People's Democratic Republic of Algeria Ministry of Higher Education and Scientific Research

Amar Telidji University of Laghouat

Faculty of Science

Department of Mathematics

Laboratory of Pure and Applied Mathematics (LMPA).

Course handout. Title:

Course on equations in mathematical physics, with corrected exercises.

Presented by: **Abdesselam Nawel**

Emails : nawelabedess@gmail.com / n.abdesselam@lagh-univ.dz

2024 - 2025

FOREWORD

PDEs play an important role in modern mathematics, particularly in geometry and analysis. This course handout on equations in mathematical physics is intended for undergraduate students. It provides an introduction to the basic concepts of partial differential equations (PDEs) and effective techniques for analyzing and solving them. It is also intended to provide students with tools for obtaining existence (and often uniqueness) results for some examples of linear PDE problems of a physical nature (elliptic, parabolic, or hyperbolic). This course handout is divided into seven chapters. Among other topics, it covers the classification of second-order linear partial differential equations, the method of separation of variables, and Fourier series. Classic problems such as the wave equation, the heat equation, and the potential equation for bounded spatial domains are studied. We conclude with a few exercises included at the end of each chapter, and the handout ends with an appendix containing assessment tests and exams with solutions. This document was presented to students at Amar Telidji University in Laghouat, Algeria, from 2017 to 2025. We invite our kind readers to send us their comments and critiques so that we can enrich this work: **n.abdesselam@lagh-univ.dz**

Contents

Symbols and Notations

Number sets:

$\mathbb{N}$: Set of natural numbers.

$\mathbb{Z}$: Set of integers.

$\mathbb{R}$: Set of real numbers.

$\mathbb{R}^2$: Set of pairs of real numbers.

Greek letters:

α : alpha	ι : iota	π : pi	φ: var phi
β: beta	ε : var epsilon	ρ : rho	χ : chi
γ : gamma	ϑ : var theta	ϱ : var rho	ψ : psi
δ : delta	κ : kappa	σ : sigma	ω : omega
ϵ : epsilon	λ : lambda	ς : var sigma	Γ : Gamma
ζ : zeta	μ : mu	τ : tau	Δ : Delta
η : eta	ν : nu	υ : upsilon	Θ : Theta
θ : theta	ξ: xi	ϕ: phi	∇ : nabla

Functional spaces:

$\mathcal{L}^p(\Omega), 1 \leq p < \infty$: Lebesgue space of functions whose p-th power is integrable on Ω.

$\mathcal{L}^\infty(\Omega)$: Lebesgue space of essentially bounded functions on Ω.

$\mathcal{C}^0(\Omega)$: space of continuous functions on Ω.

$\mathcal{C}_0(\Omega), n \in [0, \infty[$: space of functions of class $\mathcal{C}(\Omega)$ with compact support.

$\mathcal{C}^\infty(\Omega)$: space of infinitely differentiable functions on Ω.

Operations on functions.

Operations on functions:

Δf : The Laplacian of f.

$\nabla f(a_1, ..., a_n)$: Gradient of f at the point a.

$\frac{\partial f}{\partial x_i}(a_1, ..., a_n)$: $i-$th partial derivative of f.

For a real function of a real variable: f' first derivative of f, f'' second derivative, $f^{(j)}$ j-th derivative.

General introduction

There are an infinite number of partial differential equations (PDEs). In this module, we will list some of the classic partial differential equations. Many fields are heavily dependent on the theory of partial differential equations . Acoustics, aerodynamics, fluid dynamics, elasticity, electrodynamics, geophysics, quantum mechanics, meteorology, oceanography, and plasma physics are just a few of these fields.

Subject: Mathematical physics equations Credits: 5

Coefficient: 2

Course objectives:

This course is intended to provide the mathematical tools used in technical sciences (mechanics, electrical engineering, geophysics) Recommended prior knowledge: Real analysis and linear algebra, topology Course content:

Chapter 1: First-order PDEs—Characteristic methods.

Linear case.

Quasi-linear case

Nonlinear case

Chapter 2: Second-order linear PDEs, characteristics, classification, standard forms.

Chapter 3: Separation of variables method (Fourier).

Chapter 4: Laplace's equation, harmonic functions, Poisson's kernel.

Chapter 5: Wave equation (Kirchhoff's formula).

Chapter 6: Heat equation (Poisson's integral).

Assessment method: Exam (60%) , continuous assessment (40%)

CHAPTER 1

First steps in theory

General information and ratings

Many natural phenomena are modeled by partial differential equations. Examples include weather forecasting (which involves a large number of parameters), seismic tremors, ocean movements, and scientific disciplines such as economics, finance, and medical sciences. Partial differential equations also arise in physical problems: in electromagnetism (Maxwell's equations), fluid mechanics (Navier-Stokes equation), and quantum mechanics.

1. Ordinary differential equations

The simplest example is when the function u depends on only one variable. Then this relationship is simply described by what is called an ordinary differential equation (ODE).

What are ODE and PDE?

To clarify the concepts, let's first review some notions about ordinary differential equations (ODEs).

Definition 1.1:

An ordinary differential equation (ODE) is defined as a relation between a real independent variable t, an unknown function $t \longmapsto y(t)$, and its derivatives $y', y'', ..., y^{(n)}, n \in \mathbb{N}^{\star}$.

The order of an (ODE) is defined as the order of the highest derivative appearing in the equation. Thus, an n^{th}-order differential equation can be written in the form

$$F(x, y, y', y'', ..., y^{(n)}) = 0 \tag{1}$$

The function F is a function of $n+2$ variables.

The unknown function $t \longmapsto y(t)$ of the real variable t takes its values in $\mathbf{R}$ or $\mathbf{R}^k; k = 2, 3, ...$

We take t in an interval I of $\mathbf{R}$ (I may be the entire set $\mathbf{R}$).

Here are a few examples

Example 1.1. *a)* $y' + ty = \exp^t$ *is a first-order differential equation.*

b) $y'' + 4ty = 0$ *is a second-order differential equation.*

c) $y^{(9)} - ty'' = t^2$ *is a differential equation of order* 9.

d) The simplest example is that of the motion of a body (considered as a point) on a line.

$$y''(x) = f(y(x)).$$

2. Partial Differential Equations

A partial differential equation is a relationship linking an unknown function of several variables u to its partial derivatives. Partial differential equations are found in applications in physics, engineering, biology, economics, etc. Indeed, in these fields, phenomena are often modeled by mathematical systems involving partial differential equations. The different processes of the phenomenon are described by determining a relationship between u and its partial derivatives.

Definition 2.1:

An equation in which a function f of several independent variables $x_1, ..., x_n$ and the partial derivatives of f with respect to these variables appear, i.e., an equation of the form:

$$F(x_1, ..., x_n, f, \frac{\partial f}{\partial x_1}, ..., \frac{\partial^2 f}{\partial x_1^2}, \frac{\partial^2 f}{\partial x_1 \partial x_2}, ..., \frac{\partial^m f}{\partial x_n^m}) = 0 \tag{2}$$

is called a partial differential equation.

Note: We will use PDE as an abbreviation for partial differential equation throughout this work.

Now, we give the definition of a first-order Partial Differential Equation (PDE). .

Definition 2.2:

A first-order partial differential equation with unknown u and n independent variables $x_1, ..., x_n$ is an equation of the form

$$F(x_1, ..., x_n, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}) = 0 \tag{3}$$

where $(x_1, ..., x_n) \in \Omega$, an open subset of $\mathbb{R}^n$.

We now move on to the definition of a second-order Partial Differential Equation (PDE).

Definition 2.3:

A second-order partial differential equation with unknown u and n independent variables $x_1, ..., x_n$ is an equation of the form

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, ..., \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}) = 0 \tag{4}$$

where $(x, y) \in \Omega$, an open subset of $\mathbb{R}^2$.

A solution of equation (2) is a function $u = u(x, y, ...)$ of the independent variables $x, y, ...$ whose partial derivatives appearing in the equation exist at the points of the interval, and such that after substituting this function and its partial derivatives into equation (2), the equation is satisfied.

EXAMPLE 2.1. *Consider the equation* $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$, *which is a PDE in the domain* $\mathbb{R}^2$, *and* $u(x, y) = (x + y)^3$, $u(x, y) = \sin(x - y)$ *are two solutions of this equation. We have, if* $u(x, y) = (x+y)^3$, *then* $\frac{\partial u}{\partial x} = 3(x+y)^2$, $\frac{\partial^2 u}{\partial x^2} = 6(x+y)$, *and also* $\frac{\partial u}{\partial y} = 3(x+y)^2$, $\frac{\partial^2 u}{\partial y^2} = 6(x+y)$, *we obtain*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 6(x + y) - 6(x + y) = 0.$$

And if $u(x, y) = \sin(x - y)$, *then we have* $\frac{\partial u}{\partial x} = \cos(x - y)$, $\frac{\partial^2 u}{\partial y^2} = -\sin(x - y)$, *hence*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = -\sin(x - y) + \sin(x - y) = 0.$$

3. Dimension and order, homogeneity, and linearity of a PDE

In this section, we will define a few concepts: order, dimension, and homogeneity, linearity.

Definition 3.1:

The dimension of a partial differential equation is the number of independent variables on which the unknown function u depends.

Definition 3.2:

The order of a partial differential equation is the highest order of differentiation appearing in the equation.

The concept of linearity for PDEs involves differential operators. A differential operator is an operator constructed from the partial derivatives of differentiable functions. A linear differential operator is one that has the following properties:

Definition 3.3:

A PDE with unknown u is said to be linear if it can be written in the form

$$Lu = f$$

where L is a linear differential operator, and f is a function of n independent variables defined on a domain of $\mathbb{R}^n$. If $f \equiv 0$, the equation is said to be homogeneous linear; otherwise, it is non-homogeneous.

To clarify ideas, we need these remarks.

Remark 3.1:

The dimension of a partial differential equation is the number of independent variables on which the unknown function u depends. Before introducing the first notions of partial differential equations, we will recall the essential basic concepts that must be known and well understood for the continuation of the program.

Thus, in what follows, an operator L will denote a transformation that associates with every **well-behaved function** $u = u(x, y, ...)$ of several variables $x, y, ...$ on a given domain, a function $Lu = Lu(x, y, ...)$ on that same domain.

The term **well-behaved** here means that Lu is well-defined. Sometimes it will be necessary to require that the partial derivatives of u exist up to a certain order. If $u = u(x, y)$, then $Lu = \frac{\partial u}{\partial x}$ is an example of an operator.

However, for Lu to be well-defined, it is necessary that the partial derivative of u with respect to x exists on the domain; this is what the term **well-behaved** means in this case.

The equation $[E]$ can therefore be written in the form $L(u) = f(x, y, ...)$, where $f(x, y, ...)$ is a function of the independent variables, L is an operator, and u is the function to be determined.

An operator L is linear if and only if:

$$L(au + bv) = aL(u) + bL(v)$$

for all real numbers a, b and all **well-behaved functions** u, v.

Implicitly, we assume that the function $au + bv$ is also a **well-behaved function**.

For example,

Example 3.1.

$$(\frac{\partial u}{\partial x})^2 - u(\frac{\partial u}{\partial y}) = 0.$$

The equation is of 1^{st} *order.*

Example 3.2.

$$(x^2 + u^2)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) = x + y + u.$$

The equation is of 1^{st} *order, linear and non-homogeneous.*

Example 3.3. *The Laplace equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

The equation is of 2^{nd} *order, linear and homogeneous.*

Poisson's equation

EXAMPLE 3.4.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = g(x, y, z).$$

The equation is of 2^{nd} *order, linear and non-homogeneous.*

EXAMPLE 3.5.

$$\frac{\partial^3 u}{\partial x^2 \partial y} + x(\frac{\partial^2 u}{\partial x^2})^2 = e^x.$$

is an EDP of order 3.

EXAMPLE 3.6.

$$u + y\frac{\partial^2 u}{\partial^2 x} + 2xy\frac{\partial^2 u}{\partial^2 y} == 1$$

is a non-homogeneous linear PDE in $\mathbb{R}$*, where* $u = u(x, y)$*. In this example,*

$$Lu = u + y\frac{\partial^2 u}{\partial^2 x} + 2xy\frac{\partial^2 u}{\partial^2 y}.$$

is a linear operator and $f(x, y) = 1$*. Indeed,* L *is linear, because if* a *and* b *are two arbitrary real numbers and* u *and* v *two functions;* u *and* v *must be good functions whose partial derivatives appearing in the definition of* Lu *exist on* $\mathbb{R}$*, then we will verify that:*

$$L(au + bv) = aL(u) + bL(v).$$

$L(au + bv) = (au + bv) + y\frac{\partial^2(au+bv)}{\partial^2 x} + 2xy\frac{\partial^2(au+bv)}{\partial^2 y}.$
$L(au + bv) = a(u + y\frac{\partial^2 u}{\partial^2 x} + 2xy\frac{\partial^2 u}{\partial^2 y}) + b(v + y\frac{\partial^2 v}{\partial^2 x} + 2xy\frac{\partial^2 v}{\partial^2 y}.$
$= aL(u) + bL(v).$

EXAMPLE 3.7. *Another PDE in* $\mathbb{R}^2$ *is*

$$(\frac{\partial u}{\partial x})(\frac{\partial^2 u}{\partial^2 x}) + xu\frac{\partial u}{\partial y} = \sin(y),$$

where $u = u(x, y)$*. However, this equation is not linear. In this case,*

$$Lu = (\frac{\partial u}{\partial x})(\frac{\partial^2 u}{\partial^2 x}) + xu\frac{\partial u}{\partial y},$$

is not a linear operator and $f(x, y) = \sin(y)$*.*
To verify that Lu *is not linear, it follows by considering* ***for example*** *the two real numbers* $a = b = 1$ *and the two functions* $u(x, y) = v(x, y) = x^2$*. With these choices, we obtain that* $L(u + v) = 16x, Lu = Lv = 4x$ *and clearly* $L(u + v)6 = Lu + Lv$*.*

Remark 3.2:

(Interpretation of semi-linearity and quasi-linearity.)

⋆ A PDE is linear if its dependence on the unknown function and its partial derivatives is linear.

⋆ A semi-linear PDE is a non-linear PDE but it is linear with respect to its derivatives, and the coefficients do not depend on u and its derivatives.

⋆ A quasi-linear PDE is a non-linear PDE but it is linear with respect to the highest-order derivatives.

⋆ A PDE is of the trivially non-linear type in the opposite case.

4. Existence of solutions

The aim is then to find a solution that satisfies certain conditions around a neighborhood of a point or in a given domain. The search for this solution can be constructed manually if necessary.

The problem of the uniqueness of the solution then arises from the choice of the type of boundary conditions. The Cauchy problem or initial value problem can be considered as a special case of this problem.

Behavior of solutions: properties of solutions, in particular that of differentiability (regularity).

Definition 4.1:

Well-posed problem (problem in the sense of Hadamard): a problem is said to be well-posed for a PDE if:

(⋆) The problem has a solution.

(⋆) The solution is unique.

(⋆) The solution depends continuously on the data.

5. Some equations from mathematical physics

Now, we present some examples concerning PDEs, these represent physical problems.

Transport equation: is used to model air pollution, the dispersion of dyes or even traffic flow.

$$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = 0 \;\; où \;\; u = u(x,t).$$

Burgers equation: is used for the study of gas dynamics.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \;\; \textit{où} \;\; u = u(x,t).$$

Heat equation: is used in the study of thermal conduction.

$$\frac{\partial u}{\partial t} - k(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) = 0 \;\; \textit{où} \;\; u = u(x,t).$$

Wave equation: is used to model small oscillations; it plays an important role in fluid dynamics and electromagnetism.

$$\frac{\partial^2 u}{\partial t^2} - c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) = 0 \;\; \textit{où} \;\; u = u(x,t).$$

Laplace or potential equation: appears notably in astronomy, electrostatics, fluid mechanics, and quantum mechanics.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \;\; \textit{où} \;\; u = u(x,t).$$

Euler-Bernoulli equation: is used in beam theory.

$$\frac{\partial^2 u}{\partial t^2} + c^4\frac{\partial^4 u}{\partial x^4} = 0 \;\; \textit{où} \;\; u = u(x,t).$$

6. Construction of a partial differential equation

6.1. First-order EDP (Transport equation). This equation is used to model air pollution, the dispersion of dyes or even traffic flow, with u representing the density of the pollutant (or dye or traffic) at the position x and at time t. For a discussion of the physical model, we consider the example proposed in [8].

EXAMPLE 6.1. *Consider a very narrow tube of length l carrying water at a constant speed c. Suppose there is a chemical substance that pollutes the water. Let $u(t,x)$ be the concentration of the chemical substance at time t and at position x.*

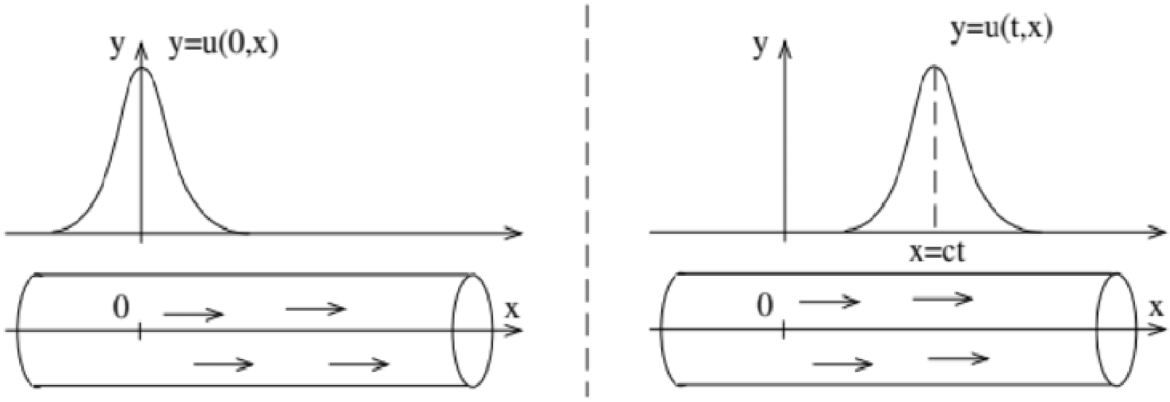


FIGURE 1. Transport of a pollutant in water

We denote by $Q(t)$ the quantity of the chemical substance at time t. The expression of Q between the points of abscissa 0 and x is given by

$$Q(t) = \int_0^x u(t, y)dy.$$

During a period h, a particle of the substance travels a distance ch. The quantity Q between the points of abscissa ch and $x+ch$ is the same quantity between 0 and x at time t. We therefore have

$$Q(t) = \int_{ch}^{x+ch} u(y, t)dy = \int_{ch}^{x+ch} u(y, t+h)dy.$$

By differentiating with respect to x, we find

$$u(x, t) = u(x + ch, t + h).$$

By differentiating this time with respect to h and setting $h = 0$, we find

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0.$$

EXAMPLE 6.2. *By considering a cylindrical tube of length L and radius a, through which a liquid flows at a constant velocity v. This tube is immersed in a tank of water at a constant temperature τ. We aim to study the evolution of the temperature of the liquid in the tube over time. The tube is considered sufficiently thin ($a << L$) (to work with a one-dimensional model). This example can be found in a slightly different form in [5].*

Let $T(x, t)$ be the temperature (expressed, for example, in degrees) at the point x in the tube and at time t. Let us examine what happens in a small (infinitesimal) slice of the tube of volume $\pi a^2 \Delta x$.

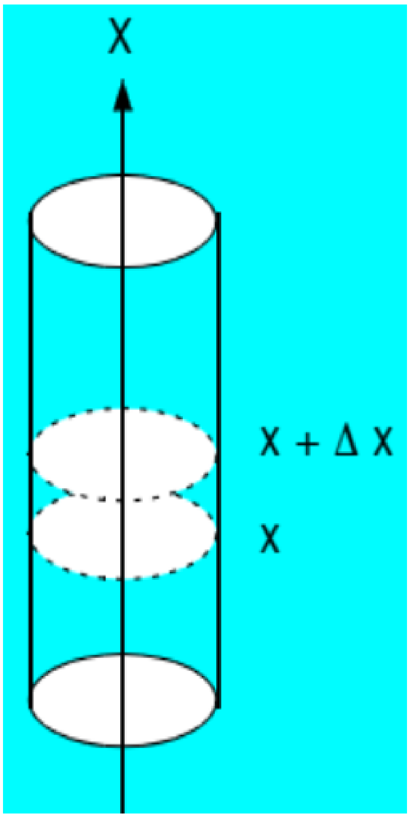


FIGURE 2. Schematic view of the tube

Now, we will present a heuristic argument to justify the wave equation in the case of a membrane:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

We will assume that the following five conditions are satisfied.

⋆ The membrane is flexible and elastic.

⋆ The tension has a constant magnitude T_0.

⋆ The membrane has a constant density ρ.

⋆ The weight of the membrane is negligible compared to the tension.

⋆ The displacement of the membrane is small compared to its minimum diameter.

The first condition implies that the tension acts in the direction of the profile of the membrane. The last condition means that the horizontal displacement of the membrane is approximately zero, and we will assume it to be zero in what follows. Let $u = u(x, y, t)$ be the vertical displacement of the membrane above the point (x, y) at time t.

Consider the portion of the membrane above the rectangle with vertices: $(x, y), (x+\Delta x, y), (x, y+\Delta y), (x+\Delta x, y+\Delta y)$. We assume that Δx and Δy are small. In Figure [1], we have illustrated the tension forces acting on this portion of the membrane. The angles α, β, γ, and δ are the angles formed

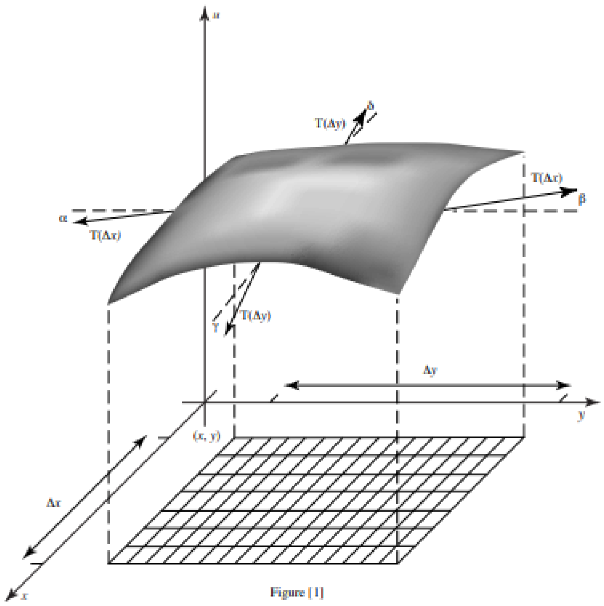


FIGURE 3. Wave equation in the case of a membrane.

by these forces with the horizontal, and the points $(x_1, y), (x_2, y + \Delta y), (x, y_1)$ and $(x + \Delta x, y_2)$ are those where the tension forces act on the edges of this portion of the membrane. The vertical force is

$$F_{vert} = T_0(\Delta x)\sin(\beta) - T_0(\Delta x)\sin(\alpha) + T_0(\Delta y)\sin(\gamma) - T_0(\Delta y)\sin(\delta).$$

It should be noted that the angles α, β, γ, and δ are almost zero. We can therefore use the following approximations: $\sin(\alpha) \approx \tan(\alpha), \sin(\beta) \approx \tan(\beta), \sin(\gamma) \approx \tan(\gamma)$ and $\sin(\delta) \approx \tan(\delta)$. After substitution, we obtain

$$F_{vert} = T_0(\Delta x)(\tan(\beta) - \tan(\alpha)) + T_0(\Delta y)(\tan() - \ tan(\gamma)) = \rho(\Delta A)\frac{\partial^2 u}{\partial t^2}$$

where $\Delta A \approx (\Delta x)(\Delta y)$ is the area of the portion of the membrane. By definition of the partial derivative and since the stress acts in the direction of the membrane profile, we have that $\tan(\alpha) = \frac{\partial u}{\partial y}(x_1, y), \tan(\beta) = \frac{\partial u}{\partial y}(x_2, y + \Delta y), tan(\gamma) = \frac{\partial u}{\partial x}(x, y_1)$ et $\tan(\gamma) = \frac{\partial u}{\partial x}(x + \Delta x, y_2)$. Substituting this into the above expression for F_{vert} that

$$F_{vert} = \rho(\Delta x)(\Delta y)\frac{\partial^2 u}{\partial t^2} = T_0(\Delta x)(\frac{\partial u}{\partial y}(x_2, y{+}\Delta y){-}\frac{\partial u}{\partial y}(x_1, y)){+}T_0(\Delta y)(\frac{\partial u}{\partial x}(x{+}\Delta x, y_2){-}\frac{\partial u}{\partial x}(x, y_1)).$$

Consequently

$$\frac{\partial^2 u}{\partial t^2} = \frac{T_0}{\rho}\frac{1}{\Delta y}(\frac{\partial u}{\partial y}(x_2, y + \Delta y) - \frac{\partial u}{\partial y}(x_1, y)) + \frac{T_0}{\rho}\frac{1}{\Delta y}(\frac{\partial u}{\partial x}(x + \Delta x, y_2) - \frac{\partial u}{\partial y}(x, y_1)).$$

Passing the limit $(\Delta x), (\Delta y) \to 0$, we then have

$$x_1, x_2 \to x; y_1, y_2 \to y$$

and

$$\frac{1}{\Delta y}(\frac{\partial u}{\partial y}(x_2, y+\Delta y) - \frac{\partial u}{\partial y}(x_1, y)) \to \frac{\partial^2 u}{\partial y^2}$$

and

$$\frac{1}{\Delta y}(\frac{\partial u}{\partial x}(x+\Delta x, y_2)\frac{\partial u}{\partial y}(x, y_1)) \to \frac{\partial^2 u}{\partial x^2}.$$

From all of the above, we can therefore conclude that $u = u(x, y, t)$ satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}), \ où \ c^2 = \frac{T_0}{\rho}.$$

7. Resolution of simple EDPs

We're going to solve the equation.

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

Right now, we put

$$v(x, y) = \frac{\partial u}{\partial x},$$

which implies for everything (x, y) that

$$\frac{\partial u}{\partial x} = 0.$$

This means that $v(x, y)$ is constant for all y fixed. So

$$v(x, y) = C(y)$$

where C is an arbitrary function of y. We are then reduced to finding u such that

$$\frac{\partial u}{\partial x} = C(y).$$

A similar reasoning leads to

$$u(x, y) = C(y)x + D(y)$$

where D is an arbitrary function.

We are interested in finding the general solution of the equation

$$\frac{\partial^2 u}{\partial x^2} + u = 0.$$

We note that for fixed y, the equation, using a change of variable

$$v(x) = u(x, y),$$

can be written as

$$v'' + v = 0.$$

This is a second-order linear ODE whose general solution is given by

$$v(x, y) = A\cos x + B\sin x,$$

where A, B are arbitrary constants. Returning to u, we obtain

$$u(x, y) = A(y) \cos x + B(y) \sin x,$$

from which the solution of the proposed equation follows, where A and B are this time arbitrary functions.

8. Method of variable substitution

We will first state the chain rule for a function $u(\xi, \rho)$ of two variables ξ, ρ themselves functions of two other variables x, y.

Theorem 8.1:

(Chain Rule): Let $w = u(\xi, \rho)$, $\xi = \xi(x, y)$ and $\rho = \rho(x, y)$ be functions, that is, w can then be considered as a function $u(\xi(x, y), \rho(x, y))$ of x and y. When the following terms are well defined, we can write

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x}$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial y}$$

EXAMPLE 8.1. *Let's determine all the functions $u \in \mathbb{R}_+^{\star 2}$ and satisfying the partial differential equation*

$$2x\frac{\partial u}{\partial x} - y\frac{\partial u}{\partial y} = 0 \tag{5}$$

We can consider the new coordinates: $\xi = x$ and $\rho = xy^2$.
It is easy to verify that

$$\xi, \rho > 0, \quad x = \xi, \quad y = \sqrt{\frac{\rho}{\xi}}$$

since $x, y > 0$. By the chain rule, we obtain

$$\frac{\partial w}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\rho}{\xi}\frac{\partial u}{\partial \rho}$$

et

$$\frac{\partial w}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \rho}\frac{\partial \rho}{\partial y} = 2\sqrt{\xi\rho}\frac{\partial u}{\partial \rho}.$$

Here it should be noted that the partial derivatives $\frac{\partial u}{\partial \xi}$ *and* $\frac{\partial u}{\partial \rho}$ *are continuous. By substituting* $x, y, \frac{\partial u}{\partial x}$ *and* $\frac{\partial u}{\partial y}$ *by their corresponding expressions in terms of* ξ *and* ρ *in (5), we obtain*

$$2\xi\frac{\partial u}{\partial \xi} = 0.$$

Since ξ *is strictly positive, we obtain* $\frac{\partial u}{\partial \xi} = 0 \to u = h(\eta)$, $h : (0, \infty) \Rightarrow \mathbb{R}$ *arbitrary and differentiable. Thus,* $u = h(xy^2)$ *is the solution of (5).*

9. Exercise series N°= 01

EXERCICE 1. *For each of the partial differential equations below, indicate its order, whether it is linear or not, and whether it is linearly homogeneous or not.*

a) $\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y} = y$;
b) $(\frac{\partial u}{\partial x})^2 + u(\frac{\partial u}{\partial x}) = 1$;
c)$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^4 u}{\partial y^4} = 0$;
d)$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = \sin(x)$;
e)$\frac{\partial^2 u}{\partial x^2} + (\frac{\partial u}{\partial x})^2 + \sin(u) = e^y$.

SOLUTION. *a) This PDE is linear, non-homogeneous, and of order* 2. *To show that the PDE is linear, let us consider the operator*

$$u \mapsto L(u) = \frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial y}.$$

This operator is linear. Indeed, let $a, b \in \mathbb{R}$ *and* u, v *be two functions. Then we need to verify that* $L(au + bv) = aL(u) + bL(v)$. *But by the properties of partial derivatives, we obtain*

$$L(au+bv) = \frac{\partial^2(au+bv)}{\partial x^2} + x\frac{\partial(au+bv)}{\partial y} = a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x^2} + ax\frac{\partial u}{\partial y} + bx\frac{\partial u}{\partial y} = aL(u) + bL(v).$$

This completes the proof that L *is a linear operator. Since our equation is of the form* $L(u) = y$, *we can conclude that the PDE is linear. Also, because of this form, the PDE is non-homogeneous. As the highest-order partial derivative is of order* 2, *the PDE is therefore of order* 2.

b) This PDE is not linear. It is of order 1. *This PDE is of the form* $T(u) = 1$ *where* T *is the operator*

$$u \mapsto T(u) = \left(\frac{\partial u}{\partial x}\right)^2 + u\left(\frac{\partial u}{\partial x}\right).$$

To verify that this PDE is not linear, it suffices to show that T *is not a linear operator. We therefore need to find two real numbers* a *and* b, *as well as two functions* u *and* v *such that*

$$T(au + bv) \neq aT(u) + bT(v).$$

Let us take $a = b = 1,\ u = (2x + y)$, *and* $v = y^2$. *We obtain*

$$T(u+v) = T(2x+y+y^2) = (\frac{\partial(2x+y+y^2)}{\partial x})^2 + (2x+y+y^2)(\frac{\partial(2x+y+y^2)}{\partial x}) =$$

$$2^2 + (2x+y+y^2)(1+2y) = 4 + 2x + y + 2y^3 + 3y^2 + 4xy;$$

$$T(u) = T(2x+y) = (\frac{\partial(2x+y)}{\partial x})^2 + (2x+y)(\frac{\partial(2x+y)}{\partial x}) = 2^2 + (2x+y) = 4 + 2x + y$$

and

$$T(v) = T(y^2) = (\frac{\partial(y^2)}{\partial x})^2 + (y^2)(\frac{\partial(y^2)}{\partial x}) = 0 + y^2(2y) = 2y^3.$$

We clearly see in this case that $T(u+v) \neq T(u) + T(v)$. *This shows that the PDE is not linear. Since the highest-order partial derivative is of order* 1, *the PDE is therefore of order* 1.

c) This PDE is linear, homogeneous, and of order 4.

d) This PDE is linear, non-homogeneous, and of order 2.

e) This PDE is not linear. It is of order 2.

EXERCICE 9.1. *Verify that the functions* $u(x,y) = x^2 - y^2$ *and* $u(x,y) = e^x \sin(y)$ *are indeed solutions of the equation*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

SOLUTION. *If* $u(x,y) = x^2 - y^2$, *then*

$$\frac{\partial u}{\partial x} = 2x, \frac{\partial^2 u}{\partial x^2} = 2, \frac{\partial u}{\partial y} = 2y, \frac{\partial^2 u}{\partial y^2} = 2 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

shows that u is indeed a solution.

If $u(x,y) = \exp\sin(y)$, *then*

$$\frac{\partial u}{\partial x} = \exp\sin(y), \frac{\partial^2 u}{\partial x^2} = \exp\sin(y), \frac{\partial u}{\partial y} = \exp\cos(y), \frac{\partial^2 u}{\partial y^2} = -\exp\sin(y) \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

shows that u is indeed a solution.

EXERCICE 2. *Determine the general solution of*

$$\frac{\partial^2 u}{\partial y^2} + u = 0$$

or $u = u(x,y)$.

SOLUTION. *Recall that the ordinary differential equation*

$$Y''(y) + Y(y) = 0$$

for which $Y = Y(y)$ *is a function of the variable* y *and* $Y''(y)$ *denotes the second derivative with respect to* y *has as general solution* $Y(y) = a\cos(y) + b\sin(y)$ *where* a *and* b *are arbitrary real numbers. We can now adapt this result. Thus the PDE*

$$\frac{\partial^2 u}{\partial x^2} + u = 0 \text{ with } u = u(x,y),$$

has as general solution $u(x,y) = a(x)\cos(y) + b(x)\sin(y)$ *where* $a(x)$ *and* $b(x)$ *are arbitrary functions of* x.

EXERCICE 9.2. *Determine the general solution of*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0.$$

Using the new coordinates: $\xi = x + y$ *and* $\rho = x - y$, *or* $u = u(x, y)$.

SOLUTION. *Determine the general solution of*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

where $u = u(x, y)$ *using the new coordinates:* $\varphi = x + y$ *and* $\eta = x - y$.

If $\xi = x + y$ *and* $\eta = x - y$, *then by using the chain rule, we obtain*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial \xi}[\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}]\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}[\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}]\frac{\partial \eta}{\partial x} = [\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \eta}](1) + [\frac{\partial^2 u}{\partial \xi \eta} + \frac{\partial^2 u}{\partial \eta^2}](1)$$

$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial \xi}[\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}]\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}[\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}]\frac{\partial \eta}{\partial y} = [\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \eta}](1) - [\frac{\partial^2 u}{\partial \xi \eta} + \frac{\partial^2 u}{\partial \eta^2}](-1)$$

$$= \frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

Consequently

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \Leftrightarrow 4\frac{\partial^2 u}{\partial \xi \eta} = 0$$

So, all you need to do is solve the PDE.

$$\frac{\partial^2 u}{\partial \xi \eta} = 0.$$

So, we have

$$\frac{\partial}{\partial \xi}[\frac{\partial u}{\partial \eta}] = 0 \Rightarrow \frac{\partial u}{\partial \eta} = f(\eta) \ et \ u(\xi, \eta) = \int f(\eta)d\eta + g(\xi) = h(\eta) + g(\xi),$$

where f, g *and* h *are arbitrary differentiable functions, and the general solution is* $u(x, y) = g(x + y) + h(x - y)$.

EXERCICE 9.3. *Let an (ideal) string of length* $l (l > 0)$ *be fixed at its two endpoints and whose initial displacement* $f(x)$ *and initial velocity* $g(x)$ *are known. We want to determine the vertical displacement* $u(x, t)$ *of this string at the point* $x, (0 < x < l)$ *at time* $t, (t > 0)$. *Mathematically we have the equation*

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0,$$

with initial conditions: $u(x,0) = f(x)$ *and* $\frac{\partial u}{\partial t}(x,0) = g(x)$ *for all* $x, (0 < x < l)$ *and boundary conditions:* $u(0,t) = 0$ *and* $u(l,t) = 0$ *for all* $t, t > 0$.
Determine the solution $u(x,t)$.

EXERCICE 9.4. *Show, using the chain rule, that the heat equation*

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

expressed in polar coordinates: $r = \sqrt{x^2+y^2}, \quad \theta = \arctan\left(\frac{y}{x}\right)$ *is*

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}\right) + \frac{1}{r}\frac{\partial u}{\partial r}$$

where $u = u(x,y,t) = u(r,\theta,t)$.

SOLUTION. *Let us recall that we have* $r = \sqrt{x^2+y^2}, \theta = \arctan(\frac{y}{x})$ *Using the chain rule, we have*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} - \frac{x}{x^2+y^2}\frac{\partial u}{\partial \theta};$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}[\frac{x}{\sqrt{x^2+y^2}}]\frac{\partial u}{\partial r} + \frac{x}{\sqrt{x^2+y^2}}[\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial \theta}{\partial x}]$$
$$-\frac{\partial}{\partial x}[\frac{y}{\sqrt{x^2+y^2}}]\frac{\partial u}{\partial \theta} - \frac{y}{\sqrt{x^2+y^2}}[\frac{\partial^2 u}{\partial \theta\partial r}\frac{\partial r}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial x}]$$

we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}\frac{\partial u}{\partial r} + \frac{2xy}{(x^2+y^2)^2}\frac{\partial u}{\partial \theta} + \frac{x^2}{(x^2+y^2)}\frac{\partial^2 u}{\partial r^2} - \frac{2xy}{(x^2+y^2)^{\frac{3}{2}}}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{y^2}{(x^2+y^2)^2}\frac{\partial^2 u}{\partial \theta^2}.$$

On the other hand, we have

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{x}{x^2+y^2}\frac{\partial u}{\partial \theta};$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}[\frac{y}{\sqrt{x^2+y^2}}]\frac{\partial u}{\partial r} + \frac{y}{\sqrt{x^2+y^2}}[\frac{\partial^2 u}{\partial r^2}\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial \theta}{\partial y}]$$
$$-\frac{\partial}{\partial y}[\frac{x}{\sqrt{x^2+y^2}}]\frac{\partial u}{\partial \theta} + \frac{x}{\sqrt{x^2+y^2}}[\frac{\partial^2 u}{\partial \theta\partial r}\frac{\partial r}{\partial y} + \frac{\partial^2 u}{\partial \theta^2}\frac{\partial \theta}{\partial y}]$$

then, we obtain

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2}{(x^2+y^2)^{\frac{3}{2}}}\frac{\partial u}{\partial r} - \frac{2xy}{(x^2+y^2)^2}\frac{\partial u}{\partial \theta} + \frac{x^2}{(x^2+y^2)}\frac{\partial^2 u}{\partial r^2} + \frac{2xy}{(x^2+y^2)^{\frac{3}{2}}}\frac{\partial^2 u}{\partial r\partial\theta} + \frac{x^2}{(x^2+y^2)^2}\frac{\partial^2 u}{\partial \theta^2}.$$

$$\frac{\partial u}{\partial t} = k(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

is equivalent to '

$$\frac{\partial u}{\partial t} = k(\frac{1}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{(x^2+y^2)}\frac{\partial^2 u}{\partial \theta^2})$$

As $r = \sqrt{x^2+y^2}$, *The heat equation in polar coordinates is*

$$\frac{\partial u}{\partial t} = k(\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2})$$

CHAPTER 2

Linear equations of order 1

In this chapter, we will begin our study by considering linear PDEs of order 1. We will restrict our discussion to linear equations with only two independent variables, that is, equations of the form

$$A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} + Cu = D \tag{6}$$

for which A, B, C, and D are functions of x and y continuously differentiable on the domain D. The method presented to solve these equations will be the method of characteristic curves. The latter can also be adapted to the case of first-order linear PDEs with more than two independent variables. These equations naturally arise in certain models. But they can also come from approximations of higher-order equations. For example, the one-dimensional wave equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

is an approximation of the diffusion equation

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + \frac{1}{2}D\frac{\partial^2 u}{\partial x^2}$$

when the diffusion coefficient D is almost zero. In some cases, it is also possible to replace a higher-order PDE with a system of first-order equations. This is another situation in which first-order equations appear naturally. We will illustrate this with two examples. First, consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0. \tag{7}$$

We can rewrite it in the following form

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = (\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}) = 0.$$

Note that, in the above, we assume that the partial derivatives of u of order $m \leq 2$ are continuous on D, and this implies that

$$\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial^2 u}{\partial t \partial x}$$

Equation (7) is therefore equivalent to the following system of first-order linear PDEs:

$$\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = v, \\ \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0 \end{cases}$$

We can determine v from the second equation and then solve the first equation in y by substituting v. As a second example, we can consider Laplace's equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{8}$$

and try to proceed as above. We can rewrite this equation in the following form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (\frac{\partial}{\partial x} + i\frac{\partial y}{\partial})(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}) = 0, \ (oi = \sqrt{-1}),$$

Assuming, as above, that the partial derivatives of u of order $m \leq 2$ are continuous on D. The drawback of this approach is that we obtain a system of PDEs in which the complex number i appears, whereas in general we prefer to obtain real solutions u to equation 8. There is another way to reduce equation 8 above. Let us set

$$v = \frac{\partial u}{\partial x} \ et \ \frac{\partial u}{\partial y} = w.$$

So, because of our assumption about the continuity of second-order partial derivatives, we have the equality of mixed partial derivatives and we obtain

$$\frac{\partial w}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial v}{\partial y} \ et \ \frac{\partial w}{\partial y} = \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2}2ux2 = -\frac{\partial v}{\partial x}.$$

as well as v and w satisfy the following PDE system

$$\begin{cases} \frac{\partial w}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial w}{\partial y} = -\frac{\partial v}{\partial x}. \end{cases}$$

These two equations are well known in the theory of complex variables. These are the Cauchy-Riemann equations. One must solve these two equations simultaneously to determine v and w. Then we must consider the system

$$\begin{cases} \frac{\partial u}{\partial x} = v, \\ \frac{\partial u}{\partial y} = w. \end{cases}$$

Remark 0.1:

It should be noted that it is not always possible to associate a system of first-order linear PDEs with a higher-order PDE.

1. Parametric curves and surfaces

1.1. Parametric curves in $\mathbb{R}^2$. Before describing the method of characteristic curves, we will first recall the concepts of a parametric curve in the open domain $D \subseteq \mathbb{R}^2$ and of the directional derivative of a function of two variables in a direction $\vec{d}$. These two notions will be essential for the remainder of this chapter.

Definition 1.1:

A parameterized curve C is the image of a function $\gamma : I \to D$ from an open interval I of $\mathbb{R}$ to the open domain D of $\mathbb{R}^2$ defined by $s \mapsto \gamma = (x(s), y(s))$ for all $s \in I$, and we then say that γ is a parameterization of C. In what follows, we assume that the functions $s \mapsto x(s)$ and $s \mapsto y(s)$ are continuously differentiable on I. In this situation, the vector $\gamma'(s_0) = (x'(s_0), y'(s_0))$ is a tangent vector to the curve C at the point $(x(s_0), y(s_0))$.

Let $f(x, y)$ be a function defined on the domain D, $(x_0, y_0) \in D$, and let $\vec{d} = (d_1, d_2)$ be a direction, i.e., $\vec{d}$ is a nonzero vector in $\mathbb{R}^2$. Then, the directional derivative of f at the point (x_0, y_0) in the direction $\vec{d}$ is

$$f'_{\vec{d}}(x_0, y_0) = \lim_{t \to 0} \frac{f(x_0 + td_1, y_0 + td_2) - f(x_0, y_0)}{t} \quad \textit{(if this limit exists).}$$

If the partial derivatives of f are continuous on D, then

$$f'_{\vec{d}}(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)d_1 + \frac{\partial f}{\partial y}(x_0, y_0)d_2 = \Delta f(x_0, y_0)\vec{d}. \tag{9}$$

Here $\nabla f(x_0, y_0)$ denotes the gradient of f at the point (x_0, y_0), that is

$$\nabla f(x_0, y_0) = (\frac{\partial f}{\partial x}(x_0, y_0), \frac{\partial f}{\partial y}(x_0, y_0)).$$

In equation (9), $\nabla f(x_0, y_0).\vec{d}$ denotes the dot product $\nabla f(x_0 y_0)$ and $\vec{d}$

Example 1.1. *A curve is plotted in the following figure. It is part of an ellipse and* $\gamma :]0, \pi[\to \mathbb{R}^2$ *is defined by* $s \mapsto (\cos(s), 2\sin(s))$ *as a parametrization. The vector* $\gamma(\frac{\pi}{6}) = (-\frac{1}{2}, \sqrt{3})$ *is tangent to this curve at the point* $\gamma(\frac{\pi}{6}) = (\sqrt{\frac{3}{2}}, 1)$. *This vector is also plotted.*

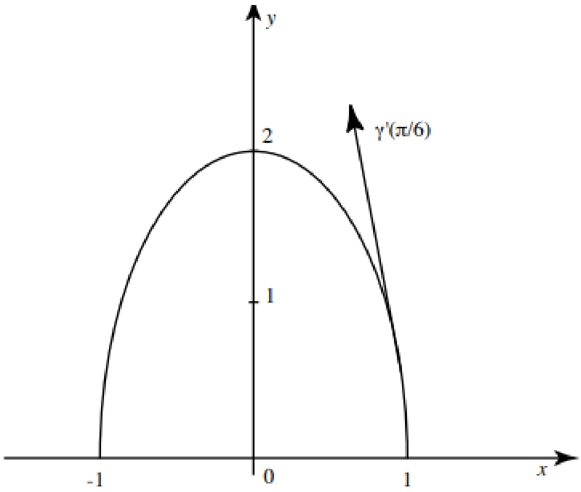
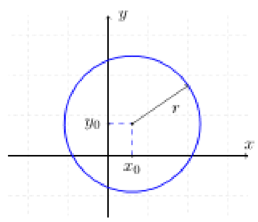


Figure [1]

Example 1.2. *The circle with center (x_0, y_0) and radius r is a curve parameterized by the function*

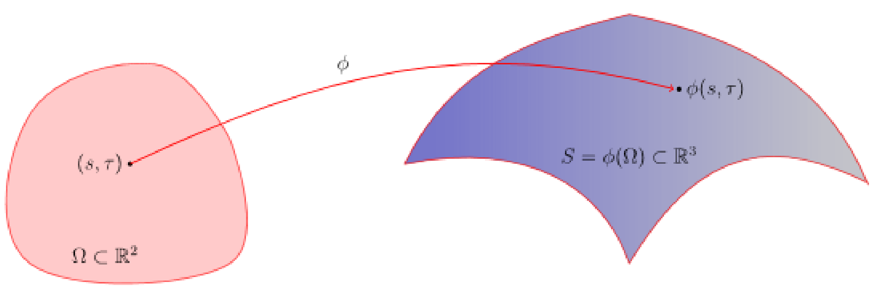
$$t \mapsto \gamma(t) = (x_0 + r\cos s, y_0 + r\sin s), 0 \leq s \leq 2\pi,$$



Cercle de centre (x_0, y_0) et de rayon r

1.2. Parametrized surfaces in $\mathbf{R}^3$. ,

u(s, τ) = $F(\tau)$. $\Rightarrow \begin{cases} s = t \\ \tau = x - cs = x - ct. \end{cases}$ *Thedesiredsolutiontotheproblemis*u = F(τ) = $F(x - ct)$ We have illustrated this in Figure [2].



In this solution $u(x,t) = F(x - ct)$, we see that the initial information $u(x, 0) = F(x)$ is transmitted along the characteristic curves, that is, $u(x,t)$ is constant for all points (x,t) such that $x - ct$ is a constant. Here the characteristic curves are of the form $x - ct = k$ where k is a constant. We have illustrated this in Figure [3].

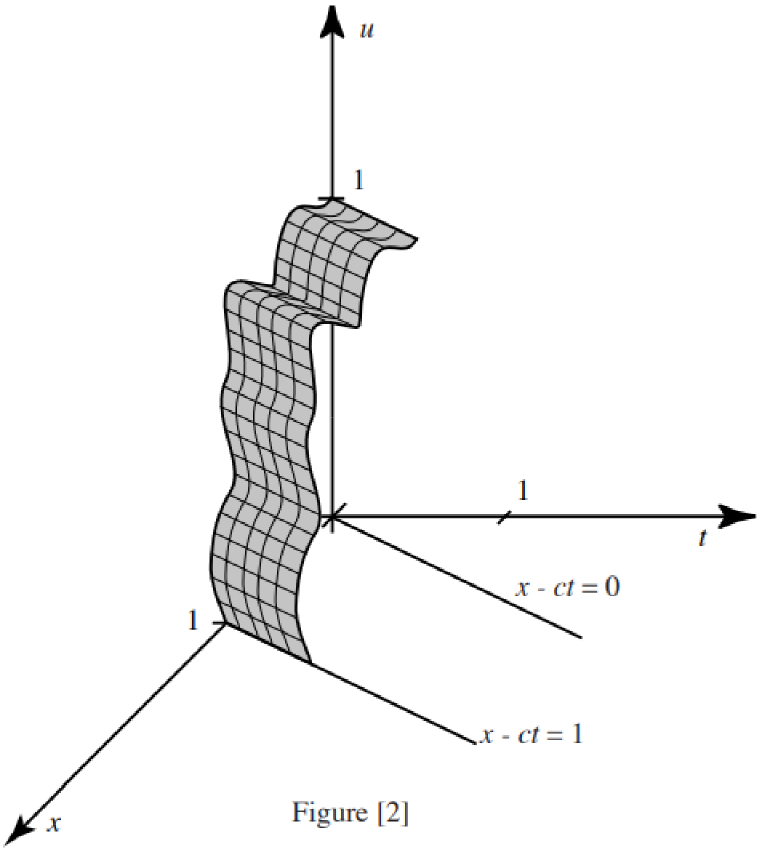


Figure [2]

Now, the general method of characteristic curves for solving the following initial value problem:

$$A\frac{\partial u}{\partial x}+B\frac{\partial u}{\partial y}+Cu=D\, avec\, u(X(\tau),Y(\tau))=F(\tau),\forall\,\tau\ \in I,$$

where A, B, C, D are continuously differentiable functions of x and $y; u = u(x, y)$ is to be determined, I is an interval and $X(\tau), Y(\tau)$ and $F(\tau)$ are continuously differentiable functions of $\tau \in I$ data. We will initially assume that $(A(x, y), B(x, y)) \neq (0, 0)$ for any point $(x, y) \in D$. Consider all parameterized curves with a parameterization : $\mathbb{R} \to \mathbb{R}^2$ defined by $s \Rightarrow ((x(s), y(s))$ for everything $s \in \mathbf{R}$ and whose tangent vector $'(s) = (x'(s), y'(s))$ is $(A(x(s), y(s)), B(x(s), y(s))$ at the point $(s) = (x(s), y(s))$ for everything $s \in \mathbb{R}$. This gives us a system of two ordinary differential equations;

$$\begin{cases} \frac{dx}{ds} = A(x,y) \\ \frac{dy}{ds} = B(x,y). \end{cases} \tag{10}$$

If we fix $x(0) = x_0$ and $y(0) = y_0$, then there is one and only one solution to this system of ordinary differential equations, because of the assumption that $(A(x, y), B(x, y)) \neq (0, 0)$ for every point $(x, y) \in D$. Note that it is generally difficult to solve such a system explicitly. If now we consider the values $u(s) = u(x(s), y(s))$ of a solution u on these curves, then we obtain by the chain rule that According to Schwarz's Theorem: Let $A \in \Omega$, where Ω is an open subset of $\mathbb{R}^2$ and f a function defined on Ω with values in $\mathbb{R}$. If the successive derivatives $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ exist and are continuous, then we have $\frac{\partial^2 f}{\partial x \partial y}(A) = \frac{\partial^2 f}{\partial y \partial x}(A)$.

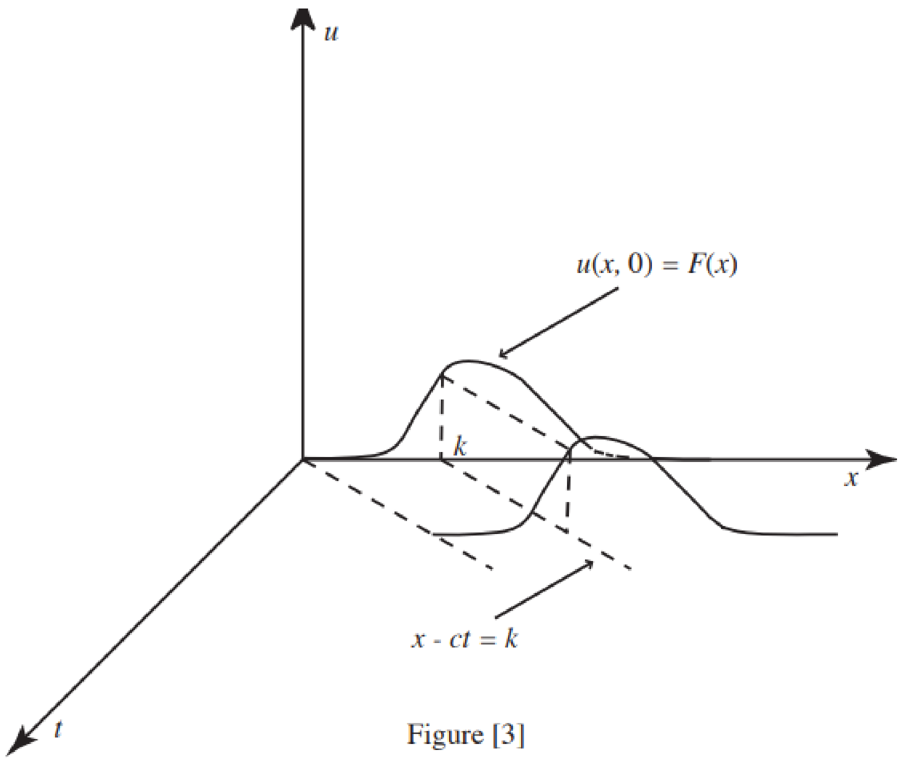


Figure [3]

2. Remark

Construction of solutions

In geometry, a surface S of R^3 has the equation $\phi(x, y, z) =$ Constant. When there exists a neighborhood where $\phi(x, y, z) = 0$ can be solved for z, we are led to the Cartesian equation $z = f(x, y)$. We interpret the solution u of (E) as a surface of $\mathbb{R}^3$.

Analytically, we seek the solutions of (E) in implicit form; that is, we look for a function ϕ of class C^1 defined on an open set V of $\mathbb{R}^3$ such that

$$((x, y, z) \in V,\ \phi(x, y, z) = Const) \Leftrightarrow ((x, y) \in \Omega\,,\ z = u(x, y)).$$

By applying the implicit function theorem, we can write it in the form

$$\frac{dx}{a(x,y,z)} = \frac{dy}{b(x,y,z)} = \frac{dz}{c(x,y,z)}. \tag{11}$$

The function ϕ is therefore a first integral of system 11. This system is called the "**characteristic system**" of the partial differential equation.

Theorem 2.1:

The set of solutions of equation (E) consists of the first integrals of system (11). Moreover, if ϕ_1 and ϕ_2 are a pair of independent first integrals, then the general solution of (E) can be written explicitly as

$$F(\phi_1(x, y, u(x, y)), \phi_2(x, y, u(x, y))) = const$$

Constant, where F denotes an arbitrary regular function of two variables.

3. Cauchy Problem

Let f be a function of a single variable and (C) the curve defined by the equation $y = f(x)$. Let $F(x, y) = f(x) - y$. Then, the vector $\overrightarrow{n} = \nabla F(x_0, f(x_0)) = (f'(x_0), -1)$ is perpendicular to (C) at $(x_0, f(x_0))$.

Definition 3.1:

The Cauchy problem related to a regular curve (C) consists in finding (if it exists) the solution of equation (16)

$$a(x,y)\frac{\partial^2 u}{\partial x^2}+b(x,y)\frac{\partial^2 u}{\partial x\partial y}+c(x,y)\frac{\partial^2 u}{\partial y^2}=F(x,y,u,\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}),$$

which satisfies $u(x,y)=g(x,y)$ on (C) and $\frac{du}{dn}(x,y)=h(x,y)$ on (C),
where $\frac{du}{dn}$ denotes the normal derivative and f,g are functions of class $\mathcal{C}^1$ and $\mathcal{C}^2$ on (C) respectively.

Theorem 3.1:

If the curve (C) is not characteristic, the Cauchy problem admits a unique solution.

Proof

First of all, we show that providing the function u and its normal derivative makes it possible to determine all the first-order partial derivatives. We assume that the curve (C) is defined by the parametric representation $(x,y)=(x_0(s),y_0(s))=m(s)$. We then have

$$\begin{cases} \overrightarrow{t(s)}=\frac{dx_0}{ds}(s)\overrightarrow{x}+\frac{dy_0}{ds}(s)\overrightarrow{y}, \\ \overrightarrow{n(s)}=-\frac{dy_0}{ds}(s)\overrightarrow{x}+\frac{dx_0}{ds}(s)\overrightarrow{y}, \end{cases}$$

with $\overrightarrow{t(s)}=1$. Given that the following functions are known

$$\frac{du}{dn}(x_0(s),y_0(s))=h(x_0(s),y_0(s))=H(s),$$

$$\frac{dG}{ds}=\frac{\partial u}{\partial s}\frac{dx_0}{ds}+\frac{\partial u}{\partial s}\frac{dy_0}{ds}=du_{m(s)}\overrightarrow{t(s)}$$

We can derive partial derivatives of order 1

$$\frac{\partial u}{\partial y}(x_0(s),y_0(s))=\frac{dx_0}{ds}H(s)+\frac{dy_0}{ds}\frac{\partial G}{\partial s},$$

$$\frac{\partial u}{\partial y}(x_0(s),y_0(s))=\frac{dx_0}{ds}\frac{\partial G}{\partial s}-\frac{dy_0}{ds}H(s).$$

□

Proof

A necessary condition for there to be a unique solution is that equation (16) allows the first-order partial derivatives of u to be uniquely defined on (C). We have

$$u(x_0(s), y_0(s)) = g(x_0(s), y_0(s)) = G(s),$$
$$\frac{\partial u}{\partial x}(x_0(s), y_0(s)) = h_1(x_0(s), y_0(s)) = H_1(s),$$
$$\frac{\partial u}{\partial y}(x_0(s), y_0(s)) = h_2(x_0(s), y_0(s)) = H_2(s).$$

From the last two equations, we obtain

$$x_0'(s)\frac{\partial^2 u}{\partial x^2}(x_0(s), y_0(s)) + y_0'(s)\frac{\partial^2 u}{\partial x \partial y}(x_0(s), y_0(s)) = H_1'(s),$$
$$x_0'(s)\frac{\partial^2 u}{\partial x \partial y}(x_0(s), y_0(s)) + y_0'(s)\frac{\partial^2 u}{\partial y^2}(x_0(s), y_0(s)) = H_2'(s).$$

Equation (16) allows us to deduce

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})$$

For this linear system of three equations with three unknowns $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}$ to have a unique solution, it is necessary and sufficient that the determinant be nonzero, hence

$$\begin{vmatrix} x_0'(s) & y_0'(s) & 0 \\ 0 & x_0'(s) & y_0'(s) \\ a & b & c \end{vmatrix} = a(y_0'(s))^2 - 2bx_0'(s)y_0'(s) + c(y_0'(s))^2 \neq 0.$$

Since $x_0'(s) = \frac{dx_0}{ds}$ and $y_0'(s) = \frac{dy_0}{ds}$, we recognize the condition defining a curve that is not characteristic. The construction of the solution is carried out as in the one-dimensional case, by a Taylor expansion of the solution in the neighborhood of the curve.

□

4. d'Alembert's Formula[1]

The Cauchy problem for the one-dimensional homogeneous wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0; -\infty < x \leq +\infty; t > 0 \tag{12}$$

$$u(x;0) = f(x), \frac{\partial u}{\partial t}(x,0) = g(x), -\infty < x \leq +\infty \tag{13}$$

A classical solution of the Cauchy problem (39)–(40) is a function u that is twice continuously differentiable for all $t > 0$, such that u and $\frac{\partial u}{\partial t}$ are continuous for $t > 0$, and such that (39)–(40) are satisfied.

[1]d'Alembert: French philosopher, writer, and mathematician, Paris 1717 – ibid., 1783.

Theorem 4.1:

The solution of the Cauchy problem (39)–(40) is given by the following formula (called d'Alembert's formula)

$$u(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

Moreover, if $f \in \mathbb{C}^2(\mathbb{R})$ and $g \in \mathbb{C}^1(\mathbb{R})$, this solution is classical such that $u \in \mathbb{C}^2(\mathbb{R}\times]0,+\infty[) \cap \mathbb{C}^1(\mathbb{R}\times[0,+\infty[)$ given by d'Alembert's formula.

To conclude this chapter, we will now describe d'Alembert's solution to the wave equation in the case of a vibrating string.

EXAMPLE 4.1. *The problem is as follows. We need to determine the function $u(x,t)$ such as*

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0, \\ u(x,0) = f(x), \\ \frac{\partial u}{\partial t}(x,0) = g(x). \end{cases}$$

Here c is a given positive real number ($c > 0$), and $f(x)$ and $g(x)$ are given functions. ***Physically,*** *$u(x,t)$ is the vertical displacement of a vibrating string at point x of this string at time t. We assume that the string is sufficiently long so that the ends do not interfere during the time interval for which we consider u. $f(x)$ is the initial displacement of the string and $g(x)$ is the (vertical) initial velocity. As we have seen, we can rewrite the PDE*

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

*in the form of the system (**??**)*

$$\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = v, \\ \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0 \end{cases} \tag{14}$$

*Thus, we will first determine v taking into account the initial conditions. For $t = 0$, we obtain from the first equation of system (**??**) that*

$$v(x,0) = \frac{\partial u}{\partial t}(x,0) - c\frac{\partial u}{\partial x}(x,0) = g(x) - cf'(x),$$

because $\frac{\partial u}{\partial t}(x,0) = g(x)$ and $u(x,0) = f(x) \Rightarrow \frac{\partial u}{\partial x}(x,0) = f'(x)$.
Thus, the function $v(x,t)$ will be a solution of the following initial value problem

$$\begin{cases} \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = 0 \\ v(x,0) = g(x) - cf'(x) \end{cases}$$

Previously, as we have seen, the solution to this problem is

$$v(x,t) = g(x-ct) - cf'(x-ct).$$

Now we need to consider the first equation of the system (??) with the solution $v(x,t)$ *above. We obtain the following new initial value problem*

$$\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = g(x-ct) - cf'(x-ct), \\ u(x,o) = f(x) \end{cases}$$

First, we must determine the characteristic curves. We therefore need to consider the ordinary differential equations

$$\frac{dx}{ds} = -c, \frac{dt}{ds} = 1, \frac{du}{ds} = g(x(s)-ct(s)) - cf'(x(s)-ct(s)).$$

Moreover, the curve of initial values is $\tau \mapsto (X(\tau), T(\tau), u(X(\tau), T(\tau)) = (\tau, 0, f(\tau))$. *From this, we obtain* $x(s,\tau) = -cs+\tau, t(s,\tau) = s$, *as well as*

$$\frac{du}{ds} = g(-cs+\tau-cs) - cf'(cs+\tau-cs) \Rightarrow u(s,\tau) = (\int_0^s g(-2c\omega+\tau) - cf'(-2c\omega+\tau)d\omega) + f(\tau).$$

Considering substitution $\lambda = -2c\omega + \omega$ *in this complete collection and because* $d\lambda = 2cd\omega$, *we obtain*

$$u(s,) = (\frac{1}{-2c}\int_\tau^{\tau-2cs} g(\lambda)cf'(\lambda)d\lambda) + f(\tau).$$

It is easy to verify that $s = t$ *and* $= x + cs = x + ct$. *Substituting this into the expression for* u *above, we obtain that*

$$u(x,t) = (\frac{1}{-2c}\int_\tau^{\tau-2cs} g(\lambda)cf'(\lambda)d\lambda) + f(x+ct) = (\frac{1}{2c}\int_{x-ct}^{x+ct} g(\lambda) - cf'(\lambda)d\lambda) + f(x+ct).$$

We can integrate $f'(\lambda)$ *in relation to* λ. *Consequently*

$$u(x,t) = (\frac{1}{2c}\int_{xct}^{x+ct} g(\lambda)d\lambda) - \frac{1}{2}(f(x+ct) - f(x-ct)) + f(x+ct).$$

Finally, we obtain the following solution:

$$u(x,t) = (\frac{1}{2c}\int_{xct}^{x+ct} g(\lambda)d\lambda) - \frac{1}{2}(f(x+ct) + f(x-ct)).$$

This is the solution obtained by d'Alembert.

For example, if the initial displacement and initial velocity are respectively

$$f(x) = \begin{cases} \sin(x), & si\ 0 \le x \le \pi \\ 0, & sinon; \end{cases}$$

and $g(x) = 0\ \forall x$, *then the vertical displacement* u *will be*

$$u(x,t) = \frac{1}{2}(f(x+ct) + f(x-ct))$$

$$= \begin{cases} 0, \; si \; si(x,t) \in A; \\ \frac{\sin(x+ct)}{2}, \; si \; (x,t) \in B; \\ \frac{\sin(x+ct)+\sin(x-ct)}{2}, \; si \; (x,t) \in C; \\ 0, \; si \; (x,t) \in D; \\ \frac{\sin(x-ct)}{2}, \; si \; (x,t) \in E; \\ 0, \; si \; (x,t) \in F; \end{cases}$$

where A, B, C, D, E and F are the regions shown in the following figure. There are thus two waves: one moving to the right and the other to the left.

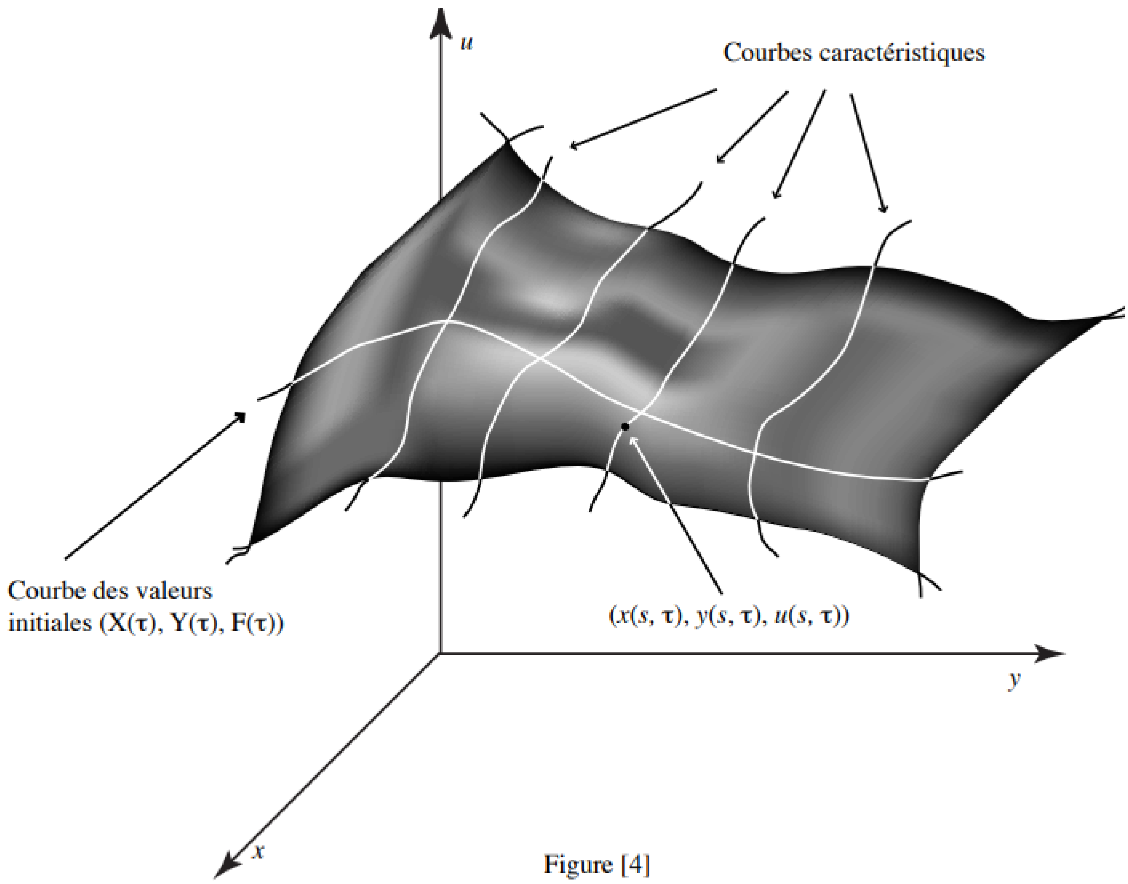


Figure [4]

5. Exercise series $N^\circ = 02$

EXERCICE 3. *Show that the PDE*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + 2\lambda \frac{\partial}{\partial t} = 0$$

can be written in the form

$$(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x} + 2\lambda)u - 2c\lambda \frac{\partial u}{\partial x} = 0$$

where $u = u(x,t)$. Conclude from this that this PDE is equivalent to the system.

$$\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = v - 2\lambda u \\ \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} - 2c\lambda\frac{\partial u}{\partial x} = 0. \end{cases}$$

SOLUTION. *a) We can write the PDE in the form*

$$(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})u = F(x,t)$$

Indeed, we obtain

$$(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})u = (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial t\partial x} + c\frac{\partial^2 u}{\partial x\partial t} - c^2\frac{\partial^2 u}{\partial x^2}$$
$$= \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F(x,t)$$

assuming that the partial derivatives of order 2 of u are continuous and consequently that

$$\frac{\partial^2 u}{\partial t\partial x} = \frac{\partial^2 u}{\partial x\partial t}$$

we do get the system

$$(\star)\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = v; \\ \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = F(x,t) \end{cases}$$

b) To determine the solution u, we must first determine v such that

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial t} = F(x,t).$$

The initial conditions $u(x,0) = f(x)$ and $\frac{\partial v}{\partial t}(x,0) = g(x)$ mean, using the first equation of $(\star)$, that

$$v(x,0) = \frac{\partial v}{\partial t}(x,0) - c\frac{\partial v}{\partial x}(x,0) = g(x) - cf'(x).$$

Thus, we have as the problem to solve $\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = F(x,t)$. with the initial condition $v(x,0) = g(x) - cf'(x)$.

We therefore need to consider the system of ordinary differential equations

$$\frac{dt}{ds} = 1 \text{ and } \frac{dx}{ds} = c.$$

These equations can be solved by separation of variables:

$$\frac{dt}{ds} = 1 \Rightarrow dt = ds \Rightarrow \int dt = \int ds + ctet = s + t_0'$$

and

$$\frac{dx}{ds} = 1 \Rightarrow dx = ds \Rightarrow \int dx = \int ds + ctex = cs + x_0'$$

If we consider v as a function of s i.e.,$v(s) = v(x(s), t(s))$, we obtain

$$\frac{dv}{ds} = \frac{\partial v}{\partial t}\frac{dx}{ds} + \frac{\partial v}{\partial t}\frac{dt}{ds} = c\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = F(x(s), t(s)) = F(cs + x_0', s + t_0').$$

By the fundamental theorem of calculus, we obtain that there is only one solution such that $v(0) = v_0$ and this one is

$$v(s) = (\int_0^s F(cw + x_0', w + t_0^{\prime)} dw) + v_0.$$

If we consider the initial values $x_0' = \tau, t_0' = 0$ and $v_0 = g(\tau) - cf'(\tau)$, we obtain for each τ the characteristic curve

$$s \Rightarrow (x(s,\tau,t(s,\tau), v(s,\tau)) = (cs + \tau, s, (\int_0^s F(cw + \tau, w)dw) + g(\tau) - cf'(\tau))$$

Now we can consider the problem. $\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial} x = v$ *with the condition* $u(x,0) = f(x)$
We must therefore consider the system of ordinary differential equations. We must therefore consider the system of ordinary differential equations.

$$\frac{dt}{ds} = 1 \ et \ \frac{dx}{ds} = c.$$

These equations can be solved by separation of variables:

$$\frac{dt}{ds} = 1 \Rightarrow dt = ds \Rightarrow \int dt = \int ds + ctet = s + t_0'$$

and

$$\frac{dx}{ds} = -c \Rightarrow dx = ds \Rightarrow \int dx = -c\int ds + ctex = -cs + x_0'$$

If we consider v as a function of s i.e.,$v(s) = v(x(s), t(s))$, we obtain

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = -c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = v(x(s), t(s)), s + t_0').$$

EXERCICE 4. *Solve the following initial value problem*

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \lambda u = 0$$

with $u(x,0) = f(x)$,where $\lambda > 0, f(x)$ is a given function and $u = u(x,t)$.

SOLUTION. *Consider the system of ordinary differential equations*

$$\frac{dt}{ds} = 1 \ et \ \frac{dx}{ds} = c.$$

We can solve these equations by separating variables:

$$\frac{dt}{ds} = 1 \Rightarrow dt = ds \Rightarrow \int dt = \int ds + constante \Rightarrow t = s + t_0$$

and

$$\frac{dx}{ds} = c \Rightarrow dx = ds \Rightarrow \int dx = \int cds + constante \Rightarrow x = cs + x_0.$$

If we consider the function u sought as a function of s, i.e. $u(s) = u(x(s), t(s))$, so using the chain rule and the fact that u is an PDE solution

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \lambda u = 0,$$

we obtain

$$\frac{du}{ds}=\frac{\partial u}{\partial x}\frac{dx}{ds}+\frac{\partial u}{\partial t}\frac{dt}{ds}=cudx+ut=-\lambda u$$

We can solve this last ordinary differential equation by separation of variables.

$$\frac{du}{ds}=-\lambda u\Rightarrow\frac{du}{u}=-\lambda ds\Rightarrow\int\frac{du}{u}=-\lambda\int ds+cte\Rightarrow ln(u)=-\lambda s+cte.$$

If we fix $u(0)=u_0$, *then we have* $u(s)=u_0e^{-\lambda s}$. *Thus, the characteristic curves are*

$$s\mapsto(x(s),t(s),u(s))=(cs+x_0,s+t_0,u_0e^{-\lambda s}).$$

If we consider the initial conditions: $x_0=\tau,t_0=0$ *and* $u_0=f(\tau),$, *then we obtain the characteristic curves:*

$$s\mapsto(x(s,\tau),t(s,\tau),u(s,\tau))=(cs+\tau,s,f(\tau)e^{-\lambda s}).$$

It is possible to determine the inverse function of $(s,\tau)\mapsto(x(s,\tau),t(s,\tau))$. *Indeed*

$$\begin{cases}cs+\tau=x\\ s=t\end{cases}\Rightarrow\begin{cases}s=t\\ \tau=x-cs=x-ct\end{cases}$$

Thus, if we express u *as a function of* x *and* t *rather than* s *and* τ, *we obtain*

$$u(x,t)=f(x-ct)e^{-\lambda t}.$$

EXERCICE 5. *Consider the d'Alembert solution of the wave equation for the following initial displacement* $f(x)$ *and initial velocity* $g(x)$*:*

a) $f(x)=x$ *and* $g(x)=0$;

b) $f(x)=0$ *and* $g(x)=x$;

c) $f(x)=\sin(x)$ *and* $g(x)=-c\cos(x)$;

d) $f(x)=\sin(x)$ *and* $g(x)=c\cos(x)$.

SOLUTION. *a) If* $f(x)=x$ *and* $g(x)=0$, *then d'Alembert's solution is*

$$u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}g(\lambda)d\lambda+\frac{1}{2}(f(x+ct)+f(x-ct))=\frac{1}{2c}\int_{x-ct}^{x+ct}0d\lambda+\frac{1}{2}((x+ct)+(x-ct))=x.$$

b) If $f(x)=0$ *and* $g(x)=x$, *then d'Alembert's solution is*

$$u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}g(\lambda)d\lambda+\frac{1}{2}(f(x+ct)+f(x-ct))=\frac{1}{2c}\int_{x-ct}^{x+ct}\lambda d\lambda+\frac{1}{2}(0+0)$$

$$=\frac{1}{2c}(\frac{\lambda^2}{2}]_{\lambda=x-ct}^{\lambda=x+ct}=\frac{1}{4c}((x+ct)^2-(x-ct)^2)=xt.$$

c) If $f(x)=\sin(x)$ *and* $g(x)=-c\cos(x)$, *then d'Alembert's solution is*

$$u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}g(\lambda)d\lambda+\frac{1}{2}(f(x+ct)+f(x-ct))$$

$$= \frac{1}{2c}\int_{x-ct}^{x+ct} -c\cos(\lambda)d\lambda + \frac{1}{2}(\sin(x+ct)+\sin(x-ct))$$
$$= \frac{1}{2c}(-c\sin(\lambda]_{x-ct}^{x+ct} + \frac{1}{2}(\sin(x+ct)+\sin(x-ct))$$
$$= \frac{1}{2}(\sin(x+ct)-\sin(x-ct)) + \frac{1}{2}(\sin(x+ct)+\sin(x-ct)) = \sin(x-ct).$$

d) If $f(x) = \sin(x)$ and $g(x) = c\cos(x)$, then d'Alembert's solution is

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\lambda)d\lambda + \frac{1}{2}(f(x+ct)+f(x-ct))$$
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} c\cos(\lambda)d\lambda + \frac{1}{2}(\sin(x+ct)+\sin(x-ct))$$
$$= \frac{1}{2c}(c\sin(\lambda]_{x-ct}^{x+ct} + \frac{1}{2}(\sin(x+ct)+\sin(x-ct))$$
$$= \frac{1}{2}(\sin(x+ct)-\sin(x-ct)) + \frac{1}{2}(\sin(x+ct)+\sin(x-ct)) = \sin(x-ct).$$

EXERCICE 6. *Solve the following initial value problem*

$$\frac{\partial u}{\partial t} + e^x\frac{\partial u}{\partial x} = 0$$

with $u(x,0) = x$ where $u = u(x,t)$.

SOLUTION. *Consider the system of ordinary differential equations*

$$\frac{dt}{ds} = 1 \ et \ \frac{dx}{ds} = e^x$$

We can solve these equations by separation of variables:

$$\frac{dt}{ds} = 1 \Rightarrow dt = ds \Rightarrow \int dt = \int ds + cte \Rightarrow t = s + t_0$$

and

$$\frac{dx}{ds} = e^x \Rightarrow \frac{dx}{e^x} = ds \Rightarrow e^{-x}dx = \int ds + cte \Rightarrow e^{-x} = s + cte$$
$$x(s) = -ln(-s+cte) \Rightarrow x(s) = -ln(-s+e_{-x_0}) = ln(\frac{1}{-s+e_{-x_0}}).$$

. *In the latter case, the initial condition was set as $x(0) = x_0$. If we now consider u as a function of s, i.e.$u(s) = u(x(s), t(s))$, then by the rule of chains and the fact that u is a solution of the PDE, we obtain*

$$\frac{du}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = e^x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0.$$

This ordinary differential equation can be easily solved. We obtain $u(s) = u_0$. If we now take as the initial condition: $x_0 = \tau, t_0 = 0$ and $u_0 = \tau$, we obtain for each τ , the characteristic curve

$$s \mapsto (x(s,\tau), t(s,\tau), u(s,\tau)) = (ln(\frac{1}{e^{-\tau}-s}), s, \tau)$$

.The function $(s,\tau) \mapsto (x(s,\tau), t(s,\tau))$ *has an inverse function. Indeed,* $s = t$ *and*

$$x = ln(\frac{1}{e^{-\tau} - s}) = ln(\frac{1}{e^{-\tau} - t}) \Rightarrow e^{-x} = (e^{-\tau} - t) \Rightarrow e^{-x} + t = e^{-\tau} \Rightarrow ln(e^{-x} + t) = -\tau$$

allows us to conclude that

$$\tau = ln(\frac{1}{e^{-x} + t}).$$

The solution to the problem is

$$u(x,t) = ln(\frac{1}{e^{-x} + t}) = ln(\frac{e^{x}}{1 + te^{x}}) = x - ln(1 + te^{x})$$

EXERCICE 7. *Consider the non-homogeneous linear PDE*

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t)$$

where c *is a positive real number,* $F(x,t)$ *is a given function and* $u = u(x,t)$.

a) Show that this PDE is equivalent to the system

$$\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = v; \\ \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = F(x,t) \end{cases}$$

b) Determine the solution to the initial value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t), \\ u(x,0) = f(x), \\ \frac{\partial u}{\partial t}(x,0) = g(x), \end{cases}$$

proceeding as we did when describing d'Alembert's solution for the wave equation. Here $f(x)$ *and* $g(x)$ *are given.*

CHAPTER 3

Second-Order Partial Differential Equations

In this chapter, several physical phenomena related to fluid dynamics, heat transfer, mechanics, and electromagnetism are described by second-order PDEs. The study of their solution methods is very important for solving real-world problems.

1. A bit of order now.

After taking our first steps into the theory, we will describe in this chapter how to classify all linear second-order PDEs. We will have three types of PDEs: hyperbolic, parabolic, and elliptic. Next, we will describe the canonical form obtained after a change of coordinates for each of these. types of PDEs. Equation (**??**) from the first chapter describes the general form of these equations with n independent variables. At first, we will restrict ourselves to the case where $n = 2$. Thus, the PDEs we will initially consider will be of the following form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{15}$$

and A, B, C, D, E, F, and G are functions of x and y that do not vanish simultaneously. We will also assume that u, A, B, C, D, E, F, and G all have at least continuous partial derivatives of order $m \leq 2$ on a domain D of the x, y plane.

In the following remark, we will give the definition of a second-order semi-linear partial differential equation.

Remark 1.1:

A second-order semi-linear partial differential equation with unknown u on an open set Ω of $\mathbb{R}^2$ is called an equation of the form

$$a(x,y)\frac{\partial^2 u}{\partial x^2} + b(x,y)\frac{\partial^2 u}{\partial x \partial y} + c(x,y)\frac{\partial^2 u}{\partial y^2} = F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \tag{16}$$

where a, b, c are given functions, and F is a function defined on an open subset of $\mathbb{R}^5$.

The equation (15) is said to be hyperbolic (respectively parabolic, elliptic) at the point $(x_0, y_0) \in D$ if and only if $(B(x_0, y_0))^2 - 4A(x_0, y_0)C(x_0, y_0)$ is positive (respectively zero, negative). If a PDE

is hyperbolic (respectively parabolic, elliptic) for all points (x_0, y_0) in the domain D, it is then said to be hyperbolic (respectively parabolic, elliptic) on D.

Remark 1.2:

The classification of second-order PDEs originates from the classification of the quadratic equation of conic sections in analytic geometry. The equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

represents a hyperbola, a parabola, or an ellipse depending on the sign of $b^2 - 4ac$ (positive, zero, or negative).

It is important to observe that this classification is preserved by any change of coordinates. This indicates that our criterion is valid. It is clear that the form of a PDE is modified by a change of coordinates, but we can relate the solutions of the equation before the change of coordinates with the solutions after it. Thus, we do not really obtain a new equation. Let us explain why our classification is preserved by any change of coordinates.
More precisely, let $\xi = \xi(x, y), \eta = \eta(x, y)$, two new variables which are functions of x and y having at least their partial derivatives of order $m \leq 2$ continuous on the domain D and such that the determinant

Determinant

$$J = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} = (\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial y}) - (\frac{\partial \xi}{\partial x})(\frac{\partial \xi}{\partial x}).$$

is never zero on the domain D. Then we obtain a new equation [2] using these new variables (ξ, η), and it is hyperbolic (respectively parabolic, elliptic) at the point $(\xi_0, \eta_0) = (\xi(x_0, y_0), \eta(x_0, y_0))$ if and only if the equation (15) is hyperbolic (respectively parabolic, elliptic) at the point (x_0, y_0).

2. Classification of Equations

The classification of second-order PDEs comes from the classification of the quadratic equation of conic sections in analytic geometry. The equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0$$

represents the hyperbola, the parabola, or the ellipse depending on the sign of $b^2 - 4ac$ (positive, zero, or negative).

Thus, the classification of equation (16) depends on the coefficients $a(x, y), b(x, y)$ and $c(x, y)$ at a given point (x, y). Consequently, we give the following definition

Definition 2.1:

Let $\Delta(x,y) = b^2(x,y) - 4a(x,y)c(x,y)$, we have the following cases:

1. If $\Delta(x,y) > 0$, the equation is said to be hyperbolic.

2. If $\Delta = 0$, the equation is said to be parabolic.

3. If $\Delta < 0$, the equation is said to be elliptic.

EXAMPLE 2.1. *The wave equation*

$$\frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial x^2} - = 0$$

is a hyperbolic equation on the domain $D = \mathbb{R}^+ \times \mathbb{R}$ *because*

$$\Delta(t,x) = b^2(t,x) - 4a(t,x)c(t,x) = 4c^2 > 0.$$

EXAMPLE 2.2. *Let's consider the equation*

$$x^2\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin(x+y), \tag{17}$$

$\Delta(x,y) = 3x^2y^2$. *So if* $x = 0$ *where* $y = 0, \Delta$ *is canceled. So on the domain*

$$D = \{(x,y) \in \mathbb{R}^2 / x = 0 \ ou \ y = 0\}.$$

This PDE is parabolic and elliptic on $\mathbb{R}^2$ D.

EXAMPLE 2.3. *For the equation*

$$x^2\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = e^x,$$

we then $A(x,y) = x^2, B(x,y) = -xy, C(x,y) = y^2$ *and*

$$B^2 - 4AC = (-xy)^2 - 4(x^2)(y^2) = -3x^2y^2.$$

So if $x_0 = 0$ *where* $y_0 = 0$, *this PDE is parabolic at the point* (x_0, y_0); *otherwise it is elliptical to the point* (x_0, y_0).

To transform equation (E) into a canonical form, we need to make a change of independent variables.

3. Change of Variables

To transform equation (15) into a canonical form, we need to make a change of independent variables.

Theorem 3.1:

We consider the change of variables $(\xi(x,y), \eta(x,y))$ assumed to be twice continuously differentiable and such that the Jacobian J does not vanish, then there exist functions a', b', c' and F' such that

$$a'(x,y)\frac{\partial^2 u}{\partial \xi^2} + b'(x,y)\frac{\partial^2 u}{\partial \xi \partial \eta} + c'(x,y)\frac{\partial^2 u}{\partial \eta^2} = F'(\xi, \eta, u, \partial\xi, \partial\eta). \tag{18}$$

Equation (18) is called the standard or canonical form of the equation. Moreover, we have

$$\Delta'(\xi,\eta) = b'^2(\xi,\eta) - 4a'(\xi,\eta)c'(\xi,\eta) = J^2\Delta(x,y).$$

Proof

We set $u(x,y) = v(\xi(x,y), \eta(x,y))$. We will now write the different derivatives in the equation in terms of the partial derivatives of the function v. For this, we use the chain rule to express the first derivatives

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial x},$$
$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y},$$

Based on the two previous equations and once again on the chain rule, we express the second derivatives as follows

$$\frac{\partial^2 u}{\partial x^2} = (\frac{\partial \xi}{\partial x})^2\frac{\partial^2 v}{\partial \xi^2} + 2(\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial x})\frac{\partial^2 v}{\partial \xi \partial \eta} + (\frac{\partial \eta}{\partial x})^2\frac{\partial^2 v}{\partial \eta^2} + (\frac{\partial^2 \xi}{\partial x^2})\frac{\partial v}{\partial \xi} + (\frac{\partial^2 \eta}{\partial x^2})\frac{\partial v}{\partial \eta},$$

$$\frac{\partial^2 u}{\partial x \partial y} = (\frac{\partial \xi}{\partial x})(\frac{\partial \xi}{\partial y})\frac{\partial^2 v}{\partial \xi^2} + [(\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial x})$$
$$+(\frac{\partial \xi}{\partial y})(\frac{\partial \eta}{\partial y})]\frac{\partial^2 v}{\partial \xi \partial \eta} + (\frac{\partial \eta}{\partial x})(\frac{\partial \eta}{\partial y})\frac{\partial^2 v}{\partial \eta^2} + (\frac{\partial^2 \xi}{\partial x \partial y})\frac{\partial v}{\partial \xi} + (\frac{\partial^2 \eta}{\partial x \partial y})\frac{\partial v}{\partial \eta},$$

$$\frac{\partial^2 u}{\partial y^2} = (\frac{\partial \xi}{\partial y})^2\frac{\partial^2 v}{\partial \xi^2} + 2(\frac{\partial \xi}{\partial y})(\frac{\partial \eta}{\partial y})\frac{\partial^2 v}{\partial \xi \partial \eta} + (\frac{\partial \eta}{\partial y})^2\frac{\partial^2 v}{\partial \eta^2} +$$
$$(\frac{\partial^2 \xi}{\partial y^2})\frac{\partial v}{\partial \xi} + (\frac{\partial^2 \eta}{\partial y^2})\frac{\partial v}{\partial \eta}.$$

Substituting these expressions into (16), the following equation is obtained

$$a'(x,y)\frac{\partial^2 v}{\partial \xi^2} + b'(x,y)\frac{\partial^2 v}{\partial \xi \partial \eta} + c'(x,y)\frac{\partial^2 v}{\partial \eta^2} = F'(\xi, \eta, v, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta}) \tag{19}$$

□

Proof

where

$$a' = a(\frac{\partial \xi}{\partial x})^2 + b(\frac{\partial \xi}{\partial x})(\frac{\partial \xi}{\partial y}) + c(\frac{\partial \xi}{\partial y})^2,$$

$$b' = 2a(\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial x}) + b[(\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial x}) + (\frac{\partial \xi}{\partial y})(\frac{\partial \eta}{\partial y})] + 2c(\frac{\partial \xi}{\partial y})(\frac{\partial \eta}{\partial y}),$$

$$c' = a(\frac{\partial \eta}{\partial x})^2 + b(\frac{\partial \eta}{\partial x})(\frac{\partial \eta}{\partial y}) + c(\frac{\partial \eta}{\partial y})^2,$$

$$F'(\xi, \eta, v, \partial\xi, \partial\eta) = a(\frac{\partial^2 \xi}{\partial y^2})\frac{\partial v}{\partial \xi} + a(\frac{\partial^2 \eta}{\partial y^2})\frac{\partial v}{\partial \eta} + b(\frac{\partial^2 \xi}{\partial x \partial y})\frac{\partial v}{\partial \xi}$$

$$+b(\frac{\partial^2 \eta}{\partial x \partial y})\frac{\partial v}{\partial \eta} + c(\frac{\partial^2 \xi}{\partial y^2})\frac{\partial v}{\partial \xi} + c(\frac{\partial^2 \eta}{\partial y^2})\frac{\partial v}{\partial \eta} + F''(\xi, \eta, u, \partial\xi, \partial\eta).$$

Equation (16) is a second-order PDE whose type is determined using the following formula $b'^2(\xi,\eta) - 4a'(\xi,\eta)c'(\xi,\eta) = J^2(b^2(x,y) - 4a(x,y)c(x,y))$, as $J \neq 0$, the two equations (16) and (19) are of the same type. □

4. Forme Canonique

Writing a second-order PDE in canonical (or standard) form allows it to be expressed in a simpler form. It consists of eliminating certain second derivatives. This procedure helps in finding solutions for certain equations (the wave equation, ...). The canonical form of a second-order PDE depends on the type of the equation. For this, we need to introduce the notion of a characteristic curve associated with equation (16).

Definition 4.1:

A characteristic of equation (16) is the curve satisfying the differential equation

$$a(\frac{dy}{dx})^2 - 2b(\frac{dy}{dx}) + c = 0. \quad (20)$$

We will deal with each case separately, and give the corresponding form for each case.

Hyperbolic Case

Let (E_h) be an equation of hyperbolic type. We know that in this case $b^2 - 4ac > 0$ and by

consequently, equation (**??**) admits two distinct solutions. Thus, there exist two real characteristic curves of the equations $\psi_1(x,y) = c_1$ and $\psi_2(x,y) = c_2$ for equation (**??**).

Theorem 4.1:

The equations of the characteristic curves are given by

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

By posing $\xi = \psi_1(x, y)$ and $\eta = \psi_2(x, y)$ where $\psi_i, i = 1, 2.$, the canonical form of (E_h) is written

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = G(\xi, \eta, v, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta})$$

Proof

We are looking for contact details (ξ, η) such as $a' = 0$ where $c' = 0$. Expressions of a' and c' are the same shape

$$a(\frac{\partial \psi}{\partial x})^2 + b(\frac{\partial \psi}{\partial x})(\frac{\partial \psi}{\partial y}) + c(\frac{\partial \psi}{\partial y})^2 = 0.$$

By factoring, we obtain

$$\frac{1}{a}[a(\frac{\partial \psi}{\partial x}) + (b - \sqrt{b^2 - 4ac})](\frac{\partial \psi}{\partial y})[a(\frac{\partial \psi}{\partial x}) + (b + \sqrt{b^2 - 4ac})] = 0.$$

Therefore, we must solve the following linear equations

$$2a(\frac{\partial \psi}{\partial x}) + (b - \sqrt{b^2 - 4ac})(\frac{\partial \psi}{\partial y}) = 0$$

and

$$2a(\frac{\partial \psi}{\partial x}) + (b + \sqrt{b^2 - 4ac})(\frac{\partial \psi}{\partial y}) = 0$$

Both equations are linear homogeneous, ψ is constant for each characteristic. The characteristics are solutions to the equation.

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These are called the characteristic equations. By finding the solutions to these two equations, we have the two characteristic curvess $\psi_i(x, y) = C_i, i = 1, 2$. By posing $\xi(x, y) = \psi_1(x, y)$ and $\eta(x, y) = \psi_2(x, y)$, we will have a good change of coordinates that will simplify equation (16) to equation (19) with $a' = b' = 0$. By dividing by b', we obtain the first standard form

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = G(\xi, \eta, v, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta})$$

□

EXAMPLE 4.1. *Consider the following equation*

$$y^2 \frac{\partial^2 u}{\partial x^2} - x^2 \frac{\partial^2 u}{\partial y^2} = 0$$

$\Delta(x,y) = b^2(x,y) - 4a(x,y)c(x,y) = 4x^2y^2$. Thus, if $x = 0$ or $y = 0$, this PDE is parabolic at the point (x,y), otherwise it is hyperbolic at the point (x,y). Let us consider a domain D for which the PDE is hyperbolic at all its points. At these points, the characteristic equations are

$$\frac{dy}{dx} = \frac{0 \pm \sqrt{b^2 - 4ac}}{a^2} = \frac{0 \pm \sqrt{4x^2y^2}}{2y^2} = \pm\frac{x}{y}.$$

Using the method of separation of variables, we obtain: $\frac{1}{2}(x^2+y^2) = C_1$ et $\frac{1}{2}(x^2-y^2) = C_2.C1$ and $C2$ are constants. The characteristic curves are

$$\begin{cases} \varphi_1(x,y) = x^2 + y^2, \\ \varphi_2(x,y) = x^2 - y^2 \end{cases}$$

We put down

$$\begin{cases} \xi = x^2 + y^2, \\ \eta = x^2 - y^2 \end{cases}$$

and $u(x,y) = v(\xi,\eta)$. Using this change of coordinates, we obtain

$$\begin{cases} x^2 = \eta - \xi, \\ {}^2 = \xi + \eta \end{cases}$$

By the chain rule, we have

$$\frac{\partial u}{\partial x} = -x\frac{\partial v}{\partial \xi} + x\frac{\partial v}{\partial \eta},$$
$$\frac{\partial u}{\partial y} = y\frac{\partial v}{\partial \xi} + y\frac{\partial v}{\partial \eta},$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(-x\frac{\partial v}{\partial \xi} + x\frac{\partial v}{\partial \eta}) = x^2\frac{\partial^2 v}{\partial \xi^2} + 2x^2\frac{\partial^2 v}{\partial \xi \partial \eta} + x^2\frac{\partial^2 v}{\partial \eta^2} - \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(y\frac{\partial v}{\partial \xi} + y\frac{\partial v}{\partial \eta}) = y^2\frac{\partial^2 v}{\partial \xi^2} + 2y^2\frac{\partial^2 v}{\partial \xi \partial \eta} + y^2\frac{\partial^2 v}{\partial \eta^2} + \frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}$$

and finally

$$y^2\frac{\partial^2 u}{\partial x^2} - x^2\frac{\partial^2 u}{\partial y^2} = -4x^2y^2\frac{\partial^2 v}{\partial \xi \partial \eta} - (x^2+y^2)\frac{\partial v}{\partial \xi} + (y^2 - x^2)\frac{\partial v}{\partial \eta} = 0$$

which implies, after simplification

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = \frac{\eta}{2(\xi^2 - \eta^2)}\frac{\partial v}{\partial \xi} - \frac{\xi}{2(\xi^2 - \eta^2)}\frac{\partial v}{\partial \eta}$$

This is the canonical form of the PDE at points where it is hyperbolic.

Parabolic Case

Let (E_p) be an equation of parabolic type. We know that in this case $b^2 - 4ac = 0$ and consequently equation (**??**) admits two coincident solutions. Thus, there exists a single real characteristic curve of the equation $\psi(x,y) = c$ for equation (**??**).

The equations of the characteristic curves are given by

$$\frac{dy}{dx} = \frac{b}{2a}.$$

By posing

$$\begin{cases} \xi = \varphi_1(x,y) \\ \eta = \varphi_2(x,y) \end{cases}$$

where φ_2 satisfied $J(\xi,\eta) \neq 0$, the canonical form of (E_p) is written

$$\frac{\partial^2 v}{\partial\xi\partial\eta} = G(\xi,\eta,v,\frac{\partial v}{\partial\xi},\frac{\partial v}{\partial\eta})$$

Proof

In this case, two functions must be found $\psi_1(x,y)$ and $\psi_2(x,y)$ such as $b'(\xi,\eta) = c'(\xi,\eta) = 0$ for everything $(x,y) \in D$. Just return $C = 0$, this implies that

$$a(\frac{\partial\eta}{\partial x})^2 + b(\frac{\partial\eta}{\partial x})(\frac{\partial\eta}{\partial y}) + c(\frac{\partial\eta}{\partial y})^2 = \frac{1}{a}[a(\frac{\partial\eta}{\partial x}) + \frac{b}{2}(\frac{\partial\eta}{\partial y})]^2 = 0.$$

As a result, η is a solution to the first-order linear equation

$$a(\frac{\partial\eta}{\partial x}) + \frac{b}{2}(\frac{\partial\eta}{\partial y}) = 0.$$

Therefore, the solution η is constant on each characteristic, that is, on a curve that is a solution to the equation

$$\frac{dy}{dx} = \frac{b}{2a}$$

By finding the solution to this equation, we obtain the first characteristic curve. $\varphi_2(x,y) = C$. To choose φ_1, it is sufficient to consider the constraint on the second independent variable, is that the $J(\xi,\eta) \neq 0$ in D. □

Example 4.2. *Let the equation Consider the following equation*

$$\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x\partial y} + 9\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$$

is parabolic on $\mathbb{R}^2$. *The characteristic equation is given by*

$$\frac{dy}{dx} = 3$$

whose solution is $3x - y = C$. *The first characteristic curve is* $\varphi_1(x,y) = 3x - y$. *We therefore set* $\xi = 3x - y$. *To obtain a second characteristic coordinate* η, *we have many possible choices. For this example, we will take* $\eta(x,y) = x$. *This function indeed has partial derivatives of order* $m \leq 2$ *that are continuous, and*

$$J = \begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0.$$

for every point in $\mathbb{R}^2$. *Using this change of coordinates*

$$\begin{cases} \xi = 3x - y, \\ \eta = x \end{cases}$$

and we pose $u(x,y) = v(\xi,\eta)$*, we obtain*

$$\begin{cases} x = \eta, \\ y = 3\eta - \xi \end{cases}$$

Using the chain rule, we obtain

$$\frac{\partial u}{\partial x} = 3\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta},$$

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}(3\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}) = 9\frac{\partial^2 v}{\partial \xi^2} + 6\frac{\partial^2 v}{\partial \xi \eta} + \frac{\partial^2 v}{\partial \eta^2},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}(-\frac{\partial v}{\partial \xi}) = -3\frac{\partial^2 v}{\partial \xi^2} - \frac{\partial^2 v}{\partial \xi \partial \eta},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \xi^2}$$

and finally

$$\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = \frac{\partial^2 v}{\partial \eta^2} - 5\frac{\partial v}{\partial \xi} - \frac{\partial v}{\partial \eta}$$

which implies

$$\frac{\partial^2 v}{\partial \eta^2} = 5\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}.$$

Elliptic Case

Let (E_e) be an elliptic equation. We know that in this case $b^2 - 4ac < 0$ and consequently equation (**??**) admits two complex solutions. Thus, the characteristic curves are defined from the real and imaginary parts of the solutions.

Let's φ consider a solution to the equation

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{b^2 - 4ac}}{2a}.$$

The canonical form of (E_e) is written

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} = G(\xi, \eta, v, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta})$$

where $i = \sqrt{-1}$ and (ξ, η) are given by $\xi = Re\varphi, \eta = Im\varphi$.

Proof

It is assumed that $a(x,y) \neq 0$ for everything $(x,y) \in \Omega$. We are looking for two functions $\varphi_1(x,y)$ and $\varphi_2(x,y)$ such as $b'(\xi,\eta) = c'(\xi,\eta)$ and $b'(\xi,\eta) = 0$ for everything $(x,y) \in \Omega$. system of two nonlinear first-order equations

$$a(\frac{\partial \xi}{\partial x})^2 + b(\frac{\partial \xi}{\partial x})(\frac{\partial \xi}{\partial y}) + c(\frac{\partial \xi}{\partial y})^2 = a(\frac{\partial \eta}{\partial x})^2 + b(\frac{\partial \eta}{\partial x})(\frac{\partial \eta}{\partial y}) + c(\frac{\partial}{\partial y})^2,$$

$$b' = 2a(\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial x}) + b[(\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial y}) + (\frac{\partial \xi}{\partial y})(\frac{\partial \eta}{\partial x})] + 2c(\frac{\partial \xi}{\partial y})(\frac{\partial \eta}{\partial y}) = 0.$$

The system can be written in the form

$$a[(\frac{\partial \xi}{\partial x})^2 - (\frac{\partial \eta}{\partial x})^2] + b[(\frac{\partial \xi}{\partial x})(\frac{\partial \xi}{\partial y}) - (\frac{\partial \eta}{\partial x})(\frac{\partial \eta}{\partial y})] + c[(\frac{\partial \xi}{\partial y})^2 - (\frac{\partial \eta}{\partial y})^2] = 0$$

$$2ia(\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial x}) + ib[(\frac{\partial \xi}{\partial x})(\frac{\partial \eta}{\partial y}) + (\frac{\partial \xi}{\partial y})(\frac{\partial \eta}{\partial x})] + 2ic(\frac{\partial \xi}{\partial y})(\frac{\partial \eta}{\partial y}) = 0.$$

Let's define the complex function $\phi(x,y) = \xi + i\eta$.This system is equivalent to the complex equation

$$a(\frac{\partial \phi}{\partial x})^2 + b(\frac{\partial \phi}{\partial x})(\frac{\partial \phi}{\partial y}) + c(\frac{\partial \phi}{\partial y})^2 = 0.$$

We have arrived at the same equation as in the hyperbolic case. But in the elliptic case, the equation admits no real solution or, in other words, elliptic equations have no characteristics. As in the hyperbolic case, we factorize the quadratic PDE above and obtain two linear equations, but they are now complex differential equations (where x and y are complex variables). Thus, we need to solve the equations

$$a\frac{\partial \phi}{\partial x} + (b \pm i\sqrt{4ac - b^2})\frac{\partial \phi}{\partial y} = 0.$$

As before, the characteristic system associated with this equation is given by

$$\frac{dy}{dx} = \frac{b \pm i\sqrt{4ac - b^2}}{2a}.$$

By finding the solutions to these two equations $\phi_1(x,y)$ and $\phi_2(x,y)$. Now, if we ask $\xi(x,y) = \phi_1(x,y)$ and $\eta(x,y) = \phi_2(x,y)$, Equation (16) takes the form

$$\frac{\partial^2 v}{\partial \xi \partial \eta} = G(\xi, \eta, v, \frac{\partial v}{\partial \xi}, \frac{\partial v}{\partial \eta})$$

This form is not the canonical form of an elliptic equation with real coefficients. To obtain it, we set $\xi(x,y) = Re\phi_1, \eta(x,y) = Im\phi_2$, where ϕ is one of the functions ϕ_1 where ϕ_2.

□

Let's illustrate this with an example.

EXAMPLE 4.3. *Let us consider the Tricomi equation*[1]

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} = 0,\ x > 0.$$

We have $b^2(x,y) - 4a(x,y)c(x,y) = -4x = 4i^2x$. *As* $x > 0$, *this PDE is elliptic on the domain* D. *At these points, the characteristic equations are*

$$\frac{dy}{dx} = \pm i\sqrt{x}.$$

Using the method of separation of variables, we obtain $\frac{3}{2}y \pm ix^{\frac{3}{2}} = C$ *where* C *is a constant. We set*

$$\begin{cases} \xi = \frac{3}{2}y, \\ \eta = -x^{\frac{3}{2}}, \end{cases}$$

and $u(x,y) = v(\xi,\eta)$. *Using the chain rule, we obtain*

$$\frac{\partial u}{\partial x} = -\frac{3}{2}x^{\frac{1}{2}}\frac{\partial v}{\partial \eta},$$

$$\frac{\partial u}{\partial y} = \frac{3}{2}\frac{\partial v}{\partial \xi},$$

$$\frac{\partial u^2}{\partial x^2} = \frac{\partial}{\partial x}(-\frac{3}{2}x^{\frac{1}{2}}\frac{\partial v}{\partial \eta}) = \frac{9}{4}x\frac{\partial v^2}{\partial \eta^2} - \frac{3}{4}x^{\frac{1}{2}}\frac{\partial v}{\partial \eta},$$

$$\frac{\partial u^2}{\partial y^2} = \frac{\partial}{\partial y}(\frac{3}{2}\frac{\partial v}{\partial \xi}) = \frac{9}{4}\frac{\partial v^2}{\partial \xi^2}$$

and finally

$$\frac{\partial u^2}{\partial x^2} + x\frac{\partial u^2}{\partial y^2} = \frac{9}{4}\frac{\partial v^2}{\partial \eta^2} - \frac{3}{4}x^{\frac{1}{2}}\frac{\partial v}{\partial \eta} + \frac{9}{4}\frac{\partial v^2}{\partial \xi^2},$$

$$= \frac{9}{4}(\frac{\partial v^2}{\partial \eta^2} + \frac{1}{3}\eta\frac{\partial v}{\partial \eta} + \frac{\partial v^2}{\partial \xi^2}).$$

The canonical form of Tricomi's equation is

$$\frac{\partial v^2}{\partial \eta^2} + \frac{\partial v^2}{\partial \xi^2} = -\frac{1}{3}\eta\frac{\partial v}{\partial \eta}$$

[1]Francesco Giacomo Tricomi (Naples,5 May 1897 – Turin, 21 November 1978) was an Italian mathematician, known for his studies on second-order partial differential equations of mixed type, special functions, and orthogonal series.

5. Exercise series $N^\circ = 03$

EXERCICE 8. *Determine for which points (x, y) of the plane each of the following linear second-order PDEs is hyperbolic, parabolic and elliptic.*

$i) x\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} ? \frac{\partial u}{\partial x} = 0,$
$ii)\ x\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} - (x+3)\frac{\partial u}{\partial y} = u,$
$iii) e^x\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + 5y\frac{\partial u}{\partial x} = e^x,$
$iv)\ x^2\frac{\partial^2 u}{\partial x^2} + 2(x-y)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$
$v)\ \frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} - (x+y)\frac{\partial^2 u}{\partial y^2} + 4\frac{\partial u}{\partial x} ? x\frac{\partial u}{\partial y} = \sin(x).$

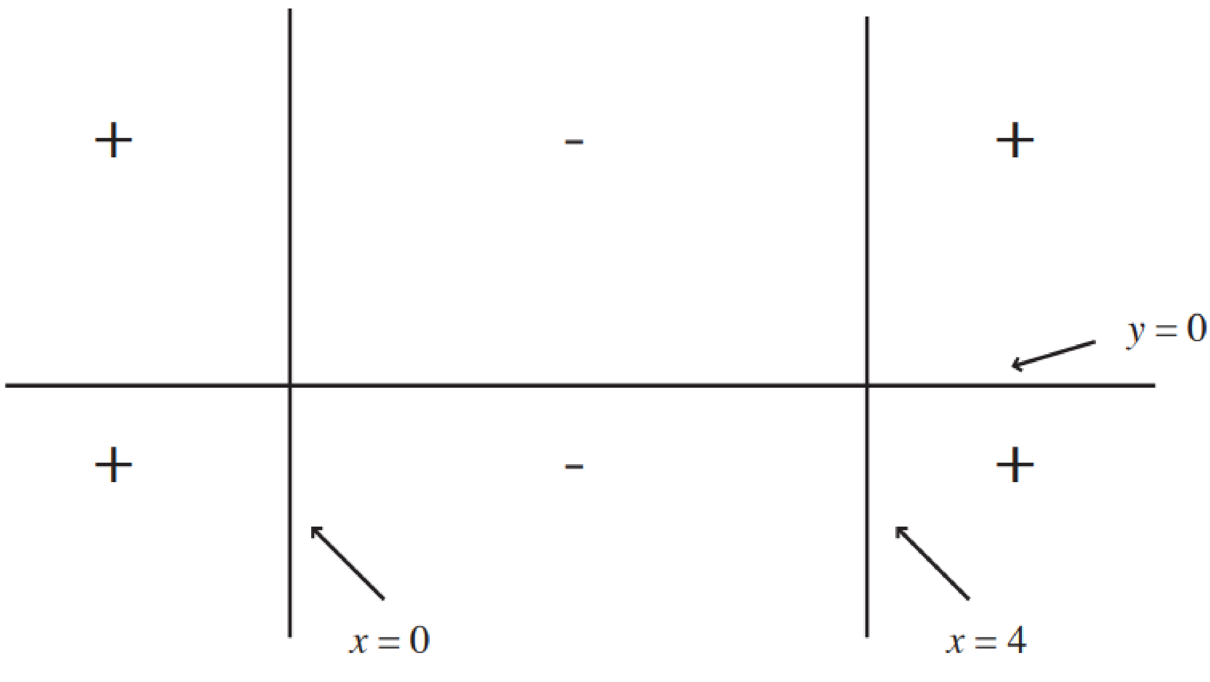
SOLUTION. *i) For the equation*

$$x\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} = 0,$$

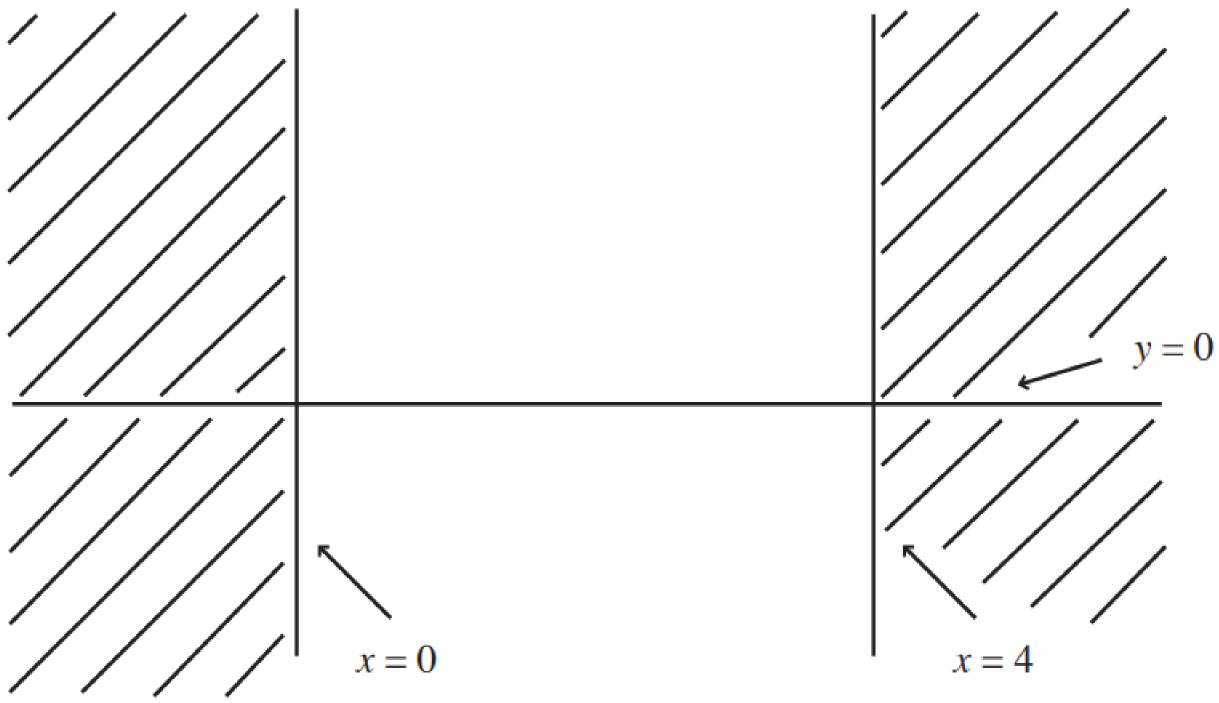
then $B^2 - 4AC = (xy)^2 - 4(x)(y^2) = x^2y^2 - 4xy^2 = xy^2(x4)$. *To determine the sign of* $B^2 - 4AC$, *First, we need to determine the points.* $(x, y) \in \mathbb{R}^2$ *such as* $B^2 - 4AC = xy^2(x-4) = 0$. *We let us obtain so that* $x = 0$, *either* $y = 0$ *or* $x = 4$
For each of the regions of

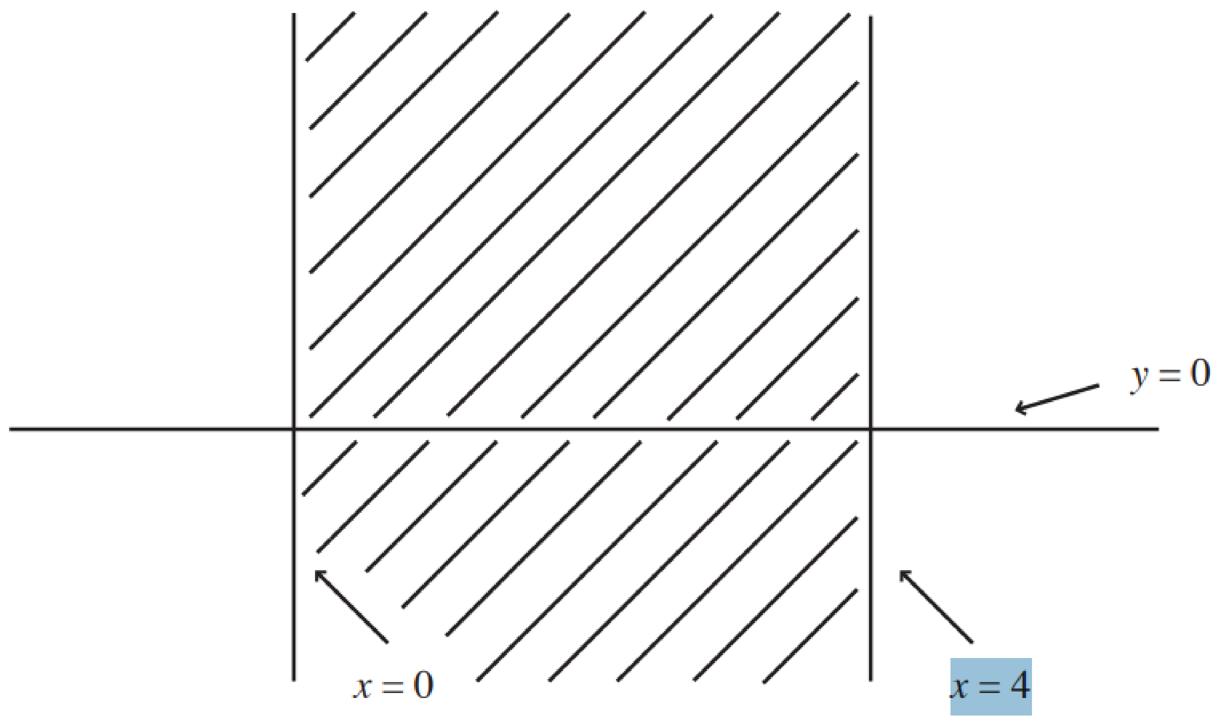
$$\mathbb{R}^2 \backslash (\{(x, y) \in \mathbb{R}^2 | x = 0\} \cup \{(x, y) \in \mathbb{R}^2 | y = 0\} \cup \{(x, y) \in \mathbb{R}^2 | x = 4\}),$$

we can determine the signs of $B^2 - 4AC$,. *We have listed these signs below.*



The equation is hyperbolic on the shaded region below.





The equation is elliptic on the shaded region below.

The equation is parabolic on the two vertical lines: $x = 0$ and $x = 4$, as well as on the horizontal line $y = 0$.

ii) For the equation

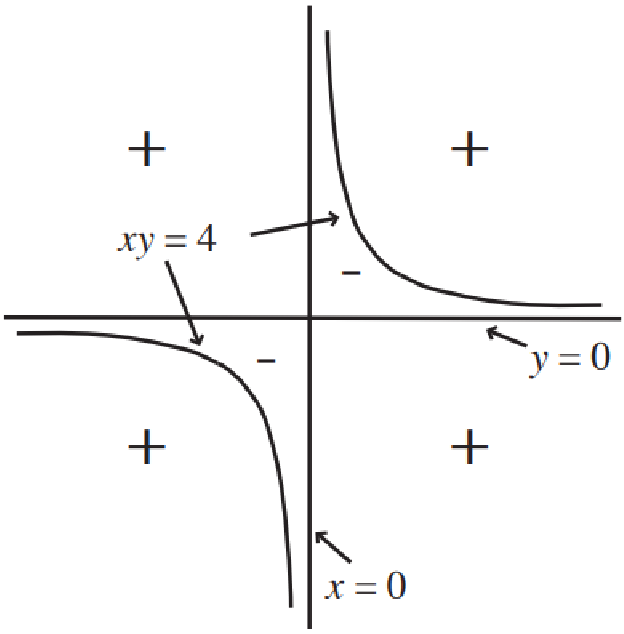
$$x\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial^2 u}{\partial y^2} - (x+3)\frac{\partial u}{\partial y} = u,$$

then $B^2 - 4AC = (xy)^2 - 4(x)(y) = xy(xy-4)$. To determine the sign of $B^2 - 4AC$, First, we need to determine the points $(x, y) \in \mathbb{R}^2$ such as$B^2 - 4AC = xy(xy-4) = 0$. We thus obtain whether $x = 0$, either $y = 0$ or $xy = 4$. The points $(x, y) \in \mathbb{R}^2$ such as $xy = 4$ is a hyperbola whose asymptotes are the axes of the x and y.

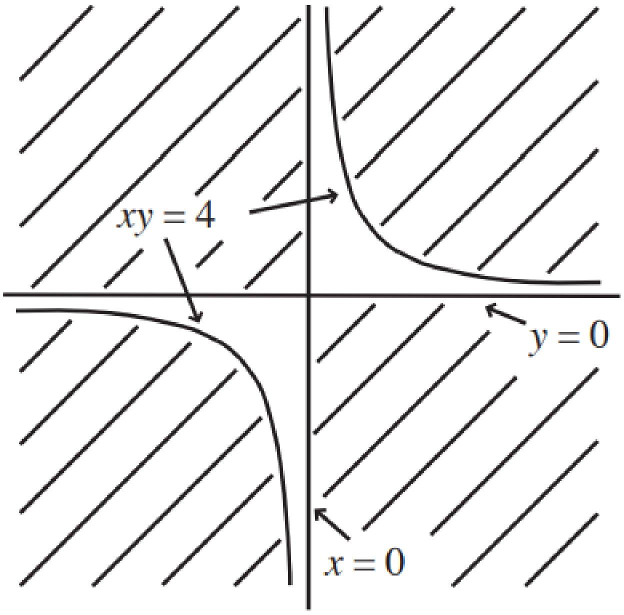
For each of the regions of

$$\mathbb{R}^2 \backslash (\{(x,y) \in \mathbb{R}^2 | x = 0\} \cup \{(x,y) \in \mathbb{R}^2 | y = 0\} \cup \{(x,y) \in \mathbb{R}^2 | xy = 4\}),$$

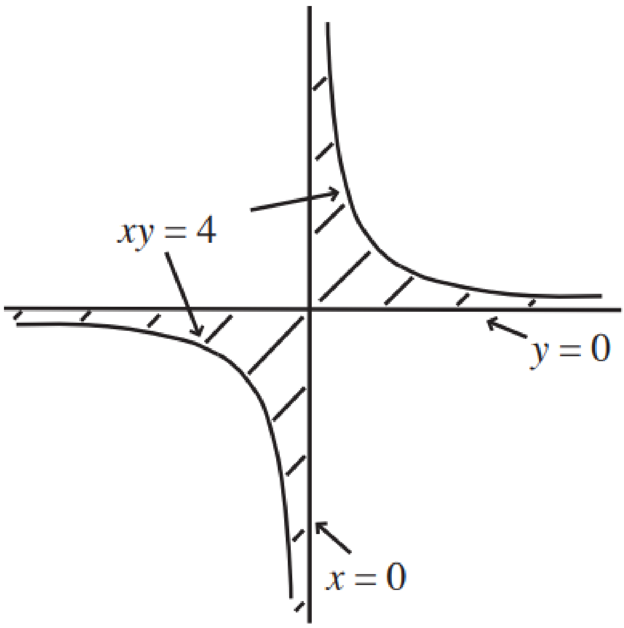
we can determine the signs of $B^2 - 4AC$. We have listed these signs below



The equation is hyperbolic on the shaded region below.



The equation is elliptic on the shaded region below.



The equation is parabolic on the vertical line.: $x = 0$, the horizontal line $y = 0$, as well as on hyperbole $xy = 4$.

iii) For the equation

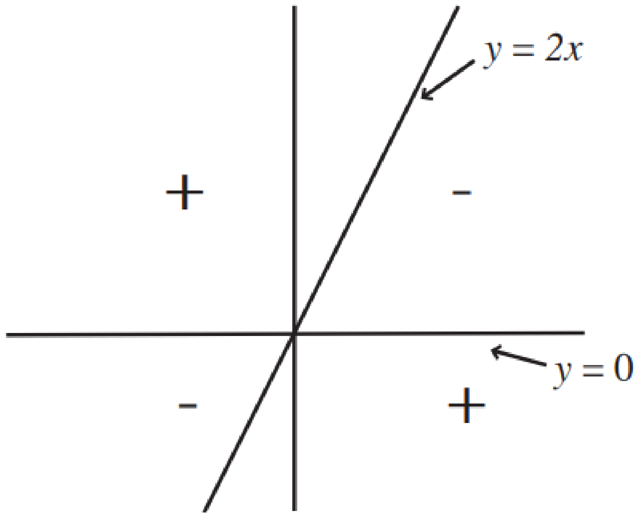
$$e^x \frac{\partial^2 u}{\partial x^2} + x \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} + 5y \frac{\partial u}{\partial x} = e^x,$$

then $B^2 - 4AC = x^2 - 4 \exp x(-1) = x^2 + 4 \exp x > 0$ for everything $(x, y) \in \mathbb{R}^2$ because $x^2 \geq 0$ and $\exp x > 0$ for everything $x \in \mathbb{R}$. So the equation is hyperbolic over the entire plane $\mathbb{R}^2$.

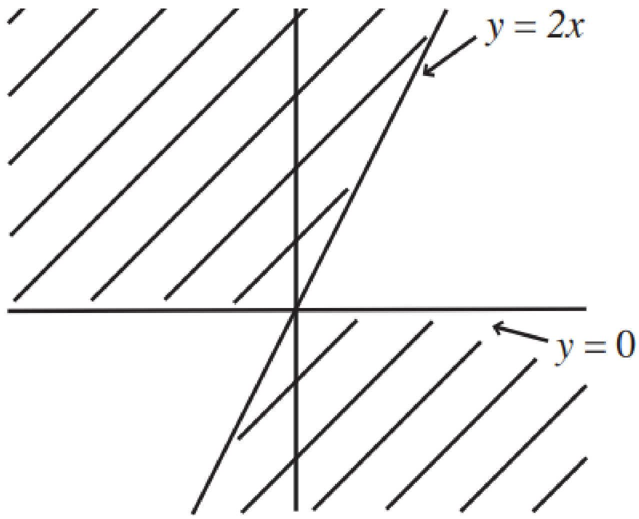
iv) For the equation

$$x^2\frac{\partial^2 u}{\partial x^2} + 2(x-y)\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

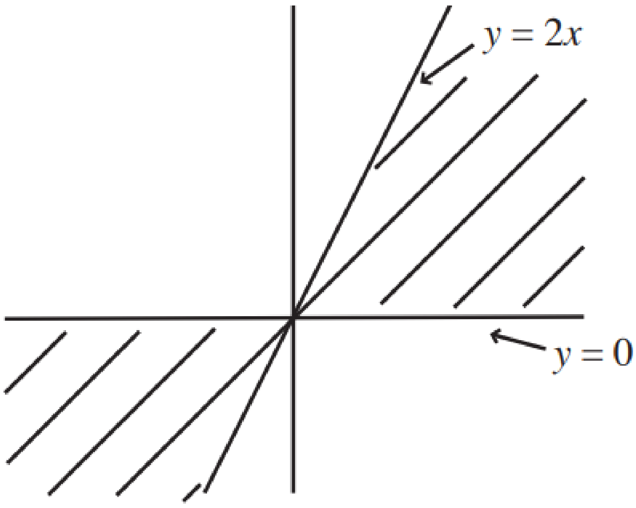
then $B^2 - 4AC = (2(x-y))2 - 4(x^2)(1) = -8xy + 4y^2 = 4y(-2x+y)$. To determine the sign of $B^2 - 4AC$, First, we need to determine the points. $(x, y) \in \mathbb{R}^2$ such as $B^2 - 4AC = 4y(-2x+y) = 0$. We thus obtain that either $y = 0$, or $(-2x+y) = 0$. For each of the regions of $\mathbb{R}^2 \backslash (\{(x,y) \in \mathbb{R}^2 | y = 0\} \cup (x,y) \in \mathbb{R}^2 | -2x+y = 0)$, we can determine the signs of $B^2 - 4AC$. We have listed these signs below.



The equation is hyperbolic on the shaded region below.



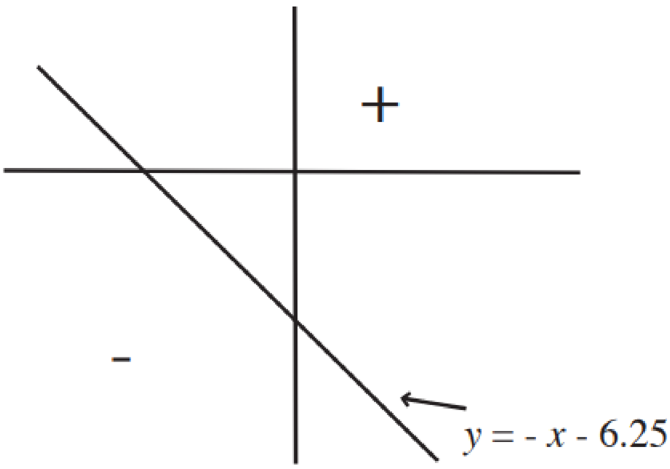
The equation is elliptic on the shaded region below.



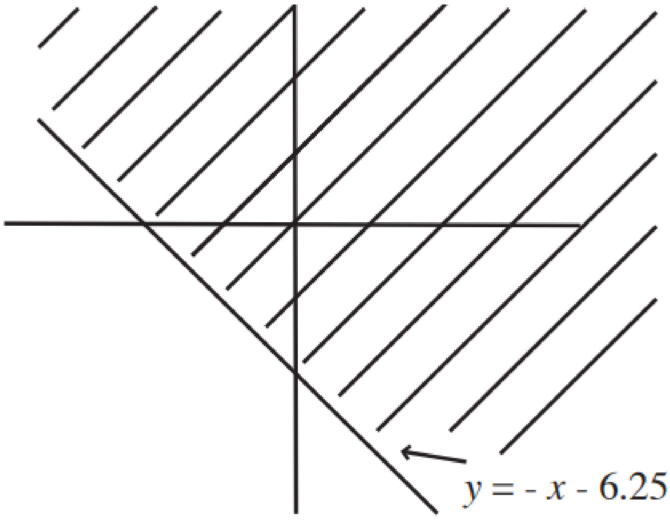
v) For the equation

$$\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} - (x+y)\frac{\partial^2 u}{\partial y^2} + 4\frac{\partial u}{\partial x}?x\frac{\partial u}{\partial y} = \sin(x).$$

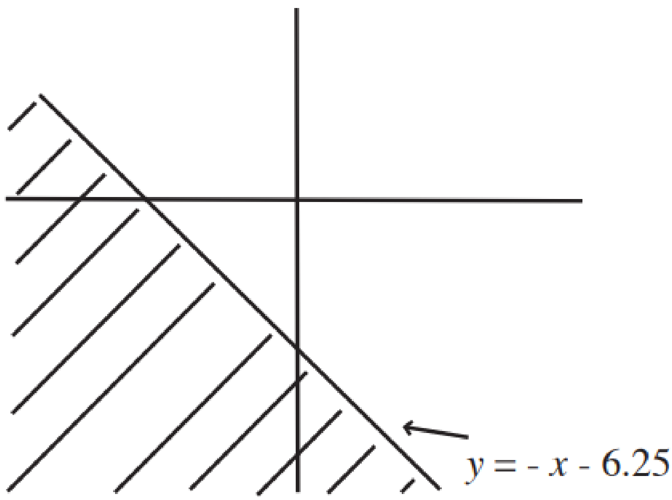
then $B^2 - 4AC = (-5)^2 - 4(1)(-(x+y)) = 25 + 4(x+y)$. To determine the sign of $B^2 - 4AC$, First, we need to determine the points. $(x, y) \in \mathbb{R}^2$ such as $B^2 - 4AC = 4x + 4y + 25 = 0$. This gives us the equation of the line $y = -x - 6.25$. For each of the regions of $\mathbb{R}^2 \backslash \{(x, y) \in \mathbb{R}^2 | 4x + 4y + 25 = 0\}$, we can determine the signs of $B^2 - 4AC$. We have listed these signs below.



The equation is hyperbolic on the shaded region below.



The equation is elliptic on the shaded region below.



The equation is parabolic on the line. $y = -x - \frac{25}{4} = -x - 6.25$.

EXERCICE 9. *For each of the following linear second-order PDEs a) determine the points of the plane x, y where these equations are hyperbolic;*
b) determine the characteristic coordinates of these equations on the domain where they are hyperbolic;
c) perform the change of coordinates for those found in b) so as to obtain the corresponding canonical equation.

i) $2y^2\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x\partial y} - x^2\frac{\partial^2 u}{\partial y^2} + 4y\frac{\partial u}{\partial x} - 3u = 0,$
ii)$x^2\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x\partial y} - 6y^2\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0.$

SOLUTION. *i) For the equation*

$$2y^2\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x\partial y} - x^2\frac{\partial^2 u}{\partial y^2} + 4y\frac{\partial u}{\partial x} - 3u = 0,$$

we have $B^2 - 4AC = (-xy)^2 - 4(2y^2)(-x^2) = 9x^2y^2$. *Thus* $B^2 - 4AC = 0$ *if and only if* $x = 0$ *or* $y = 0$. *Moreover, since* $9x^2y^2 \geq 0$, *then* $B^2 - 4AC > 0$ *if and only if* $x \neq 0$ *and* $y \neq 0$. *Therefore the equation is hyperbolic at the points* (x, y) *where* $x \neq 0$ *and* $y \neq 0$. *In other words, the equation is hyperbolic for all points that are not on the* x*- and* y*-axes. The characteristic equations are*

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{(-xy + \sqrt{9x^2y^2})}{(2y^2)} = \frac{x}{2y}$$

and

$$\frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{(-xy - \sqrt{9x^2y^2})}{(2y^2)} = -\frac{x}{y}.$$

We can solve these two ordinary differential equations using the method of separation of variables.

$$\frac{dy}{dx} = \frac{x}{2y} \Rightarrow 2ydy = xdx \Rightarrow \int 2ydy = \int xdx \Rightarrow y^2 = \frac{x^2}{2} + c.$$

Let us consider the characteristic coordinate $\xi(x, y) = y^2 - \frac{x^2}{2}$.

$$\frac{dy}{dx} = -\frac{x}{y} \Rightarrow ydy = -xdx \Rightarrow \int ydy = -\int xdx \Rightarrow \frac{y^2}{2} = -\frac{x^2}{2} + c.$$

We can therefore consider the characteristic coordinate $\eta(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$. *The characteristic coordinates are* $\xi(x, y) = y^2 - \frac{x^2}{2}$ *and* $\eta(x, y) = \frac{x^2}{2} + \frac{y^2}{2}$.. *Now we perform the variable change. By the chain rule, we have*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = -x\frac{\partial u}{\partial \xi} + x\frac{\partial u}{\partial \eta}, \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = 2y\frac{\partial u}{\partial \xi} + y\frac{\partial u}{\partial \eta}.$$

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= -\frac{\partial u}{\partial \xi} + (-x)(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi\partial \eta}\frac{\partial \eta}{\partial x}) + \frac{\partial u}{\partial \eta} + x(\frac{\partial^2 u}{\partial \xi\partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x})\\ &= x^2\frac{\partial^2 u}{\partial \xi^2} + (-2x^2)\frac{\partial^2 u}{\partial \xi\eta} + x^2\frac{\partial^2 u}{\partial \eta^2} - \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\end{aligned}$$

and we have

$$\frac{\partial^2 u}{\partial x\partial y} = -x(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta\partial \xi}\frac{\partial \eta}{\partial y}) = -2xy\frac{\partial^2 u}{\partial \xi^2} + xy\frac{\partial^2 u}{\partial \xi\partial \eta} + xy\frac{\partial^2 u}{\partial \eta^2}$$

$$\frac{\partial^2 u}{\partial y^2} = 2\frac{\partial u}{\partial \xi} + 2y(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi\partial \eta}\frac{\partial \eta}{\partial y}) + \frac{\partial u}{\partial \eta} + y(\frac{\partial^2 u}{\partial \xi\partial \eta}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial y})$$

so, we have

$$\frac{\partial^2 u}{\partial y^2} = 4y^2\frac{\partial^2 u}{\partial \xi^2} + 4y^2\frac{\partial^2 u}{\partial \xi\partial \eta} + y^2\frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}.$$

It should be noted that $y^2 = \frac{2(\xi+\eta)}{3}$ *and* $x^2 = \frac{-2\xi+4\eta}{3}$. *Substituting this into the equation, we obtain*

$$-9x^2y^2\frac{\partial^2 u}{\partial\xi\partial\eta} + (-2y^2 - 2x^2 - 4xy)\frac{\partial u}{\partial\xi} + (2y^2 - x^2 + 4xy)\frac{\partial u}{\partial\eta} - 3u = 0 \Rightarrow$$

$$\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{2(x+y)^2}{9x^2y^2}\frac{\partial u}{\partial\xi} - \frac{(2y^2 - x^2 + 4xy)}{9x^2y^2}\frac{\partial u}{\partial\eta} + \frac{1}{3x^2y^2}u = 0$$

Now, we will write the canonical equation for the first quadrant of the plane, namely the points (x, y) *such as* $x > 0$ *and* $y > 0$. *So*

$$x = \sqrt{\frac{(-2\xi+4\eta)}{3}} \text{ et } y = \sqrt{\frac{2(\xi+\eta)}{3}}.$$

For the other quadrants, simply adjust the signs in front of the radicals. Finally

$$\frac{\partial^2 u}{\partial\xi\partial\eta} + \frac{(\frac{-2\xi+4\eta}{3} + \frac{2(\xi+\eta)}{3})^2}{(\xi+\eta)(-2\xi+4\eta)}\frac{\partial u}{\partial\xi} - \frac{2\xi + 4\sqrt{\frac{2(\xi+\eta)(-2\xi+4\eta)}{9}}}{2(\xi+\eta)(-2\xi+4\eta)}\frac{\partial u}{\partial\eta} + \frac{3}{2(-2\xi+4\eta)(\xi+\eta)} = 0.$$

If we had used the coordinates $\alpha = \xi + \eta = \frac{3y^2}{2}3y2/2$ *and* $\beta = \xi - \eta = (\frac{y^2}{2}) - x^2$, *then we obtain the second form of the canonical equation. In this case, we would have*

$$\frac{\partial^2 u}{\partial\alpha^2} - \frac{\partial^2 u}{\partial\beta^2} + \frac{1}{2\alpha}\frac{\partial u}{\partial\alpha} + \frac{(\sqrt{\alpha-3\beta} + \sqrt{2\alpha}}{6\alpha\sqrt{\alpha-3\beta}}\frac{\partial u}{\partial\beta} + \frac{3}{2\alpha(\alpha-3\beta)}u = 0.$$

ii) For the equation

$$x^2\frac{\partial^2 u}{\partial x^2} - xy\frac{\partial^2 u}{\partial x\partial y} - 6y^2\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0$$

we have $B^2 - 4AC = (-xy)^2 - 4(x^2)(-6y^2) = 25x^2y^2$. *Thus* $B^2 - 4AC = 0$ *if and only if* $x = 0$ *or* $y = 0$. *Moreover, since* $25x^2y^2 \geq 0$, *then* $B^2 - 4AC > 0$ *if and only if* $x \neq 0$ *and* $y \neq 0$. *Therefore, the equation is hyperbolic at the points* (x, y) *where* $x \neq 0$ *and* $y \neq 0$. *On the other hand, the equation is hyperbolic for all points that are not on the* x- *and* y-*axes. The characteristic equations are*

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{2A} = \frac{(-xy) + \sqrt{25x^2y^2}}{2x^2} = \frac{2y}{x}, \frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{2A} = \frac{(-xy) - \sqrt{25x^2y^2}}{2x^2} = \frac{-3y}{x}$$

So, we can solve these characteristic equations using the method of separation of variables.

$$\frac{dy}{dx} = \frac{2y}{x} \Rightarrow \frac{dy}{2dx} = \frac{y}{x} \Rightarrow \int \frac{dy}{2dx} = \int \frac{y}{x} \Rightarrow \frac{1}{2}ln(y) = ln(x) + cte.$$

$$y = x^2c \Rightarrow x^{-2}y = c \text{ où } c \text{ est une constante.}$$

We consider the characteristic coordinate $\xi(x, y) = x^{-2}y$.

$$\frac{dy}{dx} = \frac{-3y}{x} \Rightarrow \frac{dy}{-3dx} = \frac{y}{x} \Rightarrow \int \frac{dy}{-3dx} = \int \frac{y}{x} \Rightarrow \frac{1}{-3}ln(y) = ln(x) + cte.$$

$$y = x^{-3}c \Rightarrow x^3y = c' \text{ où } c' \text{ est une constante.}$$

So, we consider the characteristic coordinate $\eta(x,y) = x^3y$. *The characteristic coordinates are* $\xi(x,y) = x^{-2}y$. *and* $\eta(x,y) = x^3y$. *By the chain rule, we obtain*

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = -2x^{-3}y\frac{\partial u}{\partial \xi} + 3x^2y\frac{\partial u}{\partial \eta}, \quad \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = x^{-2}\frac{\partial u}{\partial \xi} + x^3\frac{\partial u}{\partial \eta}$$

CHAPTER 4

Method of Separation of Variables

The method of separation of variables, or the Fourier method, is widely used to solve boundary value problems related to PDEs. It consists of seeking particular solutions of the separable form $u(x,y) = X(x)Y(y)$, where X and Y are functions of x and y, respectively. In many cases, the PDE is reduced to two ordinary differential equations for X and Y. Thus, we obtain boundary value problems involving ODEs. However, the question of whether a PDE can be separated into two or more ordinary differential equations is not always possible. In this chapter, we will apply this method to boundary value problems related to linear PDEs.

1. Problem Situation

We now describe the method of separation of variables and examine the conditions of applicability of the method to problems that involve second-order PDEs with two independent variables.

Definition 1.1:

We therefore consider the boundary value problem whose unknown is $u(x,t)$, defined on a domain of the form $I \times J$, where I and J are intervals of $\mathbb{R}$ such that $I = [a,b]$, $-\infty < a < b < +\infty$.

$$\begin{cases} a_1(x)\frac{\partial^2 u}{\partial x^2} + b_1(t)\frac{\partial^2 u}{\partial t^2} + a_2(x)\frac{\partial u}{\partial x} + b_2(t)\frac{\partial u}{\partial t} + (a_3(x) + b_3(t))\, u(x,t) = h(x,t), \quad (x,t) \in I \times J \\ u(x,0) = f(x), \quad x \in I : \textit{Initial condition,} \\ \textit{Boundary condition} \end{cases}$$

where $a_1, b_1, a_2, b_2, a_3, b_3, h(x,t), f(x)$ are given functions.

1.1. Boundary condition: We distinguish different types of boundary conditions (B.C)

⋆ **Dirichlet conditions:** u is fixed on the boundary of I, we have

$$u(a,t) = u(b,t) = 0.$$

⋆ **Neumann conditions:** The normal derivative of u is fixed on I

$$u_x(a,t) = u_x(b,t) = 0.$$

⋆ **Robin or mixed conditions**

$$c_1(x)u(a,t) + c_2(x)u_x(b,t) = 0.$$

⋆ **Periodic conditions**

$u(a,t) = u(b,t)$ and $u_x(a,t) = u_x(b,t)$.

Let us now outline the main steps of this method.

Remark 1.1:

1. We look for separated solutions of (E). These solutions have the special form

$$u(x,t) = X(x)T(t)$$

and satisfy the boundary conditions and the initial condition. It turns out that X and T must be solutions of the linear boundary value problems (P1) and (P2) related to X and T, respectively.

2. We solve the problems (P_1) and (P_2). The solutions of (P_1) allow us to construct a Hilbert basis $(X_i)_{i\in\mathbb{N}}$. We denote by $(T_i)_{i\in\mathbb{N}}$ the solutions of (P_2).
3. We use the principle of general superposition to generate, from $(X_i)_{i\in\mathbb{N}}$ and $(T_i)_{i\in\mathbb{N}}$, a more general solution of the problem in the form of an infinite series of separated solutions.

4. In the final step, we calculate the coefficients of this series and study its convergence.
Principle of Superposition
If u_1 and u_2 satisfy a homogeneous linear PDE, then an arbitrary linear combination $c_1u_1 + c_2u_2, c_1, c_2 \in \mathbb{R}$ also satisfies the same equation.

Next, we will review some essential concepts about Fourier series that will be used in the rest of this chapter.

2. Rappel sur les séries de Fourier

In this section, we will present the most important results concerning the convergence of these series. To continue, it is necessary to study the convergence of Fourier series in order to solve certain PDE problems. We want to answer the following questions:

(1) Does the Fourier series of $f(x)$ converge?

If so, to what value does it converge?

(2) Can we differentiate the Fourier series of $f(x)$ term by term?

If so, is the resulting Fourier series the series of $f'(x)$, the derivative of $f(x)$? (assuming $f(x)$

is differentiable.) The theory began with the works of Joseph Fourier[1] on heat. Lejeune-Dirichlet[2] studied the convergence of Fourier series rigorously in order to justify Fourier's results.

Definition 2.1:

A sequence of functions $\{\phi_n(x)\}_n, n \geq 0$ defined on an interval $[a, b]$, describes an orthonormal system with respect to the weight function $q(x)$ if and only if

$$\forall n \geq 0, \forall m \geq 0, \int_a^b \phi_n(x)\phi_m(x)q(x)dx = \begin{cases} 1 \ si \ m \neq n \\ 0 \ si \ m = n \end{cases}$$

EXAMPLE 2.1. *We will see that the cos and sin functions can form an orthonormal basis. To do this, the set of functionss* $\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin 2(x)}{\sqrt{\pi}}, ..., \frac{\cos(nx)}{\sqrt{\pi}}, \frac{\sin(nx)}{\sqrt{\pi}} ...\}$ *est un système orthogonal functions on the interval* $[-\pi, \pi]$ *with respect to the weight function* $q \equiv 1$. *This is confirmed by the following calculations*

If $m, n \geq 1$, *on a*

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \int_{-\pi}^{\pi} \frac{\cos((m-n)x) - \cos((m+n)x)}{2} = \begin{cases} 1 \ si \ m \neq n \\ 0 \ si \ m = n \end{cases}$$

If $m, n \geq 0$, *we have*

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \int_{-\pi}^{\pi} \frac{\cos((m-n)x) + \cos((m+n)x)}{2} = \begin{cases} 0 \ si \ m \neq n \\ \pi \ si \ m = n \\ 2\pi \ si \ m = n = 0 \end{cases}$$

If $m \geq 0$ *and* $n \geq 1$, *we have*

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \int_{-\pi}^{\pi} \frac{\cos((m-n)x) - \cos((m+n)x)}{2} = 0.$$

Definition 2.2:

A function $f(x)$ is said to be piecewise smooth on the interval $[a, b]$ if and only if the interval can be subdivided into m subintervals $[x_i, x_{i+1}]$, where $i = 0, 1, 2, \ldots, (m-1)$, with $a = x_0 < x_1 < x_2 < \cdots < x_m = b$ such that the function $f(x)$ and its derivative $f'(x)$ are continuous on each open subinterval $]x_i, x_{i+1}[$ and the left-hand (respectively right-hand) limits of $f(x)$ and $f'(x)$ at the points x_i for $i = 1, 2, \ldots, m$ (respectively for $i = 0, 1, 2, \ldots, (m-1)$) exist.

[1] French mathematician and physicist, Auxerre, 1768 - Paris, 1830

[2] German mathematician, Dùren, Prussian Rhine, 1805 - Gottingen, 1850

Example 2.2. *Consider the function* $f(x) = \sqrt[3]{x}$ *on the interval* $[-1, 1]$. *Then* $f'(x) = (\frac{1}{3})x^{-\frac{3}{2}}$ *if* $x \neq 0$ *and does not exist at* $x = 0$. *Since the left-hand and right-hand limits of* $f'(x) = (\frac{1}{3})x^{-\frac{3}{2}}$ *do not exist at* $x = 0$, $f(x)$ *is not piecewise smooth on* $[-1, 1]$.

Definition 2.3:

Let $f(x)$ be a function defined on the interval $[-\pi, \pi]$. Its periodic extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$ with period 2π is the function defined by

$$\tilde{f}(x) = \begin{cases} f(x), & si \ -\pi < x < \pi \ et \ k \in \mathbb{Z}; \\ \frac{f(\pi)+f(-\pi)}{2}, & x = -\pi, \pi \ et \ k \in \mathbb{Z}. \end{cases}$$

Example 2.3. *We will plot in Figure [1] the graph of* $\tilde{f}(x)$ *in the case of the function*

$$f(x) = \begin{cases} 0, & si \ -\pi \leq x \leq 0; \\ x, & si \ 0 < x \leq \pi. \end{cases}$$

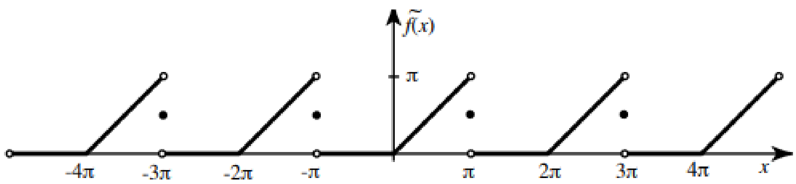


Figure [1]

Definition 2.4:

Let f be a function and D_f its domain. The function f is said to be periodic if there exists a nonzero real number p satisfying the following property: If $x \in D_f$ then $x + p \in D_f$ and $f(x + p) = f(x)$. The number p is called the period of the function f. Fourier series and Fourier coefficients.

3. Fourier Series and Fourier Coefficients

The purpose of this section is to express a piecewise continuous and 2π-periodic function $f(x)$ in the following form. Let $f(x)$ be a function defined and integrable on the interval $[-\pi, \pi]$ and its Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx)),$$

where a_n, b_n for $n \geq 0$ are called the Fourier coefficients associated with the function f.

Definition 3.1:

The Fourier coefficients associated with the function f are given by

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx\ si\ n \geq 1.$

EXAMPLE 3.1. *Either* $f(x) = x + x^2$ *over the interval* $[-\pi, \pi]$, *then*

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2\pi^2}{3}.$$

If $n \geq 1$, *then*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos(nx)dx = \frac{1}{n\pi}(x+x^2)\sin(nx)]_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi}(1+2x)\sin(nx)dx$$

$$= -\frac{1}{n^2\pi}\{(-(1+2x)\cos(nx)]_{-\pi}^{\pi} + \frac{1}{n}(2\sin(nx)]_{-\pi}^{\pi}\} = \frac{4\cos(n)}{n^2} = \frac{4(-1)^n}{n^2}$$

If

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin(nx)dx = \frac{2(-1)^{n+1}}{2}$$

So the Fourier series of $f(x)$ *is*

$$\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}(\frac{4(-1)^n}{n^2})\cos(nx) + (\frac{2(-1)^{n+1}}{2})\sin(nx)).$$

An important note to facilitate calculations.

Remark 3.1:

1. If f is an even function, then its Fourier series is

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos(nx)\ avec\ a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx\ pour\ n \geq 0.$$

2. If $f(x)$ is an odd function, then its Fourier series is

$$\sum_{n=1}^{\infty}(b_n\sin(nx)\ avec\ b_n = \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx\ pour\ n \geq 0.$$

4. Applications

4.1. Heat Equation. We consider the problem on the interval $[0, L]$ with $L > 0$, consisting of the heat equation

(21) $$\frac{\partial u}{\partial t} - k^2\frac{\partial^2 u}{\partial x^2} = 0,\ 0 < x < L,\ t > 0,$$

Dirichlet boundary conditions

(22) $$u(0,t) = u(L,t) = 0,\ t \geq 0,$$

and the initial condition

$$u(x,0) = f(x), 0 \leq x \leq L, \tag{23}$$

where f is a given function and k is a positive constant. This problem models heat conduction in a rod of length L. The temperature is assumed to be zero at both ends of the rod and equal to $f(x)$ at time $t = 0$. The boundary conditions imply that

$$f(0) = u(0,0) = u(L,t) = 0 \ et \ f(L) = u(L,0) = u(L,0) = 0.$$

These two conditions are called the compatibility conditions.

Step 1: We begin by looking for solutions of (21) in the form

$$u(x,t) = X(x)T(t), \tag{24}$$

which satisfy the conditions (22) where X and T are functions of x and t, respectively. In this step, we do not take into account the initial condition (23) and we are not interested in the zero solution $u(x,t) = 0$. We therefore look for functions X and T that do not vanish identically. By differentiating (24) with respect to t and twice with respect to x and substituting into (21), we obtain

$$XT'(t) = kX''(x)T(t), \ 0 < x < L, \ t > 0.$$

We can rewrite

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)}, \ 0 < x < L, \ t > 0, \tag{25}$$

Since x and t are independent variables, this relation implies that there exists a constant λ (called the separation constant) such that

$$\frac{T'(t)}{T(t)} = k\frac{X''(x)}{X(x)} = -\lambda, \ 0 < x < L, \ t > 0, \tag{26}$$

Since we are looking for solutions that do not vanish identically, there exists $t_0 \in \mathbf{R}$ such that $T(t) \neq 0$. Consequently, we obtain

$$\begin{cases} u(0,t_0) = X(0)T(t_0) = 0 \\ u(L,t_0) = X(L)T(t_0) = 0 \end{cases} \Rightarrow X(0) = X(L) = 0$$

Equation (**??**) leads to the following system of ODEs

$$\begin{cases} \frac{d^2X}{dx^2} + = 0, \ 0 < x < L, \\ X(0) = X(L) = 0 \end{cases}$$

and

$$\frac{dT}{dt} + \lambda T = 0, \ t > 0, \tag{27}$$

where λ is a constant.

Step 2: We first begin by solving the system (4.1). A nontrivial solution of (4.1) is called an eigenfunction with eigenvalue λ. We distinguish 3 cases:

Case 1: $\lambda = -\mu < 0$, then

$$X(x) = \alpha e^{-\mu x} + \beta e^{\mu x}$$

where α, β are arbitrary real numbers. The boundary conditions give

$$\begin{cases} \alpha + \beta = 0, \\ \alpha e^{-\mu L} + \beta e^{\mu L} = 0. \end{cases}$$

From the first equation, we have $\alpha = -\beta$. The second equation then implies $\alpha e^{-\mu L} = \alpha e^{\mu L}$, so if $\alpha \neq 0$ we get $e^{2\mu L} = 1$. This is not possible since μ and L are nonzero, and consequently $\alpha = \beta = 0$. Thus, in this case $X \equiv 0$ and $u(x,t) = 0$ for all $0 \leq x \leq L$ and $t \geq 0$. Therefore, we must exclude the case $\lambda < 0$.

Case 2: If $\lambda = 0$, we obtain

$$X(x) = \alpha + \beta x,$$

where α, β are arbitrary real numbers. The boundary conditions imply

$$\begin{cases} \alpha + \beta = 0, \\ \alpha + \beta L = 0. \end{cases}$$

Since $L \neq 0$, it is clear that $\alpha = \beta = 0$. Thus, in this case $X \equiv 0$ and $u(x,t) = 0$ for all $0 \leq x \leq L$ and $t \geq 0$. Therefore, we must exclude the case $\lambda = 0$.

Case 3: If $\lambda = \mu^2 > 0$, we obtain

$$X(x) = \alpha \cos(\mu x) + \beta \sin(\mu x)$$

where α, β are arbitrary real numbers. The boundary conditions imply that

$$\begin{cases} \alpha = 0, \\ \beta \sin(\mu L) = 0. \end{cases}$$

To avoid the trivial solution $X \equiv 0$, we assume that $\beta \neq 0$. This implies that $\sin(\mu L) = 0$. Consequently,

$$\mu L = n\pi,\ \lambda = \left(\frac{n\pi}{L}\right)^2,\ n \in \mathbb{Z}.$$

It follows that

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

are the eigenvalues and the functions $X_n(x) = \beta_n \sin\left(\frac{n\pi x}{L}\right)$ are the eigenfunctions of the problem (4.1). Since $\sin(-x) = -\sin(x)$ for all $x \in \mathbb{R}$, it suffices to consider

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad X_n(x) = \beta_n \sin\left(\frac{n\pi x}{L}\right),\ n \in \mathbb{N}^\star.$$

Now, it remains to solve the problem (27), whose solution is given by

$$T(t) = \gamma_n e^{-k(\frac{n\pi}{L})2t},\ n \in \mathbb{N}^\star.$$

At the end of this step, we can consider that we have successfully constructed a Hilbert basis $(X_i)_{i\in\mathbb{N}^\star}$.

Step 3: We use the general superposition principle to generate, from $(X_i)_{i\in\mathbb{N}^\star}$ and $(T_i)_{i\in\mathbb{N}^\star}$, a more general solution of the problem, in the form of an infinite series of separated solutions. We thus obtain the following sequence of separated solutions

$$u_n(x,t) = \gamma_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t},\ n \in \mathbb{N}^\star.$$

By the superposition principle, any linear combination

$$u(x,t) = \sum_{n=1}^{N} \gamma_n \sin(\frac{n\pi x}{L}) e^{-k(\frac{n\pi}{L})^2 t}$$

of separated solutions is also a solution of the heat equation that satisfies the Dirichlet boundary conditions. Now consider the initial condition. Suppose it has the form

$$f(x) = \sum_{n=1}^{N} \gamma_n \sin\left(\frac{n\pi x}{L}\right),$$

that is, it is a linear combination of the eigenfunctions. Then, a solution to the heat problem (21) - (23) is given by

$$u(x,t) = \sum_{n=1}^{N} \gamma_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}.$$

Fourier's brilliant idea was that it is possible to represent an arbitrary function f satisfying the boundary conditions (22) as a unique infinite linear combination of

eigenfunctions $\sin\left(\frac{n\pi x}{L}\right)$. In other words, it is possible to find constants γ_n such that

$$f(x) = \sum_{n=1}^{N} \gamma_n \sin\left(\frac{n\pi x}{L}\right).$$

Such a series is called a (generalized) Fourier series (or extension) of the function f with respect to the eigenfunctions of the problem, and $\gamma_n, n = 1, 2, \ldots$ are called the (generalized) Fourier coefficients of the series. In this case, the generalized superposition principle implies that the formal expression

$$u(x,t) = \sum_{n=1}^{N} \gamma_n \sin(\frac{n\pi x}{L}) e^{-k(\frac{n\pi}{L})^2 t}$$

is a natural candidate for a generalized solution of the problem (21) - (23). We now explain how to represent an **arbitrary** function f in the form of a Fourier series. In other words, how to calculate the coefficients γ_n. Notice

$$\int_0^L \sin(\frac{m\pi x}{L}) f(x)dx = \int_0^L \sin(\frac{m\pi x}{L}) \sin(\frac{n\pi x}{L})dx = \begin{cases} 0, & m \neq n \\ \frac{L}{2}, & m = n \end{cases}$$

Therefore, the Fourier coefficients are given by

$$\gamma_n = \frac{\int_0^L \sin(\frac{n\pi x}{L}) f(x)dx}{\int_0^L \sin^2(\frac{n\pi x}{L})dx} = \frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L}) f(x)dx$$

We obtain the explicit formula for the formal solution, which is given by

$$u(x,t) = \sum_{n=1}^{N} \gamma_n \sin(\frac{n\pi x}{L}) e^{-k(\frac{n\pi}{L})^2 t}$$

where

$$\gamma_n = \frac{2}{L}\int_0^L \sin(\frac{n\pi x}{L}) f(x)dx.$$

4.2. Heat equation with Neumann boundary conditions. Consider the following heat conduction problem in a finite interval:

$$\frac{\partial u}{\partial t} - k^2 \frac{\partial^2 u}{\partial x^2} = 0, \; 0 < x < L, \; t > 0, \tag{28}$$

Dirichlet boundary conditions

$$u_x(0,t) = u_x(L,t) = 0, \; t \geq 0, \tag{29}$$

and the initial condition

$$u(x,0) = f(x), 0 \leq x \leq L, \tag{30}$$

where f is a given initial condition and k is a positive constant. To make (29) consistent with (30), we assume the compatibility condition: $f_x(0) = f_x(L) = 0$. This problem corresponds to the evolution of the temperature $u(x,t)$ in a homogeneous one-dimensional conducting rod of length L, whose initial temperature (at time $t = 0$) is known, and such that there is no heat flux through the boundaries (heat neither enters nor leaves the system).

We start by looking for solutions of the special form

$$u(x,t) = X(x)T(t) \tag{31}$$

where X and T are functions of the variables x and t, respectively. Differentiating the separated solution (31) and substituting into the PDE, we obtain

$$XT_t = kX_{xx}T$$

We can rewrite it using λ (called the separation constant) such that

$$\frac{T_t}{T} = k\frac{X_{xx}}{X} = -\lambda \tag{32}$$

λ is a real constant. Since we are looking for solutions that do not vanish identically, there exists $t_0 \in \mathbb{R}$ such that $T(t) \neq 0$. Consequently, we obtain

$$\begin{cases} X'' + \lambda X = 0, \; 0 < x < L, \\ X'(0) = X'(L) = 0 \end{cases}$$

and

$$T' + \lambda kT = 0, \; t > 0, \tag{33}$$

where λ is a constant. We start by first solving the system (4.2).

Case 1: $\lambda = -\mu^2 < 0$, then $X(x) = \alpha e^{-\mu x} + \beta e^{\mu x}$ and $X'(x) = \mu[-\alpha e^{-\mu x} + \beta e^{\mu x}]$ where α, β are arbitrary real numbers. But the conditions

$$\begin{cases} X'(0) = 0, \\ X'(L) = 0 \end{cases} \Rightarrow \begin{cases} \mu(-\alpha + \beta) = 0, \\ \mu(-\alpha e^{-\mu L} + \beta e^{\mu L}) = 0 \end{cases} \Rightarrow \begin{cases} \alpha = \beta), \\ -\alpha e^{-\mu L} + \beta e^{\mu L} = 0 \end{cases}$$

$\Rightarrow \beta = \alpha = 0 \Rightarrow X \equiv 0 \; et \; u(x;t) = 0.foreverything 0 \leq x \leq L$ and $t \geq 0$. We must therefore exclude the case $\lambda < 0$.

Case 2: If $\lambda = 0, X(x) = \alpha + \beta x$ and $X'(x) = \beta$, where α, β are arbitrary real numbers. But

$$\begin{cases} X'(0) = 0, \\ X'(L) = 0 \end{cases} \Rightarrow \beta = 0,$$

So $\lambda = 0$ is an eigenvalue with the eigenfunction $X(x) = \alpha$ (constant).

Case 3: If $\lambda = \mu^2 > 0$, then $X(x) = \alpha\cos() + \beta\sin()$ and $X'(x) = \mu[-\alpha\sin(\mu x) + \beta cos(\mu x)]$ where α, β are arbitrary real numbers. But

$$\begin{cases} X'(0)=0, \\ X'(L)=0 \end{cases} \Rightarrow \begin{cases} \beta\cos(0)=0, \\ -\alpha e^{-\mu x}+\beta e^{\mu x}=0 \end{cases} \Rightarrow \begin{cases} \beta=0, \\ -\alpha e^{-\mu L}+\beta e^{\mu L}=0 \end{cases}$$

$\Rightarrow \beta = 0 \; et \; \alpha \sin\mu x = 0. Assin(\mu l) = 0$, then $\sin(\mu L) = 0$. Consequently $\mu L = n\pi$ and $\lambda = (n/L)^2$ with $n \in \mathbb{Z}^\star$. We obtain $\lambda_n = (n/L)^2$ and $X_n(x) = \alpha_n \cos(\frac{n\pi x}{L})$. Because $\cos(-x) = \cos(x)$ for everything $x \in \mathbb{R}$, and we take into account the value $\lambda = 0$, 1The values and functions are defined by: $\lambda_n = (n/L)^2$ and $X_n(x) = \alpha_n \cos(\frac{n\pi x}{L}), n \in \mathbb{N}$. We now move on to equation (27) whose general solution is given by

$$T(t) = \gamma_n e^{-k(\frac{n\pi x}{L})^2 t}, n \in \mathbb{N}.$$

This yielded the following sequence of separate solutions

$$u_n(x,t) = \delta_n \cos(\frac{n\pi x}{L}), n \in \mathbb{N}.$$

The principle of superposition implies that any linear combination

$$u(x,t) = \sum_{n=1}^{N} \delta_n \cos(\frac{n\pi x}{L}), n \in \mathbb{N}.$$

Suppose we have

$$f(x) = \frac{\delta_0}{2} + \sum_{n=1}^{N} \delta_n \cos(\frac{n\pi x}{L})$$

In this case, the generalized superposition principle implies the formal solution

$$u(x,t) = \frac{\delta_0}{2} + \sum_{n=1}^{\infty} \delta_n \cos(\frac{n\pi x}{L}) e^{-k(\frac{n\pi x}{L})^2 t},$$

We know that if $m, n \geq 0$, we have

$$\int_0^L \cos(mx)\cos(nx)dx = \int_0^L \frac{\cos((m-n)x)+\cos((m+n)x)}{2} = \begin{cases} 0 \; si \; m \neq n \\ \frac{L}{2} \; si \; m = n \\ L \; si \; m = n = 0 \end{cases}$$

We will now explain how to calculate the coefficients. $\delta_m, m = 0, 1, 2, ...$ For $m = 0$, let's multiply f by $\cos(\frac{0\pi x}{L})$ and we integrate on $(0, L)$, we obtain

$$\int_0^L \cos(\frac{0\pi x}{L}) f(x)dx = \frac{\delta_0}{2}\int_0^L \cos(\frac{0\pi x}{L})dx + \sum_{n=1}^{\infty} \delta_n \int_0^L \cos(\frac{0\pi x}{L})\cos(\frac{n\pi x}{L})dx$$

$$= \frac{\delta_n L}{2} + 0$$

Because $\int_0^L \cos(\frac{0\pi x}{L})\cos(\frac{n\pi x}{L})dx = 0$ since $n \neq (m = 0)$. then

$$\delta_0 = \frac{2}{L}\int_0^L \cos(\frac{0\pi x}{L}) f(x)dx = \frac{2}{L}\int_0^L f(x)dx$$

For $m \neq 0$, let's multiply f by $\cos(\frac{m\pi x}{L}), m \neq 0$ and we integrate on $(0, L)$, we obtain

$$\int_0^L \cos(\frac{0\pi x}{L}) f(x)dx = \frac{\delta_0}{2}\int_0^L \cos(\frac{m\pi x}{L})dx + \sum_{n=1}^{\infty}\delta_n \int_0^L \cos(\frac{m\pi x}{L})\cos(\frac{n\pi x}{L})dx$$
$$= \frac{\delta_0 L}{2m\pi}[\sin(0) - \sin(m)] + \delta_m \int_0^L \cos(\frac{m\pi x}{L})\cos(\frac{m\pi x}{L})dx$$
$$= \delta_m \frac{L}{2}$$

which gives

$$\delta_m = \frac{2}{L}\int_0^L \cos(\frac{m\pi x}{L}) f(x)dx$$

and

$$f(x) = \frac{1}{L}\int_0^L f(x)dx + \frac{2}{L}\sum_{n=1}^{\infty}\cos(\frac{n\pi x}{L})\int_0^L \cos(\frac{n\pi x}{L}) f(x)dx$$

Therefore, the formal solution is now given by

$$u(x,t) = \frac{1}{L}\int_0^L f(x)dxx + \frac{2}{L}\sum_{n=1}^{\infty}\cos(\frac{n\pi x}{L})e^{-k(\frac{n\pi}{L})^2 t)}\int_0^L \cos(\frac{n\pi x}{L}))f(x)dx$$

Finally, we consider the solution to the following problem:

Example 4.1.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \ \textit{où} \ \ u = u(x,t),\ x \in [0,\pi], t \geq 0$$

with $u(0,t) = 0, u(\pi,t) = 0$ *for everything* $t \geq 0$ *and* $u(x,0) = x(\pi - x)$ *for everything* $x \in [0,\pi]$. *So, the solution will be*

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(nx) exp(-n^2 t)$$

with

$$a_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin(nx)dx = \frac{4((-1)n + 1 + 1)}{\pi n^3}$$

by integrating in parts. So the solution is

$$u(x,t) = 4\sum_{n=1}^{\infty}\frac{4((-1)n + 1 + 1)}{\pi n^3}\sin(nx)exp(-n^2 t).$$

We will plot the graph of $u(x,t)$ *for* $x \in [0,\pi]$ *and* $t \in [0,1]$ *in Figure 1*

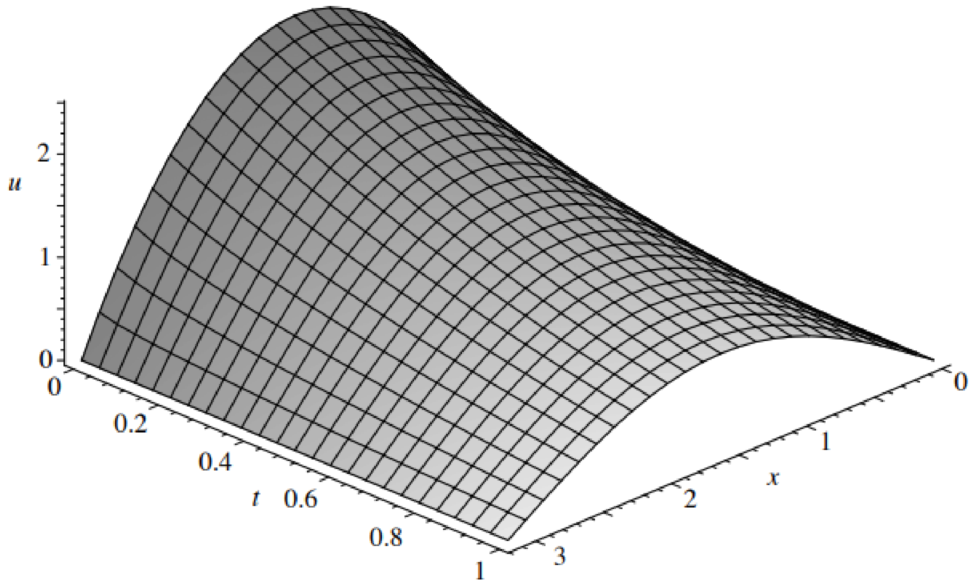


Figure [1]

CHAPTER 5

Laplace Equation

We will now consider an elliptic PDE. Thus, in this chapter, we will present the Laplace equation. We introduce two of its important properties, the maximum principle and rotational invariance. We start with these remarks

Remark 0.1:

1-The Laplace equation in $\mathbb{R}^n$ is

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + ... + \frac{\partial^2 u}{\partial x_n^2} = 0$$

2-The equation in $\mathbb{R}^3$ is the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

3-The quantity Δ is called the Laplacian of u, and in two dimensions

$$\nabla.\nabla = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

4-In one dimension

$$\frac{\partial^2 u}{\partial x^2} = 0.$$

5-A solution of the Laplace equation is called a harmonic function. In one dimension, harmonic functions are $u(x) = A + Bx$. The non-homogeneous Laplace equation is $\Delta u = f$, with f a given function, which is called the Poisson equation.

1. Principe de maximum

Let Ω be a bounded open set in $\mathbb{R}^2$ or $\mathbb{R}^3$. Let $u(x, y)$ or $u(x, y, z)$ be a harmonic function in Ω continuous on $\bar{\Omega} = \Omega \cup \partial\Omega$ (boundary of Ω). Then the maximum and minimum values of u are attained on $\partial\Omega$ and nowhere inside (unless u is constant). We use the shorthand notation $X = (x, y)$ in two dimensions or $X = (x, y, z)$ in three dimensions. Furthermore, the radial coordinate is written as: $|X| = (x^2 + y^2)^{\frac{1}{2}}$ or $|X| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.

The maximum principle states that there exist points X_M and X_m on $\partial\Omega$ such that $\forall x \in \bar{\Omega}$, we have

$$u(X_m) \leq u(x) \leq u(X_M)$$

Proof

Let $\epsilon > 0$. We set

$$v(x) = u(x) + \epsilon |x|^2$$

Since u is harmonic, we have $\Delta v(x) = \Delta u(x) + \epsilon \Delta (x+y)^2 = 0 + 4\epsilon > 0$ in Ω On $\bar{\Omega}$, the continuous function v attains a maximum, reached at a point $M_0(x_0, y_0)$ which necessarily belongs to $\partial\Omega$. Otherwise, (x_0, y_0) belongs to the open set Ω and:

a. The function $x \to u(x, y_0)$ attains its maximum at x_0 on an open interval centered at x_0.

b. The function $y \to u(x_0, y)$ attains its maximum at y_0 on an open interval centered at y_0, so that we have the relations

$$\begin{cases} \frac{\partial v}{\partial x}(x_0, y_0) = 0 & , \frac{\partial^2 v}{\partial x^2}(x_0, y_0) \leq 0 \\ \frac{\partial v}{\partial y}(x_0, y_0) = 0 & , \frac{\partial^2 v}{\partial y^2}(x_0, y_0) \leq 0 \end{cases}$$

We deduce that $\Delta v(x) = 4\epsilon \leq 0$, which is contradictory and therefore proves that the function v attains its maximum at a point $M_0(x_0, y_0)$ which, as stated, belongs to the boundary $\partial\Omega$. Thus, for any point M belonging to

$$u(x) < v(x) \leq v(M_0) = u(M_0) + \epsilon |M_0|^2 \leq u(M_0) + \epsilon l^2$$

where l is the maximal distance between $\partial\Omega$ and the origin. Letting ϵ tend to 0, we obtain: $u(x) \leq u(M_0)$, which establishes that u attains its maximum on $\partial\Omega$. The existence of a minimum point X_m is demonstrated in the same way.

□

2. Invariance in two dimensions

A translation by the vector (a, b) is given by

$$\begin{cases} x' &= x + a \\ y' &= y + a \end{cases}$$

A rotation by an angle α is given by

$$\begin{cases} x' &= x \cos\alpha + y \sin\alpha \\ y' &= x \sin\alpha + y \cos\alpha \end{cases}$$

The Laplace operator is invariant under translation or rotation

Proof

Invariance means that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial x'^2}+\frac{\partial^2 u}{\partial y'^2}$$

For translation, the proof is simple using the chain rule. For rotation, we have

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial x'}\cos\alpha-\frac{\partial u}{\partial y'}\sin\alpha$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial x'}\sin\alpha+\frac{\partial u}{\partial y'}\cos\alpha$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x'}(\frac{\partial u}{\partial x'}\cos\alpha-\frac{\partial u}{\partial y'}\sin\alpha)\cos\alpha-\frac{\partial}{\partial y'}(\frac{\partial u}{\partial x'}\sin\alpha+\frac{\partial u}{\partial y'}\cos\alpha)\sin\alpha$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial x'}(\frac{\partial u}{\partial x'}\sin\alpha+\frac{\partial u}{\partial y'}\ pha)\sin\alpha+\frac{\partial}{\partial y'}(\frac{\partial u}{\partial x'}\cos\alpha+\frac{\partial u}{\partial y'}\sin\alpha)\cos\alpha$$

We obtain

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=(\frac{\partial^2 u}{\partial x'^2}+\frac{\partial^2 u}{\partial' y^2})(\cos^2\alpha+\sin^2\alpha)+0.\frac{\partial^2 u}{\partial x\partial y}$$

$$=\frac{\partial^2 u}{\partial x'^2}+\frac{\partial^2 u}{\partial' y^2}$$

□

3. Invariance in polar coordinates

When looking for special harmonic functions that are themselves invariant under rotation. In two dimensions, this means using polar coordinates (r,θ) and seeking solutions depending only on r. Let the transformation

$$\begin{cases} x &= r\cos\theta,\ r>0,\ \theta\in[0,2\pi[\\ y &= r\sin\theta, \end{cases}$$

has the Jacobian matrix

$$J=\begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}$$

with the inverse matrix

$$J^{-1}=\begin{pmatrix}\cos\theta & \frac{-\sin\theta}{r}\\ \sin\theta & \frac{\cos\theta}{r}\end{pmatrix}$$

So, by the chain rule, we have

$\frac{\partial u}{\partial x}=\frac{\partial}{\partial r}\cos\theta-\frac{\partial}{\partial\theta}\frac{\sin\theta}{r}$

$\frac{\partial u}{\partial y}=\frac{\partial}{\partial r}\sin\theta+\frac{\partial}{\partial\theta}\frac{\sin\theta}{r}$

$\frac{\partial^2 u}{\partial x^2}=\cos^2\theta\frac{\partial^2}{\partial r^2}-2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}-2\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial}{\partial r}$

$\frac{\partial^2 u}{\partial y^2}=\sin^2\theta\frac{\partial^2}{\partial r^2}+2\frac{\sin\theta\cos\theta}{r}\frac{\partial^2}{\partial r\partial\theta}-\frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial\theta^2}+2\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial}{\partial r}$

We add the last two quantities together, and we get

$$\Delta_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \tag{34}$$

If u does not depend on θ, we obtain

$$0 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

This ordinary differential equation is easy to solve

$$u = c_1 \log r + c_2.$$

where c_1 and c_2 are arbitrary constants.

4. The Laplace equation on a rectangle

We consider the PDE, the following Laplace equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{35}$$

and the boundary conditions $u(x, 0) = f_1(x),\ u(x, N) = f_2(x),\ u(0, y) = g_1(y)$, and $u(M, y) = g_2(y)$ for $0 \le x \le M,\ 0 \le y \le N$. We want to determine the functions $u(x, y)$ for $0 \le x \le M$ and $0 \le y \le N$ that are satisfied. The function $u = u(x, y)$ is defined on the rectangle consisting of points $(x, y) \in \mathbb{R}^2$ such that $0 \le x \le M$ and $0 \le y \le N$.

It is necessary to point out these important remarks:

Remark 4.1:

We have:

(1) This problem is different from the previous ones (wave, heat) because both variables belong to bounded intervals, whereas the variable t is not bounded for the wave and heat equations.

(2) There are no initial conditions.

(3) The boundary conditions are not homogeneous. The latter

This difference presents a difficulty that will need to be overcome.

(4) It is possible to give a physical meaning to a solution $u(x, y)$ of the previous problem. We can consider $u(x, y)$ as the equilibrium temperature (i.e., when $t \to \infty$) for the heat problem

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \ \textit{où}\ u = u(x, y, t)$$

for which conditions at the border are

$$u(x, 0, t) = f_1(x), \ u(x, N, t) = f_2(x), \ u(0, y, t) = g_1(y), \ u(M, y, t) = g_2(y)$$

for $0 \leq x \leq M, 0 \leq y \leq N$ and $t \geq 0$.

(5) There are also initial conditions at $t = 0$, but we will not specify them. At equilibrium

$$\frac{\partial u}{\partial t} = 0.$$

After these remarks, we obtain the initial problem to solve 35, with

$$u(x, 0) = f_1(x) \ \textit{et}\ u(x, N) = f_2(x) \ \textit{pour}\ x \in [0, M],$$

$$u(0, y) = g_1(y) \ \textit{et}\ u(M, y) = g_2(y) \ \textit{pour}\ y \in [0, N],$$

can be decomposed into four problems, each having a single non-homogeneous condition rather than four.

Problem [1]:

$$\frac{\partial^2 u_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} = 0 \ \textit{avec}$$
$$u_1(x, 0) = f_1(x) \ \textit{et}\ u_1(x, N) = 0 \ \textit{pour}\ x \in [0, M],$$

$$u_1(0, y) = 0 \ \textit{et}\ u_1(M, y) = 0 \ \textit{pour}\ y \in [0, N],$$

Problem [2]:

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0 \ \textit{avec}$$

$$u_1(x,0)=0 \ et \ u_1(x,N)=f_2(x) \ pour \ x\in[0,M],$$

$$u_1(0,y)=0 \ et \ u_1(M,y)=0 \ pour \ y\in[0,N],$$

Problem [3]:

$$\frac{\partial^2 u_3}{\partial x^2}+\frac{\partial^2 u_3}{\partial y^2}=0 \ \ avec$$

$$u_1(x,0)=0 \ et \ u_1(x,N)=0 \ pour \ x\in[0,M],$$

$$u_1(0,y)=g_1(x) \ et \ u_1(M,y)=0 \ pour \ y\in[0,N],$$

Problem [4]:

$$\frac{\partial^2 u_4}{\partial x^2}+\frac{\partial^2 u_4}{\partial y^2}=0 \ \ avec$$

$$u_1(x,0)=0 \ et \ u_1(x,N)=0 \ pour \ x\in[0,M],$$

$$u_1(0,y)=0 \ et \ u_1(M,y)=g_2(x) \ pour \ y\in[0,N],$$

So, to continue solving this problem, we need these comments:

Remark 4.2:

We have :

(1) If u_i is a solution of **problem [i]** for $i=1,2,3,4$, then it is easy to show that $u=u_1+u_2+u_3+u_4$ is a solution of the initial problem.

(2) We can seek to solve problems [1] to [4].

(3) We illustrate the method of separation of variables to solve **problem [1]**. The same type of analysis can be done for the other three problems.

We use **the method of separation of variables** to determine the solution $u_1(x,y)$ of **problem [1]**, so the first step is to consider the intermediate problem:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \tag{36}$$

under these conditions

$$u_1(x,N)=0, \ u_1(0,y)=0 \ u_1(M,y)=0 \ pour \ x\in[0,M], \ y\in[0,N] \tag{37}$$

For now, we have dropped the non-homogeneous condition $u_1(x,0)=f_1(x)$.

Thus, we need to determine non-trivial solutions $u_1(x,y)$ of this problem (36)(37) which are of the form

$$u_1(x,y)=X(x)Y(y).$$

Then, by substituting this solution into the PDE and separating the variables, we obtain

$$X''Y+XY''=0\Rightarrow\frac{X''}{X}=-\frac{Y''}{Y}$$

where X'' is the second derivative of X with respect to x and Y'' is the second derivative of Y with respect to y. In the last equation, the left-hand side is a function of x only, whereas the right-hand side is a function of y only. Therefore, these two expressions must be constant, and we can write

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

Thus, we have the system of two ordinary differential equations:

$$X'' - \lambda X = 0 \text{ and } Y'' + \lambda Y = 0$$

If now, we consider the boundary conditions of the problem $(36 - 37)$ and because we are seeking non-trivial solutions, i.e., the functions X and Y are not identically zero, then we obtain

$$u_1(x, N) = X(x)Y(N) = 0, \ \forall x \in [0, M] \Rightarrow Y(N) = 0;$$

$$u_1(0, y) = X(0)Y(y) = 0, \ \forall y \in [0, N] \Rightarrow X(0) = 0;$$

$$u_1(M, y) = X(M)Y(y) = 0, \ \forall y \in [0, N] \Rightarrow X(M) = 0.$$

In principle, the following problem must be studied:

$$X'' - \lambda X = 0 \text{ avec } X(0) = 0, \ X(M) = 0, \ \text{et}$$

$$Y'' + \lambda Y = 0 \text{ avec } Y(N) = 0.$$

The first equation $X'' - \lambda X = 0$ with $X(0) = 0$ and $X(M) = 0$ has already appeared for the wave and heat equations. If $\lambda \geq 0$, we obtain that $X \equiv 0$ and we can exclude these values for λ because we want non-trivial solutions.
If $\lambda = -\nu^2 < 0$, then the general solution of $X'' - \lambda X = 0$ is $X(x) = A\cos(\nu x) + B\sin(\nu x)$. Since $X(0) = A = 0$ and

$$X(M) = A\cos(\nu M) + B\sin(\nu M) = 0, \ \Rightarrow \ B\sin(\nu M) = 0.$$

Since we are seeking non-trivial solutions and $A = 0$, we can assume that $B \neq 0$. Therefore, we can deduce that

$$\sin(\nu M) = 0 \Rightarrow \nu = \frac{n\pi}{M}, \ \lambda_n = -(\frac{n\pi}{M})^2) \ etX_n(x) = B_n \sin(\frac{n\pi x}{M}) \ \text{où} \ n \in \mathbb{Z}, \ n \neq 0.$$

Since $\sin(-x) = -\sin(x)$ and $\lambda_{-n} = \lambda_n$, we can restrict ourselves to the case where $n \in \mathbb{N}, \ n \neq 0$ in the above. For each integer $n \geq 1$, we have a solution $X_n = B_n \sin(\frac{n\pi x}{M})$ of the equation $X'' - \lambda X = 0$ *with* $X(0) = 0, \ X(M) = 0$ for the value $\lambda = \lambda_n = -(\frac{n\pi x}{M})^2)$.

If we now consider the second equation $Y'' + \lambda Y = 0$ with $Y(N) = 0$ in the case where $\lambda = \lambda_n = -(\frac{n\pi x}{M})^2)$ where $n \in \mathbb{N},\ n \geq 1$, then the general solution is of the form

$$Yn(y) = Cexp(\frac{n\pi y}{M}) + Dexp(-\frac{n\pi y}{M}).$$

But we also have

$$Y_n(N) = 0 \Rightarrow Cexp(\frac{n\pi y}{M}) + Dexp(-\frac{n\pi y}{M}) = 0 \Rightarrow D = -Cexp(\frac{2n\pi N}{M}).$$

By substituting this into the solution Y_n, we obtain

$$Y_n = Cexp(\frac{n\pi y}{M}) - Dexp(-\frac{2n\pi y}{M})Cexp(\frac{-n\pi N}{M})$$
$$Y_n = Cexp(\frac{n\pi N}{M})[exp(\frac{n\pi(y-N)}{M}) - exp(-\frac{n\pi(y-N)}{M})]$$
$$Y_n = 2Cexp(\frac{n\pi N}{M})\sinh(\frac{n\pi(y-N)}{M})$$

To continue, we need this remark:

Remark 4.3:

(1) Recall that the function $\sinh(\theta)$ denotes the hyperbolic sine, i.e.,

$$\sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2} \ \textit{pour tout}\ \theta \in \mathbb{R}.$$

From the above, we have

$$a_n \exp\left(\frac{n\pi N}{M}\right)\sinh\left(\frac{n\pi(y-N)}{M}\right)$$

is a solution of the problem (36_{37}). Note that we have replaced

$$B_n 2C \exp\left(\frac{n\pi N}{M}\right)$$

by a_n. Using **the principle of superposition**, we obtain that

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{M}\right)\sinh\left(\frac{n\pi(y-N)}{M}\right)$$

is also a solution of the problem $(36 - 37)$.

If we return to **Problem [1]**, then

$$u_1(x,y) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{M})\sinh(\frac{n\pi(y-N)}{M})$$

is a solution if and only if

$$u_1(x,0) = \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{M})\sinh(-\frac{n\pi N)}{M}) = f_1(x)\ \forall x \in [0,M];$$

that is to say $\sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{M}) \sinh(-\frac{n\pi N)}{M})$ is the odd Fourier series of $f_1(x)$, and we have:

$$a_n = (\frac{2}{M\sinh(\frac{-n\pi N}{M}}) \int_0^M f_1(x)\sin(\frac{n\pi x}{M}dx \ pour\ tout\ n \in \mathbb{N}, n \geq 1.$$

Thus, we obtain a formal solution of **Problem [1]**. By proceeding in a similar way, we obtain formal solutions for **problems [2], [3]**, and **Problem [4]**. These solutions are
For **problem [2]**, we have

$$u_2(x,y) = \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{M}) \sinh(\frac{n\pi(y-N)}{M}) \ avec\ b_n = (\frac{2}{M\sinh(\frac{-n\pi N}{M}}) \int_0^M f_2(x)\sin(\frac{n\pi x}{M})dx.$$

For **problem [3]**, we have

$$u_3(x,y) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi y}{N}) \sinh(\frac{n\pi(x-M)}{N}) \ avec\ c_n = (\frac{2}{N\sinh(\frac{-n\pi N}{M}}) \int_0^M g_1(x)\sin(-\frac{n\pi x}{N})dx.$$

For **problem [4]**, we have

$$u_4(x,y) = \sum_{n=1}^{\infty} d_n \sin(\frac{n\pi y}{N}) \sinh(\frac{n\pi x}{N}) \ avec\ d_n = (\frac{2}{N\sinh(\frac{-n\pi N}{M}}) \int_0^M g_1(x)\sin(\frac{n\pi x}{N})dx.$$

Thus, the formal solution to the initial problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with

$$u(x,0) = f_1(x) \ et\ u(x,N) = f_2(x),\ pour\ x \in [0,M];$$

$$u(0,y) = g_1(y),\ et\ u(M,y) = g_2(y),\ pour\ y \in [0,N]$$

is

$$u(x,y) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{M})[a_n \sin(\frac{n\pi(y-N)}{M}) + b_n \sin(\frac{n\pi x}{M})] + \sum_{n=1}^{\infty} \sin(\frac{n\pi y}{M})[c_n \sin(\frac{n\pi(x-M)}{M}) + d_n \sin(\frac{n\pi x}{N})]$$

with a_n, b_n, c_n and d_n as before.

Theorem 4.1:

If the functions $f_1(x)$ and $f_2(x)$ are continuous and piecewise smooth on the interval $[0, M]$, the functions $g_1(y)$ and $g_2(y)$ are continuous and piecewise smooth on the interval $[0, N]$, $f_1(0) = f_2(0) = f_1(M) = f_2(M) = 0$, $g_1(0) = g_2(0) = g_1(N) = g_2(N) = 0$, then the formal solution above is a true solution of the initial problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with

$$u(x, 0) = f_1(x) \text{ et } u(x, N) = f_2(x), \text{ pour } x \in [0, M];$$

$$u(0, y) = g_1(y), \text{ et } u(M, y) = g_2(y), \text{ pour } y \in [0, N],$$

Now, we will determine the solution $u(x, y)$ for the following problem.

EXAMPLE 4.1.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ et } u = u(x, y),\ 0 \leq x \leq \pi \text{ et } 0 \leq y \leq \pi$$

with

$$u(x, 0) = x(\pi - x) \text{ et } u(x, N) = 0, \text{ pour } x \in [0, M];$$

$$u(0, y) = x(\pi - y), \text{ et } u(M, y) = 0, \text{ pour } y \in [0, N],$$

Thus, it is easy to evaluate the coefficients a_n, b_n, c_n and d_n by integrating by parts. We obtain that

$$a_n = c_n = \frac{4(1 + (-1)^{n+1})}{\pi n^3 \sinh(-n\pi)} \text{ et } b_n = d_n = 0 \text{ si } n \geq 1.$$

So the solution to the problem is

$$u(x, y) = \sum_{n=1}^{\infty} \left(\frac{4(1 + (-1)^{n+1})}{\pi n^3 \sinh(-n\pi)}\right)[\sin(nx)\sinh(n(y - \pi)) + sin(ny)sinh(n(x - \pi)].$$

The graph of $u(x, y)$ for $x \in [0, \pi]$ and $y \in [0, \pi]$ is shown in Figure 1

5. Exercise series $N^\circ = 05$

EXERCICE 10. *Determine the formal solution to Laplace's equation.*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ où } u = u(x, y),\ 0 \leq x \leq M,\ 0 \leq x \leq N$$

when the initial conditions for $0 \leq x \leq M$ and $0 \leq y \leq N$ are

a) $u(x, 0) = 0, u(x, N) = 0, \frac{\partial u}{\partial y}(0, y) = 0, \frac{\partial u}{\partial y}(M, y) = f(y);$

b) $\frac{\partial u}{\partial x}(x, 0) = g(x), \frac{\partial u}{\partial y}(x, N) = 0, u(0, y) = \frac{\partial u}{\partial y}(0, y), u(M, y) = \frac{\partial u}{\partial y}(M, y);$

c) $\frac{\partial u}{\partial x}(x, 0) = 0, \frac{\partial u}{\partial y}(x, N) = 0, u(0, y) = 0, u(M, y) = h(y).$

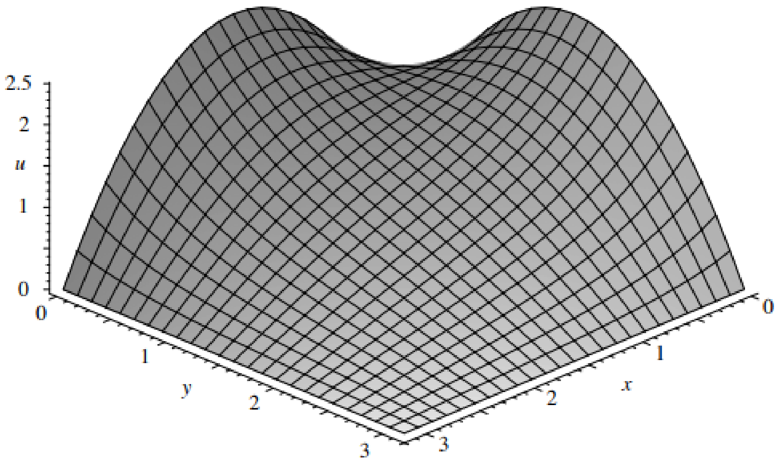


Figure [1]

CHAPTER 6

Wave Equation

In this chapter, we present the wave equation.

$$\frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \tag{38}$$

where c is a constant. In the one-dimensional case, the canonical form of the equation will be used to show that the Cauchy problem is well-posed. Moreover, we will derive simple explicit formulas for the solutions. We also discuss some important properties of the solutions of the wave equation, which are also typical for more general hyperbolic problems.

1. Modeling and Physical Derivation

In mechanics and physics, the wave equation serves as a simplified model for oscillations on a vibrating string ($n = 1$), a membrane ($n = 2$), or an elastic solid ($n = 3$). In these models, $u(x,t)$ is the displacement of the mass at a point in certain directions from the point x at time $t \geq 0$. In (38), f models the external force.

For example, suppose an elastic solid occupies a region Ω in $\mathbb{R}$ without external force. For any subregion $G \subset \Omega$, $F(x,t)$ is the contact force density acting on G through the boundary ∂G. The mass density is normalized to one. Let ν be the outward unit normal vector of ∂G. The acceleration in G is

$$\frac{d^2}{dt^2}\int_G u dx = \int_G \frac{d^2}{dt^2} u dx$$

such that the net contact force is

$$-\int_G \vec{F}\nu dS$$

According to Newton's law, we obtain

$$\int_G \frac{d^2}{dt^2} u\, dx = -\int_G \vec{F}\nu dS = \int_G \nabla \cdot \vec{F}\, dx$$

where we applied the divergence theorem[1], we conclude

[1] The integral of the divergence of a vector field $\vec{F}$ over a volume V in $\mathbb{R}^3$ is equal to the flux of this field through the boundary of the volume

$$\int_V \nabla \cdot \vec{F}\, d\vec{V} = \int_{\partial V} \vec{F}\, d\vec{S}$$

where $\vec{S}$ is the normal vector to the surface, pointing outward.

$$\int_G \frac{d^2}{dt^2} u dx = \nabla . \vec{F}$$

If the elastic body, $\vec{F} = \vec{F}(\nabla u)$, then

$$\frac{d^2}{dt^2} u + \nabla \cdot \vec{F}(\nabla u) = 0.$$

In the case of small oscillations, $|\nabla u|$ is very small, so $\vec{F}(\nabla u) \approx -c^2 \nabla u$, therefore,

$$\frac{d^2 u}{dt^2} - c^2 \Delta u = 0.$$

2. Cauchy Problem and d'Alembert's Formula

Without going into all the details, it should be noted that we have already seen this in the chapter; but to fix the ideas, we will present the Cauchy problem for the one-dimensional homogeneous wave equation, which is given by

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \ -\infty < x \geq +\infty, t > 0, \tag{39}$$

$$u(x, 0) = f(x), \ \ u_t(x, 0) = g(x), \ \ -\infty < x \geq +\infty \tag{40}$$

A classical solution of the Cauchy problem (39)- (40) is a function u that is twice continuously differentiable for all $t > 0$, such that u and $\frac{\partial u}{\partial t}$ are continuous for $t > 0$, and such that (39)-(40) are satisfied, according to theorem (**??**).

The following example illustrates the use of d'Alembert's formula.

Example 2.1. *Consider the Cauchy problem*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0, \ \ -\infty < x \geq +\infty, t > 0,$$

$$u(x, 0) = f(x) = \begin{cases} 0, & -\infty < x < -1, \\ x + 1, & -1 \leq x \leq 0, \\ 1 - x, & 0 \leq x \leq 1, \\ 0, & 1 < x < \infty, \end{cases}$$

$$u_t(x, 0) = g(x) = \begin{cases} 0, -\infty < x < -1, \\ 1, \ \ -1 \leq x \leq 1, \\ 0, \ \ 1 < x < \infty, \end{cases}$$

Give $u(1, \frac{1}{2})$ and discuss the regularity of the solution u. d'Alembert's formula implies

$$u(1, \frac{1}{2}) = \frac{1}{2}[f(\frac{3}{2}) + f(\frac{1}{2})] + \frac{1}{2} \int_{\frac{1}{2}}^{\frac{3}{2}} g(s) ds$$

$$= \frac{1}{2}(\frac{1}{2} + \int_{\frac{1}{2}}^{1} g(s)ds + \int_{1}^{frac32} g(s)ds$$

$$= \frac{1}{2}(\frac{1}{2} + \int_{\frac{1}{2}}^{1} 1.ds) = \frac{1}{2}$$

Since g and f' are not continuous at the points $x = -1, 0, 1$, then u may not be of class $\mathcal{C}^1$. The problem is posed when $x \pm t = \text{Const}$. The solution is regular in the neighborhood of the point $(1, \frac{1}{2})$, which allows us to compute $(1, \frac{1}{2})$.

3. The Non-Homogeneous Cauchy Problem

Consider the following Cauchy problem

(41) $$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = F(x,t), \ -\infty < x \geq +\infty, t > 0,$$

(42) $$u(x,0) = f(x), \ u_t(x,0) = g(x), \ -\infty < x \geq +\infty$$

This problem models, for example, the vibration of a very long chain in the presence of an external force F.

Theorem 3.1:

Let $f \in \mathcal{C}^2(\mathbb{R}), g \in \mathcal{C}^1(\mathbb{R})$ and $F, F_x \in C(\mathbb{R}^2)$, so the problem (41)- (42) admits a single solution $u \in \mathcal{C}^2(\mathbb{R} \times (0,\infty)) \cap \mathcal{C}^1(mathbbR \times [0,\infty))$ given by the formula

(43)
$$u(x,t) = \frac{1}{2}[f(x+ct) + f(xct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds + \frac{1}{2c}\int_{=0}^{=t}\int_{xi=xc(t-\tau)}^{xi=xc(t+\tau)} F(\xi,\tau)d\xi d\tau.$$

Example 3.1. *Consider Cauchy's problem*

$$\frac{\partial^2 u}{\partial t^2} - 9\frac{\partial^2 u}{\partial x^2} = e^x - e^{-x}, \ -\infty < x \geq +\infty, t > 0,$$

$$u(x,0) = x, -\infty < x \leq +\infty,$$

$$u(x,0) = \sin x, -\infty < x \leq +\infty,$$

Using d'Alembert's formula (43), we have

$$u(x,t) = \frac{1}{2}[x + 3t + x - 3t] + \frac{1}{6}\int_{x-ct}^{x+3t} g(s)ds + \frac{1}{6}\int_{=0}^{=t}\int_{xi=x3(t-\tau)}^{xi=x3(t+\tau)} e^{\xi} - e^{-\xi} d\xi d\tau.$$

$$= x + \frac{1}{3}\sin x \sin 3t - \frac{2}{9}\sinh x + \frac{2}{9}\sinh x \cosh 3t.$$

4. Kirchhoff's Formula[2] (Cauchy Problem in $\mathbb{R}^3$)

We consider the following problem

$$-\frac{\partial^2 u}{\partial t^2} - c^2\Delta u = 0, u(x,0) = f(x), x \in \mathbb{R}^3, u_t(x,0) = g(x), x \in \mathbb{R}^3 \tag{44}$$

where $\Delta u(x,t) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}(x,t)$. We introduce S_r and B_r as the sphere and the ball of $S_r = \{x \in \mathbb{R}^3 : ||x|| = r\}$ and $B_r = \{x \in \mathbb{R}^3 : ||x|| \leq r\}$ For $u : \mathbb{R}^3 \to \mathbb{R}$, the average of u is given by

$$\tilde{u}(r,t) = \frac{1}{S.(B(x;r))} \int_{S_r} u(y,t)dS(y) = \frac{1}{4\pi r^2} \int_{S_r} u(y,t)dS(y)$$

where $dS(y)$ denotes the surface measure of B_r

LEMME 6.1. *If u solves (44), then $\tilde{u}$ solves*

$$-\frac{\partial^2 \tilde{u}}{\partial t^2} - \frac{\partial^2 \tilde{u}}{\partial r^2} - \frac{(n-1)}{r}\frac{\partial \tilde{u}}{\partial r} = 0, \quad 0 < r < 1, \, t \geq 0, \tag{45}$$

with $\tilde{u}(r,0) = \tilde{f}(r)$, *et* $\tilde{u}_t(r,0) = \tilde{g}(r)$,

To obtain the equation satisfied by $\tilde{u}$, we apply Green's formula on B_r

$$\int_{B_r} \Delta u = \int_{S_r} \frac{\partial u}{\partial \nu}$$

where ν is the unit normal vector on S_r, pointing outward from B_r. Using the main equation, we find

$$\int_{B_r} \frac{\partial^2 u}{\partial t^2} = c^2 \int_{S_r} \frac{\partial u}{\partial \nu}$$

Using spherical coordinates $\begin{cases} x = r\cos\theta\cos\varphi, \\ y = r\sin\theta\sin\varphi, \\ z = r\cos\theta, \end{cases}$ where $\theta \in]0,\pi[$ and $\varphi \in]0,2\pi[$. We therefore obtain

$$\int_0^r \int_0^{2\pi} \int_0^{\pi} \frac{\partial^2 u}{t^2} r^2 \sin\theta d\theta d\varphi dr = c^2 r^2 \int_{S_r} \frac{\partial u}{\partial r} d\theta d\varphi$$

On the other hand

$$\tilde{u}(r,t) = \int_0^{2\pi} \int_0^{\pi} u \sin\theta d\theta d\varphi$$

As a result,

$$\int_0^r \frac{\partial^2 \tilde{u}}{\partial t^2} dr = c^2 r^2 \frac{\partial \tilde{u}}{\partial t}.$$

By differentiating with respect to r, we obtain the equation in (45). For the initial conditions

[2]Gustav Kirchhoff (pronounced in German [kç.hf]) (born March 12, 1824 in Königsberg, in the province of East Prussia and died in Berlin on October 17, 1887))

$$\tilde{u}(u,0) = \frac{1}{4\pi r^2}\int_{S_r} u(y,0)dS(y) = \frac{1}{4\pi r^2}\int_{S_r} f(y)dS(y) = \tilde{f}$$

and

$$\frac{\partial \tilde{u}}{\partial t}(u,0) = \frac{1}{4\pi r^2}\int_{S_r} u(y,0)dS(y) = \frac{1}{4\pi r^2}\int_{S_r} g(y)dS(y) = \tilde{g}$$

Theorem 4.1:

Let $f \in \mathcal{C}^2(\mathbb{R}^3)$ and $g \in \mathcal{C}^3(\mathbb{R}^3)$, then the problem (44) admits a unique solution

$$u(x,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{\|x-x_0\|=ct} f(s)\,dS(s)\right] + \frac{1}{4\pi c^2 t}\int_{\|x-x_0\|=ct} g(s)\,dS(s)$$

Proof

In (45), we set $v := r\tilde{u}$, and we obtain the following system:

$$\begin{cases} \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial r^2} = 0, \\ v(r,0) = r\tilde{f}, \\ \frac{\partial v}{\partial t}(r,0) = r\tilde{g}. \end{cases}$$

According to D'Alembert's formula, the solution to this problem is given by:

$$v(r,t) = \frac{1}{2}(r+ct)\tilde{f}(r+ct) + (r-ct)\tilde{f}(r-ct) + \frac{1}{2c}\int_{r-ct}^{r+ct} \tilde{g}(s)\,ds.$$

This expression can be rewritten as follows:

$$v(r,t) = \frac{\partial}{\partial r}\left[\frac{1}{2c}\int_{r-ct}^{r+ct} s\tilde{f}(s)\,ds\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} s\tilde{g}(s)\,ds.$$

By differentiating with respect to r, we obtain $u(0,t) = \tilde{u}(0,t)$ since $\tilde{u}(0,t) = \frac{\partial v}{\partial r}(0,t)$. Moreover,

$$\left(\frac{\partial}{\partial r}\left[\frac{1}{2c}\int_{r-ct}^{r+ct} s\tilde{g}(s)\,ds\right]\right)_{r=0} = t\tilde{g}(ct) = \frac{1}{4\pi c^2 t}\int_{S_{st}} g(s)\,dS(s),$$

and the same formula applies for f. Thus, we deduce:

$$u(0,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{S_{ct}} f(s)\,dS(s)\right] + \frac{1}{4\pi c^2 t}\int_{S_{ct}} g(s)\,dS(s).$$

This formula is valid even when performing a translation from 0 to x_0. Finally, we obtain the general formula, called Kirchhoff's formula:

$$u(x,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{\|x-x_0\|=ct} f(s)\,dS(s)\right] + \frac{1}{4\pi c^2 t}\int_{\|x-x_0\|=ct} g(s)\,dS(s).$$

□

EXAMPLE 4.1. *We will conclude this chapter by presenting an example. Let us determine the vertical displacement $u(x,t)$ of an ideal string of length π as a solution of the wave equation*

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

with $c = 1$, if the initial vertical displacement is

$$u(x,0) = f(x) = x^3 - 3\pi^2 x^2 + \frac{\pi^2}{2} x$$

for $x \in [0,\pi]$ and the initial velocity is zero, i.e., $\frac{\partial u}{\partial t}(x,0) = g(x) = 0$ for $x \in [0,\pi]$. Note that the function $f(x)$ satisfies conditions (C.1) and (C.2), but not (C.3). Indeed, $f''(x) = 6x - 3\pi$ and $f''(0) = -3\pi \neq 3\pi = f''(\pi)$. Nevertheless, we can still present the solution in a generalized sense. The solution is

$$u(x,t) = \sum_{n\geq 1} \sin(\frac{n\pi x}{l})[a_n \cos(\frac{cn\pi t}{l}) + b_n \sin(\frac{cn\pi t}{l})$$

where

$$\sum_{n=1}^{\infty} a_n \sin(\frac{cn\pi t}{l}) \ et \ \sum_{n=1}^{\infty} (\frac{cn\pi}{l}) b_n \sin(\frac{cn\pi t}{l})$$

are the odd Fourier series of $f(x)$ and $g(x)$, i.e., we have:

$$a_n = \frac{2}{l}\int_0^l f(x)\sin(\frac{n\pi x}{l})dx \ et \ (\frac{c\pi n}{l} b_n = \frac{2}{l}\int_0^l g(x)\sin(\frac{n\pi x}{l})dx.$$

Since $g(x) = 0$ for all $x \in [0,\pi]$, we obtain $b_n = 0$ for all $n \geq 1$. By also integrating by parts, we get

$$a_n = \int_0^{\pi} \frac{2}{l}\int_0^l (x^3 - 3\pi^2 x^2 + \frac{\pi^2}{2}x)\sin(nx)dx = \frac{6((-1)^n + 1)}{n^3} \ \textbf{\textit{pour tout}} \ \ n \geq 1.$$

So, we obtain that the solution is

$$u(x,t) = \sum_{n=1}^{\infty} (\frac{6((-1)^n + 1)}{n^3}) \sin(nx)\cos(nt),$$

for all $x \in [0,\pi]$ and $t \geq 0$. We will plot the graph of $u(x,t)$ for $x \in [0,\pi]$ and $t \in [0,1]$ in Figure [1]. For each value of t, we have the vertical displacement of the string at that instant.

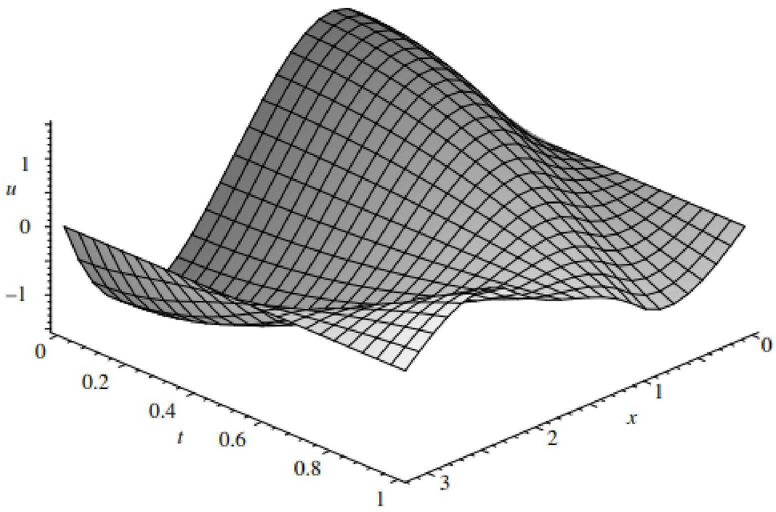
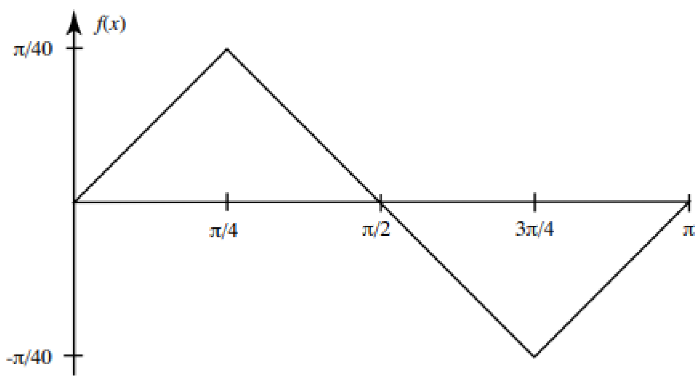


Figure [1]

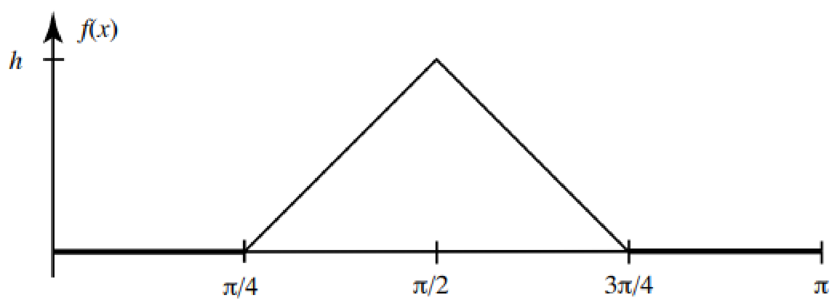
5. Exercise series $N^\circ = 06$

EXERCICE 5.1. *Determine the vertical displacement $u(x,t)$ of a vibrating (ideal) string fixed at both ends, with length $l = \pi$, initial velocity zero (i.e., $g(x) = 0$ for all $x \in [0, l]$), and initial displacement $f(x)$ represented by the graph:*

a)



b)



Solution. *a) We have*

$$f(x) = \begin{cases} \frac{x}{10}, & si\ 0 \le x \le \frac{\pi}{4}; \\ \frac{-(2x-\pi)}{20}, & si\ \frac{\pi}{4} \le x \le \frac{3\pi}{4}; \\ \frac{(x-\pi)}{10}, & si\ \frac{3\pi}{4} \le x \le \pi; \end{cases}$$

and $g(x) \equiv 0$ *for everything* $x \in [0, \pi]$*, because of the general solution to the problem and that the rope is of length* $l = \pi$*, we have*

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)[a_n \cos(cnt) + b_n \sin(cnt)].$$

In addition, we have

$$\sum_{n=1}^{\infty} a_n \sin(nx)\ si\ \sum_{n=1}^{\infty} cnb_n \sin(nx)$$

are respectively the odd Fourier series of $f(x)$ *and* $g(x)$*. Since* $g(x) \equiv 0$ *for all* $x \in [0, \pi]$*, we obtain* $b_n = 0$ *for all* $n \ge 1$*. It is therefore sufficient to calculate the* a_n*.*

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) dx$$

$$= \frac{2}{\pi}\{\int_0^{\frac{\pi}{4}} \frac{x}{10} \sin(nx)dx + \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{\pi - 2x}{20} \sin(nx)dx + \int_{\pi}^{\frac{3\pi}{4}} \frac{\pi - 2x}{20} \sin(nx)dx\}$$

$$= \frac{4}{10\pi n^2}(\sin(\frac{n\pi}{4}) - \sin(\frac{3n\pi}{4}))$$

So the solution to the problem is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{4}{10\pi n^2}(\sin(\frac{n\pi}{4}) - \sin(\frac{3n\pi}{4})) \sin(nx) \cos(cnt).$$

b) We have

$$f(x) = \begin{cases} 0, & si\ 0 \le x \le \frac{\pi}{4}; \\ \frac{(h(4x-\pi)}{\pi}, & si\ \frac{\pi}{4} \le x \le \frac{\pi}{2}; \\ \frac{(h(3\pi-4x)}{\pi}, & si\ \frac{\pi}{4} \le x \le \frac{3\pi}{4}; \\ 0, & si\ \frac{3\pi}{4} \le x \le \pi; \end{cases}$$

and $g(x) \equiv 0$ *for everything* $x \in [0, \pi]$*, because of the general solution to the problem and that the rope is of length* $l = \pi$*, on a*

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)[a_n \cos(cnt) + b_n \sin(cnt)].$$

In addition, we have

$$\sum_{n=1}^{\infty} a_n \sin(nx)\ si\ \sum_{n=1}^{\infty} cnb_n \sin(nx)$$

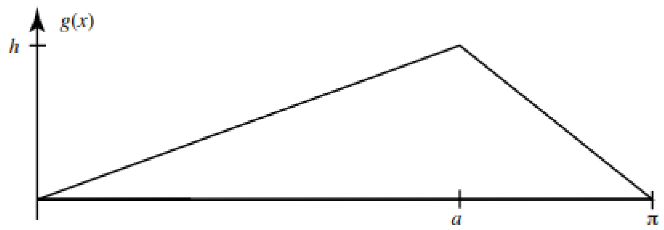
are respectively the odd Fourier series of $f(x)$ and $g(x)$. Since $g(x) \equiv 0$ for all $x \in [0, \pi]$, we obtain $b_n = 0$ for all $n \geq 1$. It is therefore enough to calculate the a_n.

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx$$
$$= \frac{2}{\pi}\{\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{\pi - 2x}{20}\sin(nx)dx + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{(h(3\pi - 4x)}{\pi}\sin(nx)dx$$
$$= \frac{8h}{n^2\pi^2}(-\sin(\frac{n\pi}{4}) + 2\sin(\frac{n\pi}{2}) - \sin(\frac{3n\pi}{4}))$$

So the solution to the problem is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8h}{n^2\pi^2}(-\sin(\frac{n\pi}{4}) + 2\sin(\frac{n\pi}{2}) - \sin(\frac{3n\pi}{4}))\sin(nx)\cos(cnt).$$

EXERCICE 11. *Determine the vertical displacement $u(x,t)$ of a vibrating (ideal) string fixed at both ends, with length $l = \pi$, initial displacement zero ($f(x) = 0$ for all $x \in [0, l]$), and initial velocity represented by the graph:*



SOLUTION. *We have $f(x) \equiv 0$ for everything $x \in [0, \pi]$ and*

$$f(x) = \begin{cases} \frac{(hx}{a}, & si\ 0 \leq x \leq a; \\ \frac{(h(\pi - x)}{\pi - a}, & si\ \leq x \leq \pi; \end{cases}$$

Due to the general solution of the problem and the fact that the string has length $l = \pi$, we have

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)[a_n\cos(cnt) + b_n\sin(cnt)].$$

In addition, we have

$$\sum_{n=1}^{\infty} a_n \sin(nx)\ si\ \sum_{n=1}^{\infty} cnb_n\sin(nx)$$

are respectively the odd Fourier series of $f(x)$ and $g(x)$. Since $g(x) \equiv 0$ for all $x \in [0, \pi]$, we obtain $a_n = 0$ for all $n \geq 1$. It is therefore enough to calculate the b_n.

$$cnb_n = \frac{2}{\pi}\int_0^{\pi} g(x)\sin(nx)dx = \frac{2}{\pi}\{\int_0^{a}\frac{hx}{a}\sin(nx)dx + \int_a^{\pi}\frac{h(\pi - x)}{\pi - a}\sin(nx)dx\} = \frac{2h\sin(na)}{n^2a(\pi - a)}$$

then

$$b_n = \frac{2h\sin(na)}{cn^3a(\pi - a)}$$

and the general solution is

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2h\sin(na)}{cn^3a(\pi-a)}\sin(nx)\sin(cnt).$$

EXERCICE 12. *Using the method of separation of variables, find solutions of each of the following equations:*

a) $x\frac{\partial u}{\partial x}-y\frac{\partial u}{\partial y}=0,$
b) $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}=2(x+y)u.$

SOLUTION. *a) We assume that* $u(x,y)=X(x)Y(y)$*, then by substituting into the PDE, we have*

$$xX'Y-yXY'=0$$

where X' *is the derivative of* X *with respect to* x *and* Y' *is the derivative of* Y *with respect to* y*. By dividing both sides of this equation by* XY*, we obtain*

$$x\frac{X'}{X}=y\frac{Y'}{Y}$$

The right-hand side of this last equation is a function of y *and the left-hand side is a function of* x*. Therefore, each of these terms must be constant. Consequently*

$$x\frac{X'}{X}=y\frac{Y'}{Y}\ \text{ où } k \text{ st une constante.}$$

This gives us a system of two ordinary differential equations as follows:

$$\begin{cases} x\frac{dX}{dx}=kX\\ y\frac{dY}{dy}=kY\end{cases}$$

These ordinary differential equations can be solved by separation of variables.

$$x\frac{dX}{dx}=kX\Rightarrow\frac{dX}{X}=k\frac{dx}{x}\Rightarrow\int\frac{dX}{X}=k\int\frac{dx}{x}\Rightarrow ln(X)=kln(x)+C_0\Rightarrow X(x)=C'x^k$$

and

$$y\frac{dY}{dy}=kY\Rightarrow\frac{dY}{Y}=k\frac{dy}{y}\Rightarrow\int\frac{dY}{Y}=k\int\frac{dy}{y}\Rightarrow ln(Y)=kln(Y)+C_1\Rightarrow Y(y)=C''y^k$$

where C_0,C_1,C' *and* C'' *are constants. So* $u(x,y)=X(x)Y(y)=C(xy)^k$ *is a solution for* $x\frac{\partial u}{\partial x}-yfrac\partial u\partial y=0$*, where* k *and* C *are constants.*
b) We assume that $u(x,y)=X(x)Y(y)$*, then by substituting into the PDE, we have* $X'Y+XY'=2(x+y)XY$ *where* X' *is the derivative of* X *with respect to* x *and* Y' *is the derivative of* Y *with respect to* y*. By dividing both sides of this equation by* XY*, we obtain*

$$\frac{X'}{X}+\frac{Y'}{Y}=2x+2y\Leftrightarrow\frac{X'}{X}-2x=-\frac{Y'}{Y}+2y.$$

The right-hand side of this last equation is a function of y *and the left-hand side is a function of* x*. Therefore, each of these terms must be constant. Consequently*

$$\frac{X'}{X} - 2x = -\frac{Y'}{Y} + 2y = k \ \textit{ where } \ k \textit{ is a constant.}$$

Thus, we obtain a system of two ordinary differential equations:

$$\begin{cases} \frac{dX}{dx} = (2x+k)X \\ \frac{dY}{dy} = (2y-k)X \end{cases}$$

These ordinary differential equations can be solved by separation of variables.

$$\frac{dX}{dx} = (2x+k)X \Rightarrow \frac{dX}{X} = (2x+k)dx \Rightarrow \int \frac{dX}{X} = \int (2x+k)dx$$
$$\Rightarrow ln(X) = x^2 + kx + C_0 \Rightarrow X(x) = C'exp(x^2 + kx)$$

and

$$\frac{dY}{dy} = (2y-k)Y \Rightarrow \frac{dY}{Y} = (2y-k)dy \Rightarrow \int \frac{dY}{Y} = \int (2y-k)dy$$
$$\Rightarrow ln(Y) = y^2 - ky + C_0 \Rightarrow Y(y) = C''exp(y^2 - ky)$$

where C_0, C_1, C' *and* C'' *are constants. So* $u(x,y) = X(x)Y(y) = Cexp(x^2 + y^2 + k(x-y))$ *is a solution for* $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u,$ *where* k *and* C *are constants.*

CHAPTER 7

The Heat Equation

In this chapter, we will study an equation that models certain evolutionary phenomena such as heat diffusion, the distribution of chemical substances, etc. We will then present the heat equation

$$\frac{\partial u}{\partial t} - k\Delta u = 0 \tag{46}$$

where u is a function defined on $\mathbb{R}^n \times \mathbb{R}_+^\star$.
With a change of variable, $T = kt$, the equation (46) takes the following form:

$$\frac{\partial u}{\partial t} - \Delta u = 0. \tag{47}$$

Thus, it is enough to study the case $k = 1$. But the question that can be asked is **How to model a reaction-diffusion equation?**

1. Modeling the Reaction-Diffusion Equation

In this section, we assume the existence of a source of particles (birth, resp. death, of insects). We will model the diffusion behavior of a population (cells, insects) or particles (chemical substances).
Let $(x, t) \in \Omega \times \mathbb{R}^n$, with Ω a bounded open set in $\mathbb{R}^n$ and $\partial\Omega$ regular. We denote

$\star$ $u(x, t)$ the particle density function (the concentration),
$\star$ $q(x, t, :::)$ the net particle creation rate (births minus deaths),
$\star$ $F(x; t; ...)$ the particle flux density, i.e., $F(x; t) \cdot n$ is the particle flux (per unit time) through a planar surface element, perpendicular to n at x and of area 1.
Throughout this chapter, we assume that u and F are regular and we consider $O \subset \Omega$ with regular boundary. The mass variation in O is due to the creation/destruction of particles in O and the flux

of particles through $\partial\Omega$.

$$\frac{\partial}{\partial t}\int_O u(x, t)dx = \int_O q(x, t)dx - \int_{\partial O} F(x, t).n(x)dSx,$$

we obtain, for all $O \subset \Omega$,

$$\int_O \frac{\partial}{\partial t} u(x,t)dx = \int_O -div(F(x,t)) + q(x,t))dx,$$

we apply the population balance law $\dfrac{\partial}{\partial t} u \operatorname{div} F + q$ in Ω. According to Fourier's law[1], heat flows from hot regions to cold regions at a speed proportional to the temperature gradient $F = -k\nabla u$ where k is the thermal conductivity constant. We assume that heat can only be lost through W and $q = 0$. Then, we have the heat equation:

$$\frac{\partial}{\partial t} u = divF = -div(k\nabla u) = k\Delta u.$$

2. Computation of a Solution

We consider the problem

$$\begin{cases} \dfrac{\partial u}{\partial t} - \partial u = 0, \; sur \, \mathbb{R}^n \times \mathbb{R}_+^\star \\ u(x,0) = g(x), \; sur \, \mathbb{R}^n \end{cases}$$

We will give the formal solution using the Fourier transform of u with respect to the spatial variables x :

$$v(\xi, t) = \hat{g}(\xi) e^{-|\xi|^2 t}$$

which gives

$$u(x,t) = \frac{1}{(2\pi} \int_{\mathbb{R}^n} \hat{g}(\xi) e^{-|\xi|^2 t} e^{ix.\xi} d\xi = \int_{\mathbb{R}^n} K(x-y,t) g(y) dy$$

where

$$K(x,t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^d} exp(i(x-y).\xi - |\xi|^2 t) d\xi = \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{x^2}{4t}}$$

where $x^2 = \sum_{i=1}^n x_i^2$ is the inner product of x with itself. The function K is called the heat kernel

Lemme 7.1. *K has the following properties ;*

1. $K \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R})$ *and for* $x \in \mathbb{R}^n$ *and* $t > 0 : (\frac{\partial}{\partial t} - \Delta) K(x,t) = 0.$

2. $\int_{\mathbb{R}^d} K(x,t)dx = 1$ *for everything* $t > 0$.

[1](1807), he described the phenomenon of thermal conductivity, Heat flux (flux) = thermal conductivity × contact surface × temperature gradient or flux

Proof

1. The proof of the first property is simple.

2. For the proof of the second property, it should be noted that

$$\int_{\mathbb{R}} e^{-x^2} dx = \sqrt{\pi}.$$

Then

$$K(x,t)dx = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{x^2}{4t}} dx$$
$$= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-\frac{x^2}{4t}} dx$$
$$= \frac{1}{\pi^{\frac{n}{2}}} \int_{\mathbb{R}} ... \int_{\mathbb{R}} e^{-x_1^2} ... e^{-x_n^2} dx_n ... dx_1$$
$$(\frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-x^2} dx)^n = 1.$$

□

3. Exercise Series $N^\circ = 07$

EXERCICE 3.1. *Solve the following problem*

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, \; x \in \mathbb{R} \; et \; t > 0. \\ u(x,0) = e^x. \end{cases}$$

CHAPTER 8

Appendix: Tests and Exams

Assessment Test $N^\circ = 01$ for Module: Mathematical Physics Equations

Duration 30 min

Exercise 1 (6 pts) :

Solve the following initial value problem:

$$c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2,\ u(x;0) = \varphi(x);$$

where $c > 0$, is a given function and $u = u(x;y)$.

Exercise 2 (9 pts.) :

Let $a \in \mathbb{R}\backslash\{1\}$ and consider the following second-order linear PDE:

$$\frac{\partial^2 u}{\partial x^2} + (1+a)\frac{\partial^2 u}{\partial x \partial y} + a\frac{\partial^2 u}{\partial y^2} = 0\ \ (E)$$

a) Determine whether the equation (E) is hyperbolic, parabolic, or elliptic.

b) Determine the characteristic equations of this equation.

c) Transform this equation into its first canonical form.

d) Deduce the general solution of (E).

e) Find the solution of (E) if $u(0;y) = y$ and $\frac{\partial u}{\partial x}(0,y) = y^2$

Model Solution of Test $N^\circ = 01$ for Module: Mathematical Physics Equations

Exercise 1 (6 pts) :

Consider the system of ordinary differential equations

(48) $$\frac{dx}{dt} = c;\ x(0) = s;$$

(49) $$\frac{dy}{dt} = 1;\ y(0) = 0;$$

The solution of this system is $x(t) = ct + s$ and $y(t) = t$: If we now consider u as a function of t, i.e. $u(t) = u(x(t);y(t))$, then by the chain rule and the fact that u is a solution of the PDE, we obtain

(50) $$\frac{du}{dt} = \frac{dx}{dt}\frac{\partial u}{\partial t} + \frac{dy}{dt}\frac{\partial u}{\partial y} = u^2\ \textit{avec}\ u(0) = u(x(0);y(0)) = u(s;0) = \varphi(s) :$$

This differential equation is equivalent to $\frac{du}{u^2} = dt$. By taking an antiderivative of each side, we obtain $-\frac{1}{u(t)} + \frac{1}{u(t)} = t$. Thus, $u(t) = \frac{u(0)}{1-t\varphi(0)} + \frac{\varphi(s)}{1-t\varphi(s)}$.
Then, for each s, the characteristic curve is

$$t \mapsto (x(t,s), y(t,s), u(t,s)) = (ct+s, t, \frac{\varphi(s)}{1-t\varphi(s)}).$$

The function $(t,s) \mapsto (x(t,s), y(t,s)) = (t+s, ct)$ has an inverse function. Indeed, $t = y$ and $s = x$

By substituting the expressions for s and t in $u(t,s)$ with their values in terms of x and y, we obtain the solution of Problem:

$$u(x,y) = \frac{\varphi(x-cy)}{1-y\varphi(x-cy)}$$

Exercise 2 (9 pts.):

a) We have $B^2 - 4AC = (1+a)^2 - 4a = 1 + 4a + a^2 - 4a = 1 + a^2 + 2a = (1+a)^2$. Since $a \neq 1$, then $B^2 - 4AC > 0$. Thus, the equation is hyperbolic on $\mathbb{R}^2$.

b) The characteristic equations are

$$\frac{dy}{dx} = \frac{B + \sqrt{B^2 - 4AC}}{A^2} = \frac{1+a+1-a}{2} = 1 \; and \; \frac{dy}{dx} = \frac{B - \sqrt{B^2 - 4AC}}{A^2} = \frac{1+a-1+a}{2} = a.$$

The solutions of these ordinary differential equations are respectively $y = x + c_1$ and $y = ax + c_2$. We can therefore consider the characteristic coordinates

$$\xi(x,y) = y - x \; and \; \eta(x,y) = y - ax.$$

By the chain rule, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = -\frac{\partial u}{\partial \xi} - a\frac{\partial u}{\partial \eta}, \frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}.$$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial}{\partial x}(\frac{\partial u}{\partial \xi}) - a\frac{\partial}{\partial x}(\frac{\partial u}{\partial \eta})$$

Exam $N^\circ = 01$ of Module: Mathematical Physics Equations

Duration 01h 30 min

Exercise 01 (10 pts):

Consider the non-homogeneous linear PDE

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F(x,t)$$

where c is a positive real number, $F(x,t)$ is a given function, and $u = u(x,t)$.

a) Show that this PDE is equivalent to the system

$$\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = v; \\ \frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = F(x,t) \end{cases}$$

b) Determine the solution to the initial value problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F(x,t), \\ u(x,0) = f(x), \\ \frac{\partial u}{\partial t}(x,0) = g(x), \end{cases}$$

proceeding as we did when describing D'Alembert's solution for the wave equation. Here $f(x)$ and $g(x)$ are given.

Exercise 02 (10 pts):

Consider the following second-order linear PDE

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0$$

a) Determine whether the equation is hyperbolic, parabolic, or elliptic.

b) Determine the characteristic equations of this equation.

c) Transform this equation into its canonical form.

Sample exam solution for the Module: Mathematical Physics Equations

Exercise 01 (10 pts):

a) We can write the PDE in the form

$$(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})u = F(x,t)$$

Indeed, we obtain

$$(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x})u = (\frac{\partial}{\partial t} + c\frac{\partial}{\partial x})(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}) = \frac{\partial^2 u}{\partial t^2} - c\frac{\partial^2 u}{\partial t \partial x} + c\frac{\partial^2 u}{\partial x \partial t} - c^2\frac{\partial^2 u}{\partial x^2}$$

$$= \frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = F(x,t)$$

assuming that the second-order partial derivatives of u are continuous and consequently that

$$\frac{\partial^2 u}{\partial t \partial x} = \frac{\partial^2 u}{\partial x \partial t}$$

we indeed obtain the system

$$(\star)\begin{cases} \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = v; \\ \frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = F(x,t) \end{cases}$$

b) To determine the solution u, we first need to determine v such that

$$\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = F(x,t).$$

The initial conditions $u(x,0) = f(x)$ and $\frac{\partial v}{\partial t}(x,0) = g(x)$ mean, using the first equation of $(\star)$, that

$$v(x,0) = \frac{\partial v}{\partial t}(x,0) - c\frac{\partial v}{\partial x}(x,0) = g(x) - cf'(x).$$

We thus have the problem to solve $\frac{\partial v}{\partial t} + c\frac{\partial v}{\partial x} = F(x,t)$ with the initial condition $v(x,0) = g(x) - cf'(x)$.

We therefore need to consider the system of ordinary differential equations

$$\frac{dt}{ds} = 1 \text{ and } \frac{dx}{ds} = c.$$

These equations can be solved by separation of variables:

$$\frac{dt}{ds} = 1 \Rightarrow dt = ds \Rightarrow \int dt = \int ds + ctet = s + t_0'$$

and

$$\frac{dx}{ds} = 1 \Rightarrow dx = ds \Rightarrow \int dx = \int ds + ctex = cs + x_0'$$

If we consider v as a function of s, i.e., $v(s) = v(x(s), t(s))$, we obtain

$$\frac{dv}{ds} = \frac{\partial v}{\partial t}\frac{dx}{ds} + \frac{\partial v}{\partial t}\frac{dt}{ds} = c\frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} = F(x(s), t(s)) = F(cs + x_0', s + t_0').$$

By the fundamental theorem of calculus, we obtain that there is a unique solution such that $v(0) = v_0$ and it is

$$v(s) = \left(\int_0^s F(cw + x_0', w + t_0')\, dw\right) + v_0.$$

If we consider the initial values $x_0' = \tau, t_0' = 0$ and $v_0 = g(\tau) - cf'(\tau), \rightarrow (00, 50)$ we obtain for each τ the characteristic curve

$$s \Rightarrow (x(s,\tau), t(s,\tau), v(s,\tau)) = (cs + \tau, s, (\int_0^s F(cw + \tau, w)\, dw) + g(\tau) - cf'(\tau))$$

We can now consider the problem

$\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial}x = v$ with the condition $u(x,0) = f(x)$

We therefore need to consider the system of ordinary differential equations

$$\frac{dt}{ds} = 1 \text{ and } \frac{dx}{ds} = c.$$

These equations can be solved by separation of variables:

$$\frac{dt}{ds} = 1 \Rightarrow dt = ds \Rightarrow \int dt = \int ds + ctet = s + t_0'$$

and

$$\frac{dx}{ds} = -c \Rightarrow dx = ds \Rightarrow \int dx = -c \int ds + ctex = -cs + x_0'$$

If we consider v as a function of s, i.e., $v(s) = v(x(s), t(s))$, we obtain

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dx}{ds} + \frac{\partial u}{\partial t}\frac{dt}{ds} = -c\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = v(x(s), t(s)), s + t_0').$$

Exercise 02 (10 pts):

Consider the following second-order linear PDE

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} = 0$$

a) Determine whether the equation is hyperbolic, parabolic, or elliptic.

b) Determine the characteristic equations of this equation.

c) Transform this equation into its canonical form.

Exam $N° = 01$ of Module: Mathematical Physics Equations

Duration 01h

Exercise 1. (4 pts) Answer true or false to the following questions. No justification is required. Grading: **+1** for a correct answer, **-1** for an incorrect answer.

1. The heat equation is a second-order PDE.

2. Laplace's equation is hyperbolic.

3. The vibrating string equation is parabolic.

4. Laplace's equation is linear.

Exercise 2. (6 pts) Solve the following Cauchy problem:

$$\begin{cases} u_t + xu_x = 0, \ u = u(x,t), u(1,t) = 2\sinh(t). \end{cases}$$

Exercise 3. (10 pts) 1. Determine the eigenvalues and the associated eigenfunctions of the following eigenvalue problem:

$$(P) \begin{cases} X'' + \lambda X = 0, \ \ 0 < x < \pi, \\ X(0) = 0, \ \ X(\pi) = 0. \end{cases}$$

2. Determine the odd Fourier series of the function $h(x) = x(\pi - x)$ on the interval $[0, \pi]$.

3. (a) By applying the method of separation of variables, solve the following mixed problem:

$$\begin{cases} u_t - u_{xx} - u = 0, \ \ 0 < x < \pi, \, t > 0, \\ u(0,t) = u(\pi,t) = 0, \ \ \ t \geq 0, \\ u(x,0) = h(x), \ \ \ 0 \leq x \leq \pi, \end{cases}$$

(b) Calculate the limit $\lim_{t \to +\infty} u(x,t)$.

Sample exam solution for the Module: Mathematical Physics Equations

Exercise 1. (04 pts)

1. True. $\rightarrow (01, 00)$

2. False.$\rightarrow (01, 00)$

3. False.$\rightarrow (01, 00)$

4. True.$\rightarrow (01, 00)$

Exercise 2. (06 pts)

By integrating the characteristic system,

$$\begin{cases} \frac{dx}{ds} = x, \ x(0) = 1, \rightarrow (01, 00) \\ \frac{dt}{ds} = 1, \quad t(0) = r, \rightarrow (01, 00) \\ \frac{du}{ds} = 0, \quad u(0) = 2\sinh(r), \rightarrow (01, 00) \end{cases}$$

we obtain

$$\begin{cases} x = e^s, \ \rightarrow (00, 50) \\ t = s + r, \quad \rightarrow (00, 50) \\ u = 2\sinh(r)., \quad \rightarrow (00, 50) \end{cases}$$

Hence, $e^t = e^s e^r = xe^r$, and thus $e^r = \frac{e^t}{x}.\rightarrow (00, 50)$ The solution is therefore

$$u(x,t) = e^r - e^t = \frac{e^t}{x} - \frac{x}{e^t}.\rightarrow (01, 00)$$

Exercise 3. (10 pts) 1. By studying the problem (P), we find

(a) $\lambda < 0 : X \equiv 0; \lambda < 0$ is not an eigenvalue.$\rightarrow (00, 25)$ **(b)** $\lambda = 0 : X \equiv 0; \lambda = 0$ is not an eigenvalue.$\rightarrow (00, 25)$ **(c)** $\lambda > 0$: for $n \geq 0$

$$\begin{cases} \lambda_n = n^2, \ \rightarrow (00, 50) \\ X_n(x) = \sin(nx), \quad \rightarrow (00, 50) \end{cases}$$

2. The odd Fourier series of the function h is given by

$$h(x) \approx \sum_{n=1}^{+\infty} b_n \sin(nx), \ \ x \in [0, \pi], \rightarrow (00, 50)$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} h(x) \sin(nx) dx, \ \ n \geq 0, \rightarrow (00, 50)$$

$$= \frac{4}{\pi n^3}[1 - (-1)^n].\rightarrow (01, 00)$$

3. (a) We look for the solution in the form $u(x,t) = X(x)T(t).\rightarrow (00, 50)$ By substituting this form into the PDE, we find $XT' - X''T - XT = 0 \Leftrightarrow \frac{T'-T}{T} = \frac{X''}{X} = -\lambda, \rightarrow (00, 50)$

$$\Rightarrow \begin{cases} X'' + \lambda X = 0, \ \rightarrow (00, 50) \\ T' + (\lambda - 1)T = 0, \quad \rightarrow (00, 50) \end{cases}$$

In addition, we have:

$$\begin{cases} u(0,t) = 0 \Rightarrow X(0) = 0, \ \rightarrow (00, 25) \\ u(\pi,t) = 0 \Rightarrow X(\pi) = 0, \ \rightarrow (00, 25) \end{cases}$$

This gives us an eigenvalue problem:

$$\begin{cases} X'' + \lambda X = 0, \ 0 < x < \pi. \\ X(0) = X(\pi) = 0 \end{cases}$$

This is the problem (P) from question 1, so the eigenfunctions are $X_n(x) = \sin(\sqrt{\lambda_n} x)$, where $\lambda_n = n^2, \ n \geq 1$.

The function $T(t)$ is determined by solving the second equation, for $n \geq 1$,

$$T_n' + (n^2 - 1)T_n = 0, \ t > 0.$$

We find: $T_n(t) = b_n e^{-(n^2-1)t}, \ n \geq 1 \rightarrow (00, 50)$ Hence, by the superposition principle, we obtain

$$u(x,t) = \sum_{n\geq 1} u_n(x,t) = \sum_{n\geq 1} X_n(x)T_n(t) = \sum_{n\geq 1} b_n e^{-(n^2-1)t} \sin(nx). \rightarrow (00, 50)$$

We have

$$u(x,0) = h(x) = \sum_{n\geq 1} b_n \sin(nx), \rightarrow (00, 50)$$

this is the odd Fourier series of the function $h(x)$ on $[0, \pi]$. The b_n are calculated in question 2. Consequently, the formal solution of the mixed problem is:

$$(\star) \ u(x,t) = \frac{4}{\pi} \sum_{n\geq 1} \frac{[1 - (-1)^n]}{n^3} \sin(nx). \rightarrow (00, 50)$$

(b) First, establish the uniform convergence of the series $(\star)$. We have for all $(x,t) \in [0, \pi] \times [0, +\infty[\rightarrow (01, 00)$

$$|u_n(x,t)| \leq \frac{8}{\pi n^3} := v_n, \ n \geq 0.$$

The numerical series $\sum v_n$ is convergent; it follows that the series $(\star)$ is normally, and thus uniformly, convergent on $[0, \pi] \times [0, +\infty[$. Then $\rightarrow (01, 00)$

$$\lim_{t\to+\infty} u(x,t) = b_1 \sin' x) + \sum_{n\geq 2} \lim_{t\to+\infty} b_n e^{-(n^2-1)t} \sin(nx) = b_1 sin(x)$$

c-‘a-d,

$$\lim_{t\to+\infty} u(x,t) = \frac{8}{\pi} \sin(x) \rightarrow (01, 00)$$

Exam $N^\circ = 01$ of Module: Mathematical Physics Equations

Duration 01h 30 min

Exercise 01 (07 pts): Let λ be a real parameter. We consider the second-order PDE, with unknown $u : \mathbb{R} \times \mathbb{R}^\star \to \mathbb{R}$, as follows

$$\frac{\partial^2 u}{\partial x^2} - 2y\frac{\partial^2 u}{\partial x \partial y} + \lambda y^2 \frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 0 \quad (E_h)$$

1. Discuss the type of the equation (E_h) according to the values of the parameter λ.

2. For $\lambda = 1$, determine the characteristic curves of (E_1).

3. Reduce the equation (E_1) to its standard form and solve the resulting equation.

4. Deduce the solutions of the equation (E_1).

Exercise 02 (08 pts): We consider the following partial differential equation with unknown $u : \mathbb{R} \times [0, +\infty \to \mathbb{R}$:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - (\frac{\partial u}{\partial x})^2 = 0 \quad (D)$$

1. Show that by setting $v = e^u$, i.e., $v(x,t) = \exp(u(x,t))$, the equation (D) reduces to the following:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \quad (H)$$

2. What type is the equation (H)?

3. Determine, using the method of separation of variables, the bounded solutions of the equation (H).

4. Deduce a class of solutions of (H).

Exercise 03 (05 pts): In polar coordinates (r, θ), Laplace's equation is given by

$$r\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0,$$

and its solution, for a circular region, is defined by

$$u(r,\theta) = A_0 + \alpha lnr + \sum_{n\geq 1}[(A_n r^n + \alpha_n r^{-n})\cos(n\theta) + (\beta_n r^n + \beta_n r^{-n})\sin(n\theta)].$$

Determine the expression of u for the unit disk in the plane such that

$$u(1,\theta) = 1 - \cos(2\theta), \;\; 0 \leq \theta \leq 2\pi.$$

Sample exam solution for the Module: Mathematical Physics Equations

Exercise 01 (07 pts):

Let λ be a real parameter. We consider the second-order PDE, with unknown $u : \mathbb{R} \times \mathbb{R}^\star \to \mathbb{R}$, as follows

$$\frac{\partial^2 u}{\partial x^2} - 2y\frac{\partial^2 u}{\partial x \partial y} + \lambda y^2 \frac{\partial^2 u}{\partial y^2} + y\frac{\partial u}{\partial y} = 0 \quad (E_h)$$

1. Type of the equation (E_h); $\rightarrow (4 \times 00,25)$

Discriminant of the equation: $\Delta = b^2 - ac = y^2 - \lambda y^2 = (1-\lambda)y^2$.

Since $y^2 > 0$, the sign of Δ depends on that of $1-\lambda$.

- If $\lambda < 1$ then $\Delta > 0$ and the equation is hyperbolic.
- If $\lambda = 1$ then $\Delta = 0$ and the equation is parabolic.
- If $\lambda > 1$ then $\Delta < 0$ and the equation is elliptic.

2. Characteristic curves of (E_h).

The characteristic curves of (E_h) are the solutions of the differential equation:

$$(\frac{dy}{dx})^2 + 2y(\frac{dy}{dx}) + 2y^2 = 0. \rightarrow (00,50)$$

$$(\frac{dy}{dx})^2 + 2y(\frac{dy}{dx}) + 2y^2 = 0 \Rightarrow (\frac{dy}{dx} + y)^2 \Rightarrow \frac{dy}{dx} + y = 0.$$

$$\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{y} = -dx \Rightarrow ln|y| = -x + A \Rightarrow y = Ce^{-x} \rightarrow (00,50)$$

where A and C are real constants.

The equation (E_1) admits a single family of characteristic curves

$$ye^x = C. \rightarrow (00,50)$$

3. Reduction of the equation (E_1) to the standard form

Let $X = ye^x \ and \ Y = y, \rightarrow (00,50)$

note that the Jacobian is not identically zero:

$$\frac{D(x,y)}{D(x,x)} = \begin{vmatrix} ye^x & ye^x \\ 0 & 1 \end{vmatrix} = ye^x \neq 0, \ \forall (x,y) \in \mathbb{R} \times \mathbb{R}^\star.$$

We have

Bibliography

[1] Baddari K, Abbassov A. Equations de la physique mathématique appliquées. OPU ; 2009.

[2] Courant R, Hilbert D. Methods of mathematical physics. Wiley -V C H , édition de 1989.

[3] Eriksson K, Estep D, Hansbo P, Johnson C. Computational Differential Equations. Cambridge University Press, New York, 1990.

[4] Nikolenko V. Équations de la physique mathématique. UM, Moscou, 1981.

[5] Pinchover, Rubenstein J. An Introduction to Partial Differential Equations. Cambridge University Press, 2005.

[6] Reinhard H. Équations aux dérivées partielles. Dunod, paris, 2001.

[7] Schwartz L. Méthodes Mathématiques pour les Sciences Physiques. Hermann, Paris, 1983.

[8] Walter A. Strauss. Partial Differential Equations: An Introduction. Wiley, 1992.