الجمهورية الجزائرية الديمقراطية الشعبية

The People's Democratic Republic of Algeria

وزارة التعليم العالي و البحث العلمي

Ministry of Higher Education and Scientific Research

جامعة عمار ثليجي – الأغواط

University Amar Telidji of Laghouat

Faculty : Technology

Departement : Mechanical Engineering

Domain : Science and Technology

Field : Mechanical Engineering

Speciality : Mechanical Engineering

Course Handout (Lectures, Tutorials, Practical Work)

Intended for students of: Mechanical Engineering, Level: 3rd Year

Bachelor's Degree

STRENGTH OF MATERIALS 2

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Academic year : 2025/2026

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Preamble

This course material, "Strength of Materials 2," is designed as a direct continuation of your foundational studies in solid mechanics. While your first course introduced the fundamental principles of stress, strain, and material behavior under simple loading, this module aims to equip you with the advanced analytical tools necessary for the design and verification of real-world mechanical and structural components.

In the context of mechanical construction, components such as shafts, beams, and frames are rarely subjected to simple, isolated loads. Instead, they must withstand combined and complex loading scenarios while maintaining structural integrity, limiting deflections, and often operating under statically indeterminate conditions.

This document is structured to guide you through these essential advanced topics:

Beam Bending and Internal Forces: We will begin with a comprehensive review of shear forces and bending moments before delving into the distribution of both normal and tangential (shear) stresses within a beam's cross-section.

Deflection of Beams: Understanding and calculating the elastic curve is critical for ensuring that deformations remain within safe and functional limits. We will explore multiple methods for determining beam deflections, including direct integration, the area-moment method, and the powerful principle of superposition.

Energy Methods: You will be introduced to the concept of strain energy and its application through Castigliano's theorem. This provides an elegant and highly efficient approach for calculating displacements and solving statically indeterminate systems, forming a cornerstone of advanced structural analysis.

Combined Loadings: Moving beyond simple bending, we will analyze more realistic scenarios, including unsymmetrical bending, eccentric (combined) axial-bending loading, and the critical case of combined bending and torsion, a fundamental analysis for any machine shaft design.

Analysis of Indeterminate Structures: Finally, we will address statically indeterminate beams and continuous structures. You will learn systematic methods such as the force superposition method and the specialized Three-Moment Theorem (Clapeyron's theorem) to resolve these practical engineering problems.

The course emphasizes a rigorous, analytical approach rooted in the principles of equilibrium, material compatibility, and energy conservation. Numerous worked examples are integrated throughout the text to illustrate theoretical concepts and their direct application to typical mechanical construction problems.

It is expected that you will actively engage with the material, practice deriving equations, and solve the proposed problems. Mastery of this content will provide you with an indispensable analytical framework for your future career in mechanical design, structural analysis, and advanced engineering simulations.

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Plane bending of symmetrical beams - Recap

1.1. Introduction:

A beam is defined as a structural element characterized by one dimension (its length) being significantly greater than the other two dimensions (its width and height). Beams are primarily subjected to bending, which may result from external actions such as applied forces and moments, or from internal actions including self-weight and temperature variations. Consequently, any structural member that experiences bending as a predominant mode of deformation can be classified as a beam.

The bending of beams refers to the transverse deformation of their initially straight longitudinal axis under applied loads. This phenomenon is characterized by the displacement of cross-sections perpendicular to the beam's neutral axis, resulting in curvature along its span. Beams are typically supported by constraints that restrict specific degrees of freedom. A critical aspect of structural analysis involves defining these supports mathematically.

- **Simple (Roller) supports**:

- Restrict translational movement perpendicular to the support surface (e.g., vertical displacement).
- Permit free rotation and longitudinal translation (axial displacement).
- Generate vertical reaction forces but no moment resistance.

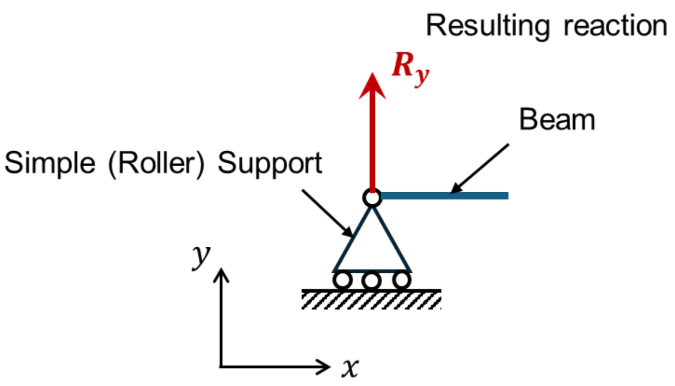


Figure 1. Simple (Roller) support.

- **Pinned support:**

- Restrict translational movement in all directions.
- Permit free rotation.
- Generate reaction forces in two directions (vatical and horizontal) but no moment resistance.

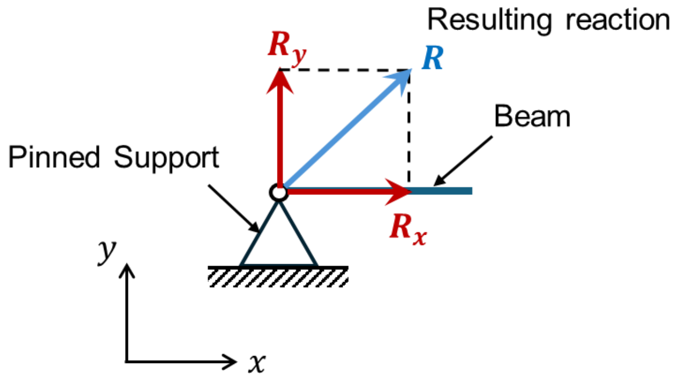


Figure 2. Pinned support.

- **Fixed support:**

- Restrict all translational and rotational displacements.
- Produce vertical, horizontal, and moment reactions.

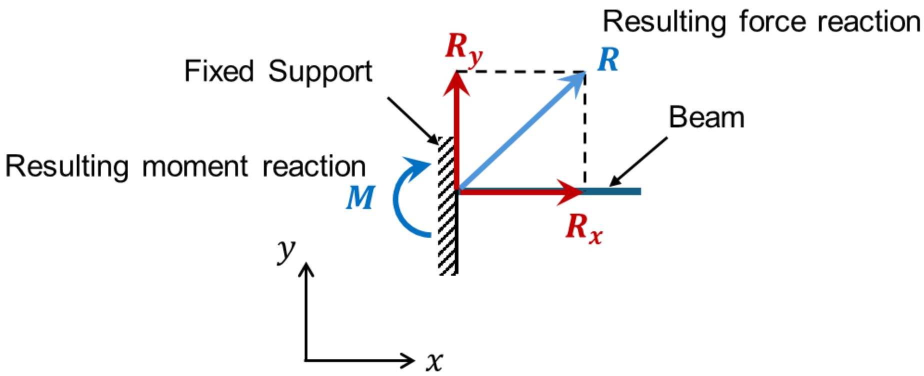


Figure 3. Fixed support.

In the example of figure 4, we find the three types of supports.

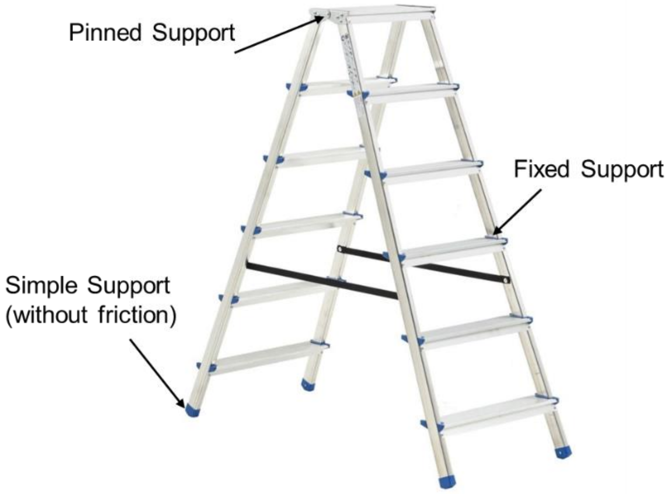


Figure 4. Three types of supports in stepladder.

1.2. Reminder bending moment - shear force:

Shear force and bending moment are fundamental internal actions in beams subjected to external loads. Their distributions along the length of a beam are essential for analyzing structural behavior and ensuring safety.

- **Shear Force (V):** The internal force acting tangentially to a cross-section of the beam, resulting from external loads. It represents the tendency of the section to slide relative to the adjacent section.
- **Bending Moment (M):** The internal moment that causes the beam to bend, resulting from external forces and moments applied to the structure.

The relationship between distributed load $q(x)$, shear force $V(x)$, and bending moment $M(x)$is given by the following differential equations:

$$\begin{cases} q(x) = -\dfrac{dV}{dx} \\ V(x) = \dfrac{dM}{dx} \end{cases} \quad (01)$$

The rate of change of shear force along the beam equals the negative value of the distributed load at that point. The rate of change of bending moment at a point equals the shear force at that location.

The variation of bending moment and shear force can be plotted in a graph called diagram (bending moment and shear force diagrams). The maximum values can be observed in the diagram (example figure 5).

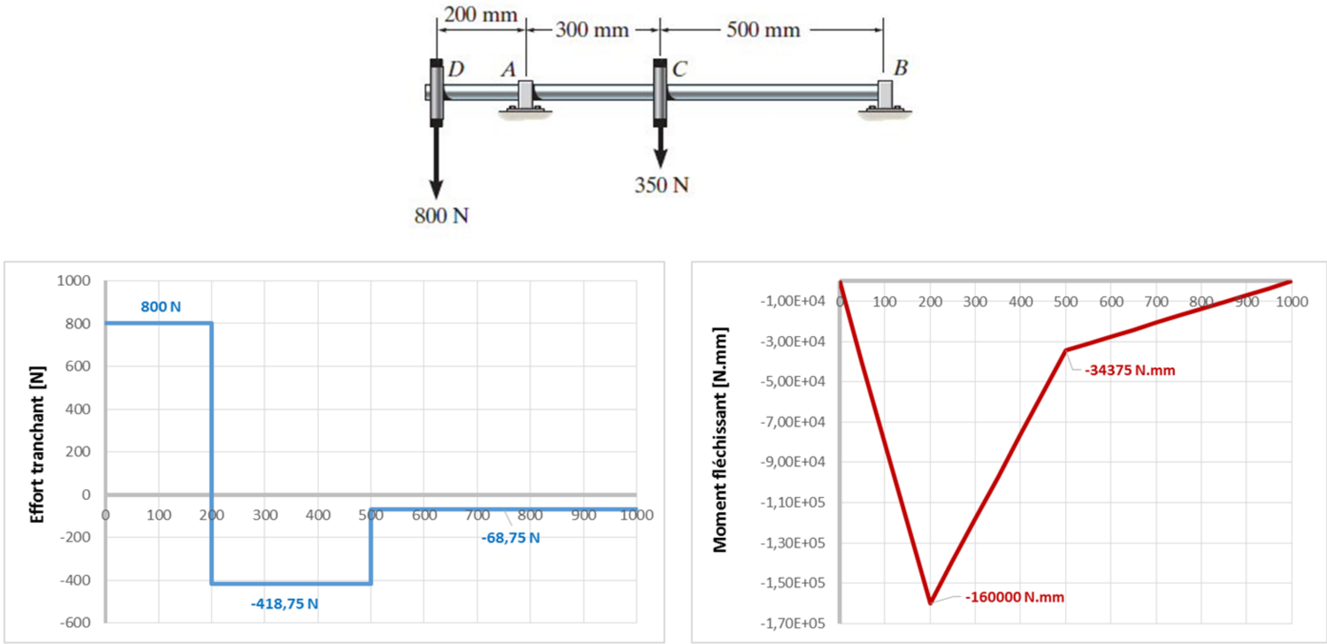


Figure 5. Beam example bending moment and shear force.

Because shear force and bending moment diagrams provide detailed information about their variation along the beam axis, they are often used by engineers to determine where to place reinforcing materials in the beam at various points along its length.

Important notes:

- Concentrated loads cause an abrupt change in shear force but do not change the bending moment instantaneously.
- Concentrated moments (couples) cause an abrupt change in the bending moment but do not affect the shear force at that point.
- The area under the load diagram between two points equals the change in shear force between those points.
- The area under the shear force diagram between two points equals the change in bending moment between those points.

Sign Conventions:

- Shear force is typically considered positive if it causes a clockwise rotation of the segment on which it acts.

- Bending moment is positive if it causes the beam to sag (concave upward or form that retains a liquid), placing the lower fibers in tension.

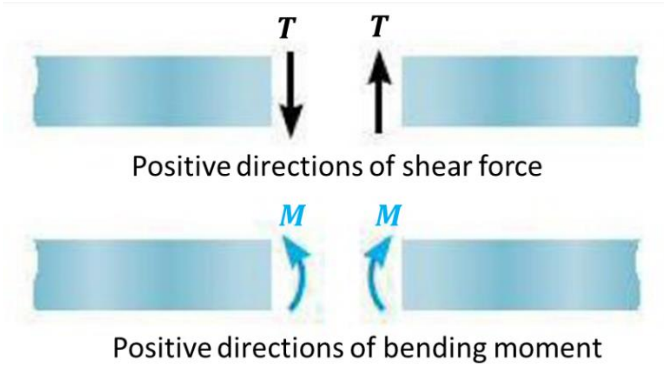
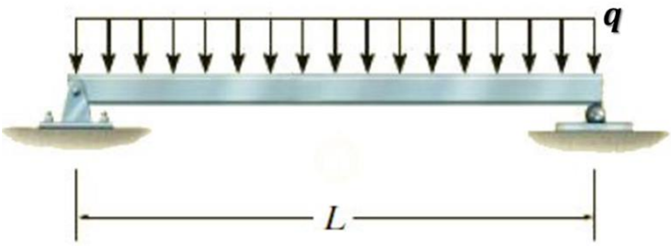


Figure 6. Sign Conventions.

Examples:

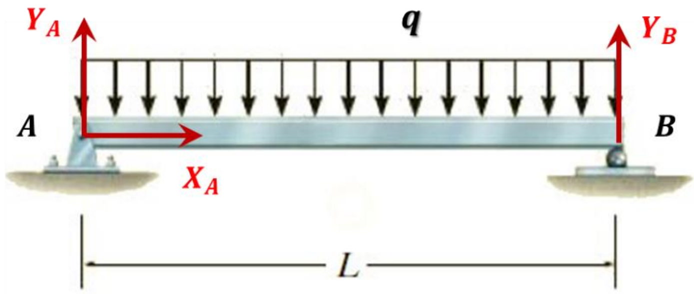
1. Draw the diagrams of the shear force and bending moment of the different beams stressed below.



Let us first determine the reactions to the supports.

The left support A is a pinned support where the direction of the reaction is opposite to the load (we will have two components, one perpendicular to the surface of the support and the other parallel to its surface).

The right support B is a simple support where the reaction is always perpendicular to the surface of the support.



a. Let us apply static equilibrium to determine the reactions (Newton's law):

$$\sum F_x = 0 \quad \rightarrow \quad X_A = 0$$

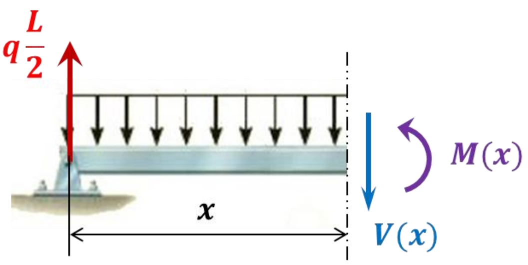
$$\sum F_y = 0 \quad \rightarrow \quad Y_A + Y_B - qL = 0$$

$$\sum M_{/A} = 0 \quad \rightarrow \quad Y_B L - qL \times \frac{L}{2} = 0 \quad \rightarrow \quad Y_B = q\frac{L}{2} \quad \text{and} \quad Y_A = q\frac{L}{2}$$

We note that the beam is symmetrical with respect to its middle, hence:

$$Y_A = Y_B = q\frac{L}{2}$$

b. Let us plot the diagrams of the shear force $V(x)$ and the bending moment $M(x)$. To do this, let us make an imaginary cut in the beam at a distance x from the left end A.

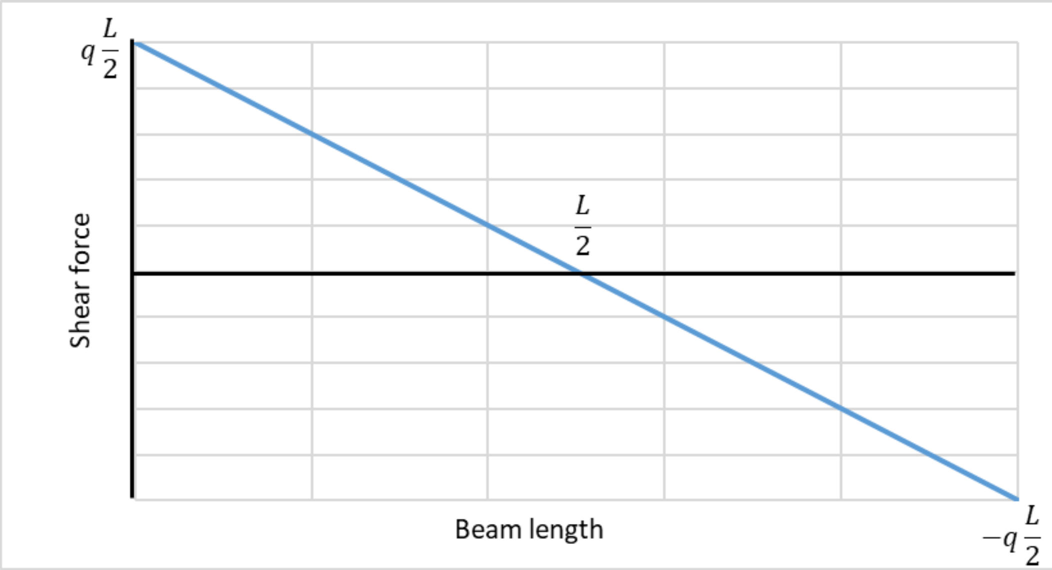


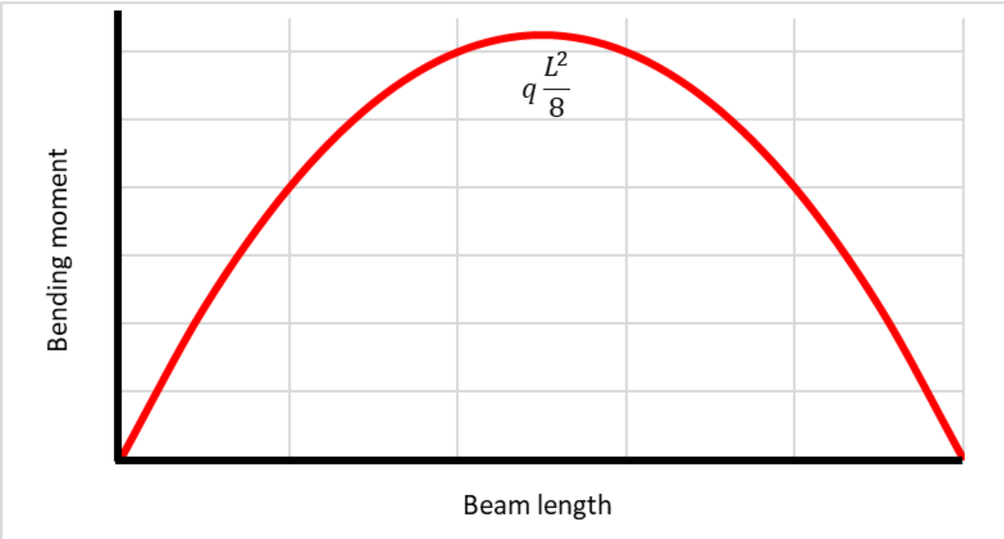
$$T(x) + qx - q\frac{L}{2} = 0 \quad \rightarrow \quad T(x) = -qx + q\frac{L}{2} \quad \rightarrow \quad \begin{cases} x = 0 \rightarrow T(0) = q\frac{L}{2} \\ x = L \rightarrow T(L) = -q\frac{L}{2} \end{cases}$$

$$M(x) + qx \times \frac{x}{2} - q\frac{L}{2}x = 0 \quad \rightarrow \quad M(x) = -q\frac{x^2}{2} + q\frac{L}{2}x \rightarrow \begin{cases} x = 0 \rightarrow M(0) = 0 \\ x = L \rightarrow M(L) = 0 \end{cases}$$

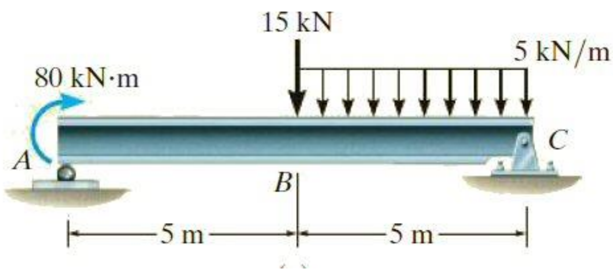
The maximum value of the bending moment and its position are determined by equating its derivative to zero, i.e.:

$$\frac{dM(x)}{dx} = 0 \quad \rightarrow \quad -qx + q\frac{L}{2} = 0 \quad \rightarrow \quad x = \frac{L}{2} \quad \rightarrow \quad M_{max} = q\frac{L^2}{8}$$





2. Draw the diagrams of the shear force and bending moment of the different beams stressed below.



Let us first determine the reactions to the supports.

The left support A is a simple support where the reaction is always perpendicular to the surface of the support.

The right support C is a pinned support where the direction of the reaction is opposite to the load (we will have two components, one perpendicular to the surface of the support and the other parallel to its surface).

a. Let us apply static equilibrium to determine the reactions:

$$\sum F_x = 0 \quad \rightarrow \quad X_C = 0$$

$$\sum F_y = 0 \quad \rightarrow \quad Y_A + Y_C - 15 - 5 \times 5 = 0 \quad \rightarrow \quad Y_A + Y_C = 40\ kN$$

$$\sum M_{/A} = 0 \rightarrow Y_C \times 10 - 5 \times 5 \times 7.5 - 15 \times 5 - 80 = 0 \rightarrow Y_C = 34.25\ kN$$

and $Y_A = 5.75\ kN$

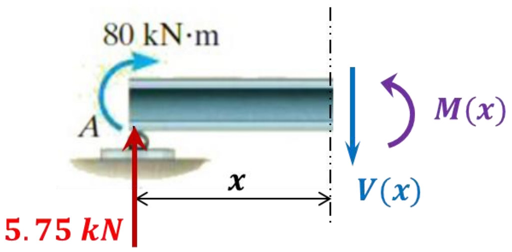
b. Let us plot the diagrams of the shear force $V(x)$ and the bending moment $M(x)$. To do this, let us make an imaginary cut in the beam at a distance x from the left end A.

- $0 \leq x \leq 5\ m$

$V(x) - 5.75 = 0 \ \rightarrow \ V(x) = 5.75\ kN$

$$M(x) - 5.75x - 80 = 0 \ \rightarrow M(x) = 5.75x + 80$$

$$\rightarrow \begin{cases} x = 0 \ \rightarrow M(0) = 80\ kN.m \\ x = 5\ m \rightarrow M(5) = 108.75\ kN.m \end{cases}$$



- $5\ m \leq x \leq 10\ m$

$$V(x) - 5.75 + 15 + 5(x-5) = 0 \ \rightarrow \ V(x) = -5x + 15.75$$

$$\rightarrow \begin{cases} x = 5\,m \ \rightarrow V(5) = -9.25\,kN \\ x = 10\,m \rightarrow V(5) = -34.25\,kN \end{cases}$$

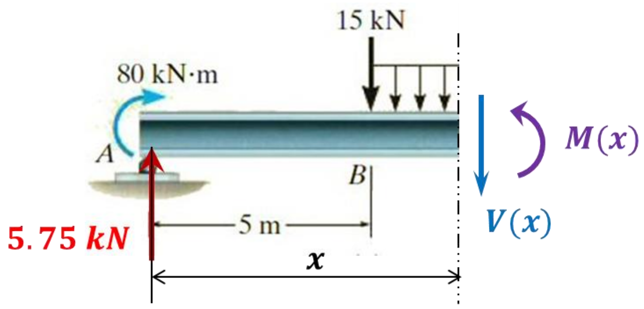
$$M(x) - 5.75x - 80 + 15(x-5) + 5 \times \frac{(x-5)^2}{2} = 0$$

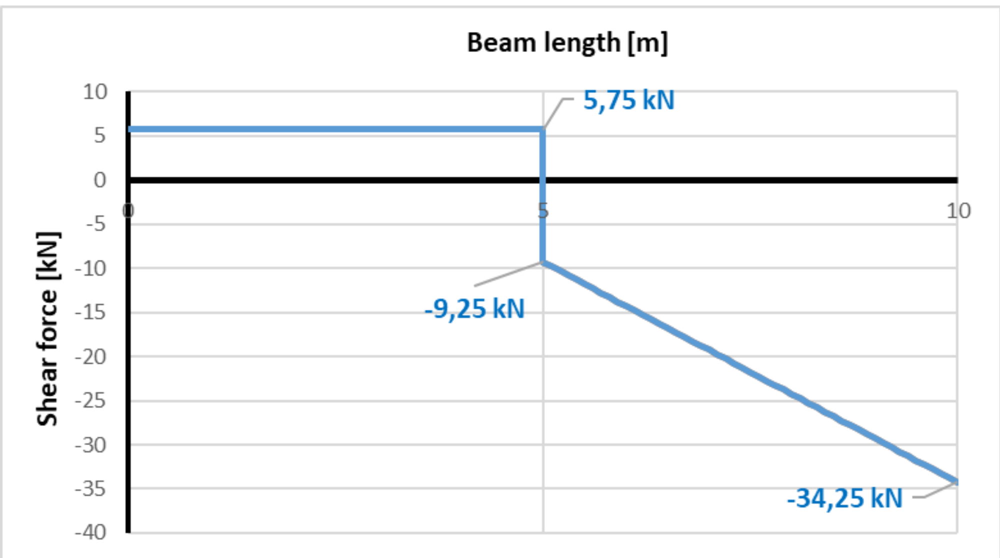
$$\rightarrow M(x) = -2.5x^2 + 15.75x + 92.5$$

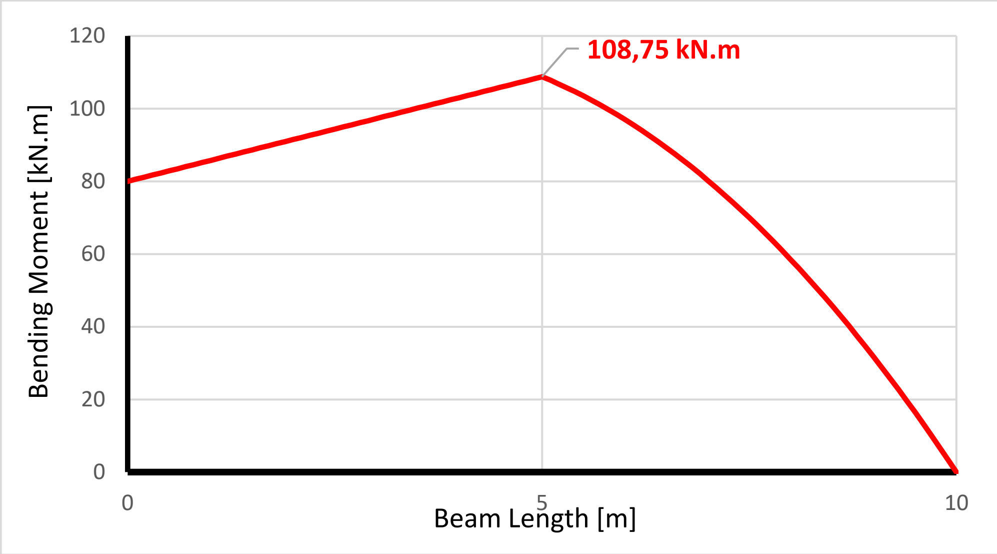
$$\rightarrow \begin{cases} x = 5\,m \ \rightarrow M(5) = 108.75\,kN.m \\ x = 10\,m \rightarrow M(10) = 0\,kN.m \end{cases}$$

The maximum value of the bending moment and its position are determined by equating its derivative to zero, i.e.:

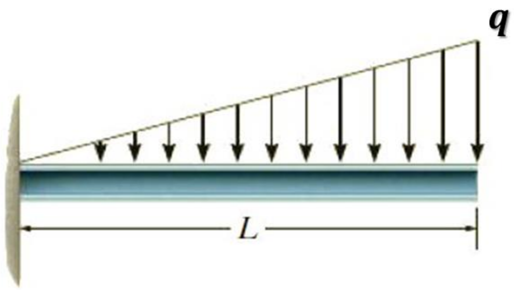
$$\frac{dM(x)}{dx} = 0 \ \rightarrow \ -5x + 15.75 = 0 \ \rightarrow \ x = 3.15\,m \ \rightarrow$$ out of range





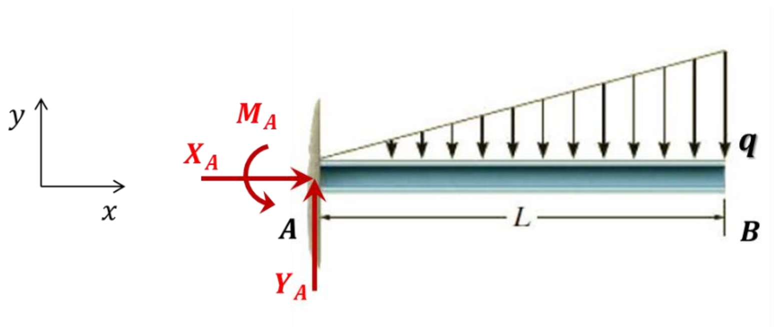


3. Draw the diagrams of the shear force and bending moment of the different beams stressed below.



Let us first determine the reactions to the supports.

The left support A is a typical embedment support where the direction of the reaction is opposite to the load (we will have two components, one perpendicular to the surface of the support and the other parallel to its surface) in addition to a moment called the embedment moment. The right end B is free, no reactions.



a. Let us apply static equilibrium to determine the reactions:

$$\sum F_x = 0 \quad \rightarrow \quad X_A = 0$$

$$\sum F_y = 0 \quad \rightarrow \quad Y_A - q\frac{L}{2} = 0 \rightarrow \quad Y_A = q\frac{L}{2}$$

$$\sum M_{/A} = 0 \rightarrow M_A - q\frac{L}{2} \times \frac{2L}{3} = 0 \rightarrow M_A = q\frac{L^2}{3}$$

b. Let us plot the diagrams of the shear force $V(x)$ and the bending moment $M(x)$. To do this, let us make an imaginary cut in the beam at a distance x from the left end A.

Let's calculate the load at the cut section:

$$\frac{q(x)}{x} = \frac{q}{L} \quad \rightarrow \quad q(x) = q\frac{x}{L}$$

The resultant of the load on the left section is:

$$Q(x) = q(x) \times \frac{x}{2} = q\frac{x^2}{2L}$$

The shear force at the cut section is:

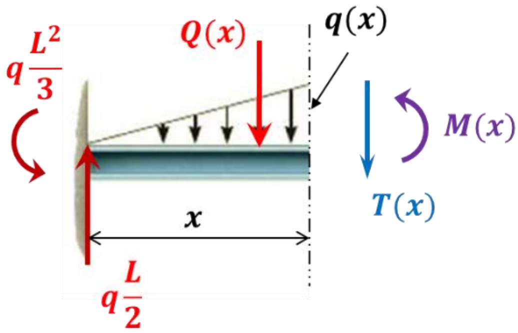
$$T(x) + Q(x) - q\frac{L}{2} = 0 \quad \rightarrow \quad T(x) = -q\frac{x^2}{2L} + q\frac{L}{2}$$

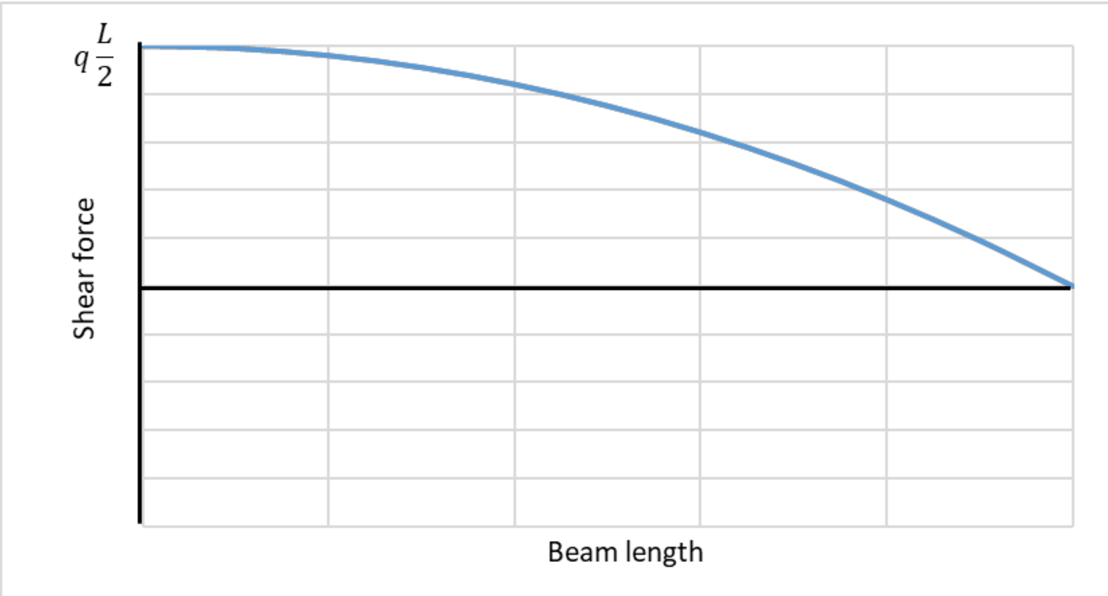
$$\rightarrow \quad \begin{cases} x = 0 \ \rightarrow \ T(0) = q\dfrac{L}{2} \\ x = L \ \rightarrow \ T(L) = 0 \end{cases}$$

The bending moment at the cut section is:

$$M(x) + Q(x)\frac{x}{3} - q\frac{L}{2}x + q\frac{L^2}{3} = 0 \quad \rightarrow \quad M(x) = -q\frac{x^3}{6L} + q\frac{L}{2}x - q\frac{L^2}{3}$$

$$\rightarrow \quad \begin{cases} x = 0 \ \rightarrow \ M(0) = -q\dfrac{L^2}{3} \\ x = L \ \rightarrow \ M(L) = 0 \end{cases}$$


$q\frac{L^2}{3}$
$Q(x)$
$q(x)$
$M(x)$
$T(x)$
x
$q\frac{L}{2}$


$q\frac{L}{2}$
Shear force
Beam length


Beam length
Bending moment
$-q\frac{L^2}{3}$

1.3. Distribution of Normal Stresses in Simple Beam Bending:

We will examine the deformations that arise when a straight beam composed of a homogeneous material is subjected to simple bending. This analysis is restricted to beams whose cross-sectional area is symmetric with respect to a principal axis. Furthermore, the bending moment is assumed to act about an axis perpendicular to this axis of symmetry.

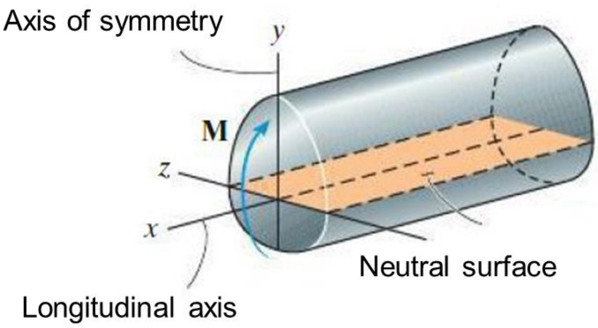


Figure 7. Beam subjected to simple bending.

It is observed that, in a beam subjected to bending, the region above the neutral surface undergoes tensile deformation (stretching), while the region below the neutral surface experiences compressive deformation (shortening). The neutral surface is neither tense nor compressed.

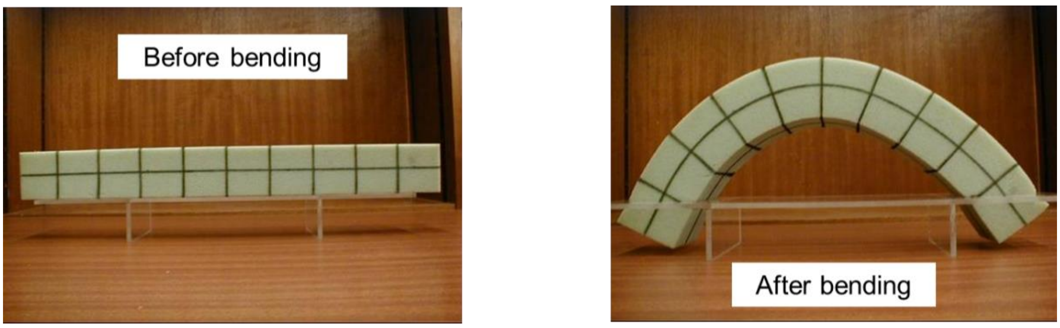


Figure 8. Longitudinal fibers before and after bending.

Based on these observations, we establish four fundamental assumptions governing stress-induced deformations in beams under bending:

a. Invariance of the Neutral Surface:

The longitudinal axis x, lying within the neutral surface, retains its original length during deformation. Applied moments induce curvature in this axis, confining it to the $x - y$ plane of symmetry.

b. Bernoulli-Navier Hypothesis:

All cross-sections remain planar and perpendicular to the deformed longitudinal axis (neutral surface) after bending. This eliminates shear deformation effects.

c. Rigid Cross-Section Assumption:

In-plane deformations of the cross-section (within the $y - z$ plane) are neglected. Cross sections rotate rigidly about the neutral axis.

d. Neutral Axis Definition:

The neutral axis (z-axis) is defined as the line within the cross-sectional plane about which the section rotates. Along this axis, longitudinal normal stresses are zero.

Consider an infinitesimal segment of the beam, positioned at an axial coordinate x along its longitudinal axis. In its undeformed state, this segment has a differential length of Δx.

Let us consider the following hypotheses:

- The x-axis aligns with the beam's longitudinal neutral axis.
- The segment Δx is sufficiently small to approximate stress and strain as constant across its length.
- The analysis assumes small deformations, consistent with linear elasticity theory.

This approach isolates the segment for detailed stress-strain analysis while maintaining compatibility with Euler-Bernoulli beam assumptions.

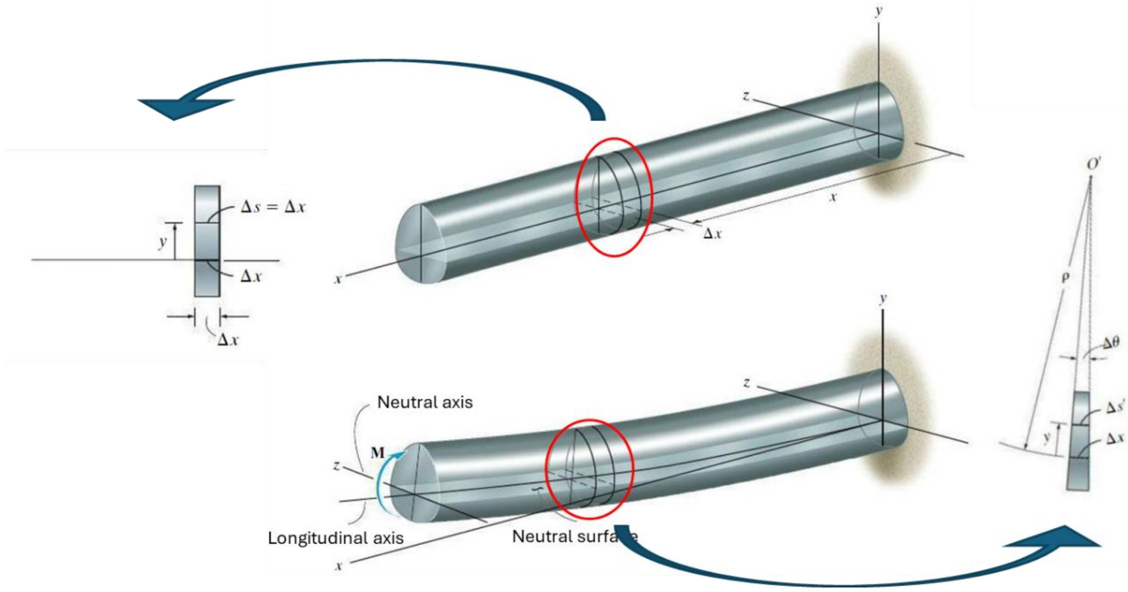


Figure 9. Beam before and after bending with isolated section for analysis.

Any segment Δx located on the neutral surface undergoes no change in its length, while any segment Δs located at a distance y from the neutral axis compresses (or stretches) and becomes $\Delta s'$ after deformation.

By definition, the strain along Δs is:

$$\varepsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s} \tag{02}$$

Before deformation, $\Delta s = \Delta x$. And after deformation Δx has a radius of curvature ρ, with a center of curvature located at O'.

$\Delta\theta$ defines the angle between the sides of the element, $\Delta x = \Delta s = \rho\Delta\theta$.

Similarly, $\Delta s' = (\rho - y)\Delta\theta$.

From where:

$$\varepsilon = \lim_{\Delta\theta \to 0} \frac{(\rho - y)\Delta\theta \ - \rho\Delta\theta}{\rho\Delta\theta} \ \rightarrow \ \varepsilon = -\frac{y}{\rho} \tag{03}$$

This important result indicates that the longitudinal deformation of any element in the beam depends on its position y on the cross-section and on the radius of curvature of the longitudinal axis of the beam.

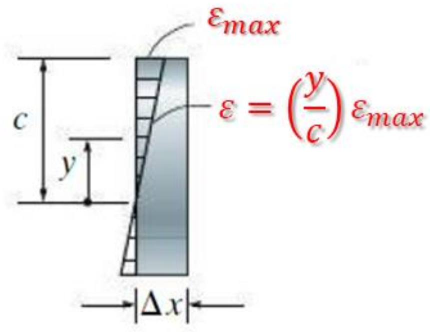


Figure 10. Longitudinal deformation on the beam cross-section.

When a moment is applied to the beam, it will only cause normal stresses in the longitudinal or x direction.

Applying Hooke's law: $\sigma = E\varepsilon$

$$\sigma = -E\left(\frac{y}{\rho}\right) = \left(\frac{y}{c}\right)\sigma_{max} \tag{04}$$

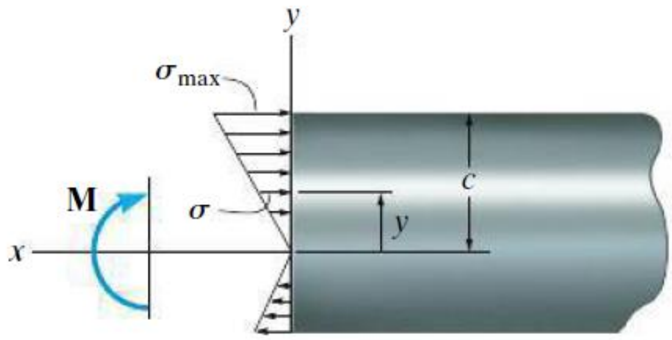


Figure 11. Normal stresses on the beam cross-section.

If a volume element of the material is selected at a specific point in the cross-section, only normal tensile or compressive stresses will act on that point.

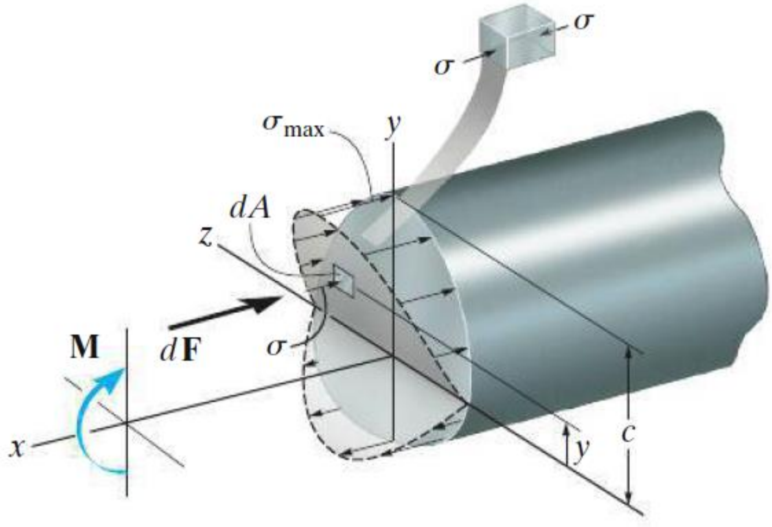


Figure 12. Normal stresses on volume element of beam material.

The position of the neutral axis on the cross-section can be located by satisfying the condition that the resultant force produced by the stress distribution on the cross-sectional area must be zero.

The force acting on the element is: $dF = \sigma dA$

So: $F_R = \sum F_x$

$$0 = \int_A dF = \int_A \sigma dA = \int_A \left(\frac{y}{c}\right)\sigma_{max} dA = \left(\frac{\sigma_{max}}{c}\right)\int_A ydA$$

$\left(\frac{\sigma_{max}}{c}\right)$ cannot be equal to zero, then $\int_A ydA = 0$

The moment of dF about the neutral axis is: dM=ydF=σydA

So: $M_{Rz} = \sum M_z$

$$M = \int_A ydF = \int_A y(\sigma dA) = \int_A y\left(\frac{y}{c}\sigma_{max}\right)dA$$

Or:

$$M = \frac{\sigma_{max}}{c}\int_A y^2 dA = \frac{\sigma_{max}}{c} I_z \qquad (05)$$

which gives:

$$\sigma_{max} = \frac{M}{I_z} c = \frac{M}{I_z} y_{max} \quad (06)$$

Knowing that:

$$\sigma = \left(\frac{y}{c}\right) \sigma_{max}$$

So:

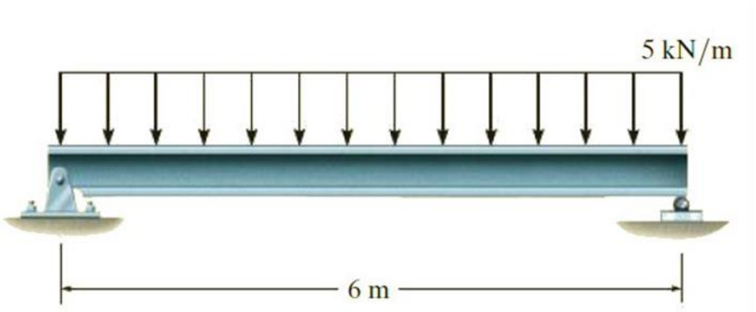
$$\sigma = \frac{M}{I_z} y \quad (07)$$

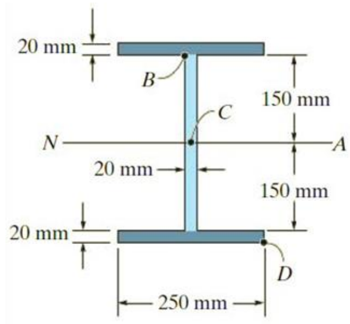
For the complete beam:

$$\sigma_{max} = \frac{M_{max}}{I_z} y_{max} \quad (08)$$

Example:

- Determine the maximum stress in the beam below.





Let us first determine the reaction forces to the supports.

We note that the beam is identical to that of example 1 (section 1.2) with $q = 5\ kN/m$ and $L = 6m$

$$Y_A = Y_B = q\frac{L}{2} = 15\ kN$$

The maximum bending moment is located in the middle of the beam with the following value:

$$M_{max} = q\frac{L^2}{8} = 22.5\ kN.m$$

The second moment of area of the cross-section with respect to the N-A axis is:

$$I_{N-A} = \frac{20 \times 300^3}{12} + 2\left(\frac{250 \times 20^3}{12}\right) = 45.33 \times 10^6 mm^4 = 45.33 \times 10^{-6} m^4$$

Then the maximum stress in the beam is (1.08):

$$\sigma_{max} = \frac{M_{max}}{I_{N-A}} y_{max} = \frac{22.5 \times 10^3}{45.33 \times 10^{-6}} \times 170 \times 10^{-3} = 84.38 \times 10^6 N/m^2 = 84.38\ MPa$$

1.4. Tangential stresses in simple bending:

In general, a beam that is bending will support both the shear force and the bending moment. The shear T results from a transverse distribution of shear stresses (tangential stresses) acting on the cross-section of the beam (figure 13). Due to the reciprocal property of shear, this stress results from longitudinal tangential stresses acting in the longitudinal planes of the beam.

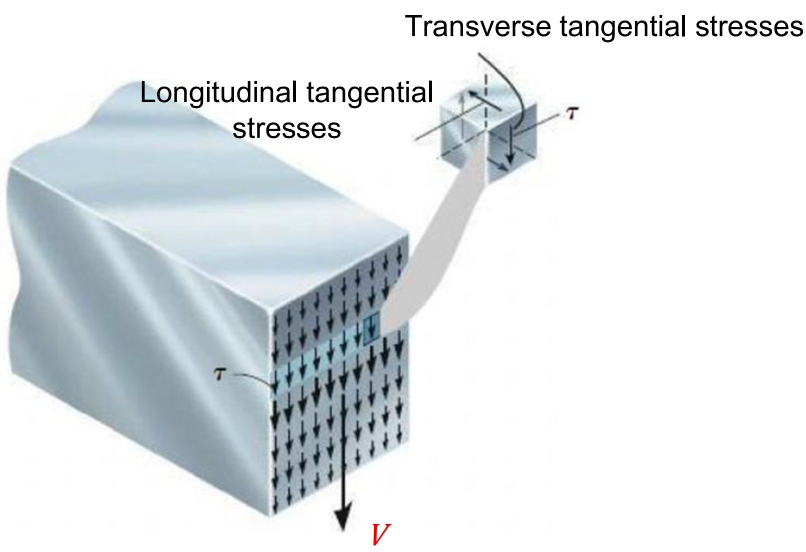


Figure 13. Tangential stresses at the cross-section of bending rectangular beam.

Under a transverse load, the longitudinal fibers of a beam do not deform uniformly (compression in the upper region and tension in the lower region), which creates a tendency for relative sliding between adjacent layers, illustrated by the slight offset of the lamellae; shear stresses then develop within the longitudinal planes to resist this sliding and ensure material cohesion, transferring forces between neighboring fibers and maintaining the mechanical continuity of the beam despite their different deformations. Example of illustration of glued (fiber) boards in figure 14.

Figure 14. Illustration of the effect of bonding the plates on the longitudinal deformation of the fibers of beams in bending.

Since the distribution of tangential stresses in bending is not easy to define, we will develop it indirectly.

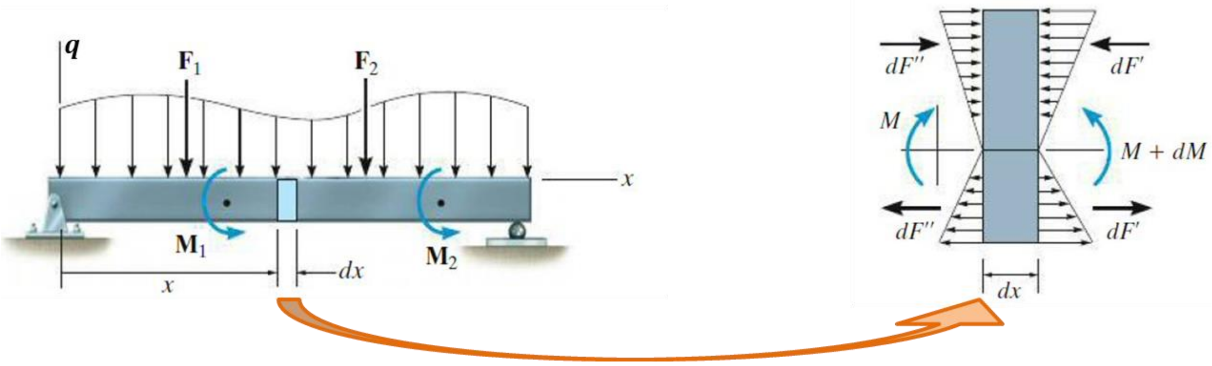


Figure 15. Forces and bending moments acting on a segment dx of the beam.

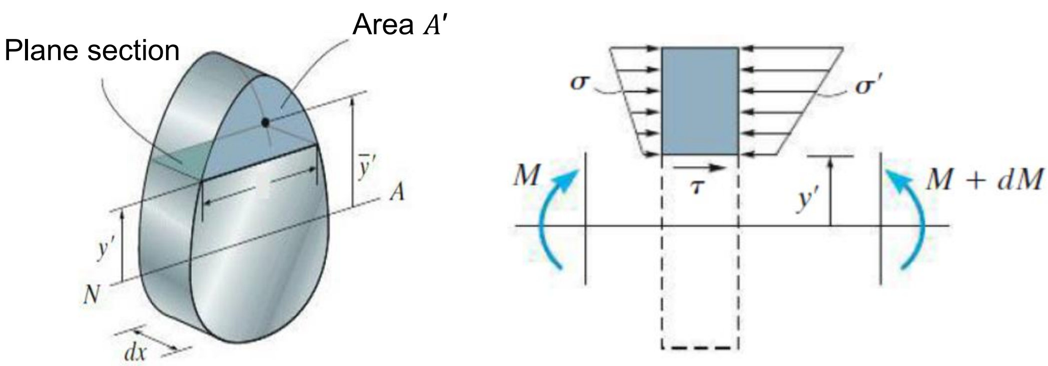


Figure 16. Static equilibrium of the cross-sectional volume A'.

Let us take the section of area A' (figure 16) and apply the equilibrium of this part:

$$\sum F_x = 0$$

$$\int_{A'} \sigma' dA' - \int_{A'} \sigma dA' - \tau(b.dx) = 0$$

$$\int_{A'} \left(\frac{M + dM}{I_z}\right) y' dA' - \int_{A'} \left(\frac{M}{I_z}\right) y' dA' - \tau(b.dx) = 0$$

$$\left(\frac{dM}{I_z}\right) \int_{A'} y' dA' = \tau(b.dx)$$

Which gives:

$$\tau = \frac{1}{bI_z} \left(\frac{dM}{dx}\right) \int_{A'} y' dA'$$

Hence:

$$\tau = \frac{VS_z}{bI_z} \qquad (09)$$

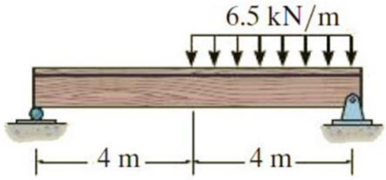
with:

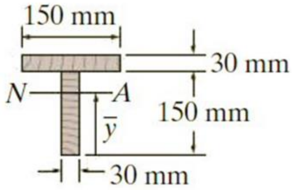
$$V = \frac{dM}{dx}$$

$S_z = \int_{A'} y' dA' = \overline{y'} A'$ Static moment of the area section A'.

Examples:

- The beam below is made of two boards. Determine the maximum shear stress in the glue required to hold the boards together along the joint where they are glued.





Let us first determine the reaction forces to the supports (R_A at the left support and R_B at the right support).

$$R_A + R_B = 6.5 \times 4 = 26\ kN$$

$$R_A \times 8 = 6.5 \times 4 \times 2 \quad \rightarrow \quad R_A = 6.5\ kN \quad \rightarrow \quad R_B = 26 - 6.5 = 19.5\ kN$$

Let's determine the center of gravity of the cross section:

$$\bar{y} = \frac{30 \times 150 \times 75 + 150 \times 30 \times 165}{30 \times 150 + 150 \times 30} = 120\ mm$$

and the second moment of area of the cross-section with respect to the N-A axis:

$$I_{N-A} = \frac{30 \times 150^3}{12} + (30 \times 150) \times (75 - 120)^2 + \frac{150 \times 30^3}{12} + (150 \times 30) \times (165 - 120)^2 = 27 \times 10^6 mm^4 = 27 \times 10^{-6} m^4$$

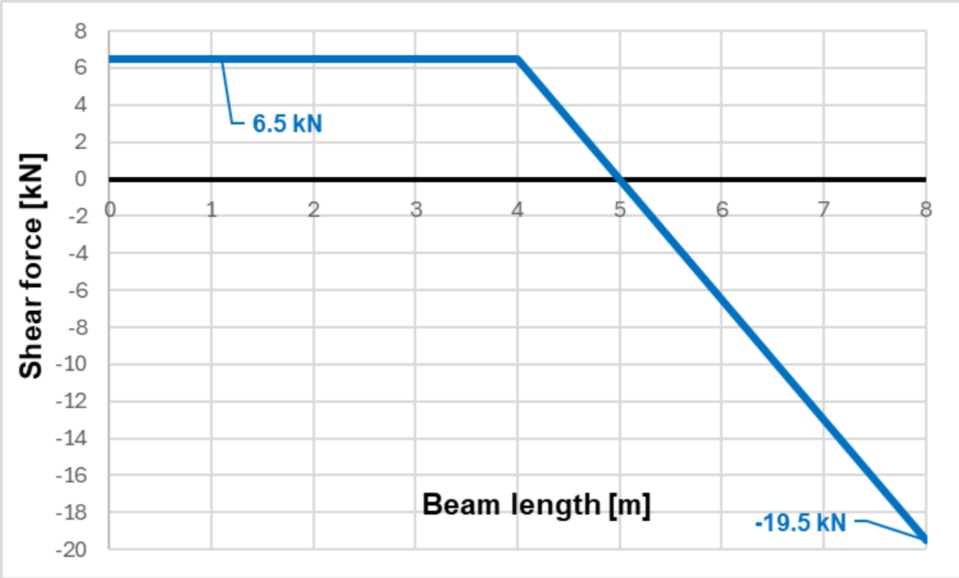
now determine the maximum shear force along the beam.

- $0 \leq x \leq 4\ m$ (from left to right):

$$T(x) = R_A = 6.5\ kN$$

- $4\ m \leq x \leq 8\ m$:

$$T(x) = R_A - 6.5(x - 4) = -6.5x + 32.5 \rightarrow \begin{cases} x = 4\ m & \rightarrow \quad T(4) = 6.5\ kN \\ x = 8m & \rightarrow \quad T(8) = -19.5\ kN \end{cases}$$



The maximum shear force is located at the section of the right-hand support for $x = 8m$ and is equal to:

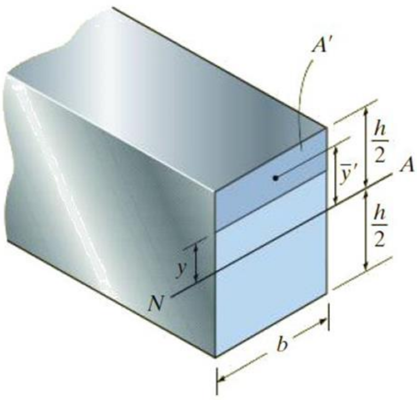
$$T_{max} = 19.5\ kN$$

Let us apply equation (1.09) to calculate the shear stress at the contact fiber between the horizontal board and the vertical board (y=150 mm):

$$\tau_{max} = \frac{T_{max} S_{N-A}}{b_{min} I_{N-A}} = \frac{19.5 \times 10^3 \times 2.025 \times 10^{-4}}{0.03 \times 27 \times 10^{-6}} = 4.875 \times 10^6 N/m^2 = 4.875\ MPa$$

with: $S_{N-A} = 150 \times 30 \times (150 - 120 + 15) = 202500\ mm^3 = 2.025 \times 10^{-4} m^4$ and $b_{min} = 30\ mm.$

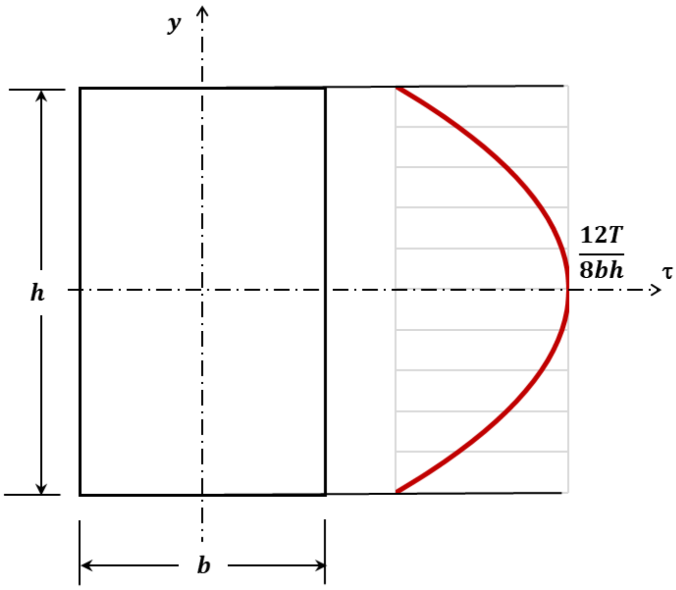
- Determine the shear stress distribution over the cross-section of the beam below.



The tangential stress at the fiber level at height y is:

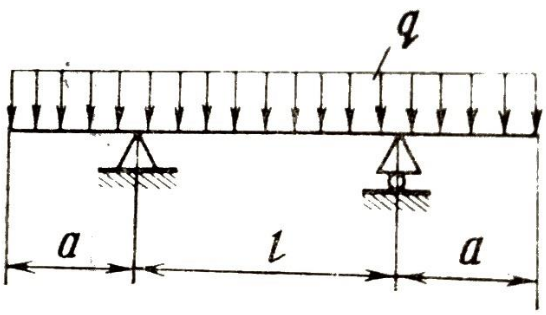
$$\tau(y) = \frac{TS_{N-A}}{bI_{N-A}} = \frac{12T}{b \times b \times h^3} b \times \left(\frac{h}{2} - y\right) \bar{y}' = \frac{12T}{bh^3}\left(\frac{h}{2} - y\right)\left(\frac{\left(\frac{h}{2} - y\right)}{2} + y\right)$$

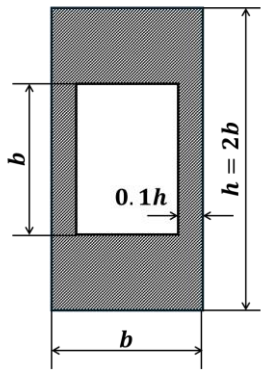
$$= \frac{12T}{bh^3}\left(\frac{h}{2} - y\right)\left(\frac{h}{4} + \frac{y}{2}\right) = \frac{12T}{bh^3}\left(\frac{h^2}{8} - \frac{y^2}{2}\right) \rightarrow \begin{cases} y = 0 \rightarrow \tau(0) = \dfrac{12T}{8bh} \\ y = \dfrac{h}{2} \rightarrow \tau\left(\dfrac{h}{2}\right) = 0 \end{cases}$$



We note that the maximum shear stress is located at the neutral fiber and that the fibers at the ends have zero stress (this is perfectly normal because they are free and there is no resistance to sliding).

- Consider the beam shown below. 1. Plot the shear force and bending moment diagrams for the beam below. 2. Determine the maximum normal stress. 3. Determine the maximum shear stress. Given: $q = 1\ kN/m$, $a = 1\ m$, $l = 2\ m$, and $b = 10\ cm$.





Reactions to support (by symmetry):

$$R_1 = R_2 = \frac{q(2a + l)}{2} = 2\ kN$$

Shear force and bending moment:

- $0 \le x \le a = 1\ m:$

$$V(x) = -qx = -x \quad \rightarrow \quad \begin{cases} x = 0 & \rightarrow \ V(0) = 0 \\ x = 1m & \rightarrow \ V(1) = -1\ kN \end{cases}$$

$$M(x) = -q\frac{x^2}{2} = -0.5x^2 \quad \rightarrow \quad \begin{cases} x = 0 & \rightarrow \ M(0) = 0 \\ x = 1m & \rightarrow \ M(1) = -0.5\ kN.m \end{cases}$$

- $1 \le x \le 3\ m:$

$$V(x) = -qx + R_1 = -x + 2 \quad \rightarrow \quad \begin{cases} x = 1\ m & \rightarrow \ V(1) = 1\ kN \\ x = 3m & \rightarrow \ V(3) = -1\ kN \end{cases}$$

$$M(x) = -q\frac{x^2}{2} + R_1(x - 1) = -0.5x^2 + 2x - 2 \quad \rightarrow \quad \begin{cases} x = 1\ m & \rightarrow \ M(1) = -0.5\ kN.m \\ x = 3\ m & \rightarrow \ M(3) = -0.5\ kN.m \end{cases}$$

$$\frac{dM}{dx} = -x + 2 = 0 \quad \rightarrow \quad x = 2\ m \quad \rightarrow \quad M(2) = 0$$

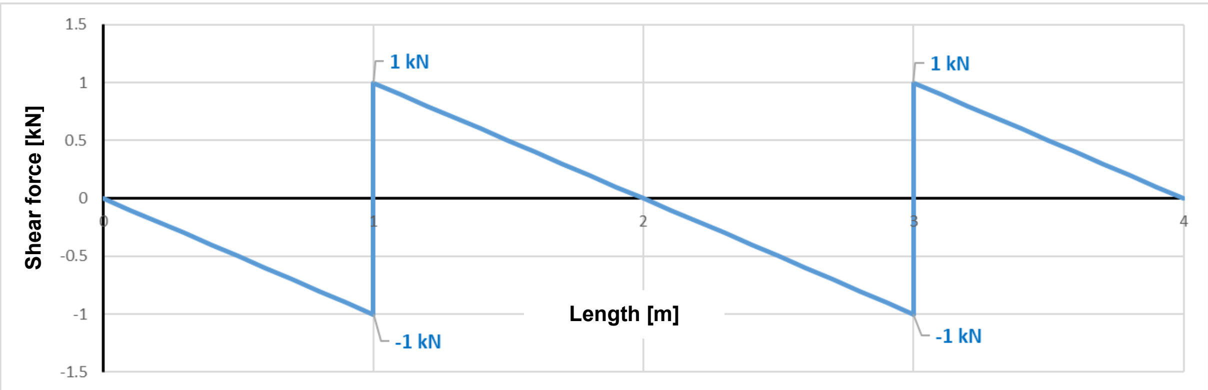
- $3 \le x \le 4\ m:$

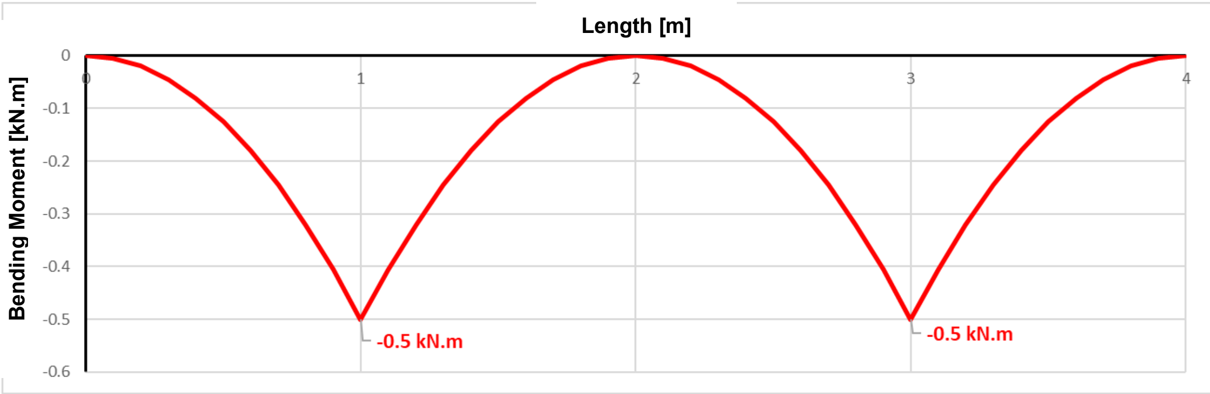
$$V(x) = -qx + R_1 + R_2 = -x + 4 \quad \rightarrow \quad \begin{cases} x = 3\ m & \rightarrow \ V(3) = 1\ kN \\ x = 4\ m & \rightarrow \ V(4) = 0 \end{cases}$$

$$M(x) = -q\frac{x^2}{2} + R_1(x - 1) + R_2(x - 3) = -0.5x^2 + 4x - 8$$

$$\rightarrow \quad \begin{cases} x = 3\ m & \rightarrow \ M(3) = -0.5\ kN.m \\ x = 4m & \rightarrow \ M(4) = 0 \end{cases}$$

Shear force and bending moment diagrams:





$$V_{max} = 1\ kN$$

$$M_{max} = 0.5\ kN.m$$

Quadratic moment:

$$I_z = \frac{bh^3}{12} - \frac{(b - 2\times 0.1h)b^3}{12} = \frac{8b^4}{12} - \frac{0.6b^4}{12} = \frac{7.4b^4}{12} = 0.6167b^4 = 6.167\times 10^{-5}m^4$$

Maximum normal stress:

$$\sigma_{max} = \frac{M_{max}}{I_z}y_{max} = \frac{0.5\times 10^3}{6.167\times 10^{-5}}\times 0.1 = 8107669\frac{N}{m^2} = 810.77\ kPa$$

Maximum shear stress:

- $S_z(0) = 0 \quad \rightarrow \quad \tau(0) = 0$
- $S_z(1) = b\times\frac{b}{2}\times\frac{3}{4}b = \frac{3}{8}b^3 = 3.75\times 10^{-4}m^3$

$$\tau(1) = \frac{V_{max}S_z(1)}{bI_z} = \frac{1\times 10^3\times 3.75\times 10^{-4}}{b\times 6.167\times 10^{-5}} = 60810\frac{N}{m^2} = 60.81\ kPa$$

- $S_z(2) = S_z(1) = 3.75\times 10^{-4}m^3$

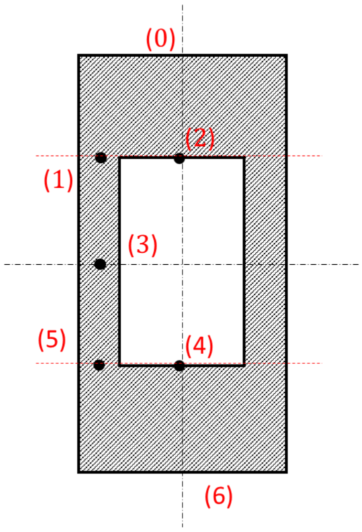
$$\tau(2) = \frac{V_{max}S_z(2)}{bI_z} = \frac{1\times 10^3\times 3.75\times 10^{-4}}{2\times 0.1h\times 6.167\times 10^{-5}} = 152020\frac{N}{m^2} = 152\ kPa$$

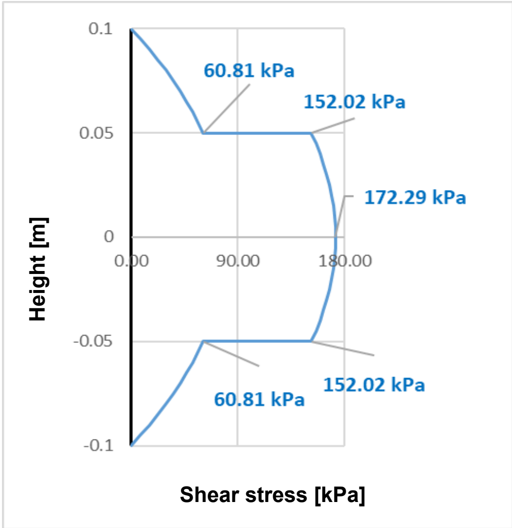
- $S_z(3) = S_z(2) + 2\times 0.1h\times\frac{b}{2}\times\frac{b}{4} = S_z(2) + \frac{0.2b^3}{4} = 0.425b^3 = 4.25\times 10^{-4}m^3$

$$\tau(3) = \frac{V_{max}S_z(3)}{bI_z} = \frac{1\times 10^3\times 4.25\times 10^{-4}}{2\times 0.1h\times 6.167\times 10^{-5}} = 172290\frac{N}{m^2} = 172.3\ kPa$$

- $\tau(4) = \tau(2)$
- $\tau(5) = \tau(1)$
- $\tau(6) = \tau(0)$

$$\tau_{max} = 172.3\ kPa$$





1.5. Conclusion:

This chapter established the foundational framework for beam analysis by defining beams as structural elements where length significantly exceeds cross-sectional dimensions. By mathematically defining supports (such as roller, pinned, and fixed) we established how boundary conditions restrict degrees of freedom and generate specific reaction forces and moments. The review of differential relationships between distributed loads, shear force, and bending moment provided the necessary tools to map internal actions along a beam's span. Furthermore, the analysis of stress distributions revealed that bending induces linear normal stresses and complex tangential stresses, the latter being essential for resisting longitudinal sliding and maintaining material cohesion.

Displacement of symmetrical beams in plane bending

2.1. Introduction:

The deflection of a beam or shaft must often be limited to ensure the integrity and stability of a structure or machine and to prevent cracking of any attached brittle material, such as concrete or glass.

Furthermore, standards restrictions often require that machine members not vibrate or deform severely in order to safely support the intended load.

The deflection curve of the longitudinal axis that passes through the center of the surface of each section of a beam is called the elastic curve (figure 17).

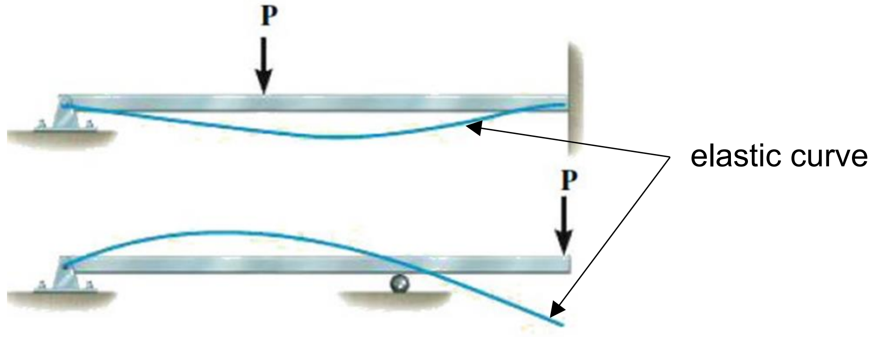


Figure 17. representation of the elastic curve of a bent beam

If the elastic curve of a beam seems difficult to establish, it is suggested to start by drawing the moment diagram for the beam using the sign convention (figure 18 and 19).

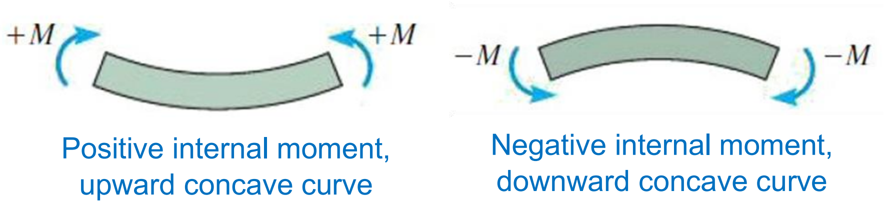


Figure 18. sign convention.

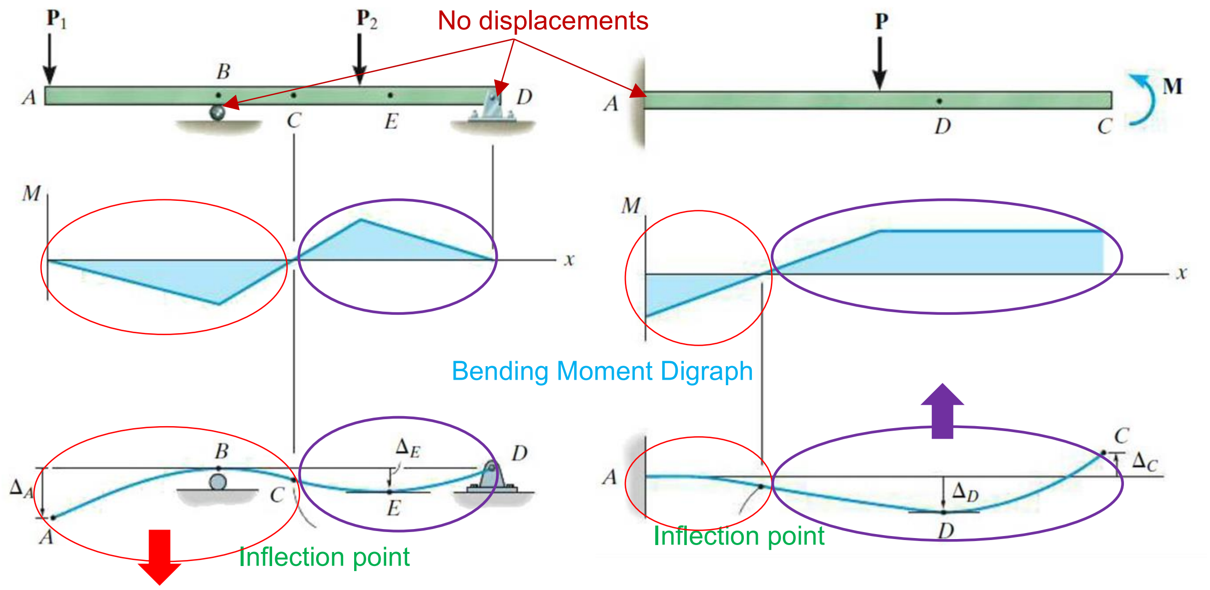


Figure 19. Relationship between the sign of the bending moment and the concavity of the elastic curve.

2.2. Relationship between moment and curvature:

Any segment Δx located on the neutral surface does not undergo any change in its length, whereas any segment Δs located at a distance y from the neutral axis is compressed (or stretched) and becomes $\Delta s'$ after deformation.

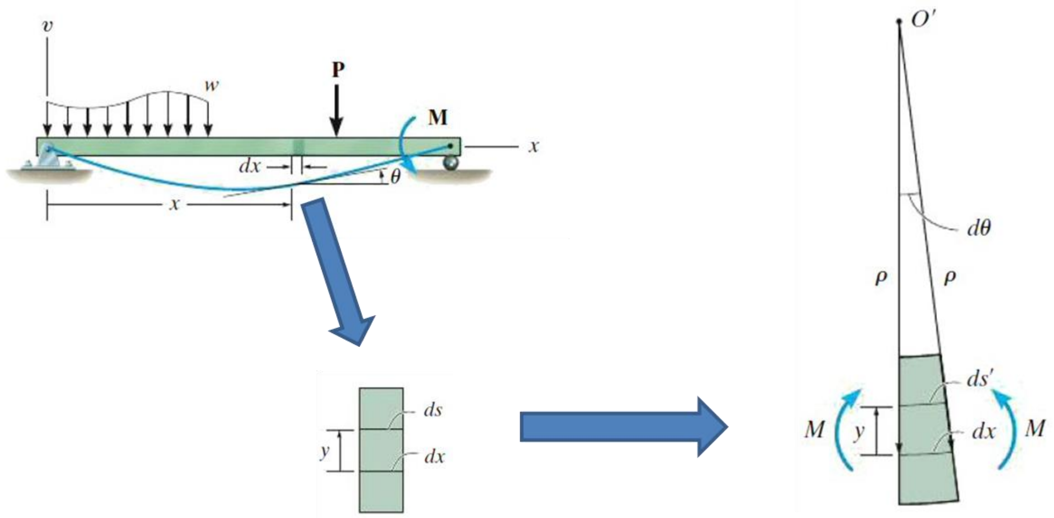


Figure 20. Segment Δx of the elastic curve.

By definition, the deformation along Δs is:

$$\varepsilon = \lim_{\Delta s \to 0} \frac{\Delta s' - \Delta s}{\Delta s}$$

Before deformation, $\Delta s = \Delta x$. And after deformation Δx has a radius of curvature ρ, with a center of curvature located at O'.

$\Delta\theta$ defines the angle between the sides of the element, $\Delta x = \Delta s = \rho\Delta\theta$

Likewise, $\Delta s' = (\rho - y)\Delta\theta$

Hence:

$$\varepsilon = \lim_{\Delta\theta\to 0}\frac{(\rho - y)\Delta\theta\ - \rho\Delta\theta}{\rho\Delta\theta} \ \rightarrow\ \varepsilon = -\frac{y}{\rho} \qquad (10)$$

If the material is homogeneous and behaves in a linear-elastic manner, then Hooke's law applies:

$$\sigma = E\varepsilon = -\frac{M}{I_z}y \qquad (12)$$

Hence:

$$\frac{1}{\rho} = \frac{M}{EI_z} \qquad (13)$$

The product EI in this equation is called flexural stiffness, and it is always a positive quantity. The equation of the elastic curve of a beam can be expressed mathematically by $\upsilon = f(x)$. The expression for the radius of curvature is:

$$\rho = \frac{\left[1 + \left(\frac{d\upsilon}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2\upsilon}{dx^2}} \qquad (14)$$

When the curvature is small, then the previous expression can be written as:

$$\frac{1}{\rho} = \frac{d^2\upsilon}{dx^2} = \frac{M}{EI_z} \qquad (15)$$

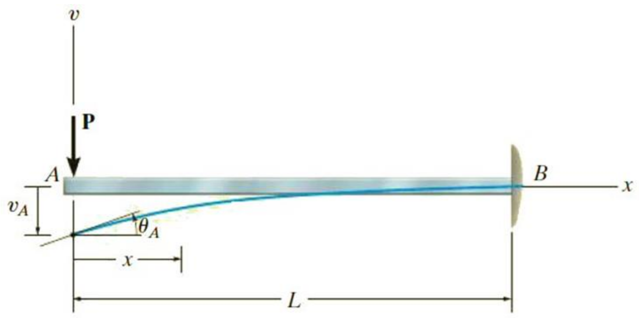
And:

$$\frac{d\upsilon}{dx} = \theta \qquad (16)$$

is the slope of the elastic curve.

Examples:

- Determine the expression for the elastic curve as well as the deflection and slope at A.



The reactions to the embedding are:

$$R_B = P$$

$$M_B = -PL$$

The expression for the bending moment along the beam is:

$$M(x) = -Px$$

Let's apply expression (2.01):

$$\frac{d^2v}{dx^2} = \frac{M}{EI_z} = -\frac{Px}{EI_z}$$

Let's integrate with respect to x:

$$\theta = \frac{dv}{dx} = \frac{1}{EI_z}\int -Px.dx = \frac{1}{EI_z}\left(-\frac{1}{2}Px^2 + C_1\right)$$

$$v = \frac{1}{EI_z}\int \left(-\frac{1}{2}Px^2 + C_1\right)dx = \frac{1}{EI_z}\left(-\frac{1}{6}Px^3 + C_1x + C_2\right)$$

Let's apply the boundary conditions:

$$\begin{cases} x = L \;\rightarrow\; \theta(L) = 0 \;\rightarrow\; -\frac{1}{2}PL^2 + C_1 = 0 \;\rightarrow\; C_1 = \frac{1}{2}PL^2 \\ x = L \;\rightarrow\; v(L) = 0 \;\rightarrow\; -\frac{1}{6}PL^3 + \frac{1}{2}PL^3 + C_2 = 0 \;\rightarrow\; C_2 = -\frac{1}{3}PL^3 \end{cases}$$

Therefore, the expression for the elastic curve along the beam is:

$$\upsilon = \frac{1}{EI_z}\left(-\frac{1}{6}Px^3 + \frac{1}{2}PL^2x - \frac{1}{3}PL^3\right)$$

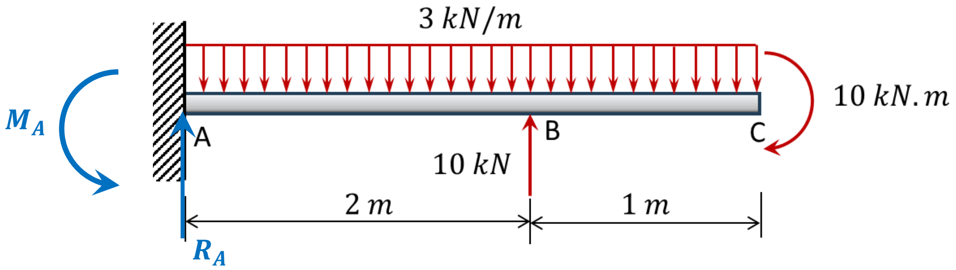
$$\theta = \frac{d\upsilon}{dx} = \frac{1}{EI_z}\left(-\frac{1}{2}Px^2 + \frac{1}{2}PL^2\right)$$

The deflection and slope at section A are ($x = 0$):

$$\upsilon_A = -\frac{PL^3}{3EI_z}$$

$$\theta_A = \frac{PL^2}{2EI_z}$$

- Consider the beam shown below: 1) Determine the slope at section A. 2) Determine the vertical displacement at section B. 3) Determine the slope at section C. Given: $E = 200\ GPa$ and $I_z = 2 \times 10^{-6} m^4$.



Support reactions:

$$R_A = 3 \times 3 - 10 = -1\ kN$$

$$M_A = 3 \times 3 \times \frac{3}{2} - 10 \times 2 + 10 = 3.5\ kN.m$$

Bending moment calculation along the beam:

- $0 \leq x \leq 2m$:

$$M(x) = -M_A + R_A x - 3\frac{x^2}{2} = -1.5x^2 - x - 3.5$$

- $2 \leq x \leq 3m$:

$$M(x) = -M_A + R_A x - 3\frac{x^2}{2} + 10(x-2) = -1.5x^2 + 9x - 23.5$$

Since the section at point A is a fixed support, then: $\theta_A = 0$.

Expression of the slope and vertical displacement:

- $0 \leq x \leq 2m$:

$$EI_z\ddot{v}_1 = M(x) = -1.5x^2 - x - 3.5$$

$$EI_z\dot{v}_1 = -0.5x^3 - 0.5x^2 - 3.5x + C_1$$

$$EI_z v_1 = -0.125x^4 - 0.1667x^3 - 1.75x^2 + C_1x + C_2$$

For $x = 0$, we have: $\dot{v}_1(0) = v_1(0) = 0$, then: $C_1 = C_2 = 0$.

- $2 \leq x \leq 3m$:

$$EI_z\ddot{v}_2 = M(x) = -1.5x^2 + 9x - 23.5$$

$$EI_z\dot{v}_2 = -0.5x^3 + 4.5x^2 - 23.5x + C_3$$

$$EI_z v_2 = -0.125x^4 1.5x^3 - 11.75x^2 + C_3x + C_4$$

For $x = 2m$, we have: $v_1(2) = v_2(2)$ and $\dot{v}_2(2) = \dot{v}_2(2)$, then: $C_3 = 20$ and $C_4 = 13.336$

We will finally:

- $0 \leq x \leq 2m$:

$$\dot{v}_1 = \frac{1}{EI_z}(-0.5x^3 - 0.5x^2 - 3.5x)$$

$$v_1 = \frac{1}{EI_z}(-0.125x^4 - 0.1667x^3 - 1.75x^2)$$

- $2 \leq x \leq 3m$:

$$\dot{v}_2 = \frac{1}{EI_z}(-0.5x^3 + 4.5x^2 - 23.5x + 20)$$

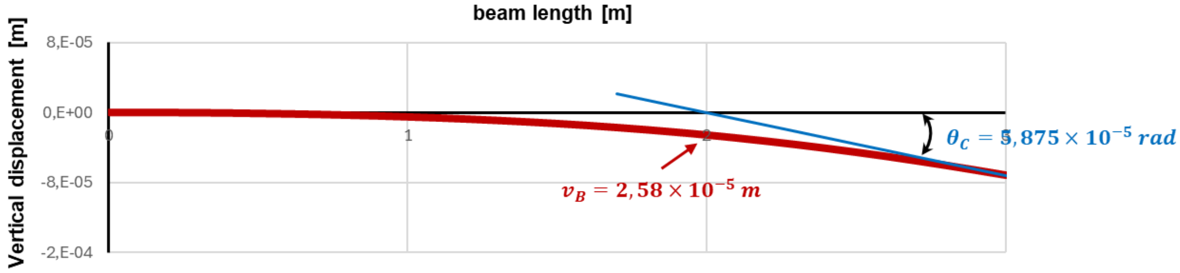
$$v_2 = \frac{1}{EI_z}(-0.125x^4 + 1.5x^3 - 11.75x^2 + 20x - 13.336)$$

The vertical displacement of section B:

$$v_B = v_1(x = 2m) = \frac{-10.336}{200 \times 10^9 \times 2 \times 10^{-6}} = -2.58 \times 10^{-5}\ m$$

The slope of the section at point C:

$$\theta_C = \dot{v}_2(x = 3m) = \frac{-23.5}{200 \times 10^9 \times 2 \times 10^{-6}} = -5.875 \times 10^{-5}\ rad$$



2.3. Beams deflection using the area moment method:

The method of moments of areas is a semi-graphical method for finding the slope and displacement at a specific point in the elastic curve of a beam. Applying this method requires calculating the area of segments of the beam's bending moment diagram. This method is based on two theorems; one is used to determine the slope and the other to determine the displacement.

- **Theorem 1:**

Consider a beam simply supported with an elastic curve (figure 21).

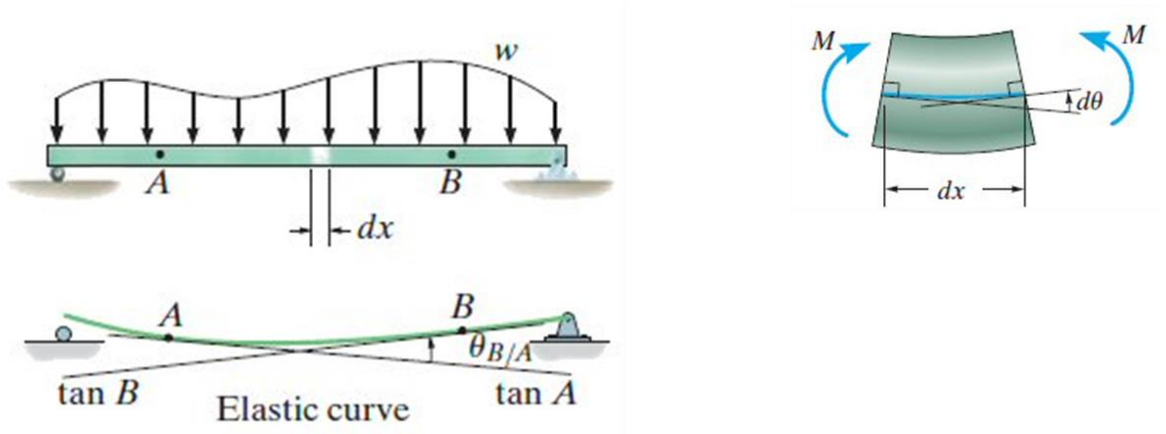


Figure 21. Proof of Theorem 1.

$$EI_z\frac{d^2v}{dx^2} = EI_z\frac{d}{dx}\left(\frac{dv}{dx}\right) = M$$

$$\mathrm{d}\theta = \frac{M}{EI_z}dx$$

If the bending moment diagram of the beam is constructed, it is divided by the bending stiffness, EI_z (figure 22). Then the previous equation indicates that $d\theta$ is equal to the area under the diagram "$\frac{M}{EI_z}$" for the segment dx.

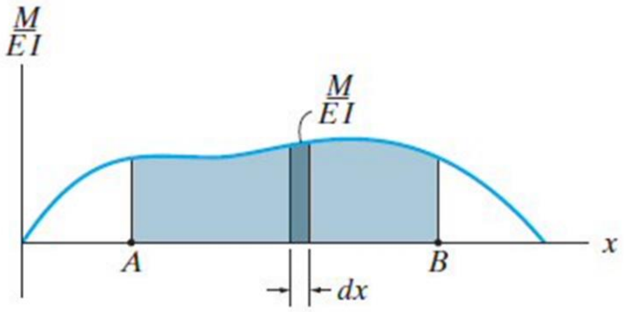


Figure 22. Diagram of the beam's bending moment divided by its bending stiffness.

Integrating from point A into the elastic curve at point B, we will have:

$$\theta_{A/B} = \int_A^B \frac{M}{EI_z} dx \quad (17)$$

- **Theorem 2:**

The second theorem of moments of areas is based on the relative deviation of the tangents of the elastic curve dt. If we assume that for minor deviations: $dt \cong ds' = xd\theta$.

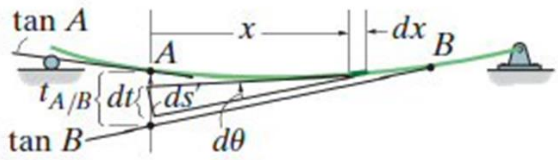


Figure 23. Proof of Theorem 2.

Therefore, the vertical deviation between the tangents at points A and B is (figure 23):

$$t_{A/B} = \int_A^B x \frac{M}{EI_z} dx \quad (18)$$

We can also write:

$$t_{A/B} = \overline{x} \int_A^B \frac{M}{EI_z} dx \quad (19)$$

With $\bar{x}$ being the center of gravity of the surface of the diagram between A and B (figure 24).

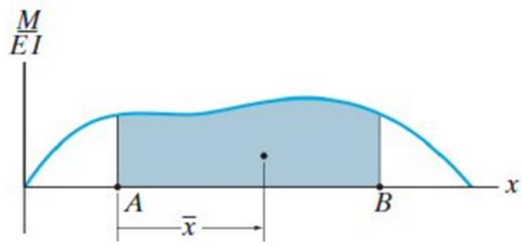


Figure 24. Center of gravity of the surface of the diagram between A and B.

$$t_{A/B} = \bar{x}\int_{A}^{B}\frac{M}{EI_z}dx \neq t_{B/A} = \bar{x}'\int_{A}^{B}\frac{M}{EI_z}dx \tag{20}$$

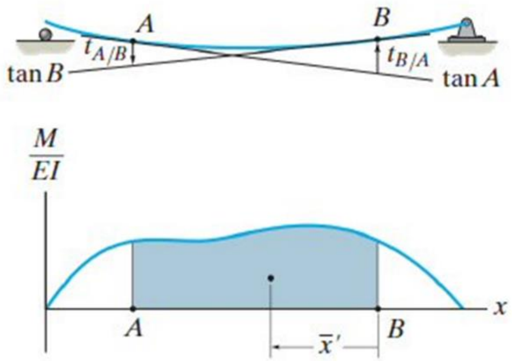
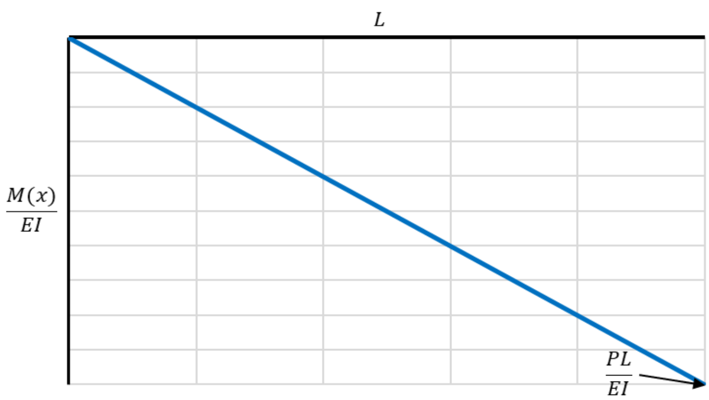


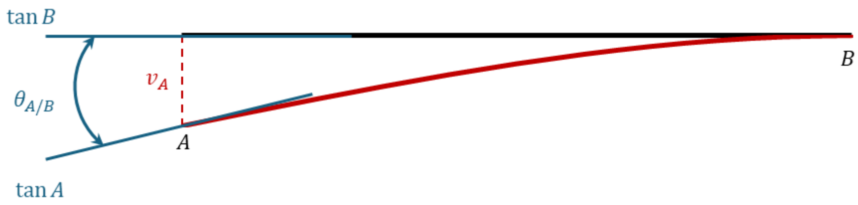
Figure 25. Center of gravity of the surface of the diagram between B and A.

Examples:

- Determine the slope and displacement at point A in the previous example.

The bending moment diagram and the elastic curve are represented below:





Note that the tangent to the elastic curve at point B is horizontal, which implies that:$\theta_B = 0$.

Let's apply expression (2.07):

$$\theta_{A/B} = \theta_A = \int_A^B \frac{M}{EI_z} dx = \frac{1}{2} \times \frac{PL}{EI_z} \times L = \frac{PL^2}{2EI_z}$$

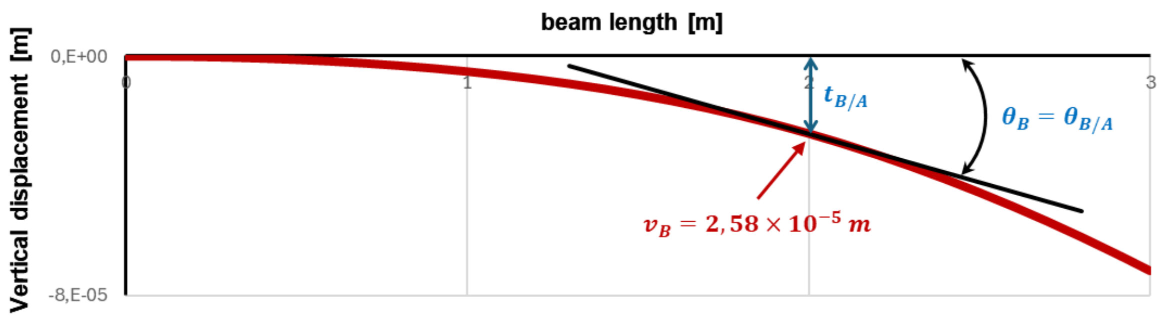
Next, let's apply expression (2.09):

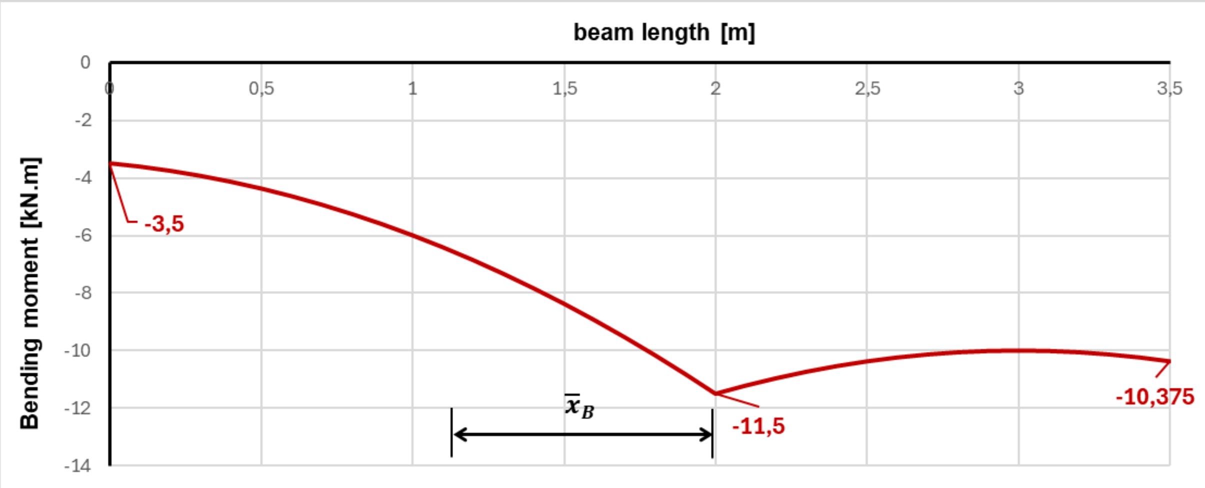
$$t_{A/B} = v_A = \overline{x} \int_A^B \frac{M}{EI_z} dx = \frac{2L}{3} \times \frac{PL^2}{2EI_z} = \frac{PL^3}{3EI_z}$$

- Determine the vertical displacement of point A and the slope at point C using the method of moments of area from the second example in the previous section.

The vertical displacement of section B can be calculated (referring to the figures below) as follows:

$$t_{B/A} = \overline{x}_B \int_A^B \frac{M(x)}{EI_z} dx$$





Let $\overline{x}_B$ denote the center of mass of the area under the graph between A and B, which can be calculated as:

$$\overline{x}_B = 4 - \frac{\int_A^B xdA}{\int_A^B dA}$$

$$\int_A^B xdA = \int_A^B xM(x)dx = \int_0^2 x(-1.5x^2 - x - 3.5)dx + \int_2^3 x(-1.5x^2 + 9x - 23.5)dx$$

$$= \int_0^2 (-1.5x^3 - x^2 - 3.5x)dx = (-0.375x^4 - 0.333x^3 - 1.75x^2)|_0^2$$

$$= - - 15.664\ kN.m^2$$

$$\int_A^B dA = \int_A^B M(x)dx = \int_0^2 (-1.5x^2 - x - 3.5)dx = (-0.5x^3 - 0.5x^2 - 3.5x)|_0^2 = -13$$

$$\overline{x}_B = 2 - \frac{-15.664\ 9}{-13} = 2 - 1.2 = 0.8\ m$$

$$t_{B/A} = \overline{x}_B \int_A^B \frac{M(x)}{EI_z}dx = 0.8 \times \frac{-13}{200 \times 10^9 \times 2 \times 10^{-6}} = -0.000026\ m$$

$$= -2.6 \times 10^{-5} m$$

The slope of section C can be calculated as follows:

$$\theta_{AC} = \theta_C = \int_A^C \frac{M(x)}{EI_z}dx$$

$$\int_A^C M(x)dx = \int_0^2 (-1.5x^2 - x - 3.5)dx + \int_2^3 (-1.5x^2 + 9x - 23.5)dx$$
$$= (-0.5x^3 - 0.5x^2 - 3.5x)|_0^2 + (-0.5x^3 + 4.5x^2 - 23.5x)|_2^3 = -23.5$$

$$\theta_C = \int_A^C \frac{M(x)}{EI_z}dx = \frac{-23.5}{200 \times 10^9 \times 2 \times 10^{-6}} = -0.00005875\ rad =$$
$$= -5.875 \times 10^{-5}\ rad$$

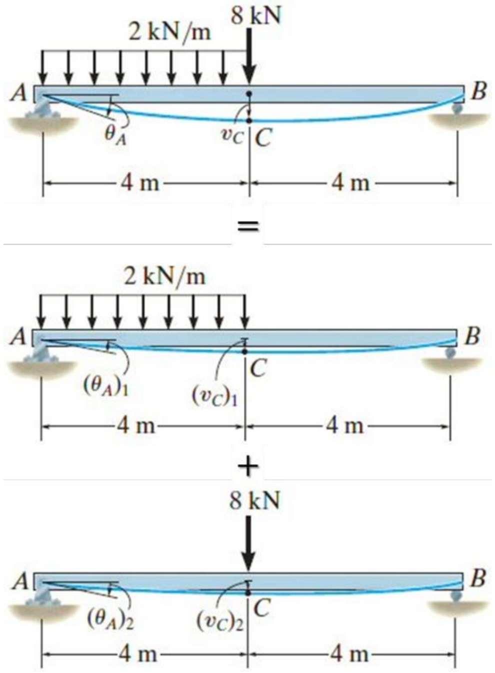
We obtain the same results as those obtained using the integral method.

2.4. Beam deflection using the force superposition method:

This method is based on the principle of superposition, which allows the decomposition of the beam's response into contributions from individual loads or redundant forces. By calculating the deflections caused by each load separately and summing them, one can determine the total displacement at any point of interest. The approach is particularly effective for beams with multiple loads or statically indeterminate conditions, as it combines equilibrium, compatibility, and energy-based relationships to provide accurate predictions of bending-induced deformations.

- **Example:**

Calculate the vertical displacement of section C and the rotation of section A of the beam using the force superposition method.



As shown in the figure above, the beam is decomposed into two simple beams, each subjected to a single load. The total displacement is obtained by summing the displacements of the two simple beams (appendix).

$$v_C = (v_C)_1 + (v_C)_2 = -\frac{5qL^4}{768EI} - \frac{PL^3}{48EI} = -\frac{5 \times 2 \times 8^4}{768EI} - \frac{8 \times 8^3}{48EI} = -\frac{136667}{EI}$$

and the total rotation of the section A is obtained by summing the rotations of the two simple beams at this section.

$$\theta_A = (\theta_A)_1 + (\theta_A)_2 = -\frac{3qL^3}{128EI} - \frac{PL^2}{16EI} = -\frac{3 \times 2 \times 4^3}{128EI} - \frac{8 \times 8^2}{16EI} = -\frac{35}{EI}$$

2.5. Conclusion:

The study of displacements transitioned the focus from internal strength to structural serviceability, emphasizing that limiting the "elastic curve" is critical for safety and function. We explored various analytical techniques, including the integration of the moment-curvature relationship and semi-graphical area-moment methods, which allow for the precise determination of slopes and deflections at specific points. A key takeaway from this chapter is the principle of superposition, which simplifies complex loading scenarios by allowing engineers to sum individual responses to find total displacement, a technique particularly useful for both simple and indeterminate structures.

General theorems of mechanical systems

3.1. Introduction:

This chapter introduces the energy-based foundations of Strength of Materials applied to the analysis of deformable mechanical systems, with particular emphasis on beams. Starting from the distinction between the initial undeformed state and the final equilibrium configuration under applied external loads (figure 26), the formulation is based on the principle of conservation of mechanical energy and the concept of elastic strain energy, assuming no dissipation. This framework enables the derivation of general expressions for elementary loading cases (tension, shear, torsion, and bending), as well as for combined loadings. The main advantage of energy methods lies in their effectiveness for determining displacements and rotations, especially in statically indeterminate structures, through powerful tools such as Castigliano's theorem and the unit load method. This theoretical approach thus provides a rigorous and efficient alternative to classical methods based solely on equilibrium equations.

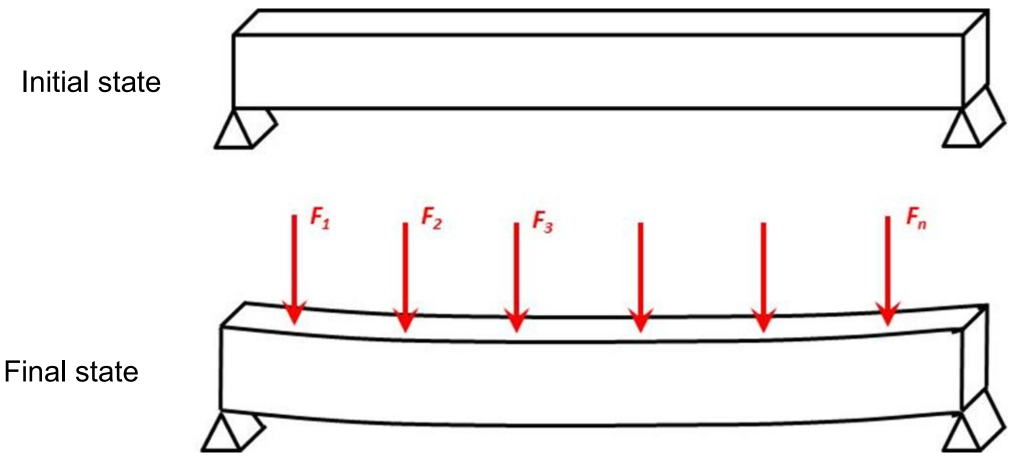


Figure 26. Initial undeformed state and final state under applied external loads.

Applying the theorem of conservation of mechanical energy between the two instants corresponding to the initial and final states gives us:

Work done by external forces + work done by internal forces = 0

$$W_{ext} + W_{int} = 0 \tag{21}$$

By definition, for a deformed material, when there is no heat dissipation or exchange:

$$U = W_{ext} = -W_{int} \tag{22}$$

Energy theorems and strain energy are essentially used, from a macroscopic point of view, to determine displacements and the lifting of hyperstaticity of a structure.

3.2. Pure tensile stress:

To calculate the strain energy, consider a section of the bar of length dx (figure 27). This section is in equilibrium under the action of forces F_0 on its ends. In this case, these forces, which are internal to the bar as a whole, are also external to the isolated section.

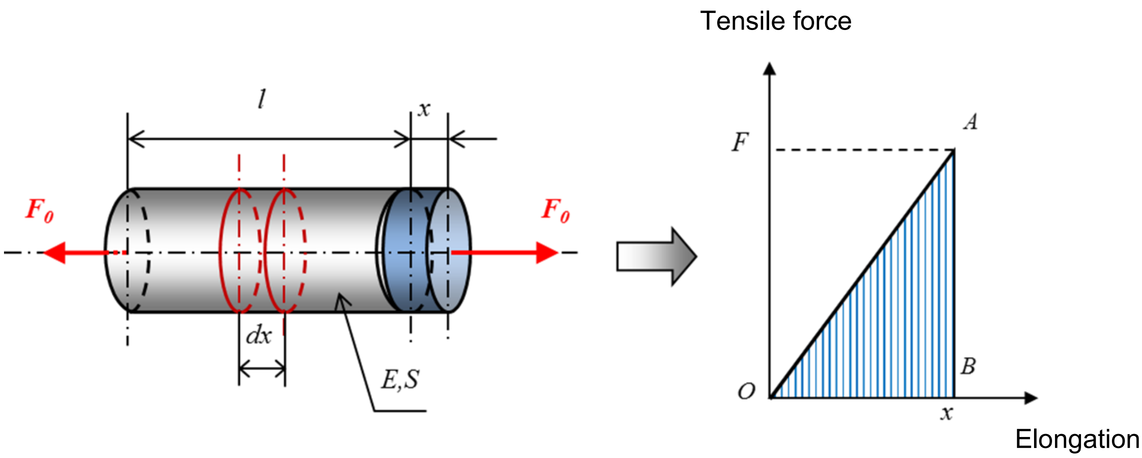


Figure 27. Pure tensile representation.

$$W_{ext} = \frac{1}{2} AB \times OB = \frac{1}{2} Fx \tag{23}$$

We can write that the deformation energy of this section is equal to the work done by these external forces:

$$dU = dW = \frac{1}{2} F \times Elongation\ of\ dx \tag{24}$$

(25)

$$dU = \frac{1}{2}F \times \frac{Fdx}{ES} = \frac{F^2dx}{2ES}$$

The strain energy stored in the entire bar is equal to the sum of the strain energies of each section.

$$U = \int_0^l \frac{F^2}{2ES}dx = \frac{F^2}{2ES}\int_0^l dx = \frac{F^2 l}{2ES} \tag{26}$$

$$W_{ext} = \frac{1}{2}Fx = \frac{1}{2}F\frac{Fl}{ES} = \frac{F^2 l}{2ES} = U \tag{27}$$

For a cubic element (dx, dy, dz) subjected to a tensile stress along the x-axis, the strain energy is:

$$dU = \frac{1}{2}(\sigma_x S)(\varepsilon_x dx) = \frac{1}{2}(\sigma_x \varepsilon_x)(S.dx) \tag{28}$$

Therefore, the unit strain energy $[J/m^3]$ is:

$$U_0 = \frac{1}{2}\sigma_x\varepsilon_x = \frac{1}{2}\frac{\sigma_x^2}{E} = \frac{1}{2}E\varepsilon_x^2 \tag{29}$$

3.3. Simple shear stress:

To calculate the strain energy in the case of simple shear, consider an infinitesimal material element subjected to a shear stress distribution (τ) on its faces, as illustrated in the figure (28). This element is in equilibrium under the action of equal and opposite shear forces acting on parallel faces, which, although internal to the body as a whole, become external when the element is isolated. These forces induce a relative sliding between adjacent layers, represented by the linear variation of shear stress shown in the diagram. The work done by these shear forces during deformation is stored as strain energy within the element, and the total strain energy of the body is obtained by integrating this contribution over the entire volume.

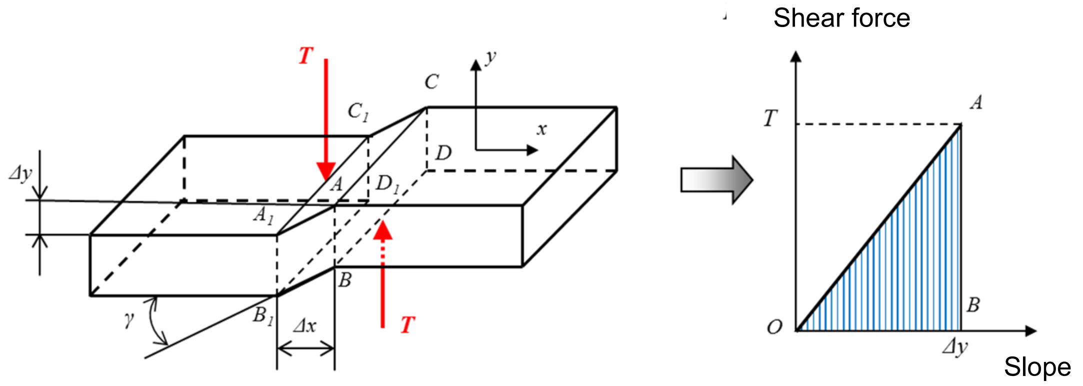


Figure 28. Simple shear representation.

$$W_{ext} = \frac{1}{2} AB \times OB = \frac{1}{2} T \Delta y \tag{30}$$

$$\Delta y = \frac{T}{GS} \Delta x \tag{31}$$

Because: $\tau = G\gamma = \frac{T}{S}$ and $\gamma = \frac{\Delta y}{\Delta x}$

$$dU = \frac{1}{2} T \times \frac{T}{GS} \Delta x = \frac{T^2}{2GS} \Delta x \tag{32}$$

$$W_{ext} = U = \frac{T^2}{2GS} \int_0^l dx \tag{33}$$

For a cubic element (dx, dy, dz) subjected to shear stresses, the strain energy is:

$$dU = \frac{1}{2}(\tau_{xy} S)(\gamma_{xy} dx) = \frac{1}{2}(\tau_{xy}\gamma_{xy})(S.dx) \tag{34}$$

Therefore, the unit strain energy [J/m^3] is:

$$U_0 = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{1}{2}\frac{\tau_{xy}^2}{G} = \frac{1}{2} G\gamma_{xy}^2 \tag{35}$$

3.4. Pure torsional stress:

To calculate the strain energy, we consider a segment of length dx (figure 29). This segment is in equilibrium under the action of two torsional moments M_t. If θ is the unit angle of twist, then we have:

$$d\alpha = \theta . dx \tag{36}$$

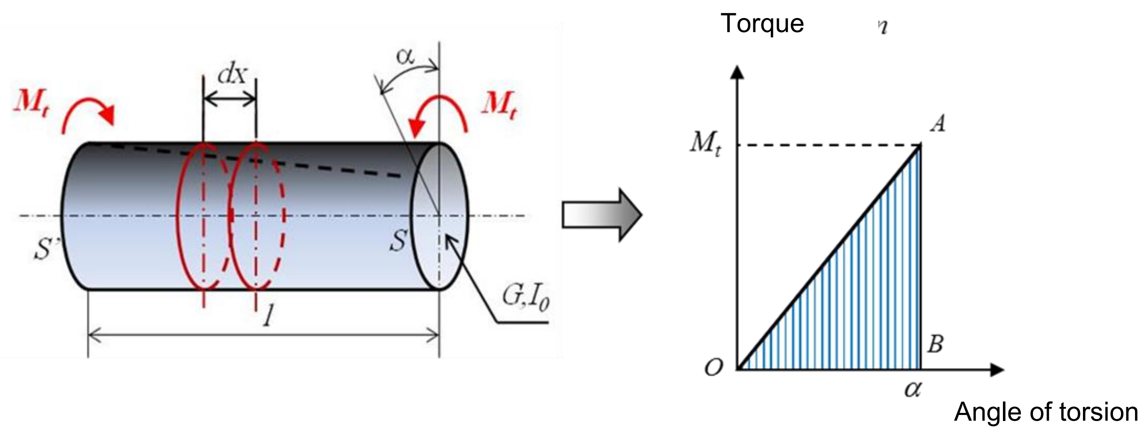


Figure 29. Pure torsion representation.

During deformation, the work done by the moment M_t, equal to the deformation energy of the section, is:

$$dW_{ext} = \frac{1}{2} M_t . d\alpha = \frac{1}{2} M_t . \theta . dx \tag{37}$$

$$dW_{ext} = \frac{1}{2} M_t . \frac{M_t}{GI_0} dx = \frac{M_t^2}{2GI_0} dx \tag{38}$$

$$U = W_{ext} = \int_0^l \frac{M_t^2}{2GI_0} dx = \frac{M_t^2 l}{2GI_0} \tag{39}$$

3.5. Simple bending stress:

To calculate the deformation energy in simple bending, consider a segment dx of a beam bent under external loads (Figure 30). This segment is in equilibrium under the action of bending moments M_f. According to the fundamental law of bending (2.05), we have:

$$\frac{d\varphi}{dx} = \frac{M_f}{EI_Z} \Rightarrow d\varphi = \frac{M_f}{EI_Z}.dx \qquad (40)$$

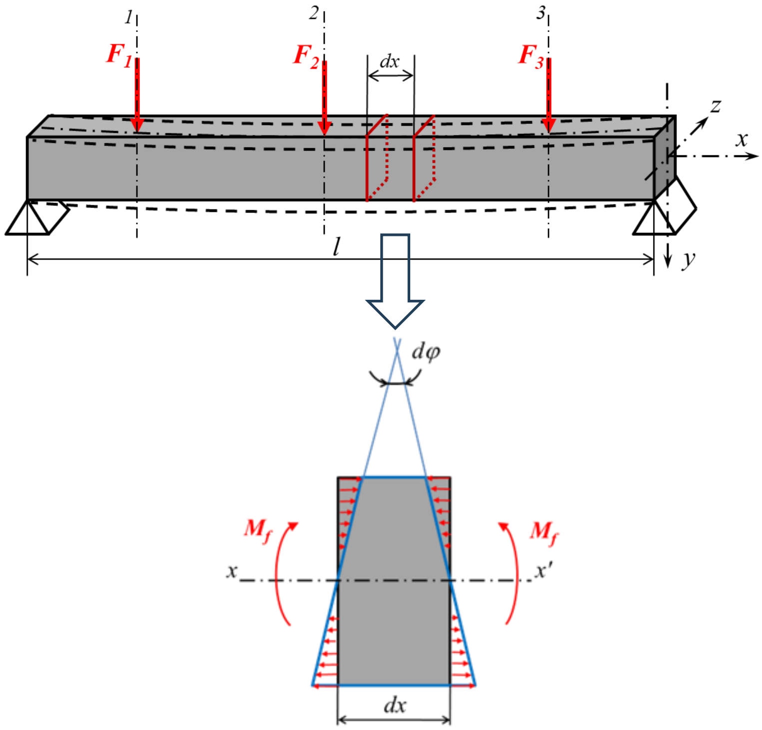


Figure 30. Simple bending representation.

$$dW_{ext} = dU = \frac{1}{2}M_f.d\varphi \qquad (41)$$

$$dU = \frac{1}{2}M_f.\frac{M_f}{EI_Z}dx = \frac{M_f^2}{2EI_Z}dx \qquad (42)$$

$$U = \frac{1}{2EI_Z}\int_0^l M_f^2 dx \qquad (43)$$

The work done by the external forces on the beam is equal to:

$$W_{ext} = \frac{1}{2}F_1 v_1 + \frac{1}{2}F_2 v_2 + \frac{1}{2}F_3 v_3 \ldots \qquad (44)$$

3.6. Complex loading conditions:

In the case of complex loading, where a solid is simultaneously subjected to a normal force N, a shear force T, a torsional moment M_t, and a bending moment M_f, the deformation energy will be:

$$U = \frac{1}{2ES}\int_0^l N^2 dx + \frac{1}{2GS}\int_0^l T^2 dx + \frac{1}{2GI_0}\int_0^l M_t^2 dx + \frac{1}{2EI_Z}\int_0^l M_f^2 dx \qquad (45)$$

In a material exhibiting linear elastic behavior, whether isotropic or not, the elastic strain energy can be calculated as if the loads were applied simultaneously. Thus, in a region of the body under the action of a stress tensor, the elastic strain energy per unit volume can be expressed as:

$$U_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}\right) \qquad (46)$$

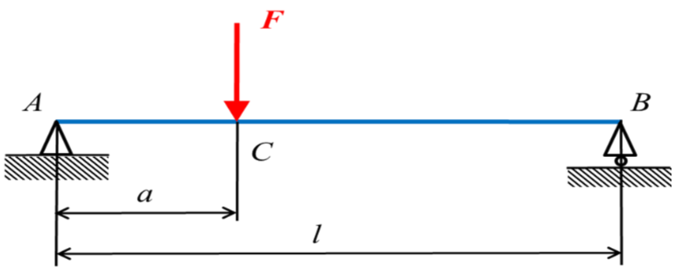
$$U_0 = \frac{1}{2E}\left[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2v\left(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z\right)\right] + \frac{1}{2G}\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right) \qquad (47)$$

For a principal stress tensor, we will have:

$$U_0 = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2v(\sigma_1\sigma_2 + \sigma_1\sigma_3 + \sigma_2\sigma_3)\right] \qquad (48)$$

Examples:

- Consider a beam with a straight section AB of length l supported on two supports at its ends and carrying a concentrated load F at C with abscissa a. Calculate the deflection of the beam in the section with abscissa a, with I the second moment of area of the straight section and E the modulus of elasticity of the material.



The beam is subjected primarily to bending (bending moment and shear force). Since the transverse modulus of elasticity was not provided, we only consider the effect of the bending moment.

$$W_{ext} = U = \frac{1}{2} F v_C \;\rightarrow\; v_C = \frac{2U}{F}$$

Let's calculate the reactions at the supports:

$$R_A + R_B = F$$

$$R_A l = F(l-a) \;\rightarrow\; R_A = \frac{F(l-a)}{l} \;\rightarrow\; R_B = \frac{Fa}{l}$$

Let's now calculate the bending moment along the beam:

- $0 \leq x \leq a$:

$$M(x) = R_A x = \frac{F(l-a)}{l} x$$

- $a \leq x \leq l$:

$$M(x) = R_A x - F(x-a) = \frac{F(l-a)}{l} x - F(x-a) = \frac{Fa(l-x)}{l}$$

and the deformation energy along the beam (3.25):

- $0 \leq x \leq a$:

$$U_1 = \frac{1}{2EI}\int_0^a M_f^2 dx = \frac{1}{2EI}\int_0^a \left(\frac{F(l-a)}{l} x\right)^2 dx = \frac{1}{2EI}\int_0^a \frac{F^2(l-a)^2}{l^2} x^2 dx = \frac{F^2 a^3 (l-a)^2}{6l^2 EI}$$

- $a \leq x \leq l$:

$$U_2 = \frac{1}{2EI}\int_a^l M_f^2 dx = \frac{1}{2EI}\int_a^l \left(\frac{Fa(l-x)}{l}\right)^2 dx = \frac{1}{2EI}\int_a^l \frac{F^2 a^2 (l-x)^2}{l^2} dx = \frac{F^2 a^2 (l-a)^3}{6l^2 EI}$$

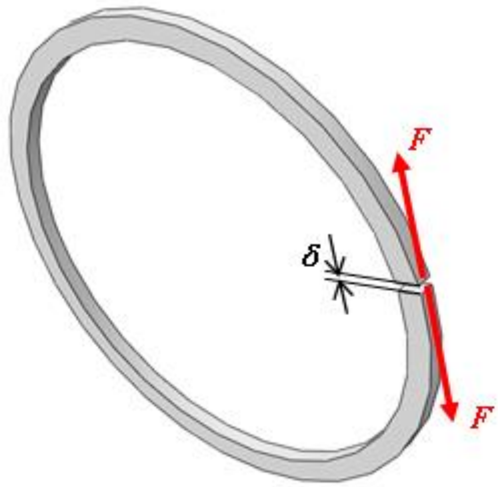
$$U = U_1 + U_2 = \frac{F^2 a^2 (l-a)^2}{6lEI}$$

Hence:

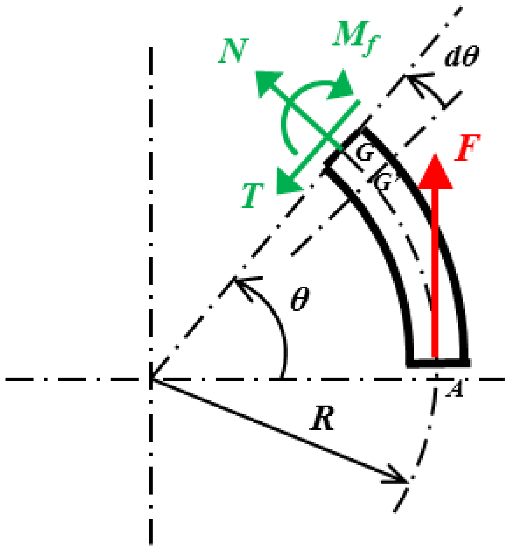
$$v_C = \frac{2U}{F} = \frac{Fa^2(l-a)^2}{3lEI}$$

- A circular ring is split radially. In the absence of external forces, the width of the split is very small. Two equal and opposite forces are applied.

Determine the increase δ in the width of the slit when the magnitude of these forces becomes equal to $F = 30\,N$. Given: $E = 1.2 \times 10^6\ MPa$, $G = 4.8 \times 10^4 MPa$, $b = 4\,mm$(thickness of the ring), $h = 6\,mm$ (height of the ring), and $R = 60\,mm$(radius of the ring's centerline).



The ring is considered to be a beam with a curved centerline. Let S be an arbitrary cross-section with center of gravity G and curvilinear abscissa $s = AG$ (Figure below). This cross-section is subjected to a normal load, shear, and a bending moment.



The deformation energy of the segment can be written as:

$$U = \frac{1}{2ES} \int_{GG'} N^2 ds + \frac{1}{2GS} \int_{GG'} T^2 ds + \frac{1}{2EI_Z} \int_{GG'} M_f^2 ds$$

From the figure, we define the external loads on the AG section:

$$\begin{cases} N = F\cos\theta \\ T = F\sin\theta \\ M_f = F.R(1-\cos\theta) \end{cases}$$

in addition to: $s = AG \rightarrow s = R.\theta \rightarrow ds = R.d\theta$

So, for the ring, the deformation energy will be given by:

$$U = \frac{1}{2ES}\int_0^{2\pi} (F\cos\theta)^2 R.d\theta + \frac{1}{2GS}\int_0^{2\pi} (F\sin\theta)^2 R.d\theta + \frac{1}{2EI_Z}\int_0^{2\pi} [F.R(1-\cos\theta)]^2 R.d\theta$$

Or:

$$U = \frac{F^2R}{2ES}\int_0^{2\pi} (\cos\theta)^2 d\theta + \frac{F^2R}{2GS}\int_0^{2\pi} (\sin\theta)^2 d\theta + \frac{F^2R^3}{2EI_Z}\int_0^{2\pi} (1-\cos\theta)^2 d\theta$$

$$U = \frac{F^2R}{2ES}\int_0^{2\pi} (\cos\theta)^2 d\theta + \frac{F^2R}{2GS}\int_0^{2\pi} (\sin\theta)^2 d\theta + \frac{F^2R^3}{2EI_Z}\int_0^{2\pi} d\theta - \frac{2F^2R^3}{2EI_Z}\int_0^{2\pi} \cos\theta\, d\theta + \frac{F^2R^3}{2EI_Z}\int_0^{2\pi} (\cos\theta)^2 d\theta$$

In the end, we will have:

$$U = \frac{F^2R}{2ES}\pi + \frac{F^2R}{2GS}\pi + \frac{3F^2R^3}{2EI_Z}\pi$$

With:

$$\begin{cases} (\cos\theta)^2 = \frac{1}{2} + \frac{\cos 2\theta}{2} \\ (\sin\theta)^2 = \frac{1}{2} - \frac{\cos 2\theta}{2} \end{cases}$$

Given that $W_{ext} = U$, then:

$$\frac{1}{2}F\delta = \frac{F^2R}{2ES}\pi + \frac{F^2R}{2GS}\pi + \frac{3F^2R^3}{2EI_Z}\pi$$

which gives us:

$$\delta = \frac{F^2R}{ES}\pi + \frac{F^2R}{GS}\pi + \frac{3F^2R^3}{EI_Z}\pi = 0.0059 + 0.147 + 21.19 = 21.35\,mm$$

It can be seen that the value of δ is due primarily to the bending moment, while the effects of normal force and shear force are negligible.

3.7. Castigliano's Theorem:

Castigliano’s theorem, introduced by Carlo Alberto Castigliano, is a fundamental result in mechanics of materials stating that, for a linearly elastic structure subjected to small deformations, the displacement at a given point in the direction of an applied force can be obtained by differentiating the total strain energy of the system with respect to that force, thus providing an efficient energy-based method for determining deflections and rotations in beams, frames, and statically indeterminate structures without directly solving the governing differential equations; mathematically, this relationship is expressed as:

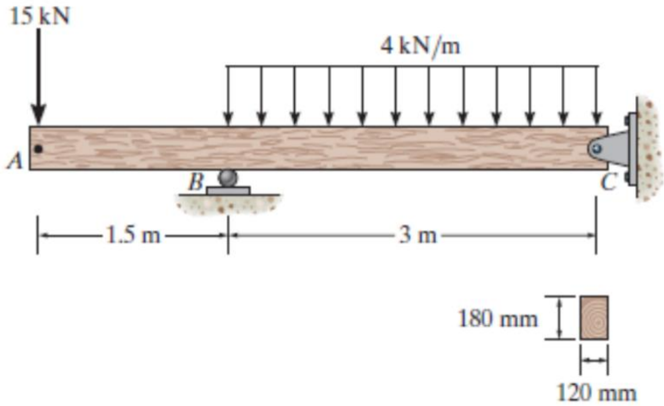
$$v_i = \frac{\partial U_i}{\partial F_i} \qquad (49)$$

or for bending (for a constant cross-section):

$$v = \frac{\partial}{\partial F}\int_0^L \frac{M_f^2}{2EI}dx = \frac{1}{EI}\int_0^L M_f\left(\frac{\partial M_f}{\partial F}\right)dx \qquad (50)$$

Example:

- Determine the vertical displacement of point A, $E = 13GPa$.



The beam is subjected primarily to bending (bending moment and shear force). Since the transverse modulus of elasticity was not provided, we only consider the effect of the bending moment.

At point A, let's replace the numerical value of the force with a letter (F) so that we can perform the partial derivative of the strain energy with respect to this force.

Let's calculate the reactions at the supports:

$$R_B + R_C = F + 4 \times 3 = F + 12\ kN$$

$$R_B \times 3 = F \times 4.5 + 4 \times 3 \times 1.5 \ \rightarrow\ R_B = (1.5F + 6)\ kN \ \rightarrow\ R_C = (-0.5F + 6)\ kN$$

Let's now calculate the bending moment along the beam:

- $0 \leq x \leq 1.5\ m$:

$$M(x) = -Fx$$

- $1.5 \leq x \leq 4.5$:

$$M(x) = -Fx + R_B(x - 1.5) - 4\frac{(x - 1.5)^2}{2} = 0.5Fx - 2.25F - 2x^2 + 12x - 13.5$$

let's apply Castigliano's theorem:

- $0 \leq x \leq 1.5\ m$:

$$v_1 = \frac{\partial}{\partial F}\int_0^{1.5} \frac{M_f^2}{2EI}dx = \frac{1}{EI}\int_0^{1.5} M_f\left(\frac{\partial M_f}{\partial F}\right)dx = \frac{1}{EI}\int_0^{1.5}(-Fx)(-x)dx = \frac{1}{EI}\int_0^{1.5} Fx^2dx$$
$$= \frac{1.125F}{EI}$$

- $1.5 \leq x \leq 4.5$:

$$v_2 = \frac{\partial}{\partial F}\int_{1.5}^{4.5} \frac{M_f^2}{2EI}dx = \frac{1}{EI}\int_{1.5}^{4.5} M_f\left(\frac{\partial M_f}{\partial F}\right)dx$$

$$= \frac{1}{EI}\int_{1.5}^{4.5}(0.5Fx - 2.25F - 2x^2 + 12x - 13.5)(0.5x - 2.25)dx$$

$$= \frac{1}{EI}(-0.25Fx^2 - 2.25Fx + 5.0625F - x^3 + 10.5x^2 - 33.75x$$

$$+ 30.375)|_{1.5}^{4.5} = \frac{1}{EI}(-3.9375F - 3.375)$$

$$v_A = v_1 + v_2 = \frac{1}{EI}(-2.8125F - 3.375)$$

we replace F with its value:

$$v_A = \frac{1}{EI}(-2.8125F - 3.375) = -\frac{45562.5}{EI} = -0.06\,m$$

With:

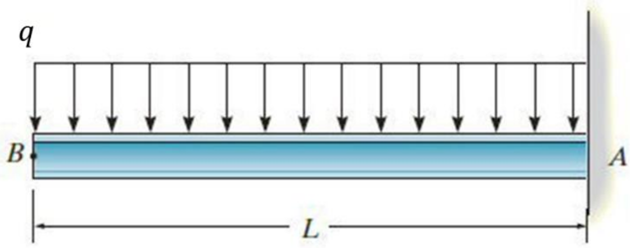
$$I = \frac{120 \times 180^3}{12} = 5.832 \times 10^7 mm^4 = 5.832 \times 10^{-5} m^4$$

3.8. Unit load method:

Applying strain energy to determine the displacement (rotation) of a point in a beam requires a force (torque) to be applied to the beam at that point and in the direction of the displacement to be determined. To generalize this method for determining the displacement (rotation) at any point and in any direction, it suffices to assume a fictitious force (torque) applied to the beam at the point considered and in the desired direction.

Example:

- Determine the vertical displacement of point B.



To apply Castigliano's theorem, a force must be present so that the strain energy can be differentiated with respect to that force. In the present case, no external force is applied at point B. To overcome this limitation, a fictitious (dummy) force F is introduced at point B during the analysis; after performing the differentiation, this force is set equal to zero in the final expression.

Let's calculate the bending moment along the beam:

$$M(x) = -Fx - q\frac{x^2}{2}$$

let's apply now Castigliano's theorem:

$$v_B = \frac{\partial}{\partial F}\int_0^L \frac{M_f^2}{2EI}dx = \frac{1}{EI}\int_0^L M_f\left(\frac{\partial M_f}{\partial F}\right)dx = \frac{1}{EI}\int_0^L\left(-Fx - q\frac{x^2}{2}\right)(-x)dx$$

$$= \frac{1}{EI}\int_0^L\left(Fx^2 + q\frac{x^3}{2}\right)dx = \frac{1}{EI}\left(\frac{Fx^3}{3} + q\frac{x^4}{8}\right)\Bigg|_0^L = \frac{1}{EI}\left(\frac{FL^3}{3} + q\frac{L^4}{8}\right)$$

$$F = 0$$

Then:

$$v_B = \frac{qL^4}{8EI}$$

3.9. Conclusion:

This chapter introduced energy methods as a sophisticated alternative to traditional static analysis. By defining strain energy for various loading states (tension, shear, torsion, and bending), we established a conservation-of-energy approach to mechanics. The application of Castigliano's Theorem and the Unit Load Method demonstrated that displacements can be efficiently calculated by differentiating total strain energy with respect to applied or "dummy" forces. These energy-based theorems provide an elegant framework that circumvents the need to solve complex differential equations directly, forming a cornerstone for advanced structural modeling.

Combined loading

4.1. Introduction:

In mechanics of materials, the present chapter introduces the fundamental notions of combined loading as part of the course material, focusing on structural elements subjected to the simultaneous action of multiple internal forces, which generate more complex stress and deformation states than those encountered under simple loading conditions. Within the scope of this lecture note, the analysis is restricted to configurations consistent with the undergraduate curriculum. It covers unsymmetrical (or oblique) bending, where bending occurs about non-principal axes of the cross-section, leading to coupled bending effects, and combined (or eccentric) bending, in which axial loads are superimposed with bending moments due to load eccentricity. The interaction between bending and torsion is introduced in a simplified manner, limited to circular shafts, without addressing the torsion of non-circular sections or warping phenomena. This presentation aims to provide students with a clear and rigorous foundation, emphasizing analytical methods and practical understanding relevant to standard engineering applications.

4.2. Unsymmetrical bending:

Unsymmetrical bending is a mode of bending in which the plane of bending moment does not coincide with one of the principal axes of inertia of the cross-section. Let us consider the bending of a beam occurring outside a principal plane (figure 31). In this case, the bending moment is inclined with respect to the principal axes, forming an angle θ with the z-axis. This moment can then be resolved into components along the y and z axes (M_z and M_y).

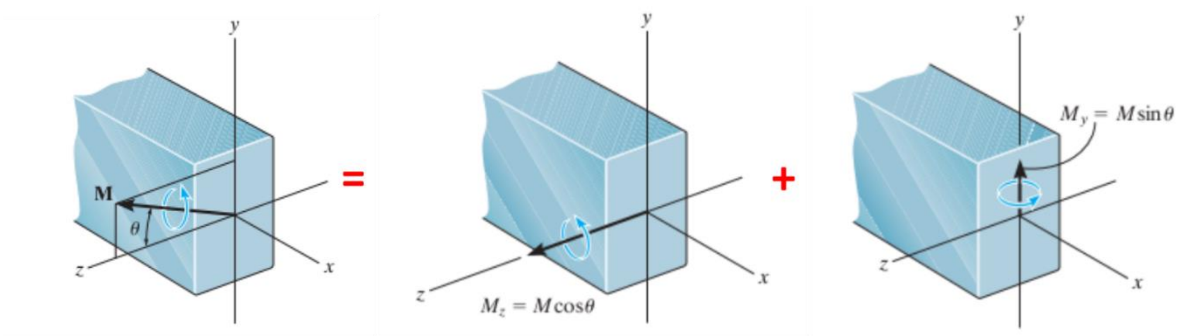


Figure 31. Decomposition of the bending moment in unsymmetrical bending.

Normal stress always acts perpendicular to the cross-section (along with the x-direction). The total stress at any point is obtained by superimposing the stresses induced by the two bending moment components (M_z and M_y):

$$\sigma = \sigma_x + \sigma_x' = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{51}$$

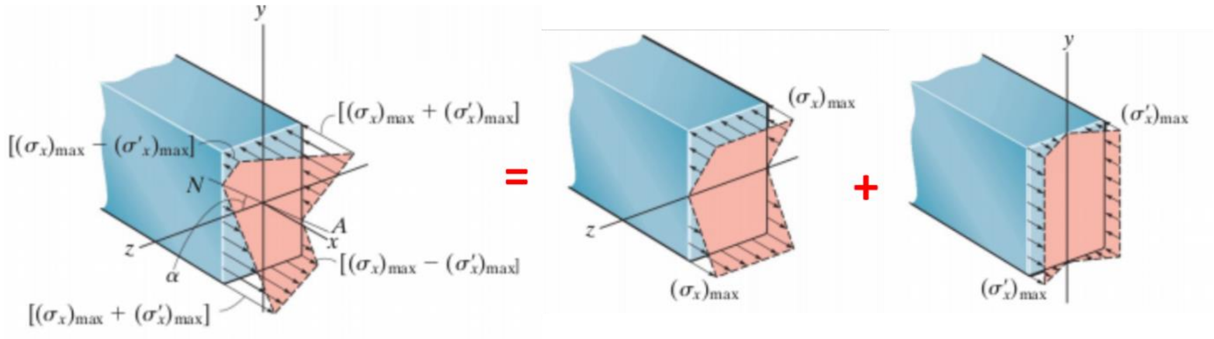


Figure 32. Normal stress distribution in unsymmetrical bending.

In this case, the neutral axis is inclined, and its orientation is determined by locating the line along which the normal stress is zero:

$$\sigma = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = 0 \tag{52}$$

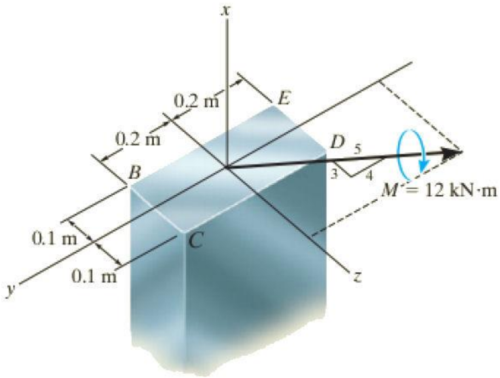
$$y = \frac{M_y I_z}{M_z I_y} z \tag{53}$$

$$y = \left(\frac{I_z}{I_y} tg\theta\right) z \tag{54}$$

$$tg\alpha = \frac{y}{z} = \frac{I_z}{I_y} tg\theta \tag{55}$$

Examples:

- Determine the distribution of normal stress in the cross-section under the action of the specified bending moment.



Let us determine the components of the moment along the two axes:

$$M_y = -\frac{4}{5} \times 12 = -9.6\ kN.m$$

$$M_z = \frac{3}{5} \times 12 = 7.2\ kN.m$$

and the corresponding second moments of area (moments of inertia) about each axis:

$$I_y = \frac{0.4 \times 0.2^3}{12} = 2.67 \times 10^{-4}\ m^4$$

$$I_z = \frac{0.2 \times 0.4^3}{12} = 10.67 \times 10^{-4}\ m^4$$

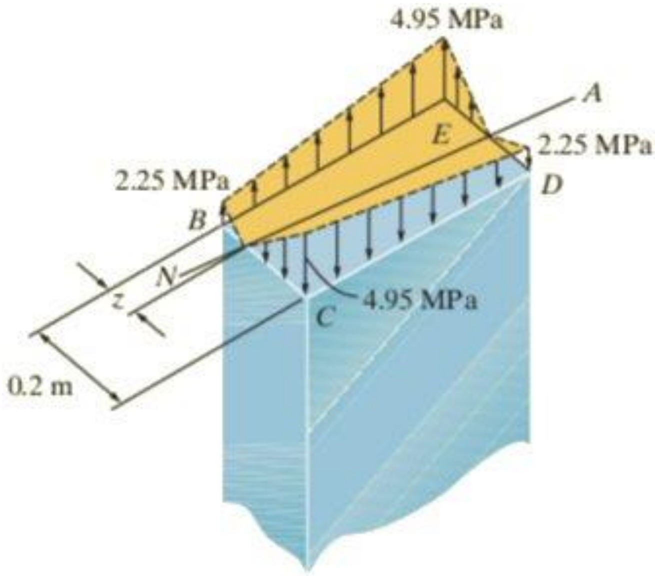
It is sufficient to calculate the stresses at the corners of the cross-section using expression (3.32):

$$\sigma_B = -\frac{7200 \times 0.2}{10.67 \times 10^{-4}} + \frac{-9600 \times (-0.1)}{2.67 \times 10^{-4}} = 2.246\ N/mm^2 = 2.246\ MPa$$

$$\sigma_C = -\frac{7200 \times 0.2}{10.67 \times 10^{-4}} + \frac{-9600 \times 0.1}{2.67 \times 10^{-4}} = -4.945\ N/mm^2 = -4.945\ MPa$$

$$\sigma_D = -\frac{7200 \times (-0.2)}{10.67 \times 10^{-4}} + \frac{-9600 \times 0.1}{2.67 \times 10^{-4}} = -2.246\ N/mm^2 = -2.246\ MPa$$

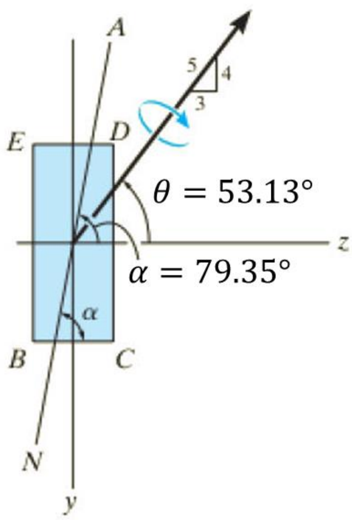
$$\sigma_E = -\frac{7200 \times (-0.2)}{10.67 \times 10^{-4}} + \frac{-9600 \times (-0.1)}{2.67 \times 10^{-4}} = 4.945\ N/mm^2 = 4.945\ MPa$$



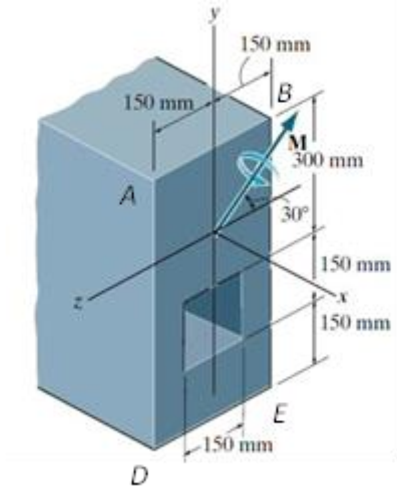
The orientation of the neutral axis is then given by:

$$tg\alpha = \frac{y}{z} = \frac{I_z}{I_y} tg\theta = \frac{10.67 \times 10^{-4}}{2.67 \times 10^{-4}} \times \left(-\frac{4}{3}\right) = -5.32$$

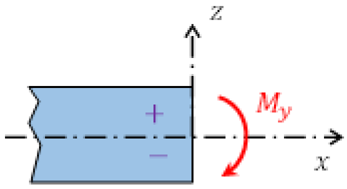
$$\alpha = -79.35°$$



- If a beam with the specified cross-section is made of a material with allowable tensile and compressive strengths of $[\sigma]_t = 125\ MPa$ and $[\sigma]_c = 150\ MPa$, respectively, determine the maximum allowable internal moment M that can be applied to the beam.

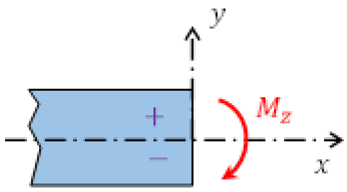


xz-plane: The moment M_y creates tensile stress in the positive z-direction (points A and D) and compressive stress in the negative z-direction (points B and E).



$$M_y = M \sin 30° = 0.5M$$

xy-plane: The moment M_z creates tensile stresses in the positive y-direction (points A and B) and compressive stresses in the negative y-direction (points D and E).



$$M_z = M \cos 30° = 0.866M$$

Let's determine the center of gravity of the cross-section along the y-direction relative to the base DE:

$$y_g = \frac{600 \times 300 \times 300 - 150 \times 150 \times 225}{600 \times 300 - 150 \times 150} = 310.7\ mm$$

So the quadratic moments can be calculated as:

$$I_z = \frac{300 \times 600^3}{12} + 300 \times 600 \times (300 - 310.7)^2 - \left[\frac{150^4}{12} + 150 \times 150 \times (225 - 310.7)^2\right]$$

$$= 52.13 \times 10^8 mm^4$$

$$I_y = \frac{600 \times 300^3}{12} - \frac{150^4}{12} = 13.07 \times 10^8 mm^4$$

Let's calculate the stresses at the four points.

Point A:

$$\sigma_A = +\frac{M_y}{I_y}z + \frac{M_z}{I_z}y = \frac{0.5M}{13.07\times10^8} \times 289.3 + \frac{0.866M}{52.13\times10^8} \times 150 = 1.35 \times 10^{-7}M \leq 125MPa$$

$$M \leq 921.9kN.m$$

Point *B* :

$$\sigma_B = -\frac{M_y}{I_y}z + \frac{M_z}{I_z}y = -\frac{0.5M}{13.07 \times 10^8} \times 289.3 + \frac{0.866M}{52.13 \times 10^8} \times 150 = -8.57 \times 10^{-8}M$$
$$\geq -150MPa$$

$$M \leq 1.75\ MN.m$$

Point *E* :

$$\sigma_E = -\frac{M_y}{I_y}z - \frac{M_z}{I_z}y = -\frac{0.5M}{13.07 \times 10^8} \times 289.3 - \frac{0.866M}{52.13 \times 10^8} \times 150 = -1.35 \times 10^{-7}M$$
$$\geq -150MPa$$

$$M \leq 1.11\ MN.m$$

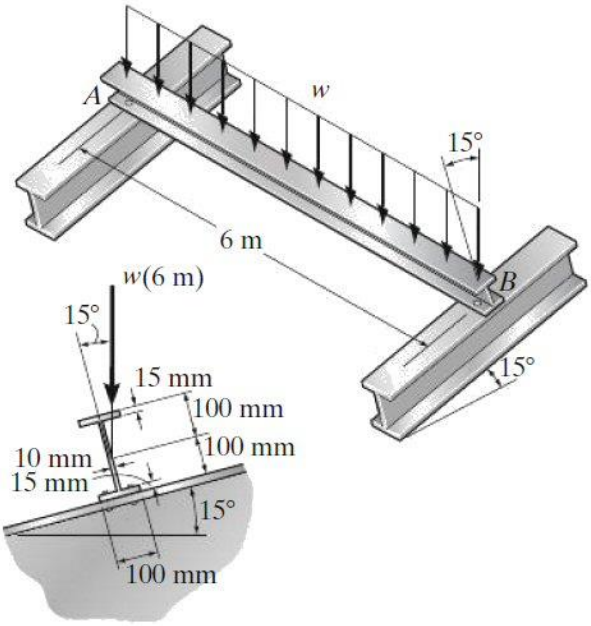
Point *D* :

$$\sigma_D = +\frac{M_y}{I_y}z - \frac{M_z}{I_z}y = +\frac{0.5M}{13.07 \times 10^8} \times 289.3 - \frac{0.866M}{52.13 \times 10^8} \times 150 = 8.57 \times 10^{-8}M \leq 125MPa$$

$$M \leq 1.46\ MN.m$$

Therefore, the torque must be equal to or less than $921.9\ kN.m$.

- Determine the maximum allowable load w that can be applied to the beam. Assume that w passes through the center of the beam's cross-section and that the beam is simply supported at points A and B. The beam is made of a material with an allowable bending stress $[\sigma] = 165\ MPa$.



The reactions at the supports are:

$$A = B = 3w$$

The bending moment is:

$$M(x) = 3wx - 0.5wx^2$$

The maximum moment occurs at the midpoint of the beam and is equal to:

$$M_{max} = 4.5w$$

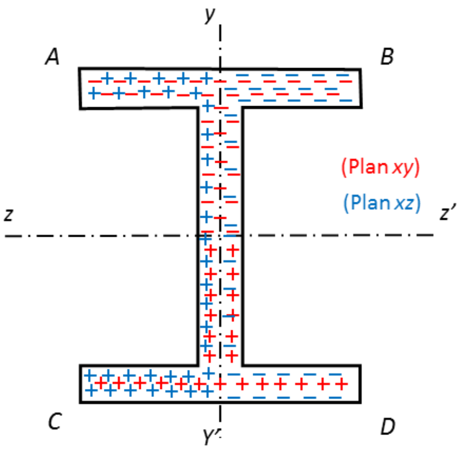
$$M_{z_{max}} = M_{max}\cos 15° = 0.966M_{max} = 4.347w$$

$$M_{y_{max}} = M_{max}\sin 15° = 0.259M_{max} = 1.165w$$

The moments of area with respect to the principal axes of the cross-section are:

$$I_z = \frac{100 \times 200^3}{12} - 2 \times \frac{45 \times 170^3}{12} = 29.8 \times 10^6 mm^4$$

$$I_y = \frac{170 \times 10^3}{12} + 2 \times \frac{15 \times 100^3}{12} = 2.5 \times 10^6 mm^4$$



Let's calculate the stresses at the four points.

Point *A* :

$$\sigma_A = +\frac{M_y}{I_y}z - \frac{M_z}{I_z}y = \frac{1.165\times 10^3 w}{2.5\times 10^6}\times 50 - \frac{4.347\times 10^3 w}{29.8\times 10^6}\times 100 = 8.71\times 10^{-3}w \le 165MPa$$

$$w \le 18937kN/mm = 18.9kN/m$$

Point *B* :

$$\sigma_B = -\frac{M_y}{I_y}z - \frac{M_z}{I_z}y = -\frac{1.165\times 10^3 w}{2.5\times 10^6}\times 50 - \frac{4.347\times 10^3 w}{29.8\times 10^6}\times 100 = -37.88\times 10^{-3}w \ge -165MPa$$

$$w \le 4355kN/mm = 4.35kN/m$$

Point *E* :

$$\sigma_C = +\frac{M_y}{I_y}z + \frac{M_z}{I_z}y = +\frac{1.165\times 10^3 w}{2.5\times 10^6}\times 50 + \frac{4.437\times 10^3 w}{29.8\times 10^6}\times 100 = 37.88\times 10^{-3}w \le 165MPa$$

$$w \le 4355kN/mm = 4.35kN/m$$

Point *D* :

$$\sigma_D = -\frac{M_y}{I_y}z + \frac{M_z}{I_z}y = -\frac{1.165\times 10^3 w}{2.5\times 10^6}\times 50 + \frac{4.347\times 10^3 w}{29.8\times 10^6}\times 100 = -8.71\times 10^{-3}w \ge -165MPa$$

$$w \le 18937kN/mm = 18.9kN/m$$

So the charge must be $\leq 4.35\ kN/m$

4.3. Combined bending:

In engineering practice, many situations arise where tensile or compressive loads are not applied through the centroid of a section, generating not only direct tension or compression but also significant bending effects. Consider the beam shown (figure 33), where the load is applied with an eccentricity e relative to a symmetry axis.

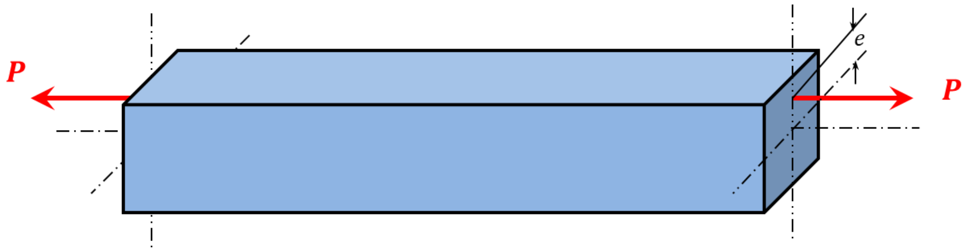


Figure 33. Combined bending.

The stress at any point is determined by calculating the bending stress according to the simple bending theory and superimposing it with the direct stress (axial load divided by the cross-sectional area), taking the appropriate sign into account.

$$\sigma = \frac{P}{A} \pm \frac{M}{I} y \tag{56}$$

with A the area of the cross section, M the moment created by the load P, I the second moment of the cross section and y the distance from the fiber considered to the neutral fiber.

$$M = Pe \tag{57}$$

$$\sigma = \frac{P}{A} \pm \frac{Pe}{I} y \tag{58}$$

It is evident that any eccentric load can be represented as an equivalent system consisting of a direct axial load applied through the centroid of the section and a bending moment about an axis passing through the centroid. The resulting stress distribution across the section is illustrated in the figure 34.

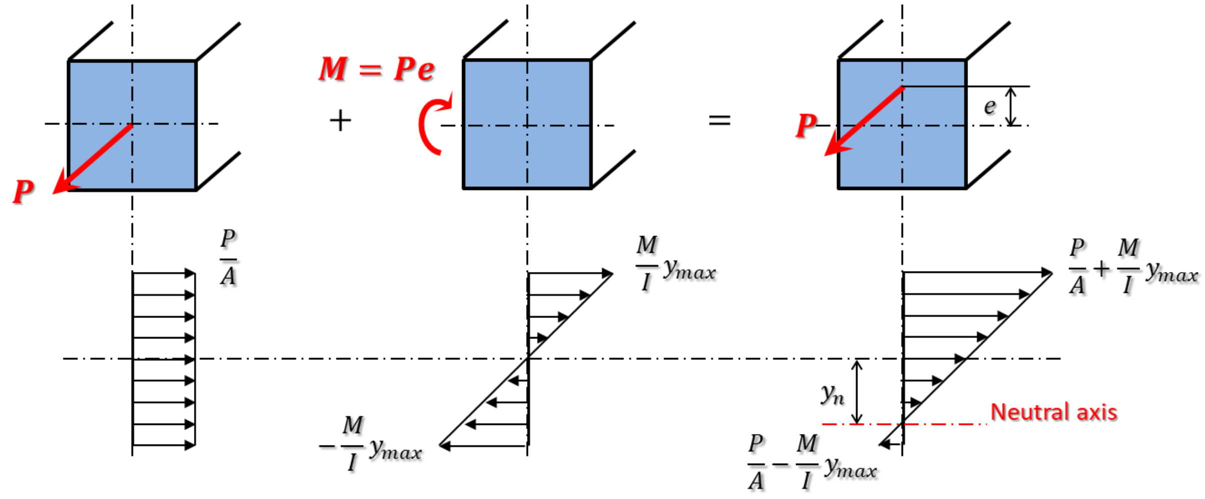


Figure 34. Normal stress distribution.

To determine the position of the neutral axis, it is sufficient to set the expression for the stress equal to zero.

$$\sigma = \frac{P}{A} \pm \frac{Pe}{I} y_n = 0 \tag{59}$$

$$\frac{P}{A} = \pm \frac{Pe}{I} y_n \tag{60}$$

$$y_n = \pm \frac{I}{Ae} \tag{61}$$

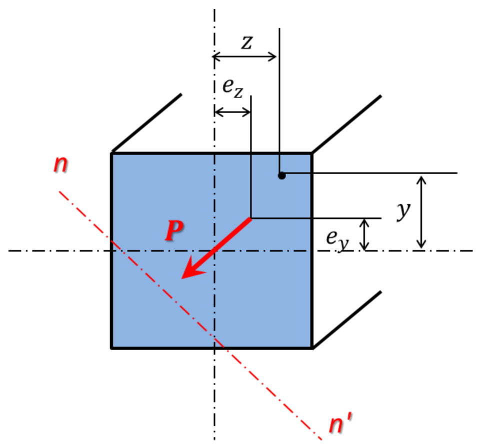


Figure 35. Combined and unsymmetrical bending.

In the case of bending about two axes (unsymmetrical bending) with an axial load, the normal stress at a point (z, y) is given by:

$$\sigma = \frac{P}{A} \pm \frac{Pe_y}{I_z} y \pm \frac{Pe_z}{I_y} z \qquad (62)$$

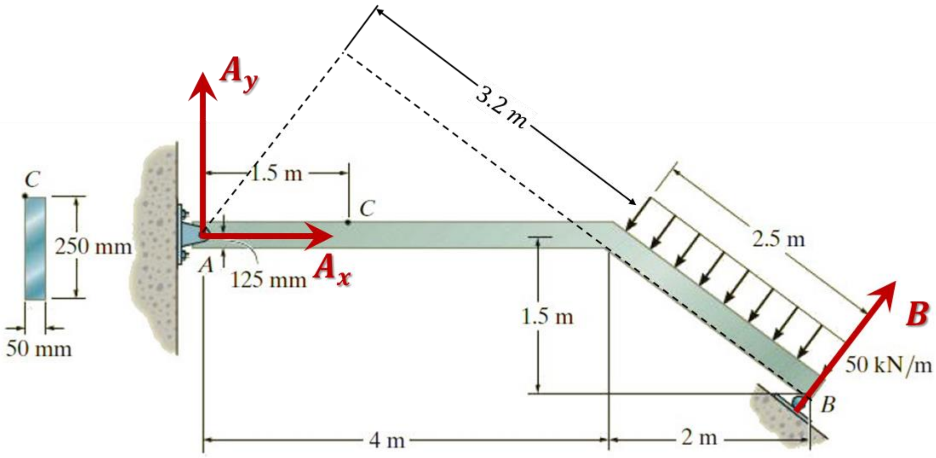
The position of the neutral axis (nn') can be determined as follows:

$$\frac{P}{A} \pm \frac{Pe_y}{I_z} y \pm \frac{Pe_z}{I_y} z = 0 \qquad (63)$$

$$\frac{Ae_y}{I_z} y \pm \frac{Ae_z}{I_y} z = \pm 1 \qquad (64)$$

Example:

- Determine the normal stress at point C.



Reaction B can be easily determined:

$$B \times (2.5 + 3.2) = 50 \times 2.5 \times \left(\frac{2.5}{2} + 3.2\right)$$

$$B \times 5.7 = 556.25$$

$$B = 97.59\ kN$$

and reaction A:

$$A + B = 50 \times 2.5 = 125\ kN$$

$$A = 27.41\ kN$$

The components of A are:

$$A_x = A\frac{1.5}{2.5} = 16.45\ kN$$

$$A_y = A\frac{2}{2.5} = 21.93\ kN$$

If the left segment AC of the member is considered, then the resultant internal loadings at the section consist of a normal force, a shear force, and a bending moment. They are:

$$N = 16.45\ kN,\quad T = 21.93\ kN \quad \text{and} \quad M_f = 21.93 \times 1.5 = 32.89\ kN.m$$

The stress components at C:

- Normal stress:

$$\sigma_x = \frac{N}{S} = \frac{16.45}{0.05 \times 0.25} = 1316\frac{kN}{m^2} = 1.32\ MPa$$

- Shear stress: The point C is located at the top of cross-section, then:

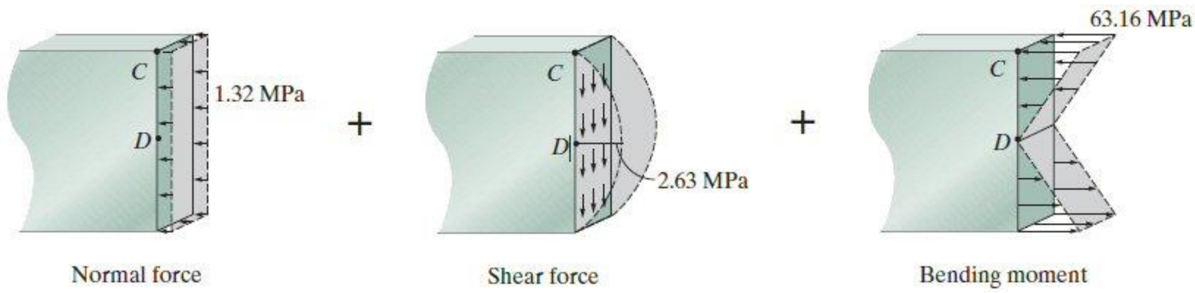
$$\tau_C = 0$$

- Normal stress does to bending moment: Point C is located at $y = 0.125\ m$ from the neutral axis, so the bending stress at C:

$$\sigma_C = \frac{M_f y}{I_z} = \frac{32.89 \times 0.125}{\frac{1}{12} \times 0.05 \times 0.25^3} = 63148.8\frac{kN}{m^2} = 63.15\ MPa$$

- Superposition: There is no shear-stress component. Adding normal stresses gives a compressive stress at C having a value of:

$$\sigma_C = 1.32 + 63.15 = 64.5\ MPa$$



4.4. Combined bending and torsion:

In mechanics of materials, the combined action of bending and torsion arises when a structural element is simultaneously subjected to bending moments and a torque, leading to a coupled stress state that involves both normal and shear components. This situation is commonly encountered in engineering applications such as shafts, beams with eccentric transverse loads, or transmission elements. Within the scope of this course, the analysis is restricted to circular cross-sections, for which torsion can be treated using classical assumptions, avoiding the complexities associated with non-circular sections. The total stress at any point of the cross-section is obtained by superimposing the normal stresses due to bending and the shear stresses induced by torsion. This section aims to provide a clear and systematic methodology for evaluating these combined effects, emphasizing practical calculation procedures and their relevance to engineering design.

Consider a shaft with a circular cross-section subjected simultaneously to a torsional moment (M_t) and a bending moment (M_f) (Figure 36). The cross-section is therefore subjected to both a distribution of shear stress and a distribution of normal stress. The maximum values of these stress distributions occur at point A, located at the outer surface of the circular section.

The equivalent stress is determined by applying an appropriate failure criterion and using Mohr's circle (Figure 37). In this course material, the Tresca criterion is adopted:

$$\sigma = \sigma_1 - \sigma_3 = \sqrt{\sigma^2 + 4\tau^2} \leq \frac{[\sigma]}{n} \tag{65}$$

where $[\sigma]$ denotes the ultimate strength of the material and n is the safety factor.

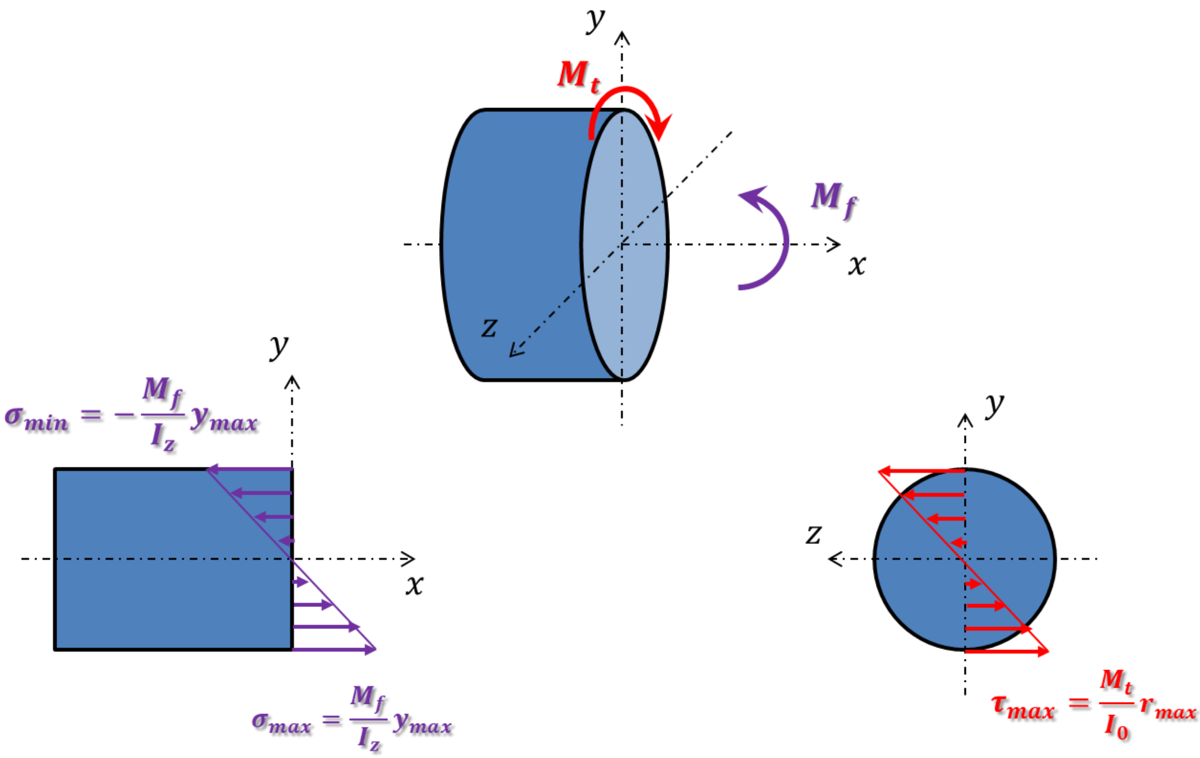


Figure 36. Stresses in a circular cross-section subjected to combined torsion and bending moments.

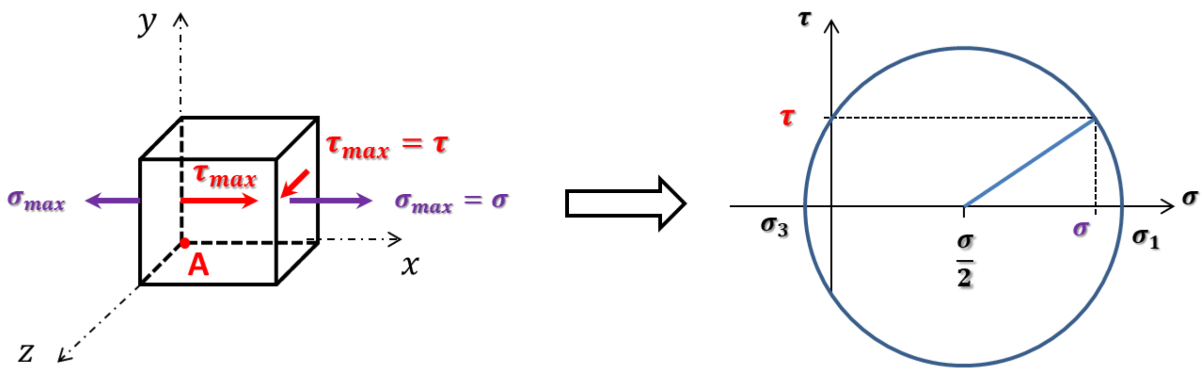
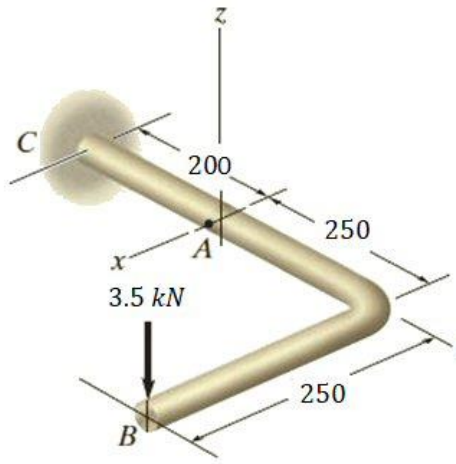


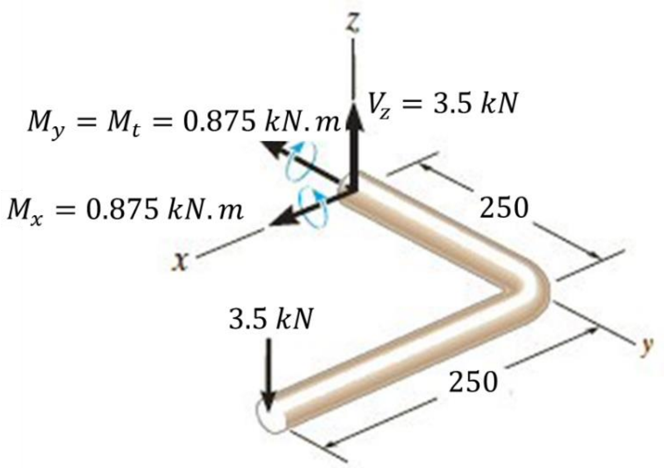
Figure 37. Mohr's circle of the stress state at point A.

Example:

- Determine the stress state at point A. Given: $d = 20\ mm$.



The rod is sectioned through point A. Using the free-body segment AB, the resultant internal loadings are determined from the equations of equilibrium. Take a moment to verify these results. The equal but opposite resultants are shown acting on segment AC



The stress components at A:

- Shear Force. The shear-stress distribution is shown in figure below. For point A, the static moment of area S_z is determined from the grey shaded semicircular area:

$$S_z = \int_{A'} y' dA' = \overline{y'}A' = \frac{2d}{3\pi}\frac{\pi d^2}{8} = \frac{d^3}{12} = 6.67 \times 10^{-7} m^3$$

$$(\tau_{yz})_A = \frac{V_z S_z}{b I_z} = \frac{3.5 \times 10^3 \times 6.67 \times 10^{-7}}{0.02 \times \frac{1}{64} \times \pi \times 0.02^4} = 14.87\ MPa$$

- Bending Moment: Since point A lies on the neutral axis, the bending stress is:

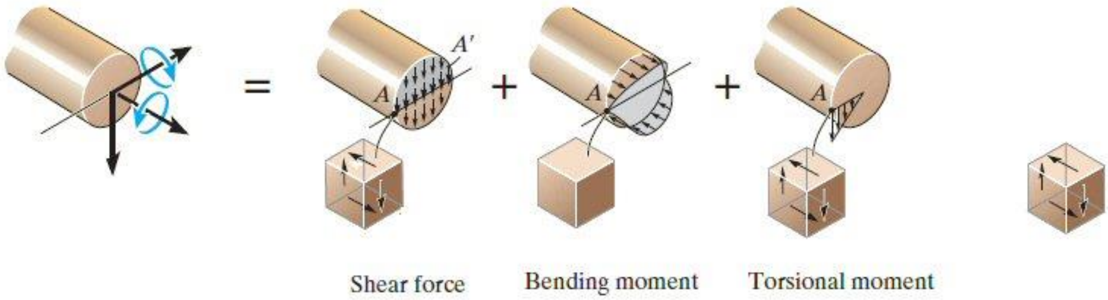
$$\sigma_A = 0$$

- Torque: At point A, r_A=10 mm. Thus, the shear stress is:

$$\left(\tau_{yz}\right)_A = \frac{M_t r_A}{I_0} = \frac{0.875 \times 10^3 \times 0.01}{\frac{1}{32} \times \pi \times 0.02^4} = 557.32\ MPa$$

- Superposition: Here the element of material at A is subjected only to a shear stress component, where:

$$\left(\tau_{yz}\right)_A = 14.87 + 557.32 = 572.19\ MPa$$



4.5. Conclusion:

Moving toward real-world engineering applications, this chapter analyzed components subjected to multiple simultaneous internal forces. We addressed unsymmetrical bending, where loads acting outside principal axes lead to coupled effects and an inclined neutral axis. Additionally, the study of combined bending (eccentric loading) and the interaction between bending and torsion, specifically in circular shafts, highlighted the necessity of superimposing stresses to determine a component's true state of loading. This comprehensive analysis is vital for the safe design of machine elements like transmission shafts, where simple loading assumptions are rarely sufficient.

Analysis of Statically Indeterminate Systems

5.1. Introduction:

This chapter focuses on the analysis of statically indeterminate structures and is organized into two main sections:

- Force Superposition Method: an energy-based approach that determines redundant forces by combining equilibrium and compatibility conditions, allowing the systematic resolution of hyperstatic structures.
- Three-Moment Theorem: a specialized method for continuous beams, which relates bending moments at three consecutive supports to applied loads and beam properties, providing an efficient tool for analyzing indeterminate beam systems.

This structure provides students with both general and beam-specific techniques for solving statically indeterminate problems.

5.2. Force Superposition Method:

The “Force Superposition Method” is a classical technique for analyzing statically indeterminate structures. It is based on the principle of superposition, which allows the decomposition of a hyperstatic system into a statically determinate primary structure and a set of redundant forces. By combining the equations of equilibrium with compatibility conditions expressed in terms of displacements or deformations, the method enables the systematic calculation of unknown reactions or internal forces. This approach is particularly valuable because it leverages energy-based tools, such as virtual work or Castigliano’s theorem, to relate applied loads and structural deformations, providing a rigorous yet practical framework for resolving complex indeterminate problems encountered in engineering practice.

Consider, for example, the beam shown in Figure 38, which is supported by a simple support at one end and a fixed support at the other. The number of reaction forces (unknowns) is 4, while the number of independent static equilibrium equations is 3 (equation). Therefore, there is one redundant unknown, and the system is classified as a statically indeterminate structure of first degree (hyperstatic of order 1).

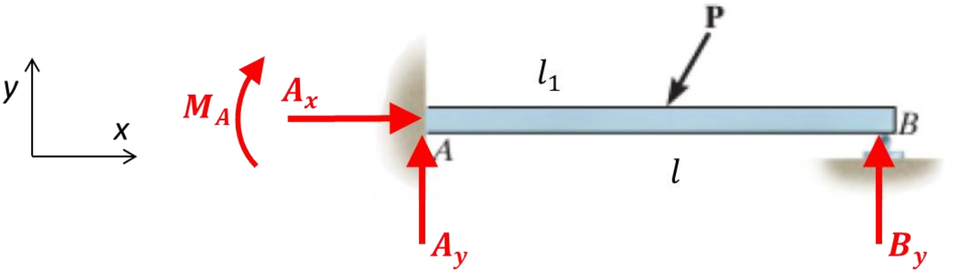


Figure 38. Hyperstatic system of order 1.

The equilibrium equations:

$$\begin{cases} \sum F_x = 0 \;\rightarrow\; A_x = P_x \\ \sum F_y = 0 \;\rightarrow\; A_y + B_y = P_y \\ \sum M_{/A} = 0 \;\rightarrow\; M_A - B_y \times l + P_y \times l_1 = 0 \end{cases} \tag{66}$$

The beam in Figure 39 rests on three simple supports and one double support. The total number of reaction forces is 5, while the number of independent static equilibrium equations is 3. Therefore, there are two redundant unknowns, and the system is classified as a statically indeterminate structure of second degree (hyperstatic of order 2).

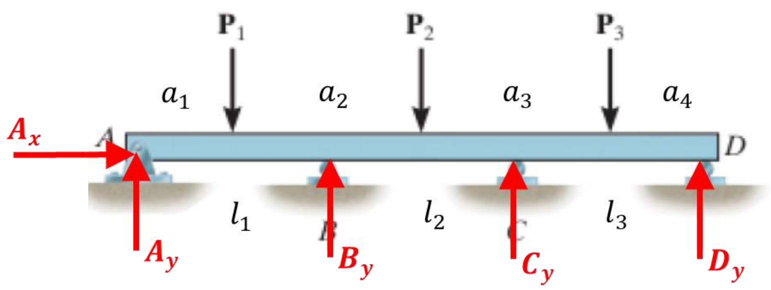


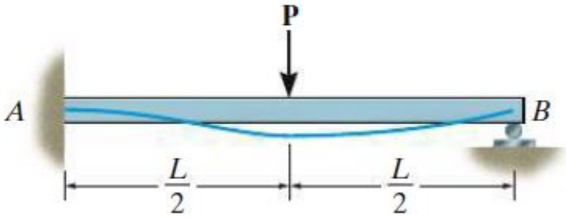
Figure 39. Hyperstatic system of order 2.

The equilibrium equations:

$$\begin{cases} \sum F_x = 0 \;\rightarrow\; A_x = 0 \\ \sum F_y = 0 \;\rightarrow\; A_y + B_y + C_y + D_y = P_1 + P_2 + P_3 \\ \sum M_{/A} = 0 \rightarrow B_y \times l_1 + C_y \times (l_1 + l_2) + D_y \times l - P_1 \times a_1 - P_2(a_1 + a_2) - P_3(l - a_4) = 0 \end{cases} \qquad (67)$$

Examples:

- Determine the reactions at the supports of the beam below.

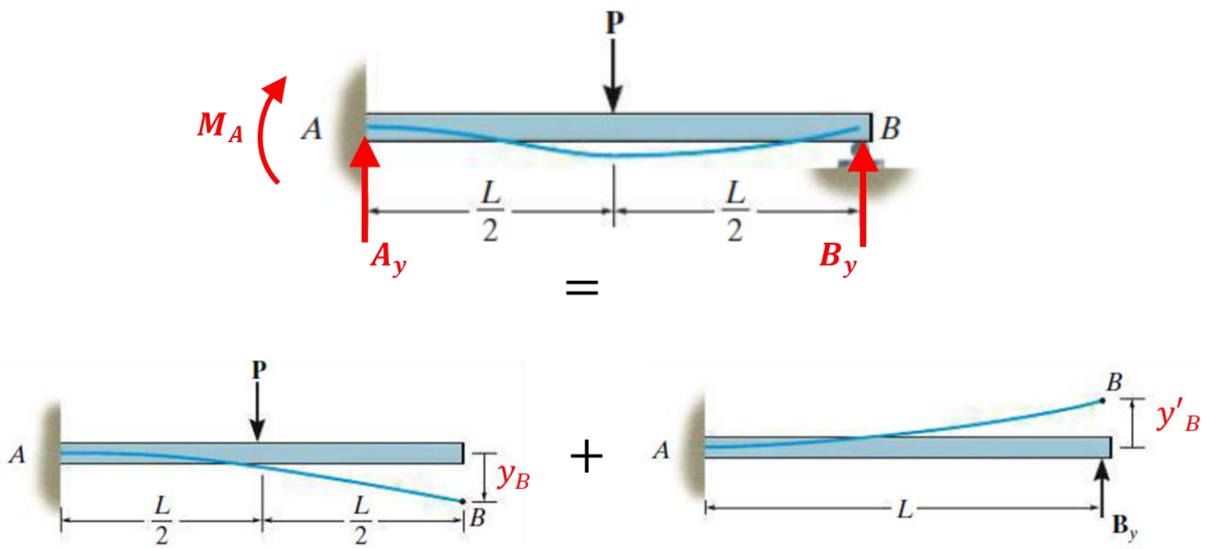


The equilibrium equations:

$$\begin{cases} \sum F_y = 0 \;\rightarrow\; A_y + B_y = P \\ \sum M_{/A} = 0 \;\rightarrow\; B_y \times L + P \times \dfrac{L}{2} = 0 \end{cases}$$

Two equilibrium equations with three unknowns correspond to a statically indeterminate system of first degree (hyperstatic of order 1).

We can use the vertical displacements of point B for two statically determined beams, respecting the condition of the initial beam where the vertical displacement of point B is zero.



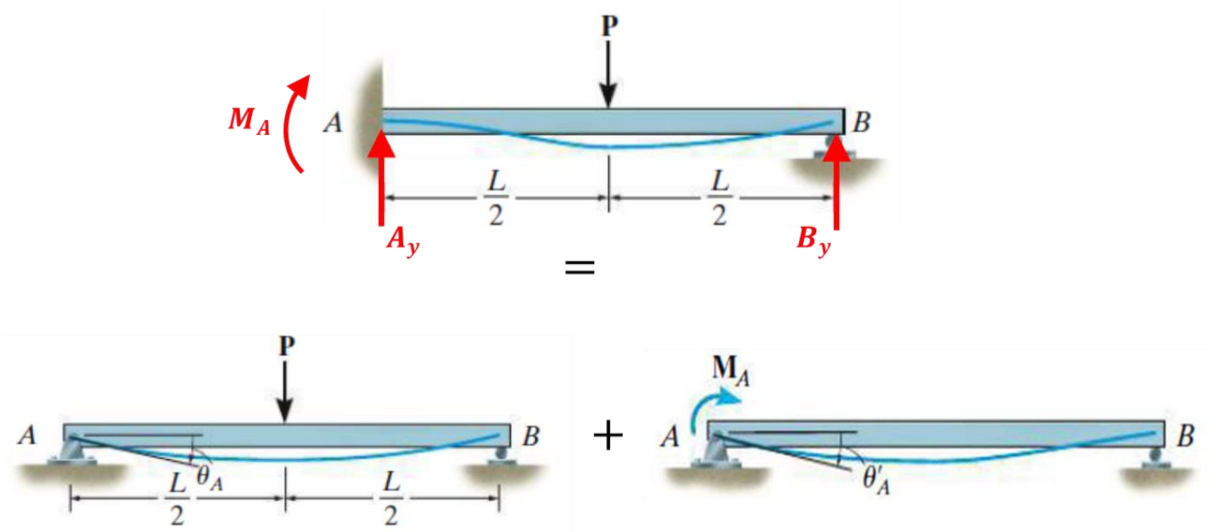
$$\vec{y}_B + \vec{y}'_B = 0$$

$$-y_B + y'_B = 0$$

$$-\frac{5PL^3}{48EI} + \frac{B_yL^3}{3EI} = 0$$

$$\begin{cases} B_y = \dfrac{5}{16}P \\ A_y = \dfrac{11}{16}P \\ M_A = \dfrac{3}{16}PL \end{cases}$$

One can also use the rotations of section A of the two statically determinate beams and, by applying the boundary condition at the fixed support A (where the section cannot rotate), the problem can be solved.



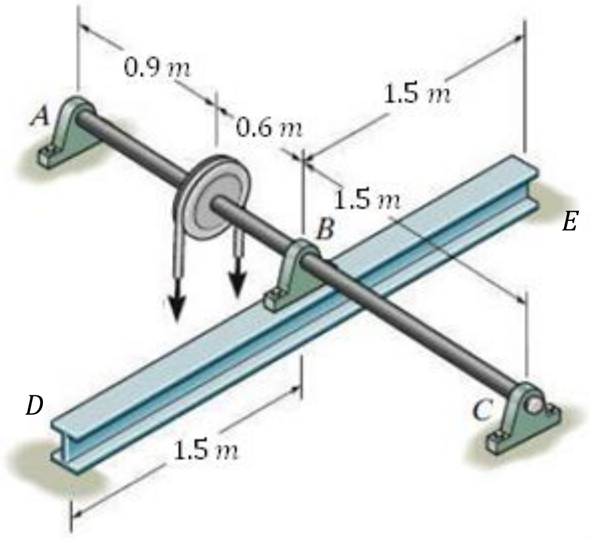
$$\vec{\theta}_B + \vec{\theta}'_B = 0$$

$$-\theta_B + \theta'_B = 0$$

$$\frac{PL^2}{16EI} + \frac{M_AL}{3EI} = 0$$

$$\begin{cases} B_y = \dfrac{5}{16}P \\ A_y = \dfrac{11}{16}P \\ M_A = \dfrac{3}{16}PL \end{cases}$$

- A shaft with a diameter of $25\ mm$ is supported by bearings at points A and C. The bearing at point B rests on a simply supported steel beam with a moment of inertia of $16.65 \times 10^7\ mm^4$. If the belt loads on the pulley are $1750\ N$ each, determine the vertical reactions at A, B, and C.



Let's write the equilibrium equations:

$$A + C + B = 3500\ N$$

$$A \times 3 + B \times 1.5 - 3500 \times 2.1 = 0$$

$$E = D = \frac{B}{2}$$ (by symmetry with respect to B)

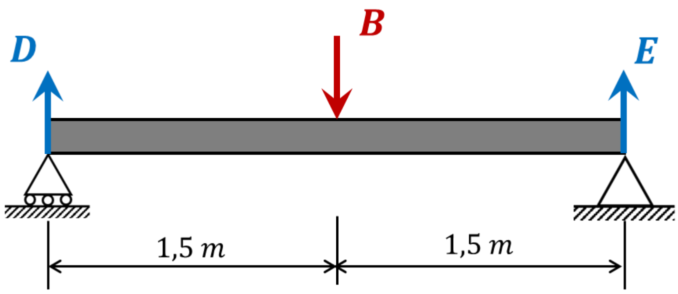
Two equilibrium equations with three unknowns correspond to a statically indeterminate system of first degree (hyperstatic of order 1).

Let's divide this system into two subsystems (the AC beam and the DE beam), while satisfying the constraint on the vertical displacement of point B (point B belongs to both subsystems).

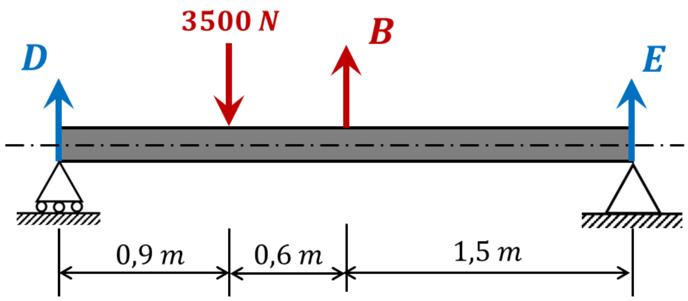
$$v_B^{AC} = v_B^{DE}$$

Beam DE is subjected to load B from beam AC, and beam AC is subjected to the same load, which resists bending.

Referring to the appendix, we have:



$$v_B^{DE} = \frac{-BL^3}{48EI}$$



$$v = \frac{-Pba}{6EIL}(L^2 - b^2 - a^2)$$

$$v_B^{AC} = -\frac{3500 \times 2.1 \times 09}{6EIL}(3^2 - 2.1^2 - 0.9^2) + \frac{BL^3}{48EI} = -\frac{1389.15}{EI} + \frac{0.5625B}{EI}$$

$$-\frac{1389.15}{EI} + \frac{0.5625B}{EI} = -\frac{BL^3}{48EI}$$

$$-1389.15 + 0.5625B + 0.5625B = 0$$

$$B = 1234.8\ N$$

$$E = D = \frac{B}{2} = 617.5\ N$$

$$A = 1832.6\ N,\ \ C = 432.6\ N$$

5.3. Three-Moment Theorem:

5.3.1. Continuous beam:

A continuous beam, as part of statically indeterminate systems, is a beam resting on more than two supports at the same level, with at least one support being a hinge. The beam is assumed to be horizontal and subjected only to vertical forces applied within its plane of symmetry.

Figure 40. Continuous Beam (Bridge).

The continuous beam consists of several intermediate spans, referred to as panels or spans. The end spans are called end spans, which may have one end free (cantilever), simply supported, or fixed.

For a continuous beam with n spans, the supports are numbered from 0 to n. Span i, of length l_i, lies between supports A_{i-1} and A_i. The number of redundant unknowns is equal to the number of intermediate supports, that is, $n-1$ (figure 41).

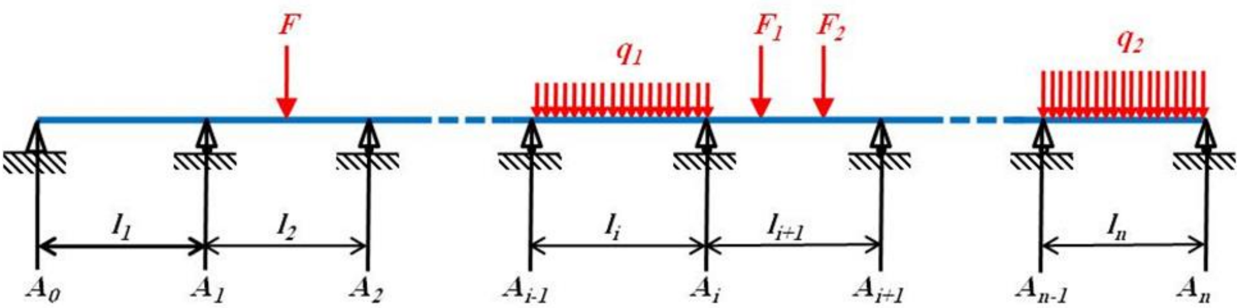


Figure 41. Elements of a Continuous Beam (Spans, Supports with Conventional Numbering…).

A continuous hyperstatic beam with n spans can be decomposed into n statically determinate (isostatic) beams. The same loads applied to the continuous beam act on these beams, with the addition of the support moments at the intermediate supports.

Indeed, this consists in selecting the (n−1) bending moments M_1, M_{1+1} ,… M_{i-1}, M_i, M_{i+1},… M_{n-1} at the intermediate supports A_1, A_{1+1}, …A_{i-1}, A_i, A_{i+1},… A_{n-1} as redundant unknowns, and introducing them as external loads when considering the statically determinate beam associated with the continuous beam.

The values of M_0 and M_n are zero when the end supports A_0 and A_n are simple supports or absent, and when no external moment is applied at these points.

For example, the three-span continuous beam shown in the figure 42 can be decomposed into three statically determinate beams.

In the general case, consider span i of a continuous beam with two adjacent spans, $(i-1)$ and $(i+1)$ (figure 43).

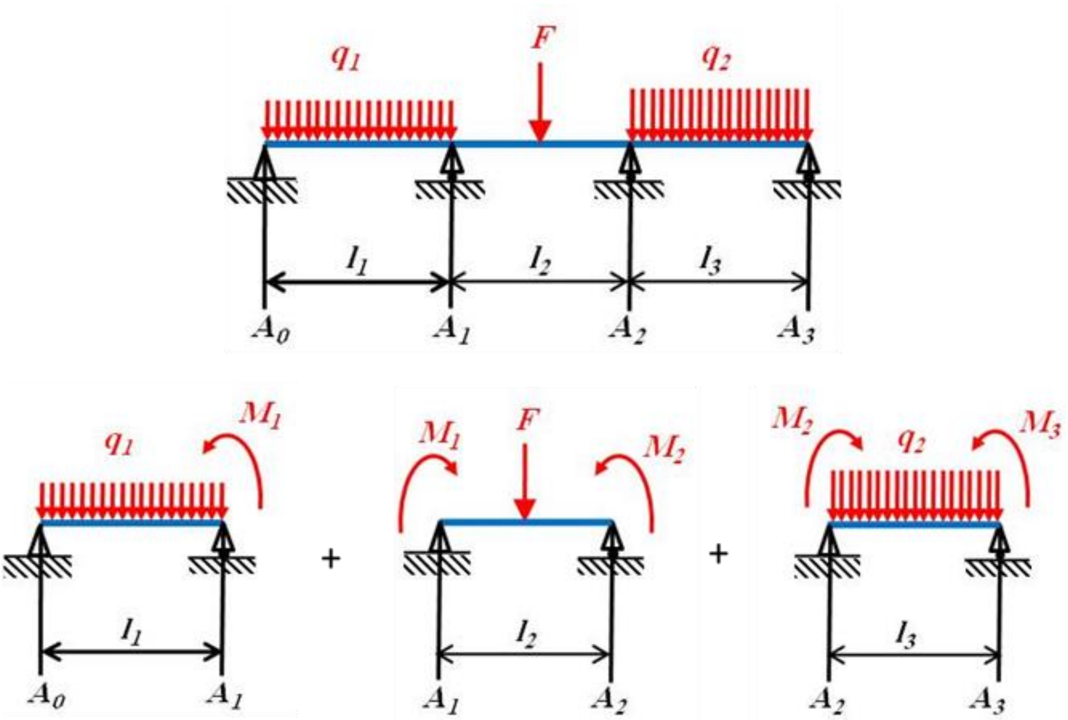


Figure 42. Three-span continuous beam.

For the associated statically determinate beams to behave equivalently to the original beam, continuity must be ensured, i.e., a common tangent at every point. Consequently, the rotations on the left and right sides of a support must be equal. For support A_i, one obtains (Figure 44):

$$\theta_i^g + \theta_i^d \tag{68}$$

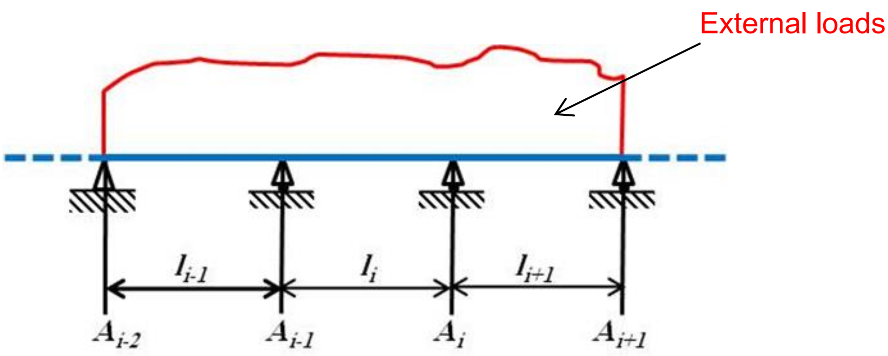


Figure 43. Span i of a continuous beam with two adjacent spans.

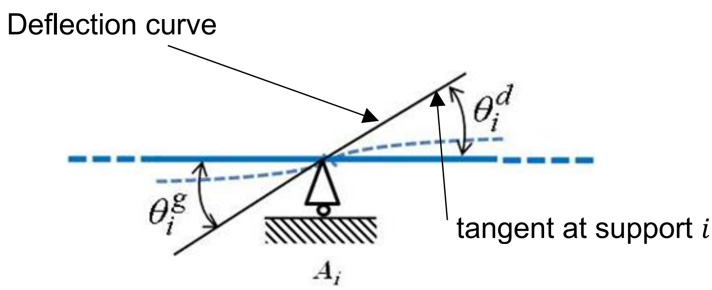


Figure 44. Beam continuity.

5.3.2. Moment–area theorem:

This theorem was addressed in Chapter 02; however, it will be further defined here within the context of continuous beams. Consider a beam of constant cross-section.

After deformation, its neutral axis is described by the equation $y(x)$. The slope at any point is given by the derivative of y. Beam theory then yields:

$$\frac{1}{\rho} = \frac{d\theta}{dx} = \frac{M(x)}{EI} \tag{69}$$

with ρ the radius of curvature of the deformed configuration's midline (neutral axis).

Therefore, we will have:

$$d\theta = \frac{M(x)}{EI} dx \tag{70}$$

The variation in slope between two points A and B of the beam is then (figure 45) :

$$\theta_{AB} = \theta = \int_A^B \frac{M(x)}{EI} dx \tag{71}$$

which represents the area under the bending moment diagram between the positions x_A and x_B.

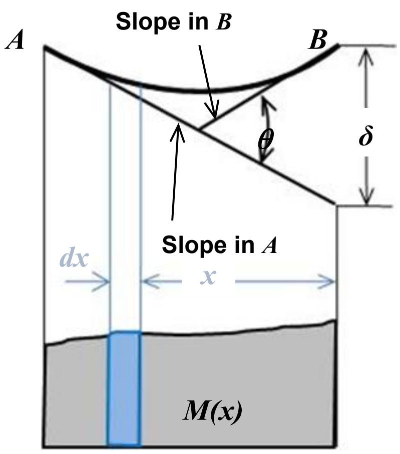


Figure 45. Representation of the deflected shape of a beam segment together with the corresponding bending moment distribution.

If δ is the vertical distance between point B on the deflected curve in Figure (45) and the tangent at point A to this curve, then:

$$\delta = \int_A^B \frac{M(x).x}{EI} dx \tag{72}$$

From which the slopes at points A and B can be determined as follows:

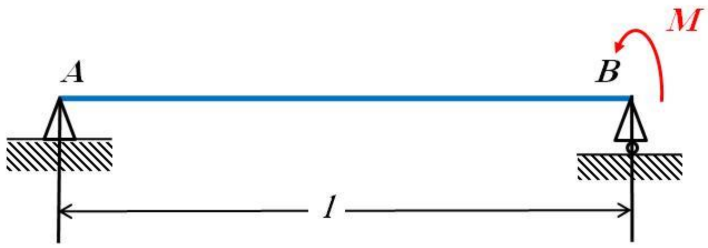
$$\theta_A = \frac{\delta}{AB} \tag{73}$$

and consequently:

$$\theta_B = \theta_A - \theta \tag{74}$$

Example:

- Consider the beam with rectangular cross-section shown, subjected to a moment applied at support B. Determine the slope at A and B as well as δ.



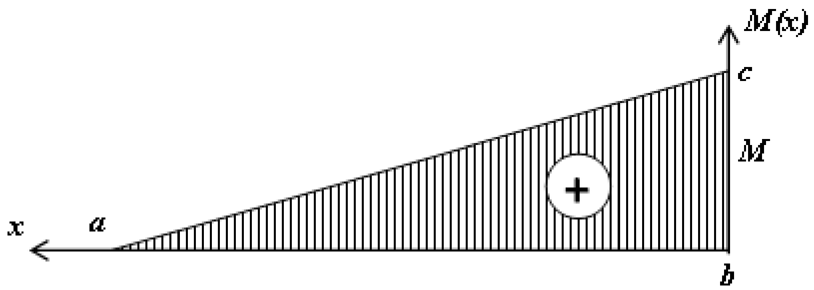
The bending moment along the beam is determined. The support reactions are easily obtained:

$$R_A = \frac{M}{l} \quad \text{and} \quad R_B = -\frac{M}{l}$$

At an arbitrary cross-section located at coordinate x between B and A, the bending moment is given by:

$$M(x) = R_B x + M = -\frac{M}{l}x + M$$

Figure below shows the bending moment distribution along the beam.



Then, according to expression (71), we obtain:

$$\theta_{AB} = \int_0^l \frac{M(x)}{EI} dx = \frac{1}{EI}\int_0^l \left(-\frac{M}{l}x + M\right) dx = \frac{Ml}{2EI}$$

which is equal to the area of triangle (abc) divided by EI.

For δ, expression (72) is applied:

$$\delta = \int_0^l \frac{M(x).x}{EI} dx = \frac{1}{EI}\int_0^l \left(-\frac{M}{l}x^2 + Mx\right) dx = \frac{Ml^2}{6EI}$$

The slope at B is determined by expression (73):

$$\theta_A = \frac{\delta}{l} = \frac{Ml}{6EI}$$

and the slope at B is given by expression (74):

$$\theta_B = \theta_A - \theta = \frac{Ml}{6EI} - \frac{Ml}{2EI} = -\frac{Ml}{3EI}$$

5.3.3. Conjugate beam:

Let us take the previous example for definition purposes. The conjugate beam is considered as a fictitious beam subjected to a distributed load defined by the bending moment diagram divided by the flexural rigidity EI. The support reactions are fictitious and are determined using Newton's laws of statics.

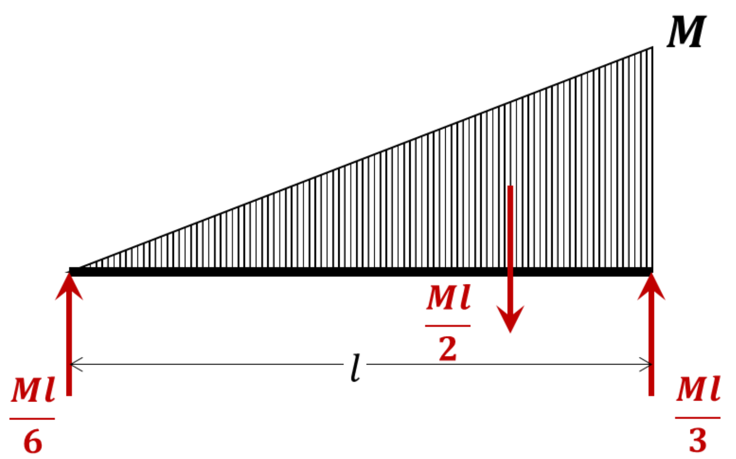
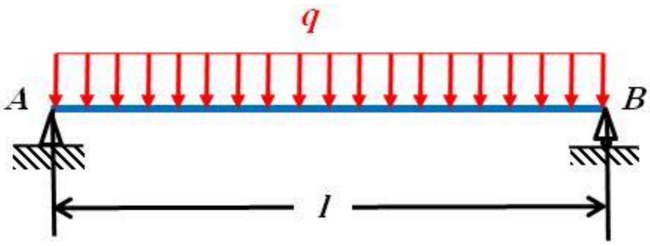


Figure 46. conjugate beam of the previous example.

By comparing these reactions with the corresponding slopes obtained in the previous exercise, it can be stated that the rotation angles θ_A and θ_B at the ends of the real beam AB are obtained by dividing, by the flexural rigidity EI, the support reactions at the ends of the corresponding fictitious beam ab. This fictitious beam is referred to as the conjugate beam.

Example:

- Determine the slopes at the ends A and B of the beam shown using the corresponding conjugate beam method.



The bending moment along the beam is determined. The support reactions are:

$$R_A = R_B = \frac{ql}{2}$$

At an arbitrary section located at coordinate x between B and A, the bending moment is:

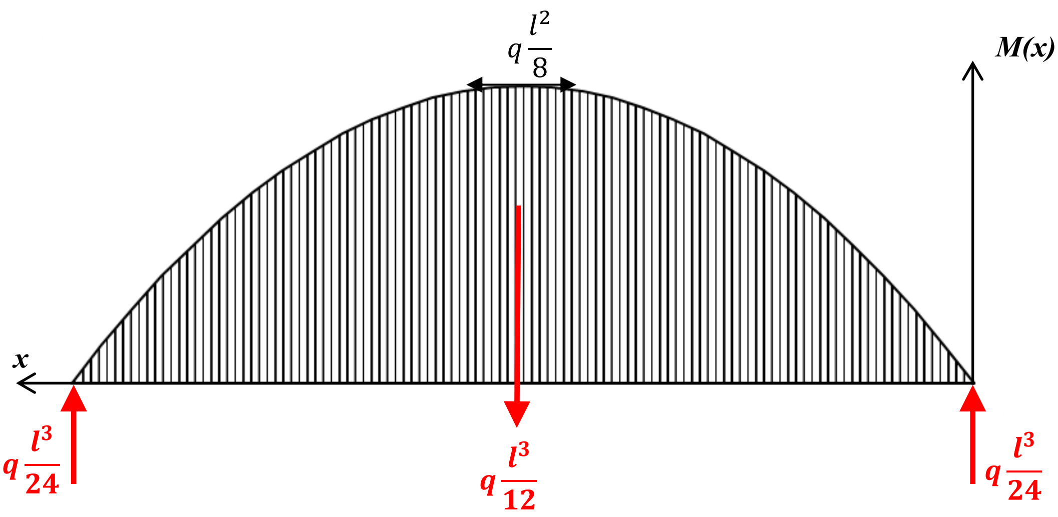
$$M(x) = R_B x - q\frac{x^2}{2} = \frac{ql}{2}x - q\frac{x^2}{2}$$

Figure below shows the bending moment distribution along the beam. This same figure also represents the conjugate beam of the real uniformly loaded beam. The area of the bending moment diagram, which represents the resultant of the fictitious load acting at the midspan of the conjugate beam, is:

$$Area\ M(x) = \int_0^l \left(\frac{ql}{2}x - q\frac{x^2}{2}\right)dx = \frac{ql^3}{12}$$

The reactions of the conjugate beam are then:

$$\bar{R}_A = \bar{R}_B = \frac{ql^3}{24}$$



hence the slopes at the ends are:

$$|\theta_A| = |\theta_B| = \frac{\bar{R}_A}{EI} = \frac{ql^3}{24EI}$$

5.3.4. Three-Moment Theorem (Clapeyron's Theorem):

Consider span i in Figure (43). The rotation angle at the left side of support i is due to the bending moment M_i, the bending moment M_{i-1}, and the external load acting on span i. Hence:

$$\theta_i^g = \theta_{M_i}^g + \theta_{M_{i-1}}^g + \bar{\theta}_i^g \tag{75}$$

With:

$$\begin{cases} \theta_{M_i}^g = -\dfrac{M_i l_i}{3EI} \\ \theta_{M_{i-1}}^g = -\dfrac{M_{i-1} l_i}{6EI} \\ \bar{\theta}_i^g = -\dfrac{\bar{R}_i^g}{EI} \end{cases} \tag{76}$$

Similarly, for the adjacent span (i+1), the rotation angle on the right side of support i is due to the bending moment M_i, the bending moment M_{i+1}, and the external load acting on span (i+1). Hence:

$$\theta_i^d = \theta_{M_i}^d + \theta_{M_{i+1}}^d + \bar{\theta}_i^d \tag{75}$$

With:

$$\begin{cases} \theta_{M_i}^d = \dfrac{M_i l_{i+1}}{3EI} \\ \theta_{M_{i+1}}^d = \dfrac{M_{i+1} l_{i+1}}{6EI} \\ \bar{\theta}_i^d = \dfrac{\bar{R}_i^d}{EI} \end{cases} \tag{76}$$

To satisfy the beam continuity condition, the rotation angles on both sides of support i must be equal. Hence:

$$\theta_i^d = \theta_i^g \tag{77}$$

Then:

$$\frac{M_i l_{i+1}}{3EI} + \frac{M_{i+1} l_{i+1}}{6EI} + \frac{\bar{R}_i^d}{EI} = -\frac{M_i l_i}{3EI} - \frac{M_{i-1} l_i}{6EI} - \frac{\bar{R}_i^g}{EI} \tag{78}$$

hence, after simplification:

$$M_{i-1} l_i + 2M_i (l_i + l_{i+1}) + M_{i+1} l_{i+1} = -6\left(\bar{R}_i^g + \bar{R}_i^d\right) \tag{79}$$

which represents the three-moment theorem. This theorem applies at support i in relation to the support moments at (i−1) and (i+1).

$\bar{R}_i^g$ and $\bar{R}_i^d$ denote, respectively, the left and right reactions at support i of the conjugate beams corresponding to spans i and $i+1$.

For a continuous beam with n spans, this theorem is applied to the internal supports numbered from 1 to (n−1). This leads to a system of (n−1) equations with (n−1) unknowns. Consequently, the theorem is not applicable to end supports, except in the case of a fixed end, where numbering starts from this support as 1 instead of 0.

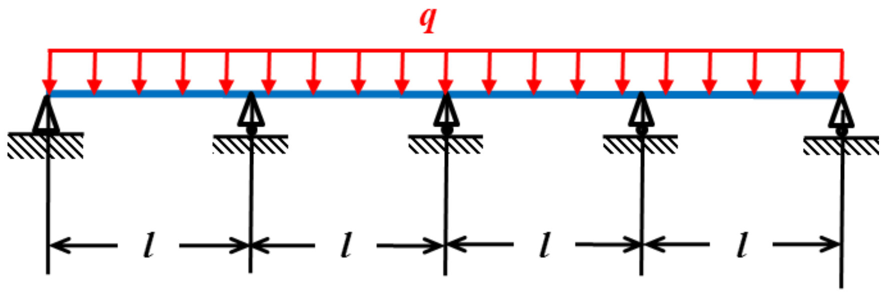
Knowing the support moments, the reactions of a continuous beam can be easily determined. They are obtained by superimposing the reactions of the corresponding isostatic spans and the reactions induced by the end moments. For example, at support i, the reaction R'_i is obtained as the sum of the reaction due to the loads acting on spans i and (i+1), and the contribution of the end moments M_{i-1}, M_i and M_{i+1}:

$$R_i = R'_i + \frac{M_{i-1} - M_i}{l_i} + \frac{M_{i+1} - M_i}{l_{i+1}} \tag{80}$$

In the case where concentrated loads are applied directly on the supports, they are transmitted directly to the corresponding supports and are added to the support reaction obtained from expression (80).

Examples:

- Consider a continuous beam consisting of four spans and five simple supports, subjected to a uniform distributed load. Plot the bending moment and shear force diagrams.



The continuous beam consists of four spans and five supports. The supports are numbered from 0 to 4, from left to right. The Three-Moment Theorem (79) is applied to the three intermediate supports (1, 2, and 3):

$$\begin{cases} n = 1 \ \rightarrow \ M_0 l_1 + 2M_1(l_1 + l_2) + M_2 l_2 = -6\left(\bar{R}_1^d + \bar{R}_1^g\right) \\ n = 2 \ \rightarrow \ M_1 l_2 + 2M_2(l_2 + l_3) + M_3 l_3 = -6\left(\bar{R}_2^d + \bar{R}_2^g\right) \\ n = 3 \ \rightarrow \ M_2 l_3 + 2M_3(l_3 + l_4) + M_4 l_4 = -6\left(\bar{R}_3^d + \bar{R}_3^g\right) \end{cases}$$

Since the spans are uniformly loaded, we obtain the same case as in previous example. Consequently, the same conjugate beams are obtained for the four spans, and therefore the support reactions of these conjugate beams are equal to:

$$\overline{R}_i^g = \overline{R}_i^d = \frac{ql^3}{24}$$

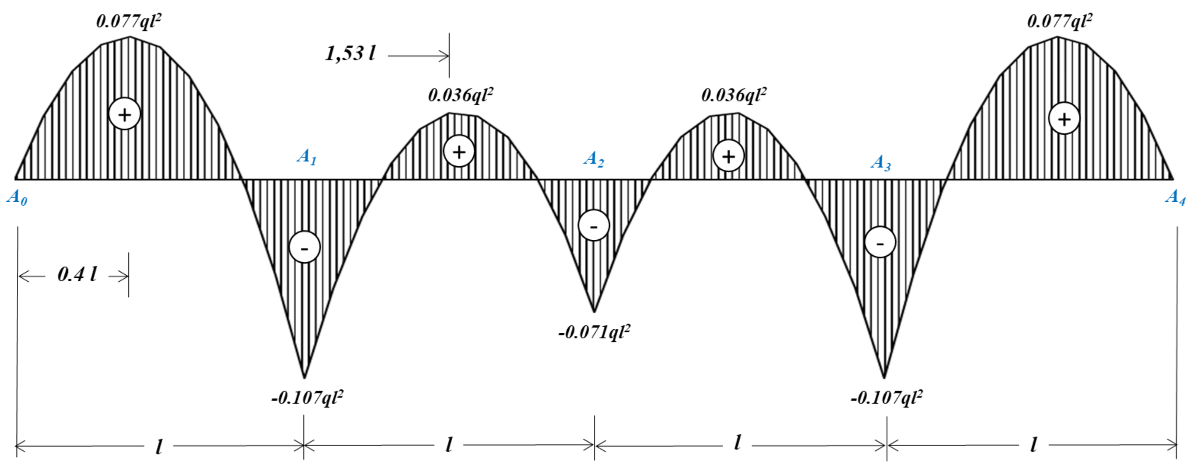
the previous system of equations takes the form:

$$\begin{cases} n = 1 \rightarrow 4M_1 l + M_2 l = -6\left(\frac{ql^3}{24} + \frac{ql^3}{24}\right) = -\frac{ql^3}{2} \\ n = 2 \rightarrow M_1 l + 4M_2 l + M_3 l = -6\left(\frac{ql^3}{24} + \frac{ql^3}{24}\right) = -\frac{ql^3}{2} \\ n = 3 \rightarrow M_2 l + 4M_3 l = -6\left(\frac{ql^3}{24} + \frac{ql^3}{24}\right) = -\frac{ql^3}{2} \end{cases}$$

This is a system of three equations with three unknowns. After solving this system, we obtain:

$$\begin{cases} M_1 = M_3 = -\frac{3}{28} ql^2 = -0.107ql^2 \\ M_2 = -\frac{1}{14} ql^2 = -0.07ql^2 \end{cases}$$

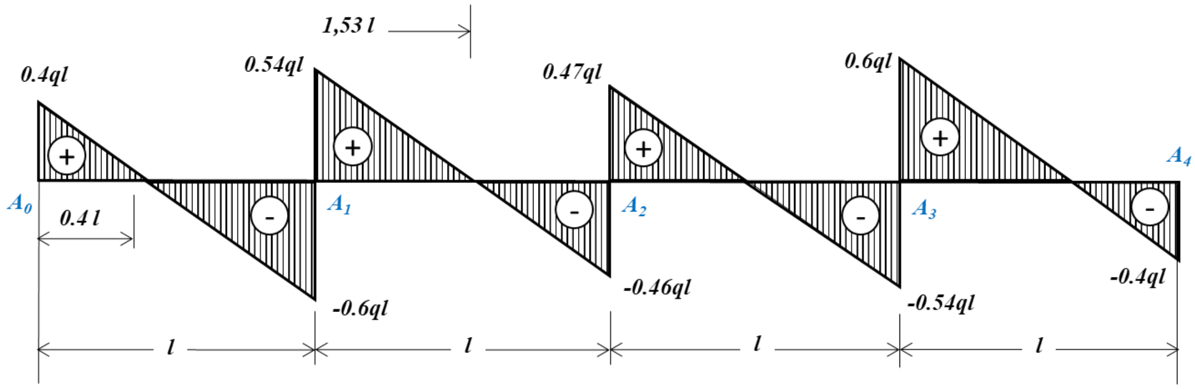
The bending moment diagram along the continuous beam is then plotted, as shown in the figure below.



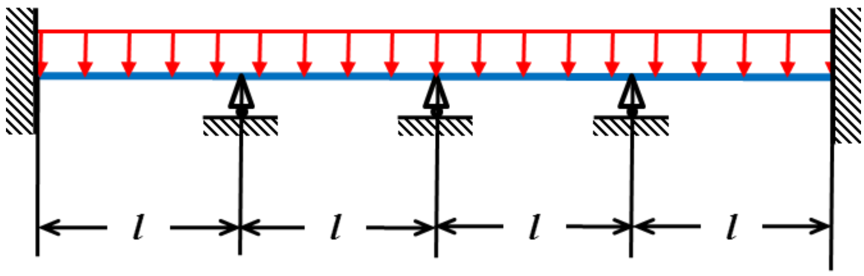
The support reactions are then calculated using expression (80):

$$\begin{cases} R_0 = R_4 = \frac{11}{28} ql = 0.4ql \\ R_1 = R_3 = \frac{16}{14} ql = 1.14ql \\ R_2 = \frac{13}{14} ql = 0.93ql \end{cases}$$

The shear force diagram along the beam is shown in the figure below.



- Consider a continuous beam consisting of four spans and five supports, with the end supports being fixed, and uniformly loaded. Plot the bending moment and shear force diagrams.



In this case, since the edge supports are fixed supports, we start numbering the supports from 1 on the left to 5 on the right. This results in five equations with five unknowns when applying the three-moment theorem.

$$\begin{cases} n = 1 \rightarrow M_0 l_1 + 2M_1(l_1 + l_2) + M_2 l_2 = -6\left(\bar{R}_1^d + \bar{R}_1^g\right) \\ n = 2 \rightarrow M_1 l_2 + 2M_2(l_2 + l_3) + M_3 l_3 = -6\left(\bar{R}_2^d + \bar{R}_2^g\right) \\ n = 3 \rightarrow M_2 l_3 + 2M_3(l_3 + l_4) + M_4 l_4 = -6\left(\bar{R}_3^d + \bar{R}_3^g\right) \\ n = 4 \rightarrow M_3 l_4 + 2M_4(l_4 + l_5) + M_5 l_5 = -6\left(\bar{R}_4^d + \bar{R}_4^g\right) \\ n = 5 \rightarrow M_4 l_5 + 2M_5(l_5 + l_6) + M_6 l_6 = -6\left(\bar{R}_5^d + \bar{R}_5^g\right) \end{cases}$$

As in the previous example, the reactions at the supports of the composite beams are equal to:

$$\overline{R}_i^g = \overline{R}_i^d = \frac{ql^3}{24}$$

Note that for the left support (support 1) $\overline{R}_1^g = 0$ and for the right support (support 5) $\overline{R}_5^d = 0$.

So the system of equations above will take the form:

$$\begin{cases} n = 1 \rightarrow 4M_1 l + M_2 l = -6\left(0 + \dfrac{ql^3}{24}\right) = -\dfrac{ql^3}{4} \\ n = 2 \rightarrow M_1 l + 4M_2 l + M_3 l = -6\left(\dfrac{ql^3}{24} + \dfrac{ql^3}{24}\right) = -\dfrac{ql^3}{2} \\ n = 3 \rightarrow M_2 l + 4M_3 l + M_4 l = -6\left(\dfrac{ql^3}{24} + \dfrac{ql^3}{24}\right) = -\dfrac{ql^3}{2} \\ n = 4 \rightarrow M_3 l + 4M_4 l + M_5 l = -6\left(\dfrac{ql^3}{24} + \dfrac{ql^3}{24}\right) = -\dfrac{ql^3}{2} \\ n = 5 \rightarrow M_4 l + 2M_5 l = -6\left(\dfrac{ql^3}{24} + 0\right) = -\dfrac{ql^3}{4} \end{cases}$$

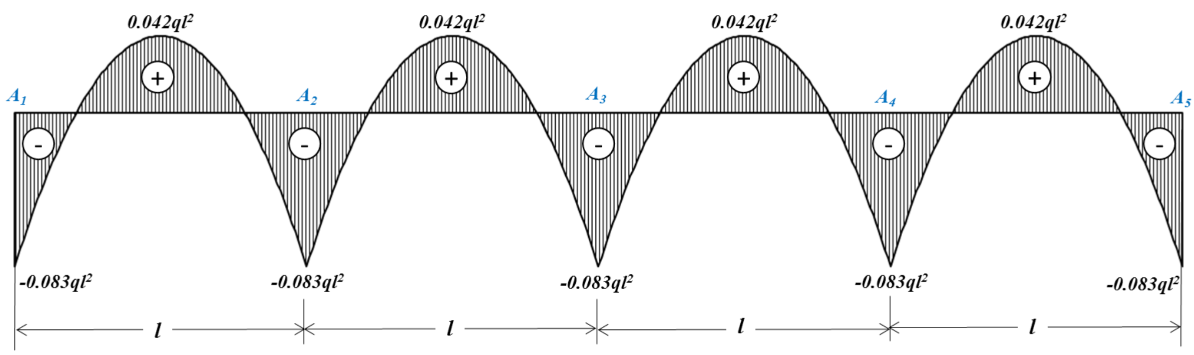
After solving this system, we find:

$$M_1 = M_2 = M_3 = M_4 = M_5 = -\frac{ql^2}{12} = -0.083ql^2$$

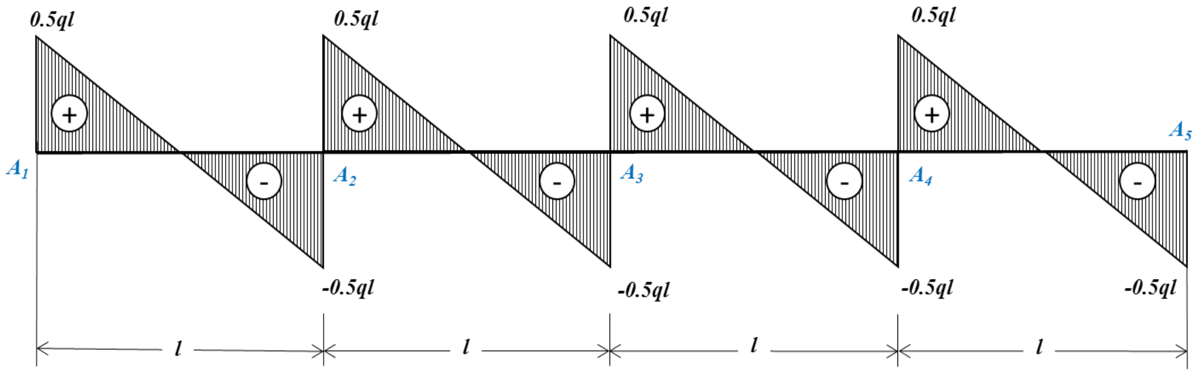
After applying equation (80), the reactions at the supports are found to be:

$$\begin{cases} R_1 = R_5 = \dfrac{1}{2}ql = 0.5ql \\ R_2 = R_3 = R_4 = ql \end{cases}$$

The bending moment diagram along the beam is shown in the following figure:



The figure below shows the shear force along the beam.



5.4. Conclusion:

The final chapter addressed the resolution of structures where static equilibrium equations alone are insufficient to determine reactions. By utilizing systematic methods such as the force superposition method and the Three-Moment Theorem (Clapeyron's Theorem), we learned to integrate material compatibility and deformation constraints into the analysis of continuous beams. Mastering these techniques allows for the design of more efficient, redundant structures common in modern mechanical and civil engineering, marking the culmination of the analytical skills developed throughout the course.

General conclusion

The comprehensive study of the "Strength of Materials 2" course marks a decisive milestone in the education of a mechanical engineer. While the first level of this discipline focused on fundamental stress analysis under simple loading, this advanced curriculum has moved into the realm of real-world complexity by integrating elastic deformations, internal energy principles, and structural indeterminacy.

1. From Static Resistance to Deformation Control

One of the primary shifts in perspective provided by this course is the transition from a purely strength-based approach (ensuring the material does not break) to a stiffness-based approach (ensuring the structure does not deform excessively). Through the study of the elastic curve (Chapter 2), it has been demonstrated that the safety and functionality of a structure depend heavily on its deflection limits. In high-precision industries, such as machine tool manufacturing or aerospace engineering, the allowable deflection is often the governing design criterion rather than the material's yield strength. Mastering the differential equations of the deflection curve allows the engineer to predict and control these critical displacements.

2. The Analytical Power of Energy Methods

The introduction of energy theorems, specifically Castigliano's Theorem and the Unit Load Method, has provided a radical alternative to traditional Newtonian mechanics. By treating a structure as a conservative system capable of storing and releasing strain energy, we gain a universal tool for solving displacements. These methods are particularly powerful because they bypass the need for solving complex systems of differential equations for every loading point. This energy-based framework serves as the theoretical bridge to modern computational methods, such as the Finite Element Method (FEM), which underpins almost all contemporary engineering simulations.

3. Realism Through Combined Loading

In industrial practice, components are rarely subjected to isolated loads. The analysis of combined loading (Chapter 4) has bridged the gap between academic theory and mechanical reality. Whether it is a transmission shaft experiencing simultaneous torsion and bending or a pillar subjected to eccentric axial compression, the ability to superimpose stress states is vital. By identifying the critical points where normal and shear stresses peak, and applying failure theories like Von Mises or Tresca, the engineer can ensure structural integrity under the most demanding multi-axial operating conditions.

4. Resolving Structural Redundancy (Hyperstaticity)

The final achievement of this course lies in the resolution of statically indeterminate (hyperstatic) systems. Most modern infrastructures, continuous bridges, complex frames, and hyperstatic trusses, are designed with redundancy to increase safety and optimize material usage. The use of compatibility equations and Clapeyron’s Three-Moment Theorem has shown that redundancy is not a calculation obstacle but a design advantage. Learning to solve these systems allows for the creation of lighter, more efficient, and more stable structures that can redistribute loads in the event of a localized failure.

5. Final Synthesis and Future Outlook

In conclusion, this course has established a rigorous methodology resting on three inseparable pillars:

Static Equilibrium: The balance of internal and external forces.

Material Constitutive Laws: The relationship between stress and strain (Hooke’s Law).

Geometric Compatibility: The continuity of deformations within the structure.

Mastering these concepts prepares the engineer to use modern Computer-Aided Engineering (CAE) tools with insight and critical judgment. Understanding the physical principles behind the software's matrix calculations is the only way to validate, critique, and optimize numerical results. This technical background constitutes the essential foundation for designing the next generation of durable, safe, and high-performance mechanical systems.

References

[1]. Richard G. Budynas et J. Keith Nisbett ; Shigley's mechanical engineering design, 9ème édition, McGraw-Hill 2011.

[2]. Giet A. et Geminard L. ; Résistance des matériaux, 5ème édition, Tome 2, Dunod, 1968.

[3]. Victor Dias da Silva ; Mechanics and strength of materials, Springer 2006.

[4]. Hearn E. J. ; Mechanics of materials 1, 3ème édition, Butterworth Heinemann 1997.

[5]. Serge Laroze et René Boudet ; Résistance des matériaux appliquée aux organes de machines, B601, Techniques de l'ingénieur

[6]. Saika mim-Das Madan Mohan ; Basic Engineering Mechanics and Strength of Materials

[7]. Jean Goulet, Jean-Pierre Boutin, Frédéric Lerouge ; Aide-mémoire, résistance des matériaux, Dunod 2014.

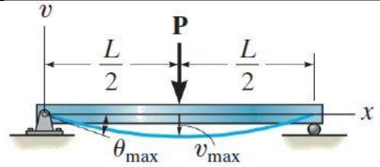
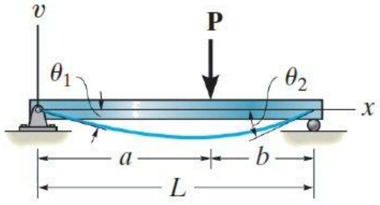
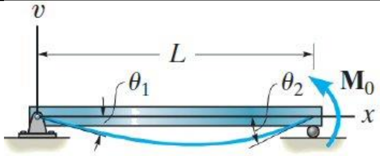
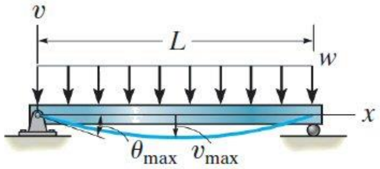
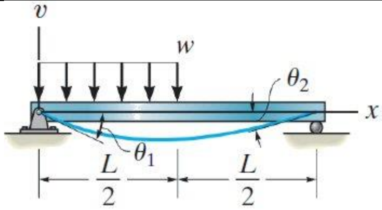
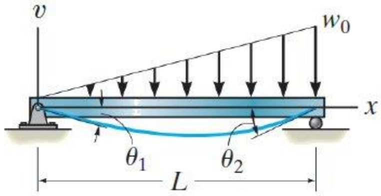
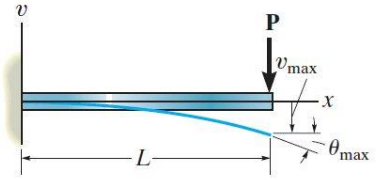
[8]. Timoshenko S. ; Résistance des matériaux, tome 1 « Théories élémentaires et problèmes », Dunod 1968.

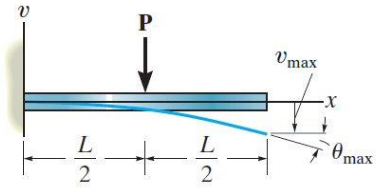
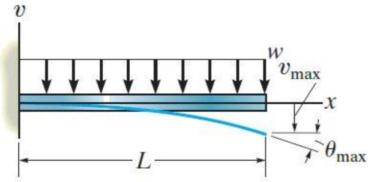
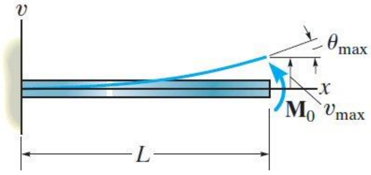
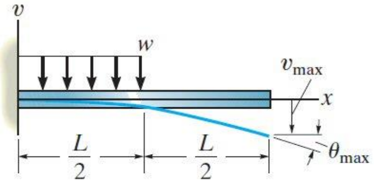
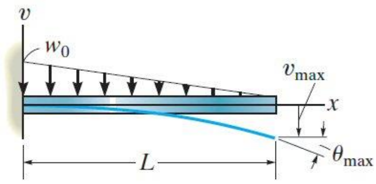
[9]. Charles Massonet et Serge Cescotto ; Mécanique des matériaux, De Boeck Supérieur 1994.

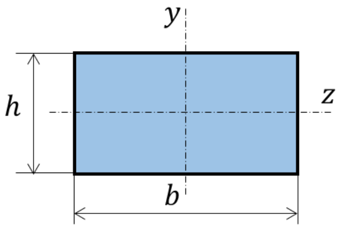
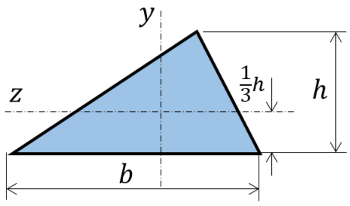
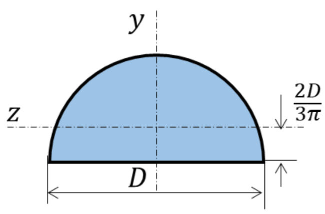
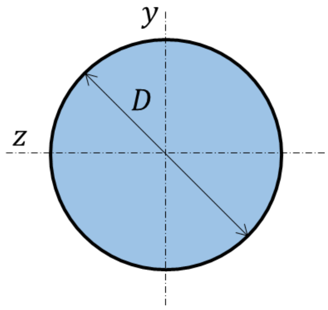
[10]. Den Hargot J. P. ; Advanced strength of materials, Dover publications 1987.

[11]. Barry Dupen ; Applied strength of materials for engineering technology, Mechanical Engineering Technology, Indiana University – Purdue University Fort Wayne 2012.

Deflections and Slopes of Simply Supported Beams

Beam	Slope	Deflection	Elastic curve
	$\theta_{max} = \dfrac{-PL^2}{6EI}$	$v_{max} = \dfrac{-PL^3}{48EI}$	$0 \le x \le L/2$ $v = \dfrac{-Px}{48EI}(3L^2 - 4x^2)$
	$\theta_1 = \dfrac{-Pab(L+b)}{6EIL}$ $\theta_2 = \dfrac{Pab(L+a)}{6EIL}$	$v\|_{x=a} = \dfrac{-Pab}{6EIL}(L^2 - b^2 - a^2)$	$0 \le x \le a$ $v = \dfrac{-Pbx}{6EIL}(L^2 - b^2 - x^2)$
	$\theta_1 = \dfrac{-M_0 L}{6EIL}$ $\theta_2 = \dfrac{M_0 L}{3EIL}$	$v_{max} = \dfrac{-M_0 L^2}{15.56EI}$ at: $x = 0.5774L$	$v = \dfrac{-M_0 x}{6EIL}(L^2 - x^2)$
	$\theta_{max} = \dfrac{-wL^3}{24EI}$	$v_{max} = \dfrac{-5wL^4}{348EI}$	$v = \dfrac{-wx}{48EI}(x^3 - 2Lx^2 - L^3)$
	$\theta_1 = \dfrac{-3wL^3}{128EI}$ $\theta_2 = \dfrac{7wL^3}{348EI}$	$v\|_{x=L/2} = \dfrac{-5wL^4}{768EI}$ $v_{max} = -0.006563\dfrac{wL^4}{EI}$ at: $x = 0.4598L$	$0 \le x \le L/2$ $v = \dfrac{-wx}{384EI}(16x^3 - 24Lx^2 + 9L^3)$ $L/2 \le x \le L$ $v = \dfrac{-wx}{384EI}(8x^3 - 24Lx^2 + 17L^2x - L^3)$
	$\theta_1 = \dfrac{-7w_0 L^3}{360EI}$ $\theta_2 = \dfrac{w_0 L^3}{45EI}$	$v_{max} = -0.00652\dfrac{w_0 L^4}{EI}$ at: $x = 0.5193L$	$v = \dfrac{-w_0 x}{384EIL}(3x^4 - 10L^2x^2 + 7L^4)$
	$\theta_{max} = \dfrac{-PL^2}{2EI}$	$v_{max} = \dfrac{-PL^3}{3EI}$	$v = \dfrac{-Px^2}{6EI}(3L - x)$

	$\theta_{max} = \frac{-PL^2}{8EI}$	$v_{max} = \frac{-5PL^3}{48EI}$	$0 \le x \le L/2$ $v = \frac{-Px^2}{12EI}(3L - 2x)$ $L/2 \le x \le 0$ $v = \frac{-PL^2}{48EI}(6x - L)$
	$\theta_{max} = \frac{-wL^3}{6EI}$	$v_{max} = \frac{-wL^4}{8EI}$	$v = \frac{-wx^2}{24EI}(x^2 - 4Lx + 6L^2)$
	$\theta_{max} = \frac{M_0 L}{EI}$	$v_{max} = \frac{M_0 L^2}{2EI}$	$v = \frac{M_0 x^2}{2EI}$
	$\theta_{max} = \frac{-wL^3}{48EI}$	$v_{max} = \frac{-7wL^4}{384EI}$	$0 \le x \le L/2$ $v = \frac{-wx^2}{24EI}(x^2 - 2Lx + 1.5L^2)$ $L/2 \le x \le L$ $v = \frac{-wL^3}{384EI}(8x - L)$
	$\theta_{max} = \frac{-w_0 L^3}{24EI}$	$v_{max} = \frac{-w_0 L^4}{30EI}$	$v = \frac{-w_0 x^2}{24EIL}(10L^3 - x^3 - 10L^2 x + 5Lx^2)$

Quadratic moments of some simple shapes with respect to axes passing through their centers of gravity		
Rectangle		$I_z = \dfrac{bh^3}{12}$ $I_y = \dfrac{hb^3}{12}$
Triangle		$I_z = \dfrac{bh^3}{36}$
Semicircle		$I_z = \dfrac{\pi D^4}{128}$ $I_y = \dfrac{\pi D^4}{128}$
Circle		$I_z = \dfrac{\pi D^4}{64}$ $I_y = \dfrac{\pi D^4}{64}$