mu)$, then $D_3(\lambda) = \mu^2(\lambda + \mu)$.

   - *We compute all the 9 minors of order 2 and their common divisor is $D_2(\lambda, \mu) = 1$.*
   - *Similarly, we calculate all the 9 minors of order 1 and thier common divisor is $D_1(\lambda, \mu) = 1$.*

We get the following invariant polynomials:

$$i_1(\lambda,\mu) = \mu^2(\lambda+\mu), \quad i_2(\lambda,\mu) = 1, \quad i_3(\lambda,\mu) = 1, \quad \textit{since} \quad D_0(\lambda,\mu) \equiv 1.$$

Thus, μ^2 is the only infinite elementary divisor. Therefore, the pencil $\lambda A + B$ is equivalent to the following canonical quasi-diagonal form:

$$\begin{pmatrix} 1 & \lambda & \vdots & 0 \\ 0 & 1 & \vdots & 0 \\ .... & .... & \vdots & .... \\ 0 & 0 & \vdots & \lambda+1 \end{pmatrix}.$$

Example 2.2. *Now, we consider matrices of higher order ($n = 7$) as follows:*

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 1 & 1 \\ 0 & 0 & 5 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

The pencil $\lambda A - B$ can be expressed in the form:

$$\lambda A - B = \begin{pmatrix} 2\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda & \lambda & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 2\lambda & 0 & 0 & -1 & 0 & 0 & 0 \\ 3\lambda & 2 & 0 & 0 & -1 & 0 & 0 \\ -2\lambda & 0 & 10 & 0 & 0 & -1 & -\lambda-1 \\ 2 & 0 & 5 & 0 & 0 & \lambda-1 & 0 \end{pmatrix},$$

and we have $\Delta_7(\lambda) = |\lambda A - B| = \lambda^4 - \lambda^2$.

- ***First step.*** *We search for the finite elementary divisors of the pencil $\lambda A - B$.*

  1. *Since $\Delta_7(\lambda)$ is the only minor of order 7, then $D_7(\lambda) = \lambda^4 - \lambda^2$.*
  2. *We compute the greatest common divisor (G.C.D) of all minors of orders* 6, 5, 4, 3, 2 *and* 1,

then we obtain:

$$D_6(\lambda) = D_5(\lambda) = D_4(\lambda) = D_3(\lambda) = D_2(\lambda) = D_1(\lambda) = 1.$$

3. *We compute the 7 invariant polynomials, then we find:*

$$i_1(\lambda) = \lambda^4 - \lambda^2 = \lambda^2(\lambda+1)(\lambda-1), \quad i_2(\lambda) = i_3(\lambda) = i_4(\lambda) = i_5(\lambda) = i_6(\lambda) = i_7(\lambda) = 1.$$

Thus, the finite elementary divisors are λ^2, $\lambda+1$ *and* $\lambda - 1$.

- ***Second step.*** *We search for the infinite elementary divisors of the pencil* $\lambda A - B$.

1. *Since* $|\lambda A - \mu B| = -2\mu^5\lambda^2 + 2\mu^3\lambda^4$. *Then,* $D_7(\lambda,\mu) = 2\mu^3\lambda^2(\mu-\lambda)(\mu+\lambda)$.
2. *Computing the G.C.D of all minors of respective orders* 6, 5, 4, 3, 2 *and* 1, *we obtain:* $D_6(\lambda,\mu) = \mu^2.$, $D_5(\lambda,\mu) = \mu$, $D_4(\lambda,\mu) = D_3(\lambda,\mu) = D_2(\lambda,\mu) = D_1(\lambda,\mu) = 1$.
3. *Consequently, we find the following invariant polynomials:*

$$i_1(\lambda,\mu) = 2\mu^2\lambda^2(\mu-\lambda)(\mu+\lambda), \quad i_2(\lambda,\mu) = \mu, \quad i_3(\lambda,\mu) = i_4(\lambda,\mu) = i_5(\lambda,\mu) = i_6(\lambda,\mu) = i_7(\lambda,\mu) = 1.$$

Therefore, μ^2, μ *are the infinite elementary divisors of the pencil* $\lambda A - \mu B$.

Finally, the pencil $\lambda A - B$ *is equivalent to:*

$$\lambda\widetilde{A} - \widetilde{B} = \begin{pmatrix} 1 & \lambda & & & & & & \\ 0 & 1 & & & & & & \\ & & 1 & & & & & \\ & & & \lambda & 1 & & & \\ & & & 0 & \lambda & & & \\ & & & & & \lambda+1 & & \\ & & & & & & \lambda-1 \end{pmatrix}$$

Case of singular pencils of matrices:

Here, we provide the Kronecker theorem, which allows us to have a quasi-diagonal form more general than that of Weierstrass.

Theorem 2.2. *[10] Any pencil $\lambda A - B$ can be reduced to the following canonical quasi-diagonal equivalent form:*

$$\begin{pmatrix} h\{\underbrace{[\ 0\ ]}g & & & & \\ & [\ L_\varepsilon\ ] & & & \\ & & [\ L'_\eta\ ] & & \\ & & \lambda A_0 - B_0 & \left\{ \begin{matrix} [\ N^{(u)}\ ] & \\ & [\ J - \lambda I\ ] \end{matrix} \right. \end{pmatrix}.$$

with

$$L_\varepsilon = \begin{pmatrix} L_{\varepsilon_{g+1}} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ & & & & L_{\varepsilon_p} \end{pmatrix}, \quad \textit{where} \quad L_{\varepsilon_i} = \left.\underbrace{\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0 \\ 0 & \lambda & 1 & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & 1 & 0 \\ 0 & \dots & \dots & \dots & \lambda & 1 \end{pmatrix}}_{\varepsilon_i + 1}\right\} \varepsilon_i$$

$$L'_\eta = \begin{pmatrix} L'_{\eta_{h+1}} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ & & & & L'_{\eta_q} \end{pmatrix}, \quad \textit{where} \quad L'_{\eta_j} = \left.\underbrace{\begin{pmatrix} \lambda & 0 & \dots & \dots & 0 \\ 1 & \lambda & & & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \lambda \\ 0 & 0 & \dots & \dots & 1 \end{pmatrix}}_{\eta_j}\right\} \eta_j + 1$$

Remark 2.1.

1. *These elements $\varepsilon_1 = \varepsilon_2 = \dots\dots = \varepsilon_g = 0$; $\eta_1 = \eta_2 = \dots\dots = \eta_h = 0$ correspond to the zero block.*

2. *We can replace the regular pencil of matrices $\lambda A_0 - B_0$ by its quasi-diagonal canonical form (Weierstrass decomposition).*

Basic steps of the reduction algorithm:

(a) Zero block. The number g (resp. h) is the maximum number of independent non-zero constant solution belonging to $Ker(\lambda A - B)$ (resp. $Ker(\lambda A' - B')$), where $\lambda A' - B'$ is the transposed pencil of $\lambda A - B$.

(b) L_ε and L'_η blocks. In this case, we are only interested in non-zero polynomial solutions in λ.

If the equation:

$$(\lambda A - B)x = 0, \tag{2.2}$$

has polynomial solutions, then we consider the one of the minimal degree $\varepsilon_1 \geq 0$:

$$x(\lambda) = x_0 - \lambda x_1 + \lambda^2 x_2 - \ldots\ldots + (-1)^{\varepsilon_1}\lambda^{\varepsilon_1} x_{\varepsilon_1}, \quad (x_{\varepsilon_1} \neq 0),$$

given rise to the birth of L_{ε_1} block.

Remark 2.2.

1. *Once the L_ε blocks are determined, it's time for the L'_η blocks, stopping at the regular block $\lambda A_0 - B_0$.*

2. *If, in the given pencil, $r = n$, i.e., if the columns of the pencil are linearly independent, then the first p diagonal blocks of the L_ε form are absent.*

3. *Similarly, if $r = m$, i.e, if the rows of $\lambda A - B$ are linearly independent, then the diagonal blocks of L'_η form are absent.*

Example 2.3. *We consider the following matrices:*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

So, $\lambda A - B = \begin{pmatrix} \lambda+1 & \lambda-1 \\ \lambda+1 & \lambda-1 \end{pmatrix}$ *is a square singular matrix of rank* 1. *Search for the Kronecker blocks:*

We have $r < m = 2$, which indecates the linear dependence among the columns and rows of the given pencil. The equation (2.2) becomes as follows for $\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}$:

$$(\lambda A - B)x = 0 \Longleftrightarrow \begin{pmatrix} \lambda+1 & \lambda-1 \\ \lambda+1 & \lambda-1 \end{pmatrix}\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longleftrightarrow \begin{cases} (\lambda+1)x_0 + (\lambda-1)x_1 = 0, \\ (\lambda+1)x_0 + (\lambda-1)x_1 = 0. \end{cases}$$

This equation has the solution

$$x(\lambda) = \begin{pmatrix} -\lambda+1 \\ \lambda+1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \end{pmatrix} = x_0 - \lambda x_1.$$

Thus, the degree of $x(\lambda)$ is equal to $\varepsilon = 1$, and it is minimal. There exists a block

$$\varepsilon\{[\lambda \quad 1]_{\varepsilon+1} = L_\varepsilon = L_1.$$

Since $m = n = 2$, then there are $n - \varepsilon - 1 = 0$ columns and $n - \varepsilon = 1$ remaining rows. Hence, the canonical form of the pencil $\lambda A - B$ becomes:

$$\lambda A - B \sim \lambda\widetilde{A} - \widetilde{B} = \begin{pmatrix} \lambda & 1 \\ 0 & 0 \end{pmatrix}.$$

Example 2.4. *Given two rectangular matrices A and B such that:*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 & -1 \\ -1 & -1 & -1 \end{pmatrix}$$

The pencil $\lambda A - B = \begin{pmatrix} \lambda & 2 & 1 \\ 1 & \lambda+1 & 2\lambda+1 \end{pmatrix}$ is singular.

By solving the equation $(\lambda A - B)x = 0$, where $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ so, we obtain the bellow system:

$$\begin{cases} \lambda x_1 + 2x_2 + x_3 = 0 \\ x_1 + (\lambda+1)x_2 + (2\lambda+1)x_3 = 0 \end{cases}$$

$\Leftrightarrow$

$$\begin{cases} x_1 = 3\lambda + 1 \\ x_2 = -2\lambda^2 - \lambda + 1 \\ x_3 = \lambda^2 + \lambda + 2 \end{cases}$$

So

$$x(\lambda) = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \lambda^2 + \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \lambda + \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix};$$

hence, $\varepsilon_1 = 2$ *describes the minimal degree and* $L_{\varepsilon_1} = L_2$ *of* $(2,3)$ *dimension. Since,* $m = 2, \quad n = 3$ *then,* $n - \varepsilon - 1 = 0$ *(columns) and* $m - \varepsilon = 0$ *(rows).*

Finally, we obtain:

$$\lambda A - B \sim \lambda \widetilde{A} - \widetilde{B} = L_{\varepsilon_1} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix}.$$

2.1.2 Application to implicit differential systems

The obtained results will be applied now to a system of m first order differential equations with constant coefficients and m unknown functions.

We consider the following implicit differential system [10]:

$$\sum_{k=1}^{n} [a_{ik} \frac{dx_k}{dt} - b_{ik} x_k] = f_i; \quad i = \overline{1,m}. \tag{2.3}$$

The matrix notation is given by:

$$\begin{cases} A \dfrac{dx(t)}{dt} - Bx(t) = f(t), \\ x(0) = x_0, \end{cases} \tag{2.4}$$

where

$$A = (a_{ik})_{i=\overline{1,m};k=\overline{1,n}} \qquad B = (b_{ik})_{i=\overline{1,m};k=\overline{1,n}}$$

$$x = (x_1, x_2, \ldots, x_n); \quad f = (f_1, f_2, \ldots, f_m).$$ [2]

We transform the equation (2.4) into the following:

$$\begin{cases} \tilde{A} \frac{dz(t)}{dt} - \tilde{B} z(t) = \tilde{f}(t), \\ z(0) = z_0, \end{cases} \tag{2.5}$$

such that

$$\tilde{A} = PAQ\ ; \quad \tilde{B} = PBQ; \quad \tilde{f}(t) = Pf(t),$$

through the bellow transformation:

[2] Let's recall that parentheses denote column vectors. Hence, $x = (x_1, x_2, \ldots, x_n)$ is a column vector with elements x_1, x_2,..., x_n.

1. First, we define $x = Qz$; where Q is a regular matrix $(det(P) \neq 0)$ and $x(0) = Qz(0)$. Then (2.4) becomes:

$$\begin{cases} AQ\dfrac{dz(t)}{dt} - BQz(t) = f(t), \\ Qz(0) = Qz_0, \end{cases} \tag{2.6}$$

2. We multiply both sides of the system (2.6) by a regular matrix P on the left hand side as follows:

$$\begin{cases} PAQ\dfrac{dz(t)}{dt} - PBQz(t) = Pf(t), \\ Qz(0) = Qz_0, \end{cases} \tag{2.7}$$

Remark 2.3. *The matrices P and Q are regular constant coefficients chosen in such a way that the pencil $\lambda A - B$ has the quasi-diagonal form of Kronecker-Weierstrass.*

Now, Let's consider the obtained equation:

$$\begin{cases} (\lambda\tilde{A} - \tilde{B})z(t) = \tilde{f}(t), \\ z(0) = z_0, \end{cases} \tag{2.8}$$

where λ plays the role of derivative. Therefore, the system (2.8) is equivalent to

$$\begin{pmatrix} h\{\underbrace{[\ 0\ ]}\}g & & & & \\ & [\ L_\varepsilon\ ] & & & \\ & & [\ L'_\eta\ ] & & \\ & & & [\ N^{(u)}\ ] & \\ & & & & [\ J - \lambda I\ ] \end{pmatrix} \begin{pmatrix} Z^{(1)} \\ Z^{(2)} \\ Z^{(3)} \\ Z^{(4)} \\ Z^{(5)} \end{pmatrix} = \begin{pmatrix} \tilde{f}^{(1)}(t) \\ \tilde{f}^{(2)}(t) \\ \tilde{f}^{(3)}(t) \\ \tilde{f}^{(4)}(t) \\ \tilde{f}^{(5)}(t) \end{pmatrix} \tag{2.9}$$

where $\tilde{z}^{(i)}$ and $\tilde{f}^{(i)}$ are vectors corresponding to the blocks of Weierstrass-Kronecker form, with

$$Z^{(1)} = z^1 = \begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix}; \quad Z^{(2)} = \begin{pmatrix} z^2 \\ \vdots \\ z^{1+p-g} \end{pmatrix}; \quad Z^{(3)} = \begin{pmatrix} z^{p-g+2} \\ \vdots \\ z^{p-g+1+q-h} \end{pmatrix}; \quad Z^{(4)} = \begin{pmatrix} z^{p-g+q-h+2} \\ \vdots \\ z^{p-g+q-h+1+s} \end{pmatrix}.$$

Here, $Z^{(5)} = Z^v$ such that $v = p - g + q - h + s + 2$. Also,

$$z^2 = \begin{pmatrix} z_{g+1} \\ \vdots \\ z_{\varepsilon_{g+1}+1} \end{pmatrix}; z^3 = \begin{pmatrix} z_{\varepsilon_{g+1}+2} \\ \vdots \\ z_{\varepsilon_{g+2}+1} \end{pmatrix}; \dots; \tilde{f}^{(1)} = \begin{pmatrix} \tilde{f}_1 \\ \vdots \\ \tilde{f}_h \end{pmatrix}; \tilde{f}^{(2)} = \begin{pmatrix} \tilde{f}_{h+1} \\ \vdots \\ \tilde{f}_{\eta_{h+1}+1} \end{pmatrix}; \quad \text{etc}\dots$$

The system (2.9) is equivalent to the following five subsystems:

$$\begin{cases} 0Z^{(1)} & = \tilde{f}^{(1)} \quad \dots\dots \quad (\mathbf{I}) \\ L_\varepsilon Z^{(2)} & = \tilde{f}^{(2)} \quad \dots\dots \quad (\mathbf{II}) \\ L'_\eta Z^{(3)} & = \tilde{f}^{(3)} \quad \dots\dots \quad (\mathbf{III}) \\ N^{(u)} Z^{(4)} & = \tilde{f}^{(4)} \quad \dots\dots \quad (\mathbf{IV}) \\ (J-\lambda I)Z^{(5)} & = \tilde{f}^{(5)} \quad \dots\dots \quad (\mathbf{V}) \end{cases} \tag{2.10}$$

By writing (2.10) explicitly and in order to avoid cluttering the notations, we'll use the symbols z and $\tilde{f}$ instead of $Z^{(i)}$ and $\tilde{f}^{(i)}$ for $i = \overline{1,5}$. So:

1. The first subsystem (**I**) is satisfied if and only if, $\tilde{f}$ is a block of zero vectors.

2. The second subsystem (**II**) is equivalent to

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 & 0 \\ 0 & 0 & \dots & \lambda & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ \\ \vdots \\ z_{\varepsilon+1} \end{pmatrix} = \begin{pmatrix} \tilde{f}_1 \\ \\ \vdots \\ \tilde{f}_\varepsilon \end{pmatrix}. \tag{2.11}$$

This subsystem has more unknowns than equations.

$$\begin{cases} z'_1 + z_2 & = \tilde{f}_1 \\ z'_2 + z_3 & = \tilde{f}_2 \\ z'_3 + z_4 & = \tilde{f}_3 \\ \vdots & = \vdots \\ z'_\varepsilon + z_{\varepsilon+1} & = \tilde{f}_\varepsilon \end{cases}$$

Also, it implies the existence of an infinite number of solutions independently of $\tilde{f}_{II}$ and the initial condition. To obtain the solutions, it is clear that we need to assign a value to one of the unknowns.

•For uniqueness, the issue is related to the presence of the L_ε block.

In order to have a unique solution to equation (2.4), it's necessary for this block to be absent.

3. The subsystem (**III**) is equivalent to

$$\begin{cases} z_1' &= \tilde{f}_1 \\ z_2' + z_1 &= \tilde{f}_2 \\ z_3' + z_2 &= \tilde{f}_3 \\ \vdots &= \vdots \\ z_\eta' + z_{\eta-1} &= \tilde{f}_\eta \\ z_\eta &= \tilde{f}_{\eta+1} \end{cases} \tag{2.12}$$

More equations than unknowns, a constraint on $\tilde{f}$ and its derivatives is imposed. Under this constraint, the solution exists and it is unique. Therefore,

$$\begin{cases} z_\eta &= \tilde{f}_{\eta+1} \\ z_{\eta-1} &= \tilde{f}_\eta - \tilde{f}'_{\eta+1} \\ \vdots &= \vdots \\ z_1 &= \tilde{f}_2 - \tilde{f}'_3 + \dots\dots\dots\dots + (-1)^{\eta-1} \tilde{f}^{\eta-1}_{\eta+1}. \end{cases} \tag{2.13}$$

So, the condition of possibility arises as follows:

$$\boxed{\tilde{f}_1 - \tilde{f}'_2 - \tilde{f}''_3 + \dots\dots\dots\dots + (-1)^\eta \tilde{f}^\eta_{\eta+1} \quad = \quad 0.}$$

4. The block (**IV**) is equivalent to:

$$\begin{cases} z_2' + z_1 &= \tilde{f}_1 \\ z_3' + z_2 &= \tilde{f}_2 \\ \vdots &= \vdots \\ z_u' + z_{u-1} &= \tilde{f}_{u-1} \\ z_u &= \tilde{f}_u \end{cases} \tag{2.14}$$

Then, we can successively determine the unique solution:

$$\begin{cases} z_u &= \tilde{f}_u \\ z_{u-1} &= \tilde{f}_{u-1} - \tilde{f}'_u \\ \vdots &= \vdots \\ z_1 &= \tilde{f}_1 - \tilde{f}'_2 + \tilde{f}''_3 + \dots\dots + (-1)^{u-1} \tilde{f}^{u-1}_u \end{cases} \tag{2.15}$$

5. The last subsystem (**V**) is equivalent to:

$$\begin{cases} Jz(t) - z'(t) &= \tilde{f}(t) \\ z(0) &= z_{0} \quad \text{given.} \end{cases} \tag{2.16}$$

The general solution of this system is unique and it is described as follows:

$$z(t) = \exp^{Jt} z_0 - \int_{0}^{t} \exp^{J(t-s)} \tilde{f}(s) ds. \tag{2.17}$$

Theorem 2.3. *[10] (**Existence and Uniqueness**)*

The system (2.4) has a solution of the form:

$$X(t) = \begin{pmatrix} X^{(1)}(t) \\ \vdots \\ X^{(5)}(t) \end{pmatrix}, \tag{2.18}$$

if and only if, the right-hand side $\mathbf{f(t)}$ *is infinitely differentiable and satisfies the following conditions:*

1. $f_{1} = f_{2} = \ldots = f_{g} = 0,$
2. $Pf_{1} - Pf'_{2} + \ldots + (-1)^{\eta} Pf^{\eta}_{\eta+1} = 0;$

where $X^{(i)} = QZ^{(i)}$ *for* $i = \overline{1,5}$ *and* $Z^{(i)}$ *satisfies (2.10).*

The transformation (transition) of matrices for the canonical form of Weierstrass-Kronecker are denoted by P *and* Q*.*

Remark 2.4.

- *If the zero block is absent, there is no need for the first condition in Theorem 2.3.*
- *If the pencil* $\lambda A - B$ *is regular, there exists a unique solution*

$$X(t) = \begin{pmatrix} X^{(4)} \\ \vdots \\ X^{(5)} \end{pmatrix}, \tag{2.19}$$

without the conditions (1) and (2) in Theorem 2.3.

Example 2.5. *In the present example, we will solve the bellow system:*

$$Ax'(t) - Bx(t) = 0, \tag{2.20}$$

where

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & -5 & 0 & 0 & -1 & 0 \end{pmatrix}.$$

The pencil $\lambda A - B$ is expressed by the following matrix:

$$\lambda A - B = \begin{pmatrix} 2\lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ -\lambda & \lambda & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2\lambda & 0 & 0 & 1 & 0 & 0 & 0 \\ 3\lambda & 2 & 0 & 0 & 1 & 0 & 0 \\ -2\lambda & 0 & 0 & 0 & 0 & 1 & -\lambda + 1 \\ 2 & 0 & 5 & 0 & 0 & \lambda + 1 & 0 \end{pmatrix}.$$

According to Example 2.2, the matrix pencil $\lambda A - B$ is equivalent to:

$$\lambda \tilde{A} - \tilde{B} = \begin{pmatrix} \begin{array}{|cc|} \hline 1 & \lambda \\ 0 & 1 \\ \hline \end{array} & & & & \\ & \boxed{1} & & & \\ & & \begin{array}{|cc|} \hline \lambda & 1 \\ 0 & \lambda \\ \hline \end{array} & & \\ & & & \boxed{\lambda + 1} & \\ & & & & \boxed{\lambda - 1} \end{pmatrix}.$$

Therefore, solving (2.20) is equivalent to:

$$\begin{pmatrix} 1 & \lambda & & & & & \\ 0 & 1 & & & & & \\ & & 1 & & & & \\ & & & \lambda & 1 & & \\ & & & 0 & \lambda & & \\ & & & & & \lambda+1 & \\ & & & & & & \lambda-1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \end{pmatrix} = \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \tilde{f}_3 \\ \tilde{f}_4 \\ \tilde{f}_5 \\ \tilde{f}_6 \\ \tilde{f}_7 \end{pmatrix}.$$

$$(2.20) \iff \begin{cases} z_1 + z_2' &= 0 \\ z_2 &= 0 \\ z_3 &= 0 \\ z_4' + z_5 &= 0 \\ z_5' &= 0 \\ z_6' + z_6 &= 0 \\ z_7' - z_7 &= 0 \end{cases} \iff \begin{cases} z_1 &= 0 \\ z_2 &= 0 \\ z_3 &= 0 \\ z_4 &= -c_5 t + c_4 \\ z_5 &= c_5 \\ z_6 &= c_6 e^t \\ z_7 &= c_7 e^{-t} \end{cases}.$$

The solution of the system: $(\lambda\tilde{A} - \tilde{B})Z(t) = \tilde{A}Z'(t) - \tilde{B}Z(t) = 0$ *is described as follows:*

$$Z(t) = \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \\ z_5(t) \\ z_6(t) \\ z_7(t) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -c_5 t + c_4 \\ c_5 \\ c_6 e^{-t} \\ c_7 e^{t} \end{pmatrix}$$

Knowing that $x(t) = QZ(t)$*, then we choose the transformation of the matrices* P *and* Q *which satisfy the conditions:*

$$PAQ = \tilde{A}, \quad PBQ = \tilde{B}.$$

where the pencil $\lambda A - B$ *will have the desired quasi-diagonal form.*

For example, if we take:

$$P = I_7 \quad (\textit{Identity}), \quad \textit{and} \quad Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix},$$

then, the general solution of the system (2.20) *will be:*

$$x(t) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -c_5 t + c_4 \\ c_5(t+1) - c_4 \\ c_5(t-1) - c_4 + c_6 e^{-t} \\ c_5(t-1) - c_4 + c_6 e^{-t} + c_7 e^{t} \end{pmatrix};$$

where $c_1, c_2, ..., c_7$ *are arbitrary constants.*

We conclude our chapter by the next flow-chart where we summarize all the possible cases.

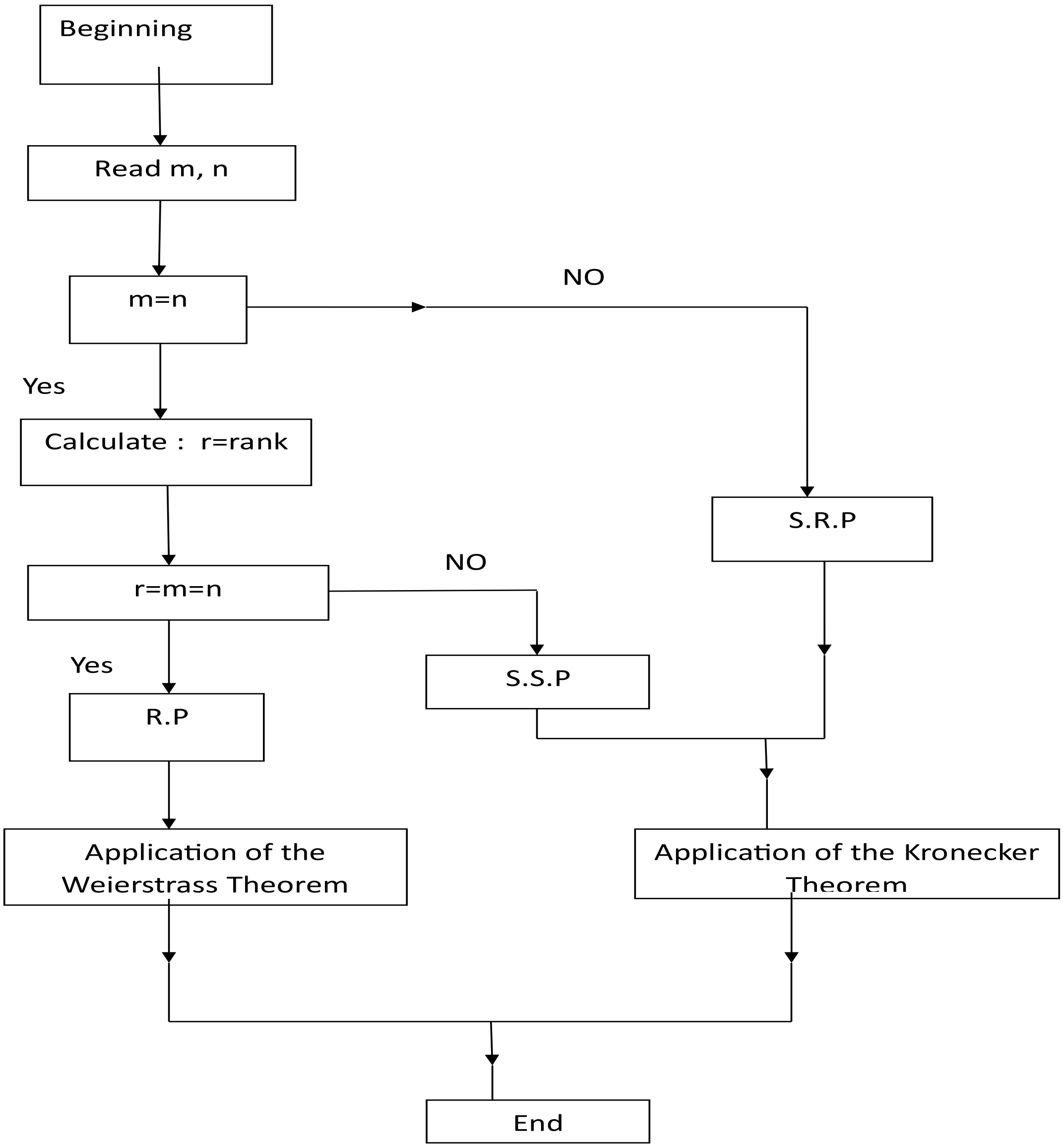

Figure 2.1: The reduction algorithm for a pencil of matrices

CHAPTER 3

S.L. GAMPBELL APPROACH

In the previous chapter, we discussed an important method to compute the solution of any implicit differential system. Now, we provide another way based on some desirable characterizations of Drazin theory of inverses (case of square singular matrices) [22].

3.1 Some Properties of Drazin inverse

In this appoach, the question arises: if a square matrix is not invertible, can we define an inverse? that is a matrix which satisfies the properties of the classical inverse (Definition 3.3). In this case, we explore a more general inverse theory known as Drazin inverse.

3.1.1 Basic preliminaries

Let $\mathbb{C}^{n\times n}$ denote the set of complex square matrices of order n.

Definition 3.1. *(Rank of a matrix). The rank of a matrix $A \in \mathbb{C}^{n\times n}$, denoted by $Rank(A)$, is the maximum number of row or column vectors that are linearly independent.*

Definition 3.2. *(Index of a matrix). For $A \in \mathbb{C}^{n\times n}$, the smallest positive integer k such that $Rank(A^{k+1}) = rank(A^k)$ is called the index of A, denoted as $ind(A)$.*

Definition 3.3. *A matrix $A \in \mathbb{C}^{n\times n}$ is said to be invertible if there exists a unique matrix $B \in \mathbb{C}^{n\times n}$, denoted by $B = A^{-1}$, such that $AB = BA = I$, where I is the identity matrix in $\mathbb{C}^{n\times n}$.*

Definition 3.4. *[5] Let $A \in \mathbb{C}^{n\times n}$, the Drazin inverse of a matrix A is defined as any matrix X that satisfies the following properties:*

(i) $AX = XA$,

(ii) $XAX = X$,

(iii) $XA^{k+1} = A^k$ *where* $k = ind(A)$.

In this case, the Drazin inverse of the matrix A *is denoted by* A^D.

Property 3.1. *[5] If* $A \in \mathbb{C}^{n\times n}$*, then we have:*

1. *If* A *is nilpotent, then* $A^D = 0$.
2. *If* A *is invertible, then* $A^D = A^{-1}$.
3. $AA^D = A^D A$.
4. $A^D AA^D = A^D$.
5. $(BAB^{-1})^D = BA^D B^{-1}$*, such that* B *is regular matrix.*

Property 3.2. *[5] For* $A \in \mathbb{C}^{n\times n}$ *the Drazin inverse* A^D *is unique.*

Property 3.3. *[5] If* $\lambda \neq 0$ *is an eigen-value of* $A \in \mathbb{C}^{n\times n}$*, then* $\frac{1}{\lambda}$ *is an eigen-value of* A^D.

The next algorithm presents a important result which suppose to carry the computing of A^D [22]

Algorithm:

Let $A \in \mathbb{C}^{n\times n}$. Assume that $\{\lambda_0, \lambda_1, \lambda_2, ..., \lambda_t\}$ are the distinct eigenvalues of A and $\lambda_0 = 0$. Let m_i denotes the algebraic multiplicity of λ_i and let $m = n - m_0 = m_1 + m_2 + ... + m_t$, $p(x)$ is the polynomial of degree $n-1$ such that $p(x) = x^{m_0}(\alpha_0 + \alpha_1 x + ... + \alpha_{m-1}x^{m-1})$ whose coefficients are the unique solutions of the following $m \times m$ system of linear equations where $(.)^{(i)}$ denotes the i-th derivative.

$$\frac{1}{\lambda_i} = p(\lambda_i),$$

$$\frac{-1}{\lambda_i^2} = p'(\lambda_i),$$

$$\vdots = \quad \text{for} \quad i = 1, 2, ..., t,$$

$$\frac{(-1)^{m_i-1}(m_i-1)!}{\lambda_i^{m_i}} = p^{(m_i-1)}(\lambda_i),$$

then $p(A) = A^D$.

Example 3.1. *Let* $A = \begin{pmatrix} 2 & 4 & 6 & 5 \\ 1 & 4 & 5 & 4 \\ 0 & -1 & -1 & 0 \\ -1 & -3 & -3 & -3 \end{pmatrix}$, *we shall use the above algorithm as follows:*

the first step *is to copmute the eigenvalues for* A *so, we obtain* $\sigma(A) = \{0, 0, 1, 1\}$.

Therefore, $m_0 = 2$ *and* $m_1 = 2$, *hence* A^D *can be expressed as* $A^D = A^2(\alpha_0 I + \alpha_1 A)$.

Since $p(x) = x^2(\alpha_0 + \alpha_1 x)$ *and* α_0, α_1 *are the solution of the system:*

$$1 = p(1) = \alpha_0 + \alpha_1,$$

$$-1 = p'(1) = 2\alpha_0 + 3\alpha_1,$$

then

$$\alpha_0 = 4, \quad \alpha_1 = -3$$

$$A^D = A^2(4I - 3A) = \begin{pmatrix} 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}.$$

3.1.2 Its application for impilicit differential systems

In the present subsection, we will use the Drazin inverse theory to provide a solution for the system (2.4) in a special case when A and B are $n \times n$ singular square matrices (systems with constant coefficients).

Theorem 3.1. *[5]*

1. *Suppose that the system* (2.4) *is consistent (solvable)*[1],

2. *The function* f *is a* k*-times continuously differentiable,*

then the general solution of (2.4) *has the following form:*

$$x(t) = e^{\tilde{A}^D \tilde{B} t} \tilde{A} \tilde{A}^D x_0 + \tilde{A}^D e^{\tilde{A}^D \tilde{B} t} \int_0^t e^{\tilde{A}^D \tilde{B} s} \tilde{f}(s) ds + (I - \tilde{A}\tilde{A}^D) \sum_{n=0}^{k-1} (-1)^n (\tilde{A}\tilde{B}^D)^n \tilde{B}^D \tilde{f}^n(0),$$

where

$$\tilde{A}_\lambda = (\lambda A - B)^{-1} A, \quad \tilde{B}_\lambda = (\lambda A - B)^{-1} B, \quad \tilde{f}_\lambda(t) = (\lambda A - B)^{-1} f(t), \quad det((\lambda A - B)^{-1}) \neq 0, \quad t \geq 0.$$

[1]The system (2.4) is said to be consistent if and only if: $\exists \lambda_0 \in \mathbb{C}$ such that $det(\lambda_0 A - B) \neq 0$, in this case the solution is unique and our system is solvable

Example 3.2. *We consider the homogeneous case of the system* (2.4) *($f(t) = 0$), where*

$$A = \begin{pmatrix} 1 & 0 & -2 \\ -1 & 0 & 2 \\ 2 & 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & -2 \\ 27 & 22 & 17 \\ -18 & -14 & -10 \end{pmatrix}.$$

Here, the matrices A and B are not commutative also, we have $det(A) = det(B) = 0$, which means that A and B are not invertible.

By taking $\lambda_0 = 1$, then $det(A_1 - B) = -45 \neq 0$. Now, we multiply our system on the left side by $(A-B)^{-1}$, then we obtain:

$$\tilde{A}x'(t) - \tilde{B}x(t) = 0;$$

where

$$\tilde{A} = (A-B)^{-1}A = \frac{1}{3}\begin{pmatrix} -3 & -5 & -4 \\ 6 & 5 & -2 \\ -3 & 2 & 10 \end{pmatrix}, \quad \tilde{B} = (A-B)^{-1}B = \frac{1}{3}\begin{pmatrix} 6 & 5 & 4 \\ -6 & -2 & 2 \\ 3 & -2 & 7 \end{pmatrix}.$$

A unique sollution will exist if and only if the initial vector $x(0)$ satisfies:

$$(I - \tilde{A}\tilde{A}^D)x(0) = 0. \tag{3.1}$$

The set of all eigen-values of the matrices $\tilde{A}$ and $\tilde{B}$ is $\{0,1,3\}$ and $\{0,1,-2\}$ respectively. By using the algorithm of computation of the Drazin inverse as in [5], then we get:

$$\tilde{A}^D = \frac{1}{27}\begin{pmatrix} -27 & -41 & -28 \\ 54 & 77 & 46 \\ -27 & -34 & -14 \end{pmatrix}, \quad \tilde{B}^D = \frac{1}{12}\begin{pmatrix} 24 & 19 & 14 \\ -24 & -16 & -8 \\ 12 & 5 & -2 \end{pmatrix}.$$

The condition (3.1) *can be replaced by: $9x_1(0) + 7x_2(0) + 5x_3(0) = 0$.*
The eigen-values of the matrix $\tilde{A}^D\tilde{B}$ are $\{0, 0, \frac{2}{3}\}$. According to Theorem 3.1, our homogeneous system has the following solution:

$$x(t) = e^{\tilde{A}^D\tilde{B}t}\tilde{A}\tilde{A}^D x_0 = \frac{1}{18}\begin{pmatrix} 18 & 1 - e^{\frac{2}{3}t} & 2(1 - e^{\frac{2}{3}t}) \\ 0 & 26 - 8e^{\frac{2}{3}t} & 16(1 - e^{\frac{2}{3}t}) \\ 0 & 13(e^{\frac{2}{3}t} - 1) & 26e^{\frac{2}{3}t} - 8 \end{pmatrix}\begin{pmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{pmatrix}.$$

Remark 3.1. *The concepts of Drazin inverse theory can be also extended naturally for implicit or degenerate differential systems such* (2.4) *with bounded operators* A[2], B *in Hilbert spaces.*

[2]This operator is not invertible.

CHAPTER 4

DISCRETE IMPLICIT SYSTEMS

This chapter is dedicated to the study of implicit discrete linear systems, that is, systems of the form:

$$\sum_{j=1}^{n} a_{ij} x_{k+1}^{j} + \sum_{j=1}^{n} b_{ij} x_{k}^{j} \quad = f_{k}^{i}(t); \tag{4.1}$$

$$i = (1, 2, ..., m), k = 0, 1, 2, ..., .$$

or in matrix notation:

$$Ax_{k+1} + Bx_k = f_k, \qquad k = 0, 1, 2, ...; \tag{4.2}$$

such that

$$A = (a_{ij}), \qquad B = (b_{ij}), \qquad i = 1, 2, ..., m; \qquad j = 1, 2, ..., n;$$

$$x_k = (x_k^1, x_k^2, ..., x_k^n), \qquad f_k = (f_k^1, f_k^2, ..., f_k^m).$$

These systems can be considered as an analogy to the implicit differential systems given by (3) (continuous case).

These systems appear in technology, physics, and applied mathematics. Implicit discrete systems find their applications in impulse chain theory, discrete control theory, finite difference schemes, and econometrics. They can also be used for approximation methods of solutions to differential equations of the form (3), Unlike explicit systems, implicit discrete systems (IDS) may not have a solution.

4.1 Solving implicit discrete systems

Now, we introduce the new unknown sequences $\overset{1}{y}_k, \overset{2}{y}_k, ..., \overset{n}{y}_k$ that are related to the original ones by a regular transformation with constant coefficients:

$$x_k = Qy_k; y_k = (\overset{1}{y}_k, \overset{1}{y}_k, \overset{2}{y}_k, ..., \overset{n}{y}_k) \qquad \det Q \neq 0.$$

Moreover, instead of the equations (4.1) we can consider m independent combinations of these equations, which is equivalent to left-multiplying the matrices A, B, f_k by a regular matrix of order m.

By substituting Qy for x in (4.2) and left-multiplying by P, we obtain:

$$\tilde{A}y_{k+1} + \tilde{B}y_k = \tilde{f}_k \tag{4.3}$$

where

$$\tilde{A} = PAQ\ ; \tilde{B} = PBQ \textit{ and } \tilde{f}_k = Pf_k = (\tilde{f}^1_k, \tilde{f}^2_k, \tilde{f}^m_k).$$

The matrices P and Q are regular with constant coefficients and are chosen such that the pencil $zA+B$ has the Kronecker-Weierstrass quasi-diagonal form:

$$z\tilde{A} + \tilde{B} = \{0; L_{\varepsilon_{g+1}}, ..., L_{\varepsilon_p}, L'_{\eta_{h+1}}, ..., L'_{\eta_q}, N^{(u_1)}, ..., N^{(u_s)}, zI + J\} \tag{4.4}$$

In accordance with the blocks of (4.4), the system of discrete equations is divided into $v = p - g + q - h + s + 2$ separate systems of the form:

$$(S) \qquad \begin{cases} 0Y^{(1)}_k & = & \tilde{f}^{(1)}_k & \ldots\ldots & \textbf{(I)} \\ L_\varepsilon Y^{(2)}_k & = & \tilde{f}^{(2)}_k & \ldots\ldots & \textbf{(II)} \\ L'_\eta Y^{(3)}_k & = & \tilde{f}^{(3)}_k & \ldots\ldots & \textbf{(III)} \\ N^{(u_i)} Y^{(4)}_k & = & \tilde{f}^{(4)}_k & \ldots\ldots & \textbf{(IV)} \\ Y^{(5)}_k + JY^{(5)}_k & = & \tilde{f}^{(5)}_k & \ldots\ldots & \textbf{(V)} \end{cases} ;$$

where:

$$Y^{(1)}_k = \overset{1}{y} = \begin{bmatrix} \overset{1}{y}_k \\ \vdots \\ \overset{g}{y}_k \end{bmatrix}; Y^{(2)}_k = \begin{bmatrix} \overset{2}{y}_k \\ \vdots \\ \overset{1+p-g}{y}_k \end{bmatrix}; Y^{(3)}_k = \begin{bmatrix} \overset{p-g+2}{y}_k \\ \vdots \\ \overset{p-g+1+q-h}{y}_k \end{bmatrix}; Y^{(4)}_k = \begin{bmatrix} \overset{p-g+q-h+2}{y}_k \\ \vdots \\ \overset{p-g+q-h+1+s}{y}_k \end{bmatrix};$$

$$Y_k^{(5)} = Y_k^{(v)},$$

such that:

$$v = p - g + q - h + s + 2;$$

and

$$y_k^2 = \begin{bmatrix} y_k^{g+1} \\ \vdots \\ y_k^{\varepsilon_{g+1}+1} \end{bmatrix}; y_k^3 = \begin{bmatrix} y_k^{\varepsilon_{g+1}+2} \\ \vdots \\ y_k^{\varepsilon_{g+2}+1} \end{bmatrix}; \ldots; \tilde{f}_k^{(1)} = \begin{bmatrix} \tilde{f}_k^1 \\ \vdots \\ \tilde{f}_k^1 \end{bmatrix} \; et \; \tilde{f}^{(2)} = \begin{bmatrix} \tilde{f}_k^{h+1} \\ \vdots \\ \tilde{f}_k^{\eta_{h+1}+1} \end{bmatrix}; etc.....$$

Therefore, solving the system (4.1) reduces to solving the system (S) above, which is of the same type. In this system, the pencil $zA + B$ has the following form:

$$\{0, L_\varepsilon, L'_\eta, N^u, zI + J\}.$$

The block $'0'$ of dimension (h, g) corresponds to the possible constant solutions of the two systems:

$$(zA + B)x_k = 0 \tag{4.5}$$

$$(zA' + B')'y_k = 0 \tag{4.6}$$

with h and g being their respective numbers.

The first subsystem **I** is possible if and only if:

$$\tilde{f}_k^1 \equiv 0, \tag{4.7}$$

In other words:

$$\tilde{f}_k^1{}_1 \equiv 0; .....; \tilde{f}_k^1{}_h \equiv 0 \tag{4.8}$$

Therefore, we can choose arbitrary elements for the first g elements of the sequence y_k. Naturally, the problem does not arise for the homogeneous equations.

The second **II** is of the form:

$$L_\varepsilon y_k^{(2)} = \tilde{f}_k^2 \tag{4.9}$$

which means that:

$$(S') \qquad \begin{cases} y^{1}_{k+1} + y^{2}_{k} &= \tilde{f}^{2}_{\substack{1\\k}} \\ y^{2}_{k+1} + y^{2}_{k} &= \tilde{f}^{2}_{\substack{2\\k}} \\ y^{3}_{k+1} + y^{4}_{k} &= \tilde{f}^{3}_{\substack{2\\K}} \\ \vdots &= \vdots \\ z^{\varepsilon}_{k+1} + y^{\varepsilon+1}_{k} &= \tilde{f}^{2}_{\substack{\varepsilon\\k}} \end{cases} .$$

Such a system is always consistent. If we choose an arbitrary element for $y_k^{\varepsilon+1}$, all the other elements of the sequence will be determined from (S'). Thus, the solution to this system is not unique. The third (**III**) is of the form:

$$L'_{\varepsilon} Y^{3}_{k} = \tilde{f}^{3}_{k}. \tag{4.10}$$

It is equivalent to:

$$\begin{cases} y^1_{k+1} &= \tilde{f}^{3}_{\substack{1\\k}} \\ y^2_{k+1} + y^1_k &= \tilde{f}^{3}_{\substack{2\\k}} \\ y^3_{k+1} + y^2_k &= \tilde{f}^{3}_{\substack{2\\K}} \\ \vdots &= \vdots \\ y^\eta_{k+1} + y^{\eta-1}_k &= \tilde{f}^{3}_{\substack{\eta\\k}} \\ y^\eta_k &= \tilde{f}^{3_{\eta+1}}_{k} \end{cases} \tag{4.11}$$

From these equations, with the exception of the first one, we can determine uniquely $y_k^{\eta}, y_k^{\eta-1}, \ldots.., y_k^1$ as follows:

$$\begin{cases} y_k^{\eta} & = \tilde{\overset{3}{f}}{}_k^{\eta+1} \\ y_k^{\eta-1} & = \tilde{\overset{3}{f}}{}_k^{\eta} - \tilde{\overset{3}{f}}{}_{k+1}^{\eta+1} \\ \vdots & = \vdots \\ y_{k+1}^1 & = \tilde{\overset{3}{f}}{}_{k+1}^{2} - \tilde{\overset{3}{f}}{}_{k+2}^{3} + \ldots\ldots\ldots\ldots + (-1)^{\eta-1} \tilde{\overset{3}{f}}{}_{k+\eta-1}^{\eta+1} \end{cases}.$$

Thus,

$$y_{k+1}^1 = \tilde{\overset{3}{f}}{}_{k+1}^{2} - \tilde{\overset{3}{f}}{}_{k+2}^{3} + \ldots\ldots\ldots\ldots + (-1)^{\eta-1} \tilde{\overset{3}{f}}{}_{k+\eta}^{\eta+1}.$$

By substituting this expression into the first equation:

$$y_{k+1}^1 = \tilde{\overset{3}{f}}{}_k^{1};$$

we obtain:

$$\boxed{\tilde{\overset{3}{f}}{}_k^{1} - \tilde{\overset{3}{f}}{}_{k+1}^{2} + (-1)^2 \tilde{\overset{3}{f}}{}_{k+1}^{3} + \ldots\ldots\ldots\ldots + (-1)^{\eta} \tilde{\overset{3}{f}}{}_{k+\eta}^{\eta+1} \quad = \quad 0.}$$

which is the feasibility condition for the system (III). The subsystem (**IV**) is of the form:

$$N^{(u)} Y_k^1 = \tilde{f}_k^4.$$

It means that:

$$\begin{cases} y_{k+1}^2 + y_k^1 & = \tilde{\overset{4}{f}}{}_k^{1} \\ y_{k+1}^3 + y_k^2 & = \tilde{\overset{4}{f}}{}_k^{2} \\ \vdots & = \vdots \\ y_{k+1}^u + y_k^{u-1} & = \tilde{\overset{4}{f}}{}_k^{u-1} \\ y_k^u & = \tilde{\overset{4}{f}}{}_k^{u} \end{cases}.$$

We successively determine the unique solutions:

$$\begin{cases} y_k^u & = \tilde{f}_k^{\overset{4}{u}} \\ y_k^{u-1} & = \tilde{f}_k^{\overset{4}{u-1}} - \tilde{f}_{k+1}^{\overset{4}{u}} \\ y_k^{u-2} & = \tilde{f}_k^{\overset{4}{u-2}} - \tilde{f}_{k+1}^{\overset{4}{u-1}} + (-1)^2 \tilde{f}_{k+2}^{\overset{4}{u}} \\ y_k^{u-3} & = \tilde{f}_k^{\overset{4}{u-3}} - \tilde{f}_{k+1}^{\overset{4}{u-2}} + (-1)^2 \tilde{f}_{k+2}^{\overset{4}{u-1}} + (-1)^3 \tilde{f}_{k+3}^{\overset{4}{u}} \\ \vdots & = \vdots \\ y_k^1 & = \tilde{f}_k^{\overset{4}{1}} - \tilde{f}_{k+1}^{\overset{4}{2}} + (-1)^2 \tilde{f}_{k+2}^{\overset{4}{3}} + ... + (-1)^{u-1} \tilde{f}_{k+u-1}^{\overset{4}{u}} \end{cases}.$$

The last subsystem **V** is of the form:

$$y_{k+1} + J y_k = \tilde{f}_k^5$$

So:

$$\begin{cases} y_{k+1} & = \tilde{f}_k^5 - J y_k \\ & = \tilde{f}_k^5 - J(\tilde{f}_{k-1}^5 - J y_{k-1}) \\ & = \tilde{f}_k^5 - J \tilde{f}_{k-1}^5 + (-1)^2 y_{k-1} \\ & = \tilde{f}_k^5 - J \tilde{f}_{k-1}^5 + (-1)^2 J^2 \tilde{f}_{k-2}^5 + (-1)^3 y_{k-2} \\ \vdots & = \vdots \\ & = \tilde{f}_k^5 - J \tilde{f}_{k-1}^5 + (-1)^2 J^2 \tilde{f}_{k-2}^5 + (-1)^3 J^3 \tilde{f}_{k-3}^5 + ... + (-1)^k J^k \tilde{f}_0^5 + (-1)^{k+1} J^{k+1} y_0 \end{cases}.$$

Hence,

$$y_k = (-1)^k J^k y_0 + \sum_{i=0}^{k-1} (-1)^i J^i \tilde{f}_{k+i-1}^5. \tag{4.12}$$

Theorem 4.1. *[2] The implicit discrete system (4.1) admits a solution if and only if the sequence* $\{\tilde{f}\}_k^\infty$ *defined by* $\tilde{f}_k = P f_k$ *satisfies:*

1. $\tilde{f}_k^1 = 0.$

2. $\tilde{f}_k^{\overset{3}{1}} - \tilde{f}_{k+1}^{\overset{3}{2}} + (-1)^2 \tilde{f}_{k+2}^{\overset{3}{3}} + ... + (-1)^\eta \tilde{f}_{k+\eta}^{\overset{3}{\eta+1}} = 0.$

4.2 Some conditions for the unicity of solutions

In the present section, we provide criteria to ensure the uniqueness of solutions as follows:

1. **The presence of a zero block.** In this case, the solution to subsystem (I) exists under condition (4.8), but it is not unique.

2. **The presence of the block** L_ε. In this case, the solution corresponding to this block exists but is not unique.

   Thus, for the solution of the **D.I.S** to be unique, both the blocks "0" and "L_ε" must be absent.

• Note that if the block "0" is absent, the condition (4.8) is unnecessary. Moreover, as seen earlier, the block "L_ε" is absent if $ker(zA + B) = 0$. We then have the following result:

Theorem 4.2. *[2] The implicit discrete system (4.1) has a solution if and only if, $\tilde{f}_{k=0}^{\infty}$ defined by $\tilde{f}_k = Pf_k$ verified:*

1. $\tilde{f}_k^1 = 0$.
2. $\overset{3}{\tilde{f}}{}_k^1 - \overset{3}{\tilde{f}}{}_{k+1}^2 + (-1)^2 \overset{3}{\tilde{f}}{}_{k+2}^3 + ... + (-1)^\eta \overset{3}{\tilde{f}}{}_{k+\eta}^{\eta+1} = 0$.

The solution becomes unique if and only if $Ker(zA + B) = \{0\}$.

Example 4.1. *We consider the system $Ax_{k+1} + Bx_k = 0$ with x_0 given by:*

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 2 & 5 \\ 3 & 2 & 6 \end{pmatrix}.$$

The pencil $zA + B$ can be written in the form:

$$zA + B = \begin{pmatrix} z+2 & z+1 & 2z+3 \\ z+3 & z+2 & 2z+5 \\ z+3 & z+2 & 3z+6 \end{pmatrix}.$$

The matrix pencil $zA + B$ is equivalent to the following z-matrix:

$$\begin{pmatrix} 1 & z & \vdots & 0 \\ 0 & 1 & \vdots & 0 \\ .... & .... & \vdots & .... \\ 0 & 0 & \vdots & z+1 \end{pmatrix}$$

Thus, solving (1) is equivalent to solving:

$$\begin{pmatrix} 1 & z & \vdots & 0 \\ 0 & 1 & \vdots & 0 \\ .... & .... & \vdots & .... \\ 0 & 0 & \vdots & z+1 \end{pmatrix} \begin{pmatrix} \tilde{x}_k \\ \tilde{y}_k \\ \\ \tilde{z}_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \\ 0 \end{pmatrix} \cdots\cdots (1')$$

With:

$$\tilde{x}_k = \begin{pmatrix} \tilde{x}_k \\ \tilde{y}_k \\ \tilde{z}_k \end{pmatrix} ;\ \tilde{x}_0 = \begin{pmatrix} \tilde{x}_0 \\ \tilde{y}_0 \\ \tilde{z}_0 \end{pmatrix}$$

$$(1') \Longleftrightarrow \begin{cases} \tilde{x}_k + \tilde{y}_{k+1} & = & 0 \\ \tilde{y}_k & = & 0 \\ \tilde{z}_{k+1} + \tilde{z}_k & = & 0 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \tilde{y}_{k+1} & = & -\tilde{x}_k \\ \tilde{y}_k & = & 0 \\ \tilde{z}_{k+1} & = & -\tilde{z}_k \end{cases}$$

$\tilde{z}_{k+1} = -\tilde{z}_k = (-1)^2\tilde{z}_{k-1} = \ldots\ldots = (-1)^{k+1} z_0$ *So, the solution of* $\tilde{A}\tilde{x}_{k+1} + \tilde{B}\tilde{x}_k = 0$, $\tilde{x}_0$ *given; is*

$$\tilde{X}_k = \begin{pmatrix} 0 \\ 0 \\ (-1)^k \tilde{z}_0 \end{pmatrix}$$

We have $X_k = Q\tilde{X}_k$, *so in order to dtermine* X_k, *we need at first to find the matrices* P *and* Q *satisfy the condition* $PAQ = \tilde{A}$ *and* $PBQ = \tilde{B}$.

Such that:

$$\tilde{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; ; \; \tilde{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Remark 4.1. *We can also introduce the notion of Drazin inverse as in* **Chapter** 02 *for the matrices A and B of a discrete implicit system to reduce it into another system admits solutions.*

GENERAL CONCLUSION

In this work, we proposed two methods for solving implicit differential systems, each one specifically designed a particular case. For example, when the spaces had finite demensions, we observed the diversity of linear algebra concepts used in general for the computation of the solution also, we give a summary algorithm that explains all possible cases.

The study of the reduced form of Weierstrass-Kronecker for a regular or singular pencil of matrices holds well in the case of implicit difference systems (discrete) described by the general abstract form as follows:

$$\begin{cases} Ax_{k+1} - Bx_k = \psi_k(x_k), \quad k = 0, 1, 2, ... \\ \\ x_0 \quad \text{is given;} \end{cases}$$

where A, B are $m \times n$ matrices with elements over the field $\mathbb{K}$, $\{\psi_k\}_{k=0}^{\infty}$ is a suitable sequence.

In infinite dimensional spaces, the study of those type of systems is based on the use of spectral theory properties for the pencil of operators $\lambda A \pm B$, See for example [1, 3, 4, 17, 18].

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