الجمهورية الجزائرية الديموقراطية الشعبية
RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE
وزارة التعليم العالي والبحث العلمي
MINISTÉRE DE L'ENSEIGEMENT SUPÉRIEUR ET DE LA RECHERCHE SCIENTIFIQUE
جامعة عمار ثليجي بالأغواط
UNIVERSITÉ AMAR TELIDJI LAGHOUAT
كلية العلوم
FACULTÉ DES SCIENCES
قسم الرياضيات
DÉPARTEMENT DE MATHÉMATIQUES

**MÉMOIRE DE MASTER**

**Domaine :** Mathématiques et Informatique

**Filière : Mathématique**

**Option : analyse fonctionnelle et applications**

**PAR :**

**SERRAYE Soumia**

# Thème

---

# LEBESGUE AND SOBOLEV SPACES WITH VARIABLES EXPONENTS

---

Devant le jury composé de :

| | | | |
|---|---|---|---|
| Mr. YAGOUB Ameur | M.C.A | Université de Laghouat | Président |
| Mr. YAZID Fares | M.C.A | Université de Laghouat | Examinateur |
| Mr. OUCHENANE Djamel | Prof. | Université de Laghouat | Encadreur |

Année universitaire : 2023-2024

# Dedication

I dedicate this modest work to my family which helped me in my study path, supported me and trusted in me. To my mother and my father, to my sisters Safa and Naziha, to my brothers Mohammed, Abd Elazize and Abd Alkader.

# Acknowledgments

First, All praise is for Allah. May Allah's peace and blessings be on Muhammad, on his family, and on his companions.

I would like to express my gratitude to my mother and my father for their materialism and morale helps and my sister Safa for her helps. I am most thankful to the supervisor Mr. Djamel Ouchenane for his guidance and his helps and to the jury members, Mr. Yagoub Ameur to accept being the president of the jury and to Mr. Fares Yazid to accept being the examiner of this thesis. Also i want to thank all my teachers for all what i learned of them.

Finaly, i express my acknowledgment to all those who have indirectly guided and helped me in preparation of this memory.

# Contents

**Contents** **iii**

**Abbreviations and Notations** **iv**

**Introduction** **1**

**1 Variable exponent Lebesgue spaces** **3**

1.1 The Lebesgue space Φ-function . . . . . 3

1.2 Basic properties . . . . . 10

1.3 Embeddings . . . . . 36

**2 Introduction to Sobolev spaces with variables exponents** **42**

2.1 Basic properties . . . . . 42

2.2 Poincaré inequalities . . . . . 47

**Conclusion** **68**

**Bibliography** **69**

# Abbreviations and Notations

| | |
|---|---|
| $\mathcal{P}(A,\mu), \mathcal{P}(\Omega)$ | Set of variable exponents. |
| $\mathcal{P}^{log}$ | Variable exponents p such that $\frac{1}{p}$ is log-Hölder continuous. |
| $p'$ | Dual exponent. |
| $p^+, p^-$ | Essential supremum and infimum of p. |
| $p_A^+, p_A^-$ | Essential supremum and infimum of p in A. |
| $\Phi(A,\mu), \Phi(\Omega)$ | Generalized Φ-functions. |
| $\varphi$ | Φ-function |
| $\varphi^*$ | conjugate Φ-function of $\varphi$. |
| $\varphi^{-1}$ | Left-continuous inverse. |
| $\widetilde{\varphi}_{p(.)}, \bar{\varphi}_{p(.)}$ | Generalized Φ-function defining $L^{p(.)}$. |
| $\varrho$ | Semimodular, modular. |
| $\varrho^*$ | Conjugate semimodular of $\varrho$. |
| $Q(x,r)$ | A cube with side length r and center at $x \in \mathbb{R}^n$. |
| $T_k$ | Averaging operator over dyadic cubes. |
| $T_Q$ | Averaging operator. |
| $W_0^{k,p(.)}$ | Sobolev space with zero boundary values. |
| $X_\varrho$ | Semimodular space or modular space. |
| sgn | The sign of the argument. |
| $N(A,\mu), N(\Omega)$ | Set of N-functions. |
| $M_Q f$ | Averge of $\|f\|$ over Q. |
| $E^{p(.)}$ | Set of finite elements of $L^{p(.)}$. |
| $L_{OC}^{\varphi}$ | Musielak-Orlicz class. |
| $L_{loc}^1$ | The space of all locally integrable functions. |
| $L_0^{p(.)}$ | The subspace of $L^{p(.)}$ consisting of function with vanishing integral. |

| | |
|---|---|
| $H_0^{k,p(.)}$ | Closure of $C_0^\infty$-functions in $W^{k,p(.)}$. |
| $c_{log}(p)$ | Log-Hölder constant of $\frac{1}{p}$. |
| $\chi_E$ | Characteristic function of the set E. |
| $L^0,\ \mathcal{M}$ | The set of all measurable functions. |
| $\nearrow$ | Limit of an increasing sequence. |
| $C_0^\infty$ | Smooth function with compact support. |
| $C_{0,0}^\infty$ | The space of compactly supported smooth functions with vanishing integral. |
| $\langle f \rangle_\Omega,\ \ _\Omega f(y)\,dy$ | The average integral over the set $\Omega$ . |
| $d(u,v)$ | The distance between u and v. |
| $diam(\Omega)$ | The diameter of $\Omega$. |

# Introduction

In this memory we talk about Lebesgue and Sobolev spaces with variable exponent, it is like the classical $L^p$, $W^{k,p}$ but p here is a function not a constant, this subject "variable exponent function spaces" has seen a big growth in recent years. For example we have 288 articles in mathematical for "variable exponent" since before 2000 then 2010. The basic properties is existent in the article [17] bu O.käváčik and Rákosnik from 1991. This article is the standard reference after X -L Fan and D. Zho [9] ten years later by the same properties and different methods. The understanding if the basics of the field has been in 2010, reached a certain stability and maturity.

Historically, the first presence of the field of variable exponent Lebesgue spaces was in a 1931 article by W.Orlicz, here the following question is asked : let $(p_i)$ $(p_i \geq 1)$ and $(x_i)$ set sequences of real numbers and $\sum_i x_i^{p_i}$ converge. what are the necessary and sufficient conditions on $(y_i)$ for $\sum_i x_i y_i$ to converge? the answer was $\sum(\lambda y_i)^{p_i'}$ should converge for $\lambda > 0$ and $p_i' = \frac{p_i}{p_i - 1}$ this is essentially Hölder's inequality in the space $l^{p(.)}$ Orlicz also proves the Hölder inequality in the variable exponent function space $L^{p(.)}$ after that he concentrated on the theory of the function space "now it bear his name" and leaved the study of variable exponent spaces. In the theory of Orlicz spaces, one defines the space $L^{\varrho}$ : $\{u : \Omega \to \mathbb{R}$ measurable functions $, \varrho(\lambda u) = \int_\Omega \varphi(\lambda |u(x)|)\, dx < \infty\}$. Abstracting this is a more general class of so-called modular function spaces, the variable exponent Lebesgue spaces is an example of the more general spaces as Nakano said. the modular spaces were investigate by several people, groups at Sapporo(Japan), voronzh(USSR), and Leiden(NetherLands), Polish mathematician for instance H.Hudzik, A,kaminska, and J.Musielak. Variable exponent Lebesgue space have been independently developed by Russian researchers. The research started by this question in a paper by I.Tsenov from 1961: the minimization of $\int_a^b |u(x) - v(x)|^{p(x)}\, dx$, u fixed function and v varies over a finite dimensional subspace of $L^{p(.)}([a, b])$, this question solved by I.Sharapudinov he also defined the Luxemburg norm for the space $L^p$ and proved that this space

is reflexive if $1 < p^- \leq p^+ < \infty$, in the mid-80's Zhikov started new line of research, that was to become profoundly to the study of variable exponent spaces, considering variational integrals with non-standard growth conditions. Another early PDE paper by O.kaváčik and the paper by O.kováčik and J.Rákodvik in the 90's [17], this paper contained many of the basic properties of $L^p$ and $W^{k,p}$ in $\mathbb{R}^n$.

There is a connection between variable exponent space and variational integrals with non-standard growth and coercivity conditions, this non-standard variational problems are related to modeling of so-called electrorheological fluids. Moreover, development in physics and engineering have made the study of fluid mechanical properties of these fluids an important issue, latter on, other applications have emerged in thermorheological fluids and image processing. Finally, this memory is contained an Introduction and two chapter, the first chapter deals with variable exponent Lebesgue spaces, and the second one deals with Introduction to sobolev spaces.

# Chapter 1

# Variable exponent Lebesgue spaces

In this chapter, we present Lebesgue spaces with variable exponents, it is a semimodular space consist of all K-valued $\mu$-measurable functions $f$ on A with $\lim_{\lambda\to 0} \int_A |\lambda f(x)|^{p(x)}\,dx = 0$ or $\int_A |\lambda f(x)|^{p(x)}\,dx < +\infty$ with $\lambda > 0$, we begin by define the appropriate $\Phi$-function and study its properties and the second section of this chapter is the Basic properties, the latter is an Embedding between spaces with different exponents.

## 1.1 The Lebesgue space $\Phi$-function

**Definition 1.1.1** *A convex, left-continuous function $\varphi : [0,\infty) \to [0,\infty]$ with $\varphi(0) = 0$, $\lim_{t\to 0^+} \varphi(t) = 0$, and $\lim_{t\to\infty} \varphi(t) = \infty$ is called a $\Phi$-function. It is called positive if $\varphi(t) > 0$ for all $t > 0$.*

**Definition 1.1.2** *Let $(A,\Sigma,\mu)$ be a $\sigma$-finite, complete measure space. A real function $\varphi : A \times [0,\infty) \to [0,\infty]$ is said to be a generalized $\Phi$-function on $(A,\Sigma,\mu)$ if :*
*(a) $\varphi(y,.)$ is a $\Phi$-function for every $y \in A$.*
*(b) $y \mapsto \varphi(y,t)$ is measurable for every $t \geq 0$.*
*If $\varphi$ is a generalized $\Phi$-function on $(A,\Sigma,\mu)$, we write $\varphi \in \Phi(A,\mu)$. If $\Omega$ is an open subset of $\mathbb{R}^n$ and $\mu$ is the n-dimensional Lebesgue measure we abbreviate this as $\varphi \in \Phi(\Omega)$ or say that $\varphi$ is a generalized $\Phi$-function on $\Omega$.*

**Definition 1.1.3** *A $\Phi$-function $\varphi$ is said to be an N-function if it is continuous and positive and satisfies $\lim_{t\to 0} \frac{\varphi(t)}{t} = 0$ and $\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty$.*
*A function $\varphi \in \Phi(A,\mu)$ is said to be a generalized N-function if $\varphi(y,.)$ is for every $y \in \Omega$ an*

*N-function.*

*If $\varphi$ is a generalized N-function on $(A,\mu)$, we write $\varphi \in N(A,\mu)$ for short. If $\Omega$ is an open subset of $\mathbb{R}^n$ and $\mu$ is the n-dimensional Lebesgue measure we abbreviate $\varphi \in N(\Omega)$.*

**Definition 1.1.4** *A function $\varphi \in N(A,\mu)$ is called uniformly convex if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$|u-v| \leq \varepsilon \max\{u,v\} \quad or \quad \varphi\left(y, \frac{u+v}{2}\right) \leq (1-\delta)\frac{\varphi(y,u)-\varphi(y,v)}{2}$$

*for all $u, v \geq 0$ and every $y \in A$.*

**Remark 1.1.1** *If $\varphi(x,t) = t^q$ with $q \in (1,\infty)$, then $\varphi$ is uniformly convex. To prove this, we have to show that for $u, v \geq 0$ the estimate $|u-v| > \varepsilon \max\{u,v\}$ implies $\left(\frac{u+v}{2}\right)^q \leq (1-\delta(\varepsilon))\frac{1}{2}(u^q+v^q)$ with $\delta(\varepsilon) > 0$ for every $\varepsilon > 0$. Without loss of generality we can assume $\varepsilon \in \left(0, \frac{1}{2}\right)$. By homogeneity it suffices to consider the case $v = 1$ and $0 \leq u \leq 1$, So we have to show that $u \in [0, 1-\varepsilon)$ implies $\left(\frac{1+u}{2}\right)^q \leq (1-\delta(\varepsilon))\frac{1}{2}(1+u^q)$. Define $f(\tau) := 2^{1-q}\frac{(1+u)^q}{(1+u^q)}$. Then $f$ is continuous on $[0,1]$ and has its maximum at 1. This proves as desired $f(u) \leq \delta(\varepsilon)$ for all $u \in [0, 1-\varepsilon)$.*
*It follows by division with $q$ that $\varphi(x,t) = \frac{1}{q}t^q$ with $1 < q < \infty$ is also uniformly convex.*

**Lemma 1.1.1** *Let $\varphi \in N(A,\mu)$ be uniformly convex. Then for every $\varepsilon_2 > 0$ there exists $\delta_2 > 0$ such that*

$$|a-b| \leq \varepsilon_2 \max\{|a|,|b|\} \;\; or \;\; \varphi\left(y, \left|\frac{a+b}{2}\right|\right) \leq (1-\delta_2)\,\frac{\varphi(y,|a|)+\varphi(y,|b|)}{2}.$$

*for all $a, b \in \mathbb{K}$ and every $y \in A$.*

**Remark 1.1.2** *If $u, v \in \mathbb{K}$ satisfies $|a-b| \leq \varepsilon_2 \max\{|a|,|b|\}$ with $\varepsilon_2 \in (0,1)$, then $\frac{|a-b|}{2} \leq \varepsilon_2 \frac{|a|+|b|}{2}$ and by the convexity of $\varphi$ follows*

$$\varphi\left(y, \frac{|a-b|}{2}\right) \leq \varepsilon_2 \frac{\varphi(y,|a|)+\varphi(y,|b|)}{2} \tag{1.1}$$

*Therefore, we can replace the first alternative in Lemma 1.1.1 by the weaker version (1.1)*

**Definition 1.1.5** *Let $\varphi \in \Phi(A,\mu)$. Then for any $y \in A$ we denote by $\varphi^*(y,.)$ the conjugate function of $\varphi(y,.)$ which is defined by*

$$\varphi^*(y,u) = \sup_{t \geq 0}(tu - \varphi(y,t))$$

*for all $u \geq 0$ and $y \in \Omega$.*

*This definition applies in particular in the case when $\varphi$ is a (non-generalized) $\Phi$-function, in which case*

$$\varphi^*(u) = \sup_{t \geq 0}(tu - \varphi(t))$$

*concurs with the Legendre transformation of $\varphi$. By definition of $\varphi^*$,*

$$tu \leq \varphi(t) + \varphi^*(u) \tag{1.2}$$

*for every $t, u \geq 0$, this inequality is called Young's inequality. If $\varphi$ is a $\Phi$-function and $\varrho(t) := \varphi(|t|)$ is its even extension to $\mathbb{R}$, then $\varrho^*(t) = \varphi^*(|t|)$ for all $t \in \mathbb{R}$.*

**Theorem 1.1.1** *If $\varphi \in N(A, \mu)$, then $\varphi^* \in N(A, \mu)$ and $(\varphi^*)' = (\varphi')^{-1}$. In particular,*

$$\varphi^*(y, t) = \int_0^t (\varphi')^{-1}(y, \tau)\, d\tau$$

*for all $y \in A$ and $t \geq 0$.*

**Proof.** *It suffices to prove the claim point-wise, and thus we may assume without loss of generality that $\varphi(y, t)$ is independent of $y$, i.e. an N-function.*

*It is easy to see that $\varphi'$ is non-decreasing, right-continuous and satisfies $(\varphi')^{-1}(0) = 0$, $(\varphi')^{-1}(t) > 0$ for $t > 0$, and $\lim_{t\to\infty}(\varphi')^{-1}(t) = \infty$. Thus,*

$$\psi(t) := \int_0^t (\varphi')^{-1}(\tau)\, d\tau$$

*for $t \geq 0$ defines an N-function, In particular, $\varphi$ and $\psi$ are finite.*

*Note that $\sigma < \varphi'(\tau)$ is equivalent to $(\varphi')^{-1}(\sigma) < \tau$. Hence, the sets*

$$\{(\tau, \sigma) \in [0, \infty) \times [0, \infty) \ : \ \sigma < \varphi'(\tau)\}$$

$$\left\{(\tau, \sigma) \in [0, \infty) \times [0, \infty) \ : \ (\varphi')^{-1}(\sigma) \geq \tau\right\}$$

*are complementary with respect to $[0, \infty) \times [0, \infty)$. Therefore, we can estimate with the help of the theorem of Fubini*

$$\begin{aligned}
0 \leq tu &= \int_0^t \int_0^u d\sigma\, d\tau \\
&= \int_{\{0\leq\tau\leq t,\ \sigma\leq u\ :\ 0\leq\sigma<\varphi'(\tau)\}} d\sigma\, d\tau + \int_{\{0\leq\tau\leq t,\ 0\leq\sigma\leq u\ :\ (\varphi')^{-1}(\sigma)\geq\tau\}} d\sigma\, d\tau \\
&= \int_0^t \int_0^{\min\{u,\varphi'(\tau)\}} d\sigma\, d\tau + \int_0^u \int_0^{\min\{t,(\varphi')^{-1}(\sigma)\}} d\tau\, \sigma \\
&\leq \int_0^t \varphi'(\tau)\, d\tau + \int_0^u (\varphi')^{-1}(\sigma)\, d\sigma \\
&= \varphi(t) + \psi(u).
\end{aligned}$$

*If $u = \varphi'(t)$ or $t = (\varphi')^{-1}(u)$. Then $\min\{u, \varphi'(\tau)\} = \varphi'(\tau)$ and $\min\{t, (\varphi')^{-1}(\sigma)\} = (\varphi')^{-1}(\sigma)$ in the integrals of the third line. So in this case we have equality in the penultimate step. Since $\varphi^*(u) = \sup_t(ut - \varphi(t))$ it follows that $\varphi^* = \psi$.*

**Definition 1.1.6** *Let $(A, \Sigma, \mu)$ be a $\sigma$-finite, complete measure space. We define $\mathcal{P}(A, \mu)$ to be the set of all $\mu$-measurable functions $p : A \to [1, \infty]$. Functions $p \in \mathcal{P}(A, \mu)$ are called variable exponents on $A$. We define $p^- := p_A^- := ess\inf_{y\in A} p(y)$ and $p^+ := p_A^+ := ess\sup_{y\in A} p(y)$. if $p^+ < \infty$, then we call $p$ a bounded variable exponent.*
*If $p \in \mathcal{P}(A, \mu)$, then we define $p' \in \mathcal{P}(A, \mu)$ by $\frac{1}{p(y)} + \frac{1}{p'(y)} = 1$, where $\frac{1}{\infty} := 0$. The function $p'$ is called the dual variable exponent of $p$.*
*In the special case that $\mu$ is the $n$-dimensional Lebesgue measure and $\Omega$ is an open subset of $\mathbb{R}^n$, we abbreviate $\mathcal{P}(\Omega) := \mathcal{P}(\Omega, \mu)$.*

**Definition 1.1.7** *For $t \geq 0$ and $1 \leq p < \infty$ we define*

$$\widetilde{\varphi}_p(t) := \frac{1}{p}t^p,$$
$$\bar{\varphi}_p(t) := t^p.$$

*Moreover we set*

$$\bar{\varphi}_\infty(t) := \widetilde{\varphi}_\infty(t) := \infty\chi_{(1,\infty)}(t) = \begin{cases} 0 & \text{if } t \in [0, 1], \\ \infty & \text{if } t \in (1, \infty). \end{cases}$$

*For variable exponent $p \in \mathcal{P}(A, \mu)$ we define for $y \in A$ and $t \geq 0$*

$$\widetilde{\varphi}_{p(.)}(y, t) := \widetilde{\varphi}_{p(y)}(t) \qquad \text{and} \qquad \bar{\varphi}_{p(.)}(y, t) := \bar{\varphi}_{p(y)}(t).$$

*It is easy to see that both $\widetilde{\varphi}_q$ and $\bar{\varphi}_q$ are $\Phi$-function if $q \in [1, \infty]$. So $\widetilde{\varphi}_{p(.)}$ and $\bar{\varphi}_{p(.)}$ are generalized $\Phi$-functions if $p \in \mathcal{P}(A, \mu)$. Even more, if $q \in (1, \infty)$ and $p \in \mathcal{P}(A, \mu)$ with $1 < p^- \leq p^+ < \infty$. then $\widetilde{\varphi}_q$ and $\bar{\varphi}_q$ are N-functions and $\widetilde{\varphi}_{p(.)}$ and $\bar{\varphi}_{p(.)}$ are generalized N-functions. If $q \in [1, \infty)$, then $\widetilde{\varphi}_q$ and $\bar{\varphi}_q$ are continuous and positive. The function $\widetilde{\varphi}_\infty = \bar{\varphi}_\infty$ is only left-continuous and it is not positive.*
*Both $\widetilde{\varphi}_p$ and $\bar{\varphi}_p$ their advantages. The advantage of $\bar{\varphi}_p$ is that the corresponding Musielak-Orlicz space $L^{\bar{\varphi}_p}$ agrees for constant $p \in [1, \infty]$ exactly with the classical $L^p$ spaces, see [7, Example 2.1.8]. In particular, for $f \in L^p(\Omega)$ we have $\|f\|_p = \|f\|_{\bar{\varphi}_p}$. Additionally, the generalized $\Phi$-function $\bar{\varphi}_{p(.)}$ has been used in the vast majority of papers on variable exponent function spaces. The advantages of $\widetilde{\varphi}_p$ are its nice properties regarding conjugation, continuity, and convexity*

*with respect to the exponent p. First, for all $t \geq 0$ the mapping $p \mapsto \widetilde{\varphi}_p(t)$ is continuous with respect to $p \in [1, \infty]$. In particular,*

$$\widetilde{\varphi}_\infty(t) = \lim_{p\to\infty} \widetilde{\varphi}_p(t)$$

*for all $t \geq 0$. This suggests that the expression $\frac{1}{p}t^p$ has for $p = \infty$ a natural interpretation, namely $\widetilde{\varphi}_\infty(t) = \infty.\chi_{(1,\infty)}(t)$. Therefore, we sometimes will just write $\widetilde{\varphi}_p(t) = \frac{1}{p}t^p$ including the case $p = \infty$.*

*Second, $\widetilde{\varphi}_p$ acts nicely with respect to conjugation. For future reference we also need the corresponding result for $(\bar{\varphi}_{p(.)})^*$.*

**Lemma 1.1.2** *If $1 \leq q \leq \infty$, then $(\widetilde{\varphi}_q)^* = \widetilde{\varphi}_{q'}$ and*

$$(\bar{\varphi}_q)^*(t) \leq \bar{\varphi}_{q'}(t) \leq (\bar{\varphi}_q)^*(2t)$$

*for all $t \geq 0$.*

**Proof.** *1)- if $1 \leq q \leq \infty$, then $(\widetilde{\varphi}_q)^* = \widetilde{\varphi}_{q'}$.*

*If $q \in (1, \infty)$ : $(\varphi_q)^*(u) = (\frac{1}{q}t^q)^*(u) = \sup(tu - \frac{1}{q}t^q)$, $(tu - \frac{1}{q}t^q)' = u = \frac{q}{q}t^{q-1} = u - t^{q-1}$.*

*$u - t^{q-1} = 0 \Leftrightarrow t^{q-1} = u \Leftrightarrow t = u^{\frac{1}{q-1}}, u - t^{q-1} > 0 \Leftrightarrow u > t^{q-1} \Leftrightarrow t < u^{\frac{1}{q-1}}, u - t^{q-1} < 0 \Leftrightarrow u < t^{q-1} \Leftrightarrow t > u^{\frac{1}{q-1}}$.*

*So: if $t \in (0, u^{\frac{1}{q-1}})$ then $u - t^{q-1} > 0$ implies $tu - \frac{1}{q}t^q$ is increasing.*

*If $t \in (u^{\frac{1}{q-1}}, \infty)$ then $u - t^{q-1} < 0$ implies $tu - \frac{1}{q}t^q$ is decreasing then because of the derivative of $tu - \frac{1}{q}t^q$ null and change it sing (positive to negative) so $\sup_{t\geq 0}(tu - \frac{1}{q}t^q) = u^{\frac{q}{q-1}} - \frac{1}{q}u^{\frac{q}{q-1}} = (1 - \frac{1}{q})u^{\frac{q}{q-1}} = \frac{1}{q'}u^{q'}$.*

$$(\widetilde{\varphi}_q)^* = \widetilde{\varphi}_{q'}, \quad q \in (1, \infty).$$

*If $q = 1$ then $(\widetilde{\varphi}_1)^*(u) = \sup_{t\geq 0}(tu - \widetilde{\varphi}_1(t)) = \sup_{t\geq 0}(tu - t) = \sup_{t\geq 0}(t(u - 1))$.*

*If $0 < u \leq 1$ then $\sup_{t\geq 0}(t(u - 1)) = 0$, if $1 < u < \infty$ then $\sup_{t\geq 0}(t(u - 1)) = \infty$. So :*

$$(\widetilde{\varphi}_1)^*(u) = \infty.\chi_{(1,\infty)}(u) = \begin{cases} 0 & \text{if } u \in [0, 1] \\ \infty & \text{if } u \in (1, \infty) \end{cases} = \widetilde{\varphi}_\infty(u).$$

*If $q = \infty$, we've got $(\widetilde{\varphi}_1)^{**} = (\widetilde{\varphi}_1)$, and $(\widetilde{\varphi}_1)^*(u) = \widetilde{\varphi}_\infty(u)$ (case precedent ).*

*$(\widetilde{\varphi}_1)^{**} = (\widetilde{\varphi}_\infty)^* \Rightarrow \widetilde{\varphi}_1 = (\widetilde{\varphi}_\infty)^*$. Finally*

$$(\widetilde{\varphi}_q)^* = \widetilde{\varphi}_{q'}, \quad \text{for all } t \in [1, \infty].$$

*2)-* $(\bar{\varphi}_q)^*(t) \leq \bar{\varphi}_{q'}(t) \leq (\bar{\varphi}_q)^*(2t), \quad \forall t \geq 0, q \in (1,\infty)$.

$\frac{1}{p'} + \frac{1}{p} = 1 \Rightarrow \frac{1}{p'} = 1 - \frac{1}{p} \Rightarrow \frac{1}{p'} = \frac{p-1}{p} \Rightarrow p' = \frac{p}{p-1} \Rightarrow p' = \frac{1}{p-1} + 1 \Rightarrow \frac{1}{p-1} = p' - 1.$

$(\bar{\varphi}_q)^*(u) = (t^q)^*(u),\ (\bar{\varphi}_q)'(t) = qt^{q-1},\ u = qt^{q-1} \Rightarrow t = \left(\frac{u}{q}\right)^{\frac{1}{q-1}}.$

$((\bar{\varphi}_q)')^{-1}(u) = \left(\frac{u}{q}\right)^{\frac{1}{q-1}} = q^{\frac{1}{q-1}} u^{q'-1}$, *integration over u :* $\int_0^u q^{\frac{1}{q-1}} t^{q-1}\,dt = q^{\frac{1}{q-1}} \int_0^u t^{q'-1}\,dt = q^{\frac{1}{1-q}} \frac{t^q}{q}\Big]_0^u =$

$q^{\frac{1}{1-q}} \frac{u^{q'}}{q'} = q^{-\frac{1}{q-1}} \frac{1}{q'} u^{q'} = q^{-q'} q(1 - \frac{1}{q}) u^{q'} = q^{-q'} q(\frac{q-1}{q}) u^{q'} = q^{-q'}(q-1)u^{q'}.$

$(\bar{\varphi}_q)^*(u) = q^{-q'}(q-1)u^q.$

*(1)-* $q^{q'} \geq q$ *due to* $q' = \frac{q}{q-1} > 1$, $q^{q'} \geq q - 1 \Rightarrow \frac{1}{q^{q'}} \leq \frac{1}{q-1} \Rightarrow \frac{q-1}{q'} \leq 1$, *considering* $t^{q'} \leq t^{q'}$ *implies* $\frac{q-1}{q'} t^{q'} \leq t^{q'} \Rightarrow (\bar{\varphi}_q)^*(t) \leq \bar{\varphi}_{q'}(t)$.

*(2)-* $\frac{\bar{\varphi}_{q'}(t)}{(\bar{\varphi}_q)^*(2t)} = \frac{t^{q'}}{q^{-q'} q q'^{-1} (2t)^{q'}} = q^{q'} q^{-1} q' 2^{-q'} = q' q^{q'-1} 2^{-q'} \leq 1 \Rightarrow \bar{\varphi}_{q'}(t) \leq (\bar{\varphi}_q)^*(2t)$.

**Lemma 1.1.3** *The function* $a \mapsto \widetilde{\varphi}_{\frac{1}{a}}(t) = at^{\frac{1}{a}}$ *is continuous and convex on* $[0,1]$ *for each* $t \geq 0$.

**Proof.** *The claim is obvious for* $t = 0$, *so assume* $t > 0$, *define* $f(a) := at^{\frac{1}{a}},\ a \in ]0,1]$ $\widetilde{\varphi}_{\frac{1}{a}}(t) = \frac{1}{\frac{1}{a}} t^{\frac{1}{a}} = at^{\frac{1}{a}} = f(a)$, $f$ *is continuous because* $f(a) = at^{\frac{1}{a}} = ae^{\frac{1}{a}\ln(t)}$, *and the functions* $a \mapsto \frac{1}{a}\ln(t)$ *and* $a \mapsto a$ *and* $a \mapsto e^a$ *functions continuous.*

*we have that :if* $f : I \to \mathbb{R}$ *a continuous function, then f is convex if and only if f is mid point convex, this is* $x, y \in I$ *implies* $f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$

*for* $a_1, a_2 \in ]0,1]$, $f\left(\frac{a_1+a_2}{2}\right) = \frac{a_1+a_2}{2} t^{\frac{1}{a_1+a_2}} = \frac{a_1 t^{\frac{1}{a_1+a_2}}}{2} + \frac{a_2 t^{\frac{1}{a_1+a_2}}}{2}$

$$\leq \frac{a_1 t^{\frac{1}{a_1}}}{2} + \frac{a_2 t^{\frac{1}{a_2}}}{2}$$
$$\leq \frac{f(a_1)}{2} + \frac{f(a_2)}{2}$$
$$\leq \frac{f(a_1) + f(a_2)}{2}$$

*So* $f(a) = at^{\frac{1}{a}}$ *is convex.*

**Remark 1.1.3** *Let* $q_0, q_1 \in [1,\infty]$. *For* $\theta \in [0,1]$ *let* $q_\theta \in [q_0, q_1]$ *be defined through* $\frac{1}{q_\theta} := \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$. *Then*

$$\widetilde{\varphi}_{q_0}(t) \leq (1-\theta)\widetilde{\varphi}_{q_0}(t) + \theta\widetilde{\varphi}_{q_1}(t),$$
$$\min\left\{\bar{\varphi}_{q_0}(t), \bar{\varphi}_{q_1}(t)\right\} \leq \bar{\varphi}_{q_\theta}(t) \leq \max\left\{\bar{\varphi}_{q_0}(t), \bar{\varphi}_{q_1}(t)\right\}$$

*for all* $t \geq 0$. *The estimate for* $\widetilde{\varphi}_{q_\theta}$ *follows by convexity* (*Lemma* 1.1.3) *and the estimate for* $\bar{\varphi}_{q_\theta}(t)$ *follows by direct calculation.*

**Lemma 1.1.4** *Let* $1 \leq q \leq \infty$. *Then*

$$\widetilde{\varphi}_q(t) \leq \bar{\varphi}_q(t) \leq \widetilde{\varphi}_q(2t) \quad \textit{for all } t \geq 0$$

**Proof.** *Since $\widetilde{\varphi}_\infty = \bar{\varphi}_\infty$, it suffices to consider the case $1 \le q < \infty$, the case $t = 0$ follows from $\widetilde{\varphi}_q(0) = 0 = \bar{\varphi}_q(0)$. 1)- for $t > 0$ we have : $t^q \le t^q \Rightarrow \frac{t^q}{q} \le t^q$ , $1 \le q < \infty$, so $\widetilde{\varphi}_q(t) \le \bar{\varphi}_q(t)$.*
*2)- $q \le 2^q$ due to : $q \le e^{q\ln(2)} \Leftrightarrow \ln(q) \le q\ln(2) \Leftrightarrow \frac{ln(q)}{q} \le \ln(2)$.*
*$\left(\frac{\ln(q)}{q}\right)' = \frac{\left(\frac{q}{q} - \ln(q)\right)}{q^2} = \frac{1-\ln(q)}{q^2}$. If $1 \le q < e$ then $1 - \ln(q) > 0$, if $e < q < \infty$ then $1 - \ln q < 0$, So: $\sup_{1\le q<\infty}\left(\frac{\ln q}{q}\right) = \frac{\ln e}{e} = \frac{1}{e} < \ln(2) \Rightarrow \frac{\ln(q)}{q} \le \ln(2)$.*
*$q \le 2^q \Rightarrow 1 \le \frac{2^q}{q}$ and we have $t^q \le t^q$ so $t^q \le \frac{1}{q}2^q t^q$ so $\bar{\varphi}_q(t) \le \widetilde{\varphi}_q(2t)$. Finally : $\widetilde{\varphi}_q(t) \le \bar{\varphi}_q(t) \le \widetilde{\varphi}_q(2t)$.*

**Remark 1.1.4** *It is also possible to show that for every $\lambda > 1$ there exists $c_\lambda \ge 1$ such that $\widetilde{\varphi}_q(t) \le \bar{\varphi}_q(t) \le c_\lambda \widetilde{\varphi}_q(\lambda t)$ for every $q \in [1, \infty]$ and $t \ge 0$.*

**Definition 1.1.8** *For a $\Phi$-function $\varphi$ we define $\varphi^{-1} : [0, \infty) \to [0, \infty)$ by*

$$\varphi^{-1}(t) := \inf\{\tau \ge 0 : \varphi(\tau) \ge t\}$$

*for all $t \ge 0$. We call $\varphi^{-1}$ the left-continuous inverse of $\varphi$.*
*For a generalized $\Phi$-function $\varphi \in \Phi(A, \mu)$ the left-continuous inverse is defined pointwise in y, i.e. for all $y \in A$ let $\varphi^{-1}(y, .) = (\varphi(y, .))^{-1}$.*
*Let us collect a few properties of the left-continuous inverse, which follow from the properties of $\varphi$. Let $\varphi$ be a $\Phi$-function. Then $\varphi^{-1}$ is non-decreasing and left-continuous on $[0, \infty)$. Moreover, $\varphi^{-1}(0) = 0$ and*

$$\varphi(\varphi^{-1}(t)) \le t \tag{1.3}$$

*for all $t \ge 0$. We also have*

$$t \le \varphi^{-1}(\varphi(t)) \tag{1.4}$$

*for all $t \ge 0$ with $\varphi(t) < \infty$.*

**Lemma 1.1.5** *If $q \in [1, \infty)$, then $\widetilde{\varphi}_q^{-1}(t) = (qt^{\frac{1}{q}})$, $\bar{\varphi}_q^{-1}(t) = t^{\frac{1}{q}}$ and $\widetilde{\varphi}_\infty^{-1}(t) = \bar{\varphi}_\infty^{-1}(t) = \chi_{(0,\infty)}(t)$ for all $t \ge 0$.*
*If $q \in [1, \infty]$ and $\frac{1}{q} + \frac{1}{q'} = 1$,then*

$$t \le \varphi_q^{-1}\varphi_{q'}^{-1}(t) \le 2t,$$

*for all $t \ge 0$.*

**Lemma 1.1.6** *If $q \in [1, \infty]$, then*

$$\frac{1}{2}\varphi_q^{-1}(t) \le \varphi_q^{-1}\left(\frac{t}{2}\right) \le \bar{\varphi}_q^{-1}(t) \le \widetilde{\varphi}_q^{-1}(t) \le 2\bar{\varphi}_q^{-1}(t)$$

*for all $t \geq 0$. Moreover,*

$$\bar{\varphi}_q^{-1}(t)\bar{\varphi}_q^{-1}\left(\frac{1}{t}\right) = 1 \quad \textit{and} \quad 1 \leq \widetilde{\varphi}_q^{-1}(t)\widetilde{\varphi}_q^{-1}\left(\frac{1}{t}\right) \leq 3$$

*for all $t > 0$.*

**Proof.** *If $q \in [1, \infty)$, then $\varphi_q^{-1}\left(\frac{t}{2}\right) = 2^{-\frac{1}{q}}\varphi_p^{-1}(t)$ and the first claim follows from $\frac{1}{2} \leq 2^{-\frac{1}{q}} \leq 1$ and Lemma* 1.1.4. *The case $q = \infty$ follows from $\varphi_\infty^{-1}(t) = \chi_{(1,\infty)}$. The second claim follows from Lemma* 1.1.5 *and $1 \leq q^{2/q} \leq e^{2/e} < 3$.*

For $a \in (0, 1]$, $\bar{\varphi}_{1/a}^{-1}(t) = t^a$. Thus we immediately obtain

**Lemma 1.1.7** *The mapping $\mapsto \bar{\varphi}_{1/a}^{-1}(t)$ is convex on $(0, 1]$ for all $t \geq 0$.*

## 1.2 Basic properties

Before we start, we must give precise formulations of some basic results which are frequently used.

**Definition 1.2.1** *Let $(A, \Sigma, \mu)$ be a $\sigma$-finite, complete measure space. Then by $L^0(A, \mu)$ we denote the space of all $\mathbb{K}$-valued, $\mu$-measurable functions on $A$. Two functions are identical, if they agree almost everywhere.*
*In the special case that $\mu$ is the n-dimensional Lebesgue measure, $\Omega$ is a $\mu$-measurable subset of $\mathbb{R}^n$, and $\Sigma$ is the $\sigma$-algebra of $\mu$-measurable subsets of $\Omega$ we abbreviate $L^0(\Omega) := L^0(\Omega, \mu)$.*

**Definition 1.2.2** *Given $\Omega$, $p(.) \in \mathcal{P}(\Omega)$ and a Lebesgue measurable function $f$, define the modular functional (or simply the modular) associated with $p(.)$ by*

$$\varrho_{p(.),\Omega}(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty)}.$$

*If $f$ is unbounded on $\Omega_\infty$ or if $f(.)^{p(.)} \notin L^1(\Omega\setminus\Omega_\infty)$, we define $\varrho_{p(.),\Omega}(f) = +\infty$. When $|\Omega_\infty| = 0$, in particular when $p^+ < \infty$, we let $\|f\|_{L^\infty(\Omega_\infty)} = 0$; when $|\Omega\setminus\Omega_\infty| = 0$, then $\varrho_{p(.),\Omega}(f) = \|f\|_{L^\infty(\Omega_\infty)}$. In situations where there is no ambiguity we will simply write $\varrho_{p(.)}(f)$ or $\varrho(f)$.*

**Definition 1.2.3** *Let $X$ be a vector space. A function $\varrho : X \to [0, \infty]$ is called left-continuous if the mapping $\lambda \mapsto \varrho(\lambda x)$ is continuous on $[0, \infty)$ in the sense that*

$$\lim_{\lambda \to 1^-} \varrho(\lambda x) = \varrho(x) \ \textit{ for every } \ x \in X.$$

*A convex and left-continuous function $\varrho : X \to [0, \infty]$ is called a semimodular on X if*

*(i) $\varrho(0) = 0$;*

*(ii) $\varrho(-x) = \varrho(x)$ for every $x \in X$;*

*(iii) if $\varrho(\lambda x) = 0$ for every $\lambda \in \mathbb{R}$, then $x = 0$.*

*A semimodular is called a modular on X if $\varrho(x) = 0$ if and only if $x = 0$.*

*A semimodular is called continuous if the mapping $\lambda \mapsto \varrho(\lambda x)$ is continuous on $[0, \infty)$ for every fixed $x \in X$.*

**Example 1.2.1** *(a) If $1 \leq p < \infty$,then*

$$\varrho_p(f) := \int_\Omega |f(x)|^p \, dx$$

*defines a continuous modular on $L^0(\Omega)$.*

*(b) Let $\varphi_\infty(t) := \infty.\chi_{(1,\infty)}(t)$ for $t \geq 0$, i.e $\varphi_\infty(t) = 0$ for $t \in [0, 1]$ and $\varphi_\infty(t) = \infty$ for $t \in (1, \infty)$. then*

$$\varrho_\infty(f) := \int_\Omega \varphi_\infty(|f(x)|) \, dx$$

*defines a semimodular on $L^0(\Omega)$ which is not continuous.*

*(c) Let $w \in L^1_{loc}(\Omega)$ with $w > 0$ almost everywhere and $1 \leq p < \infty$. Then*

$$\varrho(f) := \int_\Omega |f(x)|^p w(x) \, dx$$

*defines a continuous modular on $L^0(\Omega)$.*

*(d) The integral expression*

$$\varrho(f) := \int_\Omega \exp(|f(x)|) - 1 \, dx$$

*defines a modular on $L^0(\Omega)$. It is not continuous: if $f \in L^2(\Omega)$ is such that $|f| > 2$ and $|f| \notin L^p(\Omega)$ for any $p > 2$, then $\varrho(\lambda \log |f|) = \infty$ for $\lambda > 2$ but $\varrho(2 \log |f|) < \infty$.*

*(e) If $1 \leq p < \infty$, then*

$$\varrho_p((x_j)) := \sum_{j=0}^{\infty} |x_j|^p \, dx$$

*defines a continuous modular on $\mathbb{R}^n$.*

*(f) For $f \in L^0(\Omega)$ we define the decreasing rearrangement, $f^* : [0, \infty) \to [0, \infty)$ by the formula $f^*(s) := \sup\{t : \, |\, |f| > t| > s\}$. For $1 \leq q \leq p < \infty$ the expression*

$$\varrho(f) := \int_0^\infty |f^*(s^{p/q})|^q \, ds$$

*defines a continuous modular on* $L^0(\Omega)$.

*Let* $\varrho$ *be a semimodular on X. Then by convexity and non-negative of* $\varrho$ *and* $\varrho(0) = 0$ *it follows that* $\lambda \mapsto \varrho(\lambda x)$ *is non-decreasing on* $[0, \infty)$ *for every* $x \in X$. *Moreover,*

$$\begin{aligned} \varrho(\lambda x) = \varrho(|\lambda| x) \leq |\lambda| \varrho(x) \quad \text{for all } |\lambda| \leq 1, \\ \varrho(\lambda x) = \varrho(|\lambda| x) \geq |\lambda| \varrho(x) \quad \text{for all } |\lambda| \geq 1. \end{aligned} \tag{1.5}$$

*In the definition of a semimodular or modular the set X is usually chosen to be larger than necessary. The idea behind this is to choose the same large set X for different modulars like in our Examples 1.2.1(a), (b), (c), (d) and (f). Then depending on the modular we pick interesting subsets from this set X.*

**Definition 1.2.4** *Let X be a vector space and let* $\varrho$ *be a semimodular or a modular on X. Then the space*

$$X_\varrho := \left\{ x \in X \ : \ \lim_{\lambda \to 0} \varrho(\lambda x) = 0 \right\}$$

*is called a semimodular space or a modular space, respectively.*

**Lemma 1.2.1** *Let* $\varphi : [0, \infty) \to [0, \infty]$ *and let* $\varrho$ *denote its even extension to* $\mathbb{R}$, *i.e.* $\varrho(t) := \varphi(|t|)$ *for all* $t \in \mathbb{R}$. *Then* $\varphi$ *is a* $\Phi$*-function if and only if* $\varrho$ *is a semimodular on* $\mathbb{R}$ *with* $X_\varrho = \mathbb{R}$. *Moreover,* $\varphi$ *is a positive* $\Phi$*-function if and only if* $\varrho$ *is a modular on* $\mathbb{R}$ *with* $X_\varrho = \mathbb{R}$.

**Proof.** " $\Rightarrow$ ": *Let* $\varphi$ *be a* $\Phi$*-function. Since* $\lim_{t \to 0^+} \varphi(t) = 0$, *we have* $X_\varrho = \mathbb{R}$. *To prove that* $\varrho$ *is a semimodular on* $\mathbb{R}$ *it remains to prove that* $\varrho(\lambda t_0) = 0$ *for all* $\lambda > 0$ *implies* $t_0 = 0$. *So assume that* $\varrho(\lambda t_0) = 0$ *for all* $\lambda > 0$. *Since* $\lim_{t \to \infty} \varphi(t) = \infty$, *there exists* $t_1 > 0$ *with* $\varphi(t_1) > 0$. *Thus there exists no* $\lambda > 0$ *such that* $t_1 = \lambda t_0$, *which implies that* $t_0 = 0$. *Hence* $\varrho$ *is a semimodular. Assume that* $\varphi$ *is additionally positive. If* $\varrho(s) = 0$, *then* $\varphi(|s|) = 0$ *and therefore* $s = 0$. *This prove that* $\varrho$ *is a modular.*

" $\Leftarrow$ ": *Let* $\varrho$ *be a semimodular on* $\mathbb{R}$ *with* $X_\varrho = \mathbb{R}$. *Since* $X_\varrho = \mathbb{R}$, *there exists* $t_2 > 0$ *such that* $\varrho(t_2) < \infty$. *From (1.5) follows that* $0 \leq \varphi(t) \leq t/t_2 \varphi(t_2)$ *for all* $t \in [0, t_2]$, *which implies that* $\lim_{t \to 0^+} \varphi(t) = 0$. *Since* $1 \neq 0$, *there exists* $\lambda > 0$ *such that* $\varrho(\lambda \cdot 1) \neq 0$. *In particular there exists* $t_3 > 0$ *with* $\varphi(t_3) > 0$ *and* $\varphi(kt_3) \geq k\varphi(t_3) > 0$ *by (1.5) for all* $k \in \mathbb{N}$. *Since k is arbitrary, we get* $\lim_{t \to \infty} \varphi(t) = \infty$. *We have proved that* $\varphi$ *is a* $\Phi$*-function. Assume additionally that* $\varrho$ *is a modular. In particular* $\varrho(t) = \varphi(|t|) = 0$ *implies* $t = 0$. *Hence by negation we get that* $t > 0$ *implies* $\varphi(t) > 0$, *so* $\varphi$ *is positive.*

**Theorem 1.2.1** *Let $\varrho$ be a semimodular on X. The $X_\varrho$ is a normed $\mathbb{K}$-vector space. The norm, called the Luxemburg norm, is defined by*

$$\|x\|_\varrho := \inf\left\{\lambda > 0 \ : \ \varrho\left(\frac{1}{\lambda}x\right) \leq 1\right\}.$$

**Lemma 1.2.2 (Norm-modular unit ball property.)** *Let $\varrho$ be a semimodular on X. Then $\|x\|_\varrho \leq 1$ and $\varrho(x) \leq 1$ are equivalent. If $\varrho$ is continuous, then also $\|x\|_\varrho < 1$ and $\varrho(x) < 1$ are equivalent, as are $\|x\|_\varrho = 1$ and $\varrho(x) = 1$.*

**Definition 1.2.5** *Let $\varrho$ be a semimodular on X. Then by $X_\varrho^*$ we denote the dual space of $(X_\varrho, \|.\|_\varrho)$. Furthermore, we define : $\varrho^* : X_\varrho^* \to [0, \infty]$ by*

$$\varrho^*(x^*) := \sup_{x \in X_\varrho}(|\langle x^*, x\rangle| - \varrho(x)).$$

*We call $\varrho^*$ the conjugate semimodular of $\varrho$.*
*Note the difference between the space $X_\varrho^*$ and $X_{\varrho^*}$ : the former is the dual space of $X_\varrho$, whereas the latter is the semimodular space defined by $\varrho^*$.*
*By definition of the functional $\varrho^*$ we have*

$$|\langle x^*, x\rangle| \leq \varrho(x) + \varrho^*(x^*) \tag{1.6}$$

*for all $x \in X_\varrho$ and $x^* \in X_\varrho^*$. This inequality is a generalized version of the classical Young inequality.*

**Definition 1.2.6** *Let $\varrho$ be a semimodular on X and $x_k, x \in X_\varrho$. Then we say that $x_k$ is modular convergent ($\varrho$-convergent) to x if there exists $\lambda > 0$ such that $\varrho(\lambda(x_k - x)) \to 0$. We denote this by $x_k \xrightarrow{\varrho} x$.*

**Lemma 1.2.3** *Let $\varrho$ be a semimodular on X and $x_k \in X_\varrho$. Then $x_k \to 0$ for $k \to \infty$ if and only if $\lim_{k\to\infty} \varrho(\lambda x_k) = 0$ for all $\lambda > 0$.*

**Proof.** *Assume that $\|x_k\|_\varrho \to 0$ and $\lambda > 0$. Then $\|K\lambda x_k\|_\varrho < 1$ for all $K > 1$ and large k. Thus $\varrho(K\lambda x_k) \leq 1$ for large k, hence*

$$\varrho(\lambda x_k) \leq \frac{1}{K}\varrho(K\lambda x_k) \leq \frac{1}{K}$$

*for large k, by* (1.5)*. This implies $\varrho(\lambda x_k) \to 0$.*
*Assume now that $\varrho(\lambda x_k) \to 0$ for all $\lambda > 0$. Then $\varrho(\lambda x_k) \leq 1$ for large k. In particular, $\|x_k\|_\varrho \leq 1/\lambda$ for large k. Since $\lambda > 0$ was arbitrary, we get $\|x_k\|_\varrho \to 0$. In other words $x_k \to 0$.*

**Lemma 1.2.4** *Let $X_\varrho$ be a semimodular space. Then modular convergence and norm convergence are equivalent if and only if $\varrho(x_k) \to 0$ implies $\varrho(2x_k) \to 0$.*

**Proof.** " $\Rightarrow$ "*: Let modular convergence and norm convergence be equivalent and let $\varrho(x_k) \to 0$ with $x_k \in X_\varrho$. Then $x_k \to 0$ and by Lemma 1.2.3 it follows that $\varrho(2x_k) \to 0$.*
" $\Leftarrow$ " *: Let $x_k \in X_\varrho$ with $\varrho(x_k) \to 0$. We have to show that $\varrho(\lambda x_k) \to 0$ for all $\lambda > 0$. For fixed $\lambda > 0$ choose $m \in \mathbb{N}$ such that $2^m \geq \lambda$. Then by repeated application of the assumption we get $\lim_{k\to 0} \varrho(2^m x_k) = 0$. Then $0 \leq \lim_{k\to\infty} \varrho(\lambda x_k) \leq \lambda 2^{-m} \lim_{k\to\infty} \varrho(2^m x_k) = 0$ by (1.5). This proves that $x_k \to 0$.*

**Corollary 1.2.1** *Let $\varrho$ be a semimodular on $X$ and $x \in X_\varrho$.*
*(a) If $\|x\|_\varrho \leq 1$, then $\varrho(x) \leq \|x\|_\varrho$.*
*(b) If $1 < \|x\|_\varrho$, then $\|x\|_\varrho \leq \varrho(x)$.*
*(c) $\|x\|_\varrho \leq \varrho(x) + 1$.*

**Definition 1.2.7** *For a given measure space $(\Omega, \mathcal{A}, \mu)$ we say that $\varphi$ is a Musielak-Orlicz (Mo if no confusion arises) function on $(\Omega, \mathcal{A}, \mu)$ (or on $\Omega$ if no confusion arises from such simplification) as the collection of all functions*

$$\varphi : \Omega \times [0, \infty) \to [0, \infty)$$

*such that*
*(i) $\varphi(., t)$ is measurable for each $t \in [0, \infty)$,*
*(ii) for a.e. $x \in \Omega$, $\varphi(x, .)$ is non-decreasing, convex, continuous and $\varphi(x, 0) = 0$.*
*(iii) $\varphi(x, t) > 0$ for $t > 0$,*
*(iv) $\varphi(x, t) \mapsto \infty$ as $t \to \infty$.*

For example if $\Omega \subseteq \mathbb{R}^n$, $\mathcal{A}$ is the Borel $\sigma$-algebra of subsets of $\Omega, \mu$ is the Lebesgue measure and p is a measurable function.

$$p : \Omega \to [1, \infty),$$

the $\varphi(x, t) = t^{p(x)}$ is a Musielak-Orlicz function on $\Omega$ is this setting. In particular then function $\varphi$ in *Definition* 1.2.7 is said to be an Orlicz function if it is independent of $x \in \Omega$.

**Definition 1.2.8** *Let $\varphi$ be an Musielak-Orlicz function on $\Omega$. When the $\sigma$-algebra $\mathcal{A}$ and the measure $\mu$ are clear from the context, which will be the case throughout this monograph, we set*

$$L^\varphi(\Omega) = \left\{ u \in \mathcal{M} \ : \ \int_\Omega \varphi(x, \lambda|u(x)|)\, dx < \infty \ \text{ for some } \lambda > 0 \right\}$$

*It is easy to see that under the prescribed hypothesis $L^\varphi(\Omega)$ is a vector space.*
*When endowed with the Luxemburg norm* ([*19*, *Proposition* 1.3.2])

$$\|u\|_\varphi = \inf\left\{\lambda > 0 \ : \ \int_\Omega \varphi\left(x, \frac{|u(x)|}{\lambda}\right) dx \leq 1\right\}.$$

*$L^\varphi(\Omega)$ becomes a normed space (in fact, a Banach space if $\mu$ is $\sigma$-finite, as will be shown in* [*19*, *Theorem* 2.1.6]*), and will be called the Musielak-Orlicz space generated by $\varphi$ (MO space for short, if there is no room for confusion).*

The Musielak-Orlicz space is also called generalized Orlicz space and it is a semimodular space.

**Lemma 1.2.5** *Let $\varphi \in \Phi(A, \mu)$ and $\mu(A) < \infty$. Then every $\|.\|_\varphi$-Cauchy sequence is also a Cauchy sequence with respect to convergence in measure.*

**Proof.** *Fix $\varepsilon > 0$ and let $V_t := \{y \in A \ : \ \varphi(y, t) = 0\}$ for $t > 0$. Then $V_t$ is measurable. For all $y \in A$ the function $t \mapsto \varphi(y, t)$ is non-decreasing and $\lim_{t\to\infty} \varphi(y, t) = \infty$, so $V_t \searrow \emptyset$ as $t \to \infty$. Therefore, $\lim_{k\to\infty} \mu(V_k) = \mu(\emptyset) = 0$, where we have used that $\mu(A) < \infty$. Thus, there exists $K \in \mathbb{N}$ such that $\mu(V_K) < \varepsilon$. Note that if $\varphi$ is positive then $V_t = \emptyset$ for all $t > 0$ and we do not need this step in the proof.*
*For a $\mu$-measurable set $E \subset A$ define*

$$\nu_K(E) := \varrho_\varphi(K\chi_E) = \int_E \varphi(y, K)\, d\mu(y).$$

*If $E$ is $\mu$-measurable with $\nu_K(E) = 0$, then $\varphi(y, K) = 0$ for $\mu$-almost every $y \in E$. Thus $\mu(E\backslash V_K) = 0$ by the definition of $V_K$. Hence, $E$ is a $\mu|_{A\backslash V_K}$-null set, which means that the measure $\mu|_{A\backslash V_K}$ is absolutely continuous with respect to $\nu_K$.*
*Since $\mu(A\backslash V_K) \leq \mu(A) < \infty$ and $\mu|_{A\backslash V_K}$ is absolutely continuous with respect to $\nu_K$, there exists $\delta \in (0, 1)$ such that $\nu_K(E) \leq \delta$ implies $\mu(E\backslash V_K) \leq \varepsilon$ (cf. [15, Theorem 30.B]). Since $f_k$ is a $\|.\|_\varphi$-Cauchy sequence, there exists $k_0 \in \mathbb{N}$ such that $\|K\varepsilon^{-1}\delta^{-1}(f_m - f_k)\|_\varphi \leq 1$ for all $m, k \geq k_0$. Assume in the following $m, k \geq k_0$, then by (1.5) and the norm-modular unit ball property (lemma 1.2.2)*

$$\varrho_\varphi(K\varepsilon^{-1}(f_m - f_k)) \leq \delta\varrho_\varphi(K\varepsilon^{-1}\delta^{-1}(f_m - f_k)) \leq \delta.$$

*Let us write $E_{m,k,\varepsilon} := \{y \in A \ : \ |f_m(y) - f_k(y)| \geq \varepsilon\}$. Then*

$$\nu_K(E_{m,k,\varepsilon}) = \int_{E_{m,k,\varepsilon}} \varphi(y, K)\, d\mu(y) \leq \varrho_\varphi(K\varepsilon^{-1}(f_m - f_k)) \leq \delta.$$

*By the choice of $\delta$, this implies that $\mu(E_{m,k,\varepsilon} \setminus V_K) \leq \varepsilon$. With $\mu(V_K) < \varepsilon$ we have $\mu(E_{m,k,\varepsilon}) \leq 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves that $f_k$ is a Cauchy sequence with respect to convergence in measure.*
*If $\|f_k\|_\varphi \to 0$, then as above there exists $K \in \mathbb{N}$ such that $\mu(\{|f_k| \geq \varepsilon\}) \leq 2\varepsilon$ for all $k \geq K$. This proves $f_k \to 0$ in measure.*

**Lemma 1.2.6** *Let $\varphi \in \Phi(A, \mu)$. Then every $\|.\|_\varphi$-Cauchy sequence $(f_k) \subset L^\varphi$ has a subsequence which converges $\mu$-almost everywhere to a measurable function $f$.*

**Proof.** *Recall that $\mu$ is $\sigma$-finite. Let $A = \bigcup_{i=1}^{\infty} A_i$ with $A_i$ pairwise disjoint and $\mu(A_i) < \infty$ for all $i \in \mathbb{N}$. Then, by Lemma 1.2.5, $(f_k)$ is a Cauchy sequence with respect to convergence in measure on $A_1$. Therefore there exists a measurable function $f : A_1 \to \mathbb{K}$ and a subsequence of $f_k$ which converges to $f$ $\mu$-almost everywhere. Repeating this argument for every $A_i$ and passing to the diagonal sequence we get a subsequence $(f_{k_j})$ and a $\mu$-measurable function $f : A \to \mathbb{K}$ such that $f_{k_j} \to f$ $\mu$-almost everywhere.*

As a semimodular space, $L^\varphi = (L^\varphi, \|.\|_\varphi)$ is a normed space, which, in fact, is complete.

**Theorem 1.2.2** *Let $\varphi \in \Phi(A, \mu)$. Then $L^\varphi(A, \mu)$ is a Banach space.*

**Proof.** *Let $(f_k)$ be a Cauchy sequence. By Lemma 1.2.6 there exists a subsequence $f_{k_j}$ and a $\mu$-measurable function $f : A \to \mathbb{K}$ such that $f_{k_j} \to f$ for $\mu$-almost every $y \in A$. This implies $\varphi(y, |f_{k_j}(y) - f(y)|) \to 0$ $\mu$-almost everywhere. Let $\lambda > 0$ and $0 < \varepsilon < 1$. Since $(f_k)$ is a Cauchy sequence, there exists $K = K(\lambda, \varepsilon) \in \mathbb{N}$ such that $\|\lambda(f_m - f_k)\|_\varphi < \varepsilon$ for all $m, k \geq N$, which implies $\varrho_\varphi(\lambda(f_m - f_k)) \leq \varepsilon$ by Corollary 1.2.1. Therefore by Fatou's lemma*

$$\begin{aligned}
\varrho_\varphi(\lambda(f_m - f)) &= \int_A \lim_{j\to\infty} \varphi\left(y, \lambda|f_m(y) - f_{k_j}(y)\right) d\mu(y) \\
&\leq \liminf_{j\to\infty} \int_A \varphi\left(y, \lambda|f_m(y) - f_{k_j}(y)|\right) d\mu(y) \\
&= \liminf_{j\to\infty} \varrho_\varphi\left(\lambda(f_m - f_{k_j})\right) \\
&\leq \varepsilon.
\end{aligned}$$

*So $\varrho_\varphi(\lambda(f_m - f)) \to 0$ for $m \to \infty$ and all $\lambda > 0$ and $\|f_k - f\|_\varphi \to 0$ by Lemma 1.2.3. Thus every Cauchy sequence converges in $L^\varphi$, as was to be shown.*

**Definition 1.2.9** *Let $\varphi \in \Phi(A,\mu)$. The set*

$$L^{\varphi}_{OC} \coloneqq L^{\varphi}_{OC}(A,\mu) \coloneqq \{f \in L^{\varphi} \ : \ \varrho_{\varphi}(f) < \infty\} \tag{1.7}$$

*is called the Musielak-Orlicz class. Let*

$$E^{\varphi} \coloneqq E^{\varphi}(A,\mu) \coloneqq \{f \in L^{\varphi} \ : \ \varrho_{\varphi}(\lambda f) < \infty \text{ for all } \lambda > 0\}. \tag{1.8}$$

*The elements of $E^{\varphi}(A,\mu)$ are called finite.*

*Let us start with a few examples :*

*(a) Let $\varphi(y,t) = t^p$ with $1 \le p < \infty$. Then $E^{\varphi} = L^{\varphi}_{OC} = L^{\varphi} = L^p$.*

*(b) Let $\varphi(y,t) = \infty.\chi_{(1,\infty)}(t)$. Then*

$$E^{\varphi} = \{0\},$$
$$L^{\varphi}_{OC} = \{f : |f| \le 1 \text{ almost everywhere }\},$$
$$L^{\varphi} = L^{\infty}.$$

*(c) Let $\varphi(y,t) = \exp(t) - 1$ and $\Omega = (0,1)$. Then $\varphi \in \Phi(\Omega)$ is positive and continuous but $E^{\varphi} \neq L^{\varphi}_{OC} \neq L^{\varphi}$. Indeed, if $f \coloneqq \sum_{k=1}^{\infty} \frac{k}{2}\chi_{(2^{-k},2^{-k+1})}$, then $f \in L^{\varphi}_{OC} \backslash E^{\varphi}$ and $2f \in L^{\varphi} \backslash L^{\varphi}_{OC}$.*

*By definition of $E^{\varphi}, L^{\varphi}_{OC}$, and $L^{\varphi}$ it is clear that $E^{\varphi} \subset L^{\varphi}_{OC} \subset L^{\varphi}$. Moreover, by convexity of $\varphi$ the set $L^{\varphi}_{OC}$ is convex and the sets $E^{\varphi}$ and $L^{\varphi}$ are linear subspaces of $L^0$. There is a special relation of $E^{\varphi}$ and $L^{\varphi}$ to $L^{\varphi}_{OC}$:*

*$E^{\varphi}$ is the biggest vector space in $L^{\varphi}_{OC}$ and $L^{\varphi}$ is the smallest vector space in $L^0$ containing $L^{\varphi}_{OC}$.*

*In some cases the inclusions $E^{\varphi} \subset L^{\varphi}_{OC} \subset L^{\varphi}$ are strict and in other cases equality holds. In fact, it is easily seen that $E^{\varphi} = L^{\varphi}_{OC} = L^{\varphi}$ is equivalent to the implication $f \in L^{\varphi}_{OC} \Rightarrow 2f \in L^{\varphi}_{OC}$. The $\Delta_2$-condition (see Definition 1.2.12) implies that $\varrho_{\varphi}(2^m f) \le K^m \varrho_{\varphi}(f)$, where $K$ is the $\Delta_2$-constant, from which we conclude that*

$$E^{\varphi}(A,\mu) = L^{\varphi}_{OC}(A,\mu) = L^{\varphi}(A,\mu).$$

**Definition 1.2.10** *A function $\varphi \in \Phi(A,\mu)$ is called locally integrable on $A$ if $\varrho(t\chi_E) < \infty$ for all $t \ge 0$ and all $\mu$-measurable $E \subset A$ with $\mu(E) < \infty$.*

**Remark 1.2.1** *Let $p \in \mathcal{P}(A,\mu)$ be a bounded exponent. Then $\varphi_{p(.)}$ is locally integrable, since*

$$\int_E \varphi_{p(y)}(\lambda)\, d\mu(y) \le \mu(E) \max\{\lambda^{p^-}, \lambda^{p^+}\}$$

*for every measurable $E \subset A$ with $\mu(E) < \infty$ and every $\lambda > 0$. However, the local integrability of $\varphi_{p(.)}$ does not imply that $p^+ < \infty$. Indeed, let $A := \mathbb{R}$ and let $E_k \subset \mathbb{R}$ be pairwise disjoint with $|E_k| = \exp(\exp(-k))$. Now, define $p(x) := k$ for $x \in E_k$ and $k \in \mathbb{N}$ and $p(x) = 1$ for $x \in \mathbb{R} \setminus \bigcup_{k=1}^{\infty} E_k$. Then for every $\lambda > 0$ and every $E \subset \mathbb{R}$ with $|E| < \infty$ we have*

$$\bar{\varphi}_{p(.)}(\lambda E) \leq \lambda |E| + \sum_{k=1}^{\infty} \lambda^k \exp\left(\exp(-k)\right) < \infty.$$

*Thus $\bar{\varphi}_{p(.)}$ is locally integrable but $p^+ = \infty$.*

Let $(A, \Sigma, \mu)$ be a $\sigma$-finite, complete measure space. Recall that a measure $\mu$ is called separable if there exists sequence $(E_k) \subset \Sigma$ with the following properties:
(a) $\mu(E_k) < \infty$ for all $k \in \mathbb{N}$.
(b) For every $E \in \Sigma$ with $\mu(E) < \infty$ and every $\varepsilon > 0$ there exists an index $k$ such that $\mu(E \Delta E_k) < \varepsilon$, where $\Delta$ denotes the symmetric difference defined through $E \Delta E_k := (E \setminus E_k) \bigcup (E_k \setminus E)$. Recall that a Banach space is separable if it contains a dense, countable subset.

**Theorem 1.2.3** *Let $\varphi \in \Phi(A, \mu)$ be locally integrable and let $\mu$ be separable. Then $E^{\varphi}(A, \mu)$ is separable.*

**Proof.** *see [7].*

Now we are ready to define the variable exponents Lebesgue space.

**Definition 1.2.11** *Let $p \in \mathcal{P}(A, \mu)$ and let either $\varphi_{p(.)} := \widetilde{\varphi}_{p(.)}$ or $\varphi_{p(.)} := \bar{\varphi}_{p(.)}$. Hence we obtain a semimodular:*

$$\varrho_{L^{p(.)}(A)}(f) = \int_A \varphi_{p(x)}(|f(x)|)\, dx$$

*We define the variable exponent Lebesgue space $L^{p(.)}(A, \mu)$ as the Musielak-Orlicz space $L^{\varphi_{p(.)}}(A, \mu)$ with the norm $\|.\|_{L^{p(.)}(A,\mu)} := \|.\|_{L^{\varphi_{p(.)}}(A,\mu)}$.*
*In particular, the variable exponent Lebesgue space $L^{p(.)}(A, \mu)$ is*

$$L^{p(.)}(A, \mu) = \left\{ f \in L^0(A, \mu) \ : \ \lim_{\lambda \to 0} \varrho_{L^{p(.)}(A)}(\lambda f) = 0 \right\}$$

*or equivalently*

$$L^{p(.)}(A, \mu) = \left\{ f \in L^0(A, \mu) \ : \ \varrho_{L^{p(.)}(A)}(\lambda f) < \infty \text{ for some } \lambda > 0 \right\}$$

*equipped with the norm*

$$\|f\|_{L^{p(.)}(A,\mu)} = \inf\left\{\lambda > 0 \ : \ \varrho_{L^{p(.)}(A)}\left(\frac{f}{\lambda}\right) \leq 1\right\}.$$

*Note that $\varrho_{L^{p(.)}(A)}$ is a modular if $p$ is finite everywhere. We abbreviate $\varrho_{L^{p(.)}(A)}$ to $\varrho_{p(.)}$ and $\|.\|_{L^{p(.)}(A,\mu)}$ to $\|.\|_{p(.)}$ if the set and the measure are clear from the context. Moreover, if $\Omega \subset \mathbb{R}^n$ and $\mu$ is the Lebesgue measure we simply write $L^{p(.)}(\Omega)$ and if $\mu$ is the counting measure on $\mathbb{Z}^n$, then we write $l^{p(.)}(\mathbb{Z}^n)$.*

*This definition seems ambiguous, since either $\varphi_{p(.)} = \widetilde{\varphi}_{p(.)}$ or $\varphi_{p(.)} = \bar{\varphi}_{p(.)}$. However, due to Lemma* 1.1.4 *it is clear that $L^{\widetilde{\varphi}_{p(.)}} = L^{\bar{\varphi}_{p(.)}}$ and*

$$\|f\|_{\widetilde{\varphi}_{p(.)}} \leq \|f\|_{\bar{\varphi}_{p(.)}} \leq 2\|f\|_{\widetilde{\varphi}_{p(.)}}. \tag{1.9}$$

*Thus, the two definitions agree up to equivalence of norms with constant at most 2.*

*Recall that we have two relevant $\Phi$-functions, $\widetilde{\varphi}_{p(.)}$ and $\bar{\varphi}_{p(.)}$. Usually, the exact norm of $L^{p(.)}$ is not important, so we just work with $\varphi_{p(.)}$ without specifying whether $\varphi_{p(.)} = \widetilde{\varphi}_{p(.)}$ or $\varphi_{p(.)} = \bar{\varphi}_{p(.)}$. If there is a difference in the choice of $\varphi_{p(.)}$, then the specific choice for $\varphi_{p(.)}$ will be specified.*

**Remark 1.2.2** *Originally, the spaces $L^{p(.)}$ have been introduced by Orlicz [24] in 1931 with $\varphi_{p(.)} = \bar{\varphi}_{p(.)}$ in the case $1 \leq p^- \leq p^+ < \infty$. The first definition of $L^{p(.)}$ including the case $p^+ = \infty$ was given by Sharpudinov [27] and then, in the higher dimensional case, by Kováčik and Rákosník [17]. For measurable $f$ they define*

$$\varrho_{KR}(f) := \bar{\varrho}_{p(.)}(f\chi_{\{p\neq\infty\}}) + \|f\chi_{\{p=\infty\}}\|_\infty.$$

*If is easy to see that $\varrho_{KR}$ is a modular on $L^0(\Omega)$, the set of measurable functions. We denote the corresponding Luxemburg norm by*

$$\|f\|_{KR} = \inf\left\{\lambda > 0 : \ \varrho_{KR}\left(\frac{1}{\lambda}f\right) \leq 1\right\}.$$

*If $\mu(\{p = \infty\}) = 0$, then $\varrho_{KR} = \bar{\varrho}_{p(.)}$. But if $\mu(\{p = \infty\}) > 0$, then $\varrho_{KR} \neq \bar{\varrho}_{p(.)}$. Note that $\varrho_{KR}$ is a modular, while our $\varrho_{p(.)}$ is a only semimodular. In particular, $\varrho_{KR}(f) = 0$ implies $f = 0$. For $\bar{\varrho}_{p(.)}$ we only have that $\bar{\varrho}_{p(.)}(\lambda f) = 0$ for all $\lambda > 0$ implies $f = 0$. This is due to the fact that $\bar{\varphi}_\infty$ is not a positive $\Phi$-function. Since we developed most of the theory in [7, Chapter 2] for semimodular spaces, we do not have to treat the set $\{p = \infty\}$ differently and we can work directly with $\varrho_{p(.)}(f)$. This includes the case $p^+ = \infty$ in a more natural way.*

*Although $\varrho_{KR}$ and $\bar{\varrho}_{p(.)}$ differ if $p^+ = \infty$, they produce the same space up to isomorphism. Let us prove this : let $f \in L^0(\Omega)$ with $\|f\|_{\varrho_{KR}} \leq 1$, so $\varrho_{KR}(f) \leq 1$ by the norm-modular unit ball property (Lemma 1.2.2). In particular, $\|f\chi_{\{p=\infty\}}\|_\infty \leq 1$. Thus $\bar{\varrho}_{p(.)}(f\chi_{\{p=\infty\}}) = 0$, since $\bar{\varphi}_\infty(t) = 0$ for all $t \in [0,1]$. This proves that*

$$\bar{\varrho}_{p(.)}(f) = \bar{\varrho}_{p(.)}(f\chi_{\{p\neq\infty\}}) + \bar{\varrho}_{p(.)}(f\chi_{\{p=\infty\}}) = \bar{\varrho}_{p(.)}(f\chi_{\{p\neq\infty\}}) \leq \bar{\varrho}_{KR}(f) \leq 1.$$

*So it follows that $\|f\|_{\bar{\varrho}_{p(.)}} \leq 1$. The scaling argument shows that $\|f\|_{\bar{\varrho}_{p(.)}} \leq \|f\|_{KR}$.*
*Assume now that $\|f\|_{\bar{\varrho}_{p(.)}} \leq 1$, so $\bar{\varrho}_{p(.)}(f) \leq 1$ by the unit ball property. In particular, $\varrho_\infty(f\chi_{\{p=\infty\}}) = \bar{\varrho}_{p(.)}(f\chi_{\{p=\infty\}}) \leq 1$ and therefore $|f| \leq 1$ almost everywhere on $\{p = \infty\}$. This proves that*

$$\varrho_{KR}(f) = \bar{\varrho}_{p(.)}(f\chi_{\{p\neq\infty\}}) + \|f\chi_{\{p=\infty\}}\|_\infty \leq \bar{\varrho}_{p(.)}(f) + 1 \leq 2.$$

*This implies $\varrho_{KR}\left(\frac{1}{2}f\right) \leq \frac{1}{2}\varrho_{KR}(f) \leq 1$, so $\|f\|_{KR} \leq 2$ by the norm-modular unit ball property. The scaling argument shows $\|f\|_{KR} \leq 2\|f\|_{\bar{\varrho}_{p(.)}}$.*
*Overall, we have shown that*

$$\|f\|_{\bar{\varrho}_{p(.)}} \leq \|f\|_{KR} \leq 2\|f\|_{\bar{\varrho}_{p(.)}},$$

*for all $f \in L^0(\Omega)$. Thus $\varrho_{KR}$ and $\varrho_{\bar{\varphi}_{p(.)}}$ define the same spaces $L^{p(.)}(\Omega)$, up to equivalence of norms. For a constant exponent the relation between the modular and the norm is clear. For a variable exponent some more work is needed. We will invoke it by mentioning the unit ball property, or, when more clarity is needed, the norm-modular unit ball property.*

**Example 1.2.2** *If $\Omega = [1,4]$, $p(x) = \frac{1}{x}$ and $u(x) = 2^{x^2}$, then show that $u \in L^{p(.)}(\Omega)$.*
*Solution. Since the function $p(x) = \frac{1}{x}$ is bounded on the domain $\Omega$, we have $|\Omega_\infty| = 0$. then*

$$\varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}\,dx = \int_1^4 \left|2^{x^2}\right|^{\frac{1}{x}} dx = \frac{2^x}{ln2}\Big|_1^4 = \frac{14}{\ln 2} < \infty$$

*implies that $u \in L^{p(.)}(\Omega)$.*

**Proposition 1.2.1** *Given $\Omega$ and $p(.) \in \mathcal{P}(\Omega)$, define*

$$\|f\|^A_{p(.)} = \inf\left\{\lambda > 0 \ : \ \lambda + \lambda\varrho_{p(.)}\left(f/\lambda\right)\right\}.$$

*Then for all $f \in L^{p(.)}(\Omega)$,*

$$\|f\|_{p(.)} \leq \|f\|^A_{p(.)} \leq 2\|f\|_{p(.)}.$$

**Definition 1.2.12** *We say that $\varphi \in \Phi(A,\mu)$ satisfies the $\Delta_2$-condition if there exists $k \geq 2$ such that*

$$\varphi(y,2t) \leq k\varphi(y,t).$$

*for all $y \in A$ and all $t \geq 0$. The smallest such k is called the $\Delta_2$-constant of $\varphi$.*
*Analogously, we say that a semimodular $\varrho$ on X satisfies the $\Delta_2$-condition if there exists $k \geq 2$ such that $\varrho(2f) \leq k\varrho(f)$ for all $f \in X_\varrho$. Again, the smallest such k is called the $\Delta_2$-constant of $\varrho$.*
*If $\varphi \in \Phi(A,\mu)$ satisfy the $\Delta_2$-condition, then $\varrho_\varphi$ satisfies the $\Delta_2$-condition with the same constant. Moreover, $\varrho_\varphi$ satisfies the weak $\Delta_2$-condition for modular, so by Lemma* 1.2.4. *modular convergence and norm convergence are equivalent; and $E \subset L^\varphi(\Omega,\mu)$ is bounded with respect to the norm if and only if it is bounded with respect to the modular, i.e. $\sup_{f\in E}\|f\| < \infty$ if and only if $\sup_{f\in E}\varrho_\varphi(f) < \infty$.*
*[7, Corollary 2.1.15] shows that a small norm implies a small modular.*

**Lemma 1.2.7** *Let $\varrho$ be a semimodular on X that satisfies the $\Delta_2$-condition. Let K be the $\Delta_2$-constant of $\varrho$. Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon,K) > 0$ such that $\varrho(f) \leq \delta$ implies $\|f\|_\varrho \leq \varepsilon$.*

**Proof.** *For $\varepsilon > 0$ choose $j \in \mathbb{N}$ with $2^{-j} \leq \varepsilon$. Let $\delta := K^j$ and $\varrho(f) \leq \delta$. Then $\varrho(2^j f) \leq K^j\varrho(f) \leq 1$ and the unit ball property yields $\|f\|_\varrho \leq 2^{-j} \leq \varepsilon$.*

**Lemma 1.2.8** *Let $\varrho$ be a semimodular on X that satisfies the $\Delta_2$-condition with constant K. Then $\varrho$ is a continuous modular and for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon,K) > 0$ such that $\varrho(f) \leq 1-\varepsilon$ implies $\|f\|_\varrho \leq 1-\delta$ for $f \in X_\varrho$.*

**Theorem 1.2.4** *Let $\varphi \in N(A,\mu)$ be uniformly convex. Then $\varrho_\varphi$ is uniformly convex.*

**Proof.** *Let $\varepsilon_2,\ \delta_2 > 0$ be as in Lemma 1.1.1 and let $\varepsilon := 2\varepsilon_2$. There is nothing to show if $\varrho_\varphi(f) = \infty$ or $\varrho_\varphi(g) = \infty$. So in the following let $\varrho_\varphi(f),\ \varrho_\varphi(g) < \infty$, which implies by convexity $\varrho\left(\frac{f+g}{2}\right),\ \varrho\left(\frac{f-g}{2}\right) < \infty$.*
*Assume that $\varrho_\varphi\left(\frac{f-g}{2}\right) > \varepsilon\frac{\varrho_\varphi(f)+\varrho_\varphi(g)}{2}$. We show that*

$$\varrho_\varphi\left(\frac{f+g}{2}\right) \leq \left(1-\frac{\delta_2\varepsilon}{2}\right)\frac{\varrho_\varphi(f)+\varrho_\varphi(g)}{2},$$

*which proves that $\varrho_\varphi$ is uniformly convex. Define*

$$E := \left\{y \in A\ :\ |f(y)-g(y)| > \frac{\varepsilon}{2}\max\{|f(y)|,|g(y)|\}\right\}.$$

*It follows from Remark 1.1.2 that (1.1) holds for almost all $y \in A\setminus E$. In particular,*

$$\varrho_\varphi\left(\chi_{A\setminus E}\frac{f-g}{2}\right) \le \frac{\varepsilon}{2}\frac{\varrho_\varphi(\chi_{A\setminus E}f)+\varrho_\varphi(\chi_{A\setminus E}g)}{2} \le \frac{\varepsilon}{2}\frac{\varrho_\varphi(f)+\varrho_\varphi(g)}{2}$$

*this and $\varrho_\varphi\left(\frac{f-g}{2}\right) > \varepsilon\frac{\varrho_\varphi(f)+\varrho_\varphi(g)}{2}$ imply*

$$\varrho_\varphi\left(\chi_E\frac{f-g}{2}\right) = \varrho_\varphi\left(\frac{f-g}{2}\right) - \varrho_\varphi\left(\chi_{A\setminus E}\frac{f-g}{2}\right) \ge \frac{\varepsilon}{2}\frac{\varrho_\varphi(f)+\varrho_\varphi(g)}{2}. \tag{1.10}$$

*on the other hand it follows by the definition of $E$ and the choice of $\delta_2$ in Lemma 1.1.1 that*

$$\varrho_\varphi\left(\chi_E\frac{f+g}{2}\right) \le (1-\delta_2)\frac{\varrho_\varphi(\chi_E f)+\varrho_\varphi(\chi_E g)}{2} \tag{1.11}$$

*We estimate*

$$\frac{\varrho_\varphi(f)+\varrho_\varphi(g)}{2} - \varrho_\varphi\left(\frac{f+g}{2}\right) \ge \frac{\varrho_\varphi(\chi_E f)+\varrho_\varphi(\chi_E g)}{2} - \varrho_\varphi\left(\chi_E\frac{f+g}{2}\right),$$

*where we have split the domain of the involved integrals into the sets $E$ and $A\setminus E$ and have used $\frac{1}{2}(\varphi(f)+\varphi(g)) - \varphi\left(\frac{f+g}{2}\right) \ge 0$ on $A\setminus E$. This, (1.11), the convexity and (1.10) imply*

$$\begin{aligned}\frac{\varrho_\varphi(f)-\varrho_\varphi(g)}{2} - \varrho_\varphi\left(\frac{f+g}{2}\right) &\ge \delta_2\frac{\varrho_\varphi(\chi_E f)+\varrho_\varphi(\chi_E g)}{2}\\ &\ge \delta_2\varrho_\varphi\left(\chi_E\frac{f-g}{2}\right)\\ &\ge \frac{\delta_2\varepsilon}{2}\frac{\varrho_\varphi(f)+\varrho_\varphi(g)}{2}.\end{aligned}$$

**Lemma 1.2.9** *If $\varrho_1, \varrho_2$ are uniformly convex semimodulars on $X$, then $\varrho := \varrho_1 + \varrho_2$ is uniformly convex.*

**Proof.** *If $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$\varrho_j\left(\frac{f-g}{2}\right) \le \varepsilon\frac{\varrho_j(f)+\varrho_j(g)}{2} \;\text{ or }\; \varrho_j\left(\frac{f+g}{2}\right) \le (1-\delta)\frac{\varrho_j(f)+\varrho_j(g)}{2}$$

*for $j = 1, 2$. We show that*

$$\varrho\left(\frac{f-g}{2}\right) \le 2\varepsilon\frac{\varrho(f)+\varrho(g)}{2} \;\text{ or }\; \varrho\left(\frac{f+g}{2}\right) \le (1-\delta\varepsilon)\frac{\varrho(f)+\varrho(y)}{2},$$

*since this proves the uniform convexity of $\varrho$. Fix $f$ and $g$ and assume that $\varrho\left(\frac{f-g}{2}\right) > 2\varepsilon\frac{\varrho(f)+\varrho(g)}{2}$. Without loss of generality, we can assume that $\varrho_1\left(\frac{f-g}{2}\right) \ge \varrho_2\left(\frac{f-g}{2}\right)$ for this specific choice of $f$ and $g$. Therefore, $\varrho_1\left(\frac{f-g}{2}\right) > \varepsilon\frac{\varrho(f)+\varrho(g)}{2} \ge \varepsilon\frac{\varrho_1(f)+\varrho_2(g)}{2}$. So the choice of $\delta$ implies*

$$\varrho_1\left(\frac{f+g}{2}\right) \le (1-\delta)\frac{\varrho_1(f)+\varrho_1(g)}{2}.$$

*Taking into account the convexity of $\varrho_2$, we obtain*

$$\varrho\left(\frac{f+g}{2}\right) \leq \frac{\varrho(f)+\varrho(g)}{2} - \delta\frac{\varrho_1(f)+\varrho_1(g)}{2}.$$

*Since* $\frac{\varrho_1(f)+\varrho_1(g)}{2} \geq \varrho_1\left(\frac{f-g}{2}\right) > \varepsilon\frac{\varrho(f)+\varrho(g)}{2}$, *this implies*

$$\varrho\left(\frac{f+g}{2}\right) \leq (1-\delta\varepsilon)\frac{\varrho(f)+\varrho(g)}{2}.$$

**Theorem 1.2.5** *Let $\varrho$ be a uniformly convex semimodular on $X$ that satisfies the $\Delta_2$-condition. Then the norm $\|.\|_\varrho$ on $X_\varrho$ is uniformly convex. Hence, $X_\varrho$ is uniformly convex.*

**Proof.** *Fix $\varepsilon > 0$. Let $x, y \in X$ with $\|x\|_\varrho,\ \|y\|_\varrho \leq 1$ and $\|x-y\|_\varrho > \varepsilon$. Then $\left\|\frac{x-y}{2}\right\| > \frac{\varepsilon}{2}$ and by Lemma 1.2.7 there exists $\alpha = \alpha(\varepsilon) > 0$ such that $\varrho\left(\frac{x-y}{2}\right) > \alpha$. By the unit ball property we have $\varrho(x),\ \varrho(y) \leq 1$, so $\varrho\left(\frac{x-y}{2}\right) > \alpha^{\frac{\varrho(x)+\varrho(y)}{2}}$. Since $\varrho$ is uniformly convex, there exists $\beta = \beta(\alpha) > 0$ such that $\varrho\left(\frac{x+y}{2}\right) \leq (1-\beta)\frac{\varrho(x)+\varrho(y)}{2} \leq 1-\beta$. Now Lemma 1.2.8 implies the existence of $\delta = \delta(K,\beta) > 0$ with $\left\|\frac{x+y}{2}\right\|_\varrho \leq 1-\delta$. This proves the uniform convexity of $\|.\|_\varrho$.*

**Theorem 1.2.6** *Let $p \in \mathcal{P}(\Omega)$ with $1 < p^- \leq p^+ < \infty$. Then $\varphi_{p(.)}$ is a uniformly convex N-function, $\varrho_{p(.)}$ is a uniformly convex semimodular and $\|.\|_{p(.)}$ is a uniformly convex norm. Hence, $L^{p(.)}(\Omega)$ is uniformly convex.*

**Proof.** *Note that $\varphi_{p(.)}$ satisfies the $\Delta_2$-condition since $p^+ < \infty$. In order to apply Theorems 1.2.4 and 1.2.5, we have to show that $\varphi_{p(.)}$ is uniformly convex. In principle we have to show this for both $\bar{\varphi}_{p(.)}$ and $\widetilde{\varphi}_{p(.)}$, since the equivalence of norms does not transfer the uniform convexity. However, since $\bar{\varphi}_{p(y)}$ and $\widetilde{\varphi}_{p(y)}$ only differ for every $y \in \Omega$ by the multiplicative constant $\frac{1}{p(y)}$, the uniform convexity of $\bar{\varphi}_{p(.)}$ is equivalent to the uniform convexity of $\widetilde{\varphi}_{p(.)}$. Thus it suffices to consider the case $\bar{\varphi}_{p(.)}$.*

*Fix $\varepsilon > 0$. Let $u, v \geq 0$ be such that $|u-v| > \varepsilon\max\{u,v\}$. It follows from Remark 1.1.1 that the mapping $t \mapsto t^{p^-}$ is uniformly convex, since $p^- > 1$. Thus there exists $\delta = \delta(\varepsilon, p^-) > 0$ such that*

$$\left(\frac{u+v}{2}\right)^{p^-} \leq (1-\delta)\frac{u^{p^-}+v^{p^-}}{2}.$$

*This and the convexity of $t \mapsto t^{\frac{p(y)}{p^-}}$ for $y \in \Omega$ imply*

$$\left(\frac{u+v}{2}\right)^{p(y)} \leq \left((1-\delta)\frac{u^{p^-}+v^{p^-}}{2}\right)^{\frac{p(y)}{p^-}} \leq (1-\delta)\frac{u^{p(y)}+v^{p(y)}}{2}.$$

*This proves that $\bar{\varphi}_{p(.)}$ is uniformly convex. The semimodular $\varrho_{p(.)}$ is uniformly convex by Theorem 1.2.4, and the norm $\|.\|_{p(.)}$ is uniformly convex by Theorem 1.2.5*

According to Theorem 1.2.2 when $\varphi = \widetilde{\varphi}_{p(.)} \in \Phi(A,\mu)$ or $\varphi = \bar{\varphi}_{p(.)} \in \Phi(A,\mu)$, $L^{p(x)}$ is a Banach space. And if $1 < p^- \leq p^+ < \infty$, $L^{p(x)}$ is uniformly convex (Theorem 1.2.6), and according [4, Theorem III.29] every Banach space uniformly convex is reflexive. This results implies this theorem :

**Theorem 1.2.7** *$L^{p(x)}(\Omega)$ space is a Banach space. Moreover, if $1 < p^- \leq p^+ < \infty$ is satisfied, then $L^{p(x)}(\Omega)$ space is reflexive.*

**Lemma 1.2.10 (Norm-modular unit ball property)** *If $p \in \mathcal{P}(\Omega)$, then $\|f\|_{p(.)} \leq 1$ and $\varrho_{p(.)}(f) \leq 1$ are equivalent. For $f \in L^{p(.)}(\Omega)$ we have*

*(a) If $\|f\|_{p(.)} \leq 1$, then $\varrho_{p(.)}(f) \leq \|f\|_{p(.)}$.*

*(b) If $1 < \|f\|_{p(.)}$, then $\|f\|_{p(.)} \leq \varrho_{p(.)}(f)$.*

**Proof.** *(a) if $\|f\|_{p(.)} \leq 1 \Rightarrow \frac{1}{\|f\|_{p(.)}} > 1$, due to $\varrho_{p(.)}$ is a semimodular in $L^{p(.)}$ and $L^{p(x)}$ is a vector space then by (1.5)* $1 \geq \varrho_{p(.)}\left(\frac{f}{\|f\|_{p(.)}}\right) \geq \frac{1}{\|f\|_{p(.)}}\varrho_{p(.)}(f) \Rightarrow 1 \geq \frac{1}{\|f\|_{p(.)}}\varrho_{p(.)}(f) \Rightarrow \|f\|_{p(.)} \geq \varrho_{p(.)}(f)$.
*(b)* $\|f\|_{p(.)} < 1$, $\|f\|_{p(.)} = \inf_{\lambda>0}\left\{\lambda > 0,\ \varrho\left(\frac{f}{\lambda}\right) \leq 1\right\}$, $\exists\lambda > 1$, $\|f\|_{p(.)} = \lambda, \lambda > 1 \Rightarrow \frac{1}{\lambda} < 1$,
$\varrho_{p(.)}\left(\frac{f}{\lambda}\right) \leq \frac{1}{\lambda}\varrho_{p(.)}(f) \Rightarrow 1 + \varrho_{p(.)}\left(\frac{f}{\lambda}\right) \leq \frac{1}{\lambda}\varrho_{p(.)}(f) + 1 \Rightarrow \lambda\left(1 + \varrho_{p(.)}\left(\frac{f}{\lambda}\right)\right) \leq \frac{1}{\lambda}\varrho_{p(.)}(f) + \lambda \Rightarrow \|f\|^A_{p(.)} \leq \varrho_{p(.)}(f)$.
*We've got :*
$\|f\|_{p(.)} \leq \|f\|^A_{p(.)} \leq 2\|f\|_{p(.)}$

$\|f\|_{p(.)} \leq \|f\|^A_{p(.)} \leq \varrho_{p(.)}(f)$

$\|f\|_{p(.)} \leq \varrho_{p(.)}(f)$
*If $p \in \mathcal{P}(\Omega)$ then $\|f\| \leq 1 \Leftrightarrow \varrho_{p(.)}(f) \leq 1$.*
$\Rightarrow) - (a) \Rightarrow \varrho_{p(.)}(f) \leq \|f\|_{p(.)} \leq 1 \Rightarrow \varrho_{p(.)}(f) \leq 1$
$\Leftarrow)-$ *Suppose $\|f\|_{p(.)} > 1$*

$$(b) \Rightarrow \|f\|_{p(.)} \leq \varrho_{p(.)}(f) \leq 1$$
$$\Rightarrow \|f\|_{p(.)} \leq 1, \quad \textit{contradiction with } \|f\|_{p(.)} > 1. \ \textit{so } \|f\|_{p(.)} \leq 1$$
$$\varrho_{p(.)}(f) \leq 1 \Rightarrow \|f\|_{p(.)} \leq 1.$$

**Theorem 1.2.8** *Let $p \in \mathcal{P}(A,\mu)$. Then the following conditions are equivalent :*
*(a) $E^{p(.)}(A,\mu) = L^{\varphi_{p(.)}}_{OC}(A,\mu)$.*
*(b) $L^{\varphi_{p(.)}}_{OC}(A,\mu) = L^{p(.)}(A,\mu)$.*

*(c)* $E^{p(.)}(A,\mu) = L^{p(.)}(A,\mu)$.

*(d)* $\varphi_{p(.)}$ *satisfies the* $\Delta_2$*-condition with constant* $2^{p^+}$.

*(e)* $p^+ < \infty$.

*(f)* $\varrho_{p(.)}$ *satisfies the weak* $\Delta_2$*-condition for modulars, i.e modular convergence and norm convergence are the same.*

*(g)* $\varrho_{p(.)}$ *is a continuous modular.*

**Lemma 1.2.11** *Let* $p \in \mathcal{P}(A,\mu)$ *be a bounded exponent and let* $\mu$ *be separable. Then* $L^{p(.)}(A,\mu)$ *is separable.*

**Proof.** *Since* $\varphi_{p(.)}$ *is locally integrable by Remark 1.2.1, we can apply Theorem 1.2.3 to show that* $E^{p(.)}$ *is separable. Since* $p^+ < \infty$*, we further have* $E^{p(.)} = L^{p(.)}$ *(Theorem 1.2.8).*

**Lemma 1.2.12** *Let* $p \in \mathcal{P}(\Omega)$ *with* $p^- < \infty$*. If* $\bar{\varrho}_{p(.)}(f) > 0$ *or* $p^+ < \infty$*, then*

$$\min\left\{\bar{\varrho}_{p(.)}(f)^{\frac{1}{p^-}}, \bar{\varrho}_{p(.)}(f)^{\frac{1}{p^+}}\right\} \leq \|f\|_{\bar{\varphi}_{p(.)}} \leq \max\left\{\bar{\varrho}_{p(.)}(f)^{\frac{1}{p^-}}, \bar{\varrho}_{p(.)}(f)^{\frac{1}{p^+}}\right\}.$$

**Proof.** *Suppose that* $p^+ < \infty$*. If* $\bar{\varrho}_{p(.)}(f) \leq 1$*, then we need to prove that*

$$\bar{\varrho}_{p(.)}(f)^{\frac{1}{p^-}} \leq \|f\|_{p(.)} \leq \bar{\varrho}_{p(.)}(f)^{\frac{1}{p^+}}.$$

*By homogeneity, the latter inequality is equivalent to* $\|f/\bar{\varrho}_{p(.)}(f)^{\frac{1}{p^+}}\|_{p(.)} \leq 1$*, which by the unit ball property is equivalent to*

$$\int_\Omega \left(\frac{|f(x)|}{\bar{\varrho}_{p(.)}(f)^{\frac{1}{p^+}}}\right)^{p(x)} dx \leq 1.$$

*But since* $\bar{\varrho}_{p(.)}(f)^{-\frac{p(x)}{p^+}} \leq \bar{\varrho}_{p(.)}(f)^{-1}$*, this is clear. The other inequality and the case* $\bar{\varrho}_{p(.)}(f) \geq 1$ *are similar.*

*Consider now* $p^+ = \infty$ *and* $\bar{\varrho}_{p(.)}(f) > 0$*. In this case the upper inequality becomes* $\|f\|_{p(.)} \leq \max\left\{\bar{\varrho}_{p(.)}(f)^{1/p^-}, 1\right\}$*. If* $\bar{\varrho}_{p(.)}(f) \leq 1$*, then* $\|f\|_{p(.)} \leq 1$*, so the inequality holds. If* $\bar{\varrho}_{p(.)}(f) > 1$*, then we need to show that*

$$\int_\Omega \left(\frac{|f(x)|}{\bar{\varrho}_{p(.)}(f)^{1/p^-}}\right)^{p(x)} dx \leq 1.$$

*Since* $\bar{\varrho}_{p(.)}(f)^{-1} < 1$*, we conclude that*

$$\bar{\varrho}_{p(.)}(f)^{\frac{-p(x)}{p^-}} \leq \begin{cases} 0, & \text{if } p(x) = \infty, \\ \bar{\varrho}_{p(.)}(f)^{-1}, & \text{if } p(x) < \infty. \end{cases}$$

*Hence*

$$\int_\Omega \left(\frac{|f(x)|}{\bar{\varrho}_{p(.)}(f)^{1/p^-}}\right)^{p(x)} dx \leq \int_\Omega \frac{|f(x)|^{p(x)}}{\varrho_{p(.)}(f)}\, dx = 1.$$

*The proof of the lower inequality is analogous.*

**Theorem 1.2.9** *If $1 \leq p(.) < \infty$ and $\frac{1}{p^-} \leq s < \infty$, then*

$$\||u|^s\|_{p(.)} = \|u\|^s_{sp(.)}$$

**Proof.** *In the definition of norm*

$$\||u|^s\|_{p(.)} = \inf\left\{\lambda > 0 \ : \ \int_\Omega \left|\frac{|u(x)|^s}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

*if we substitute $\lambda = \mu^s$, it follows that*

$$\begin{aligned}
\||u|^s\|_{p(.)} &= \inf\left\{\mu^s > 0 \ : \ \int_\Omega \left|\frac{|u(x)|^s}{\mu^s}\right|^{p(x)} dx \leq 1\right\} \\
&= \inf\left\{\mu^s > 0 \ : \ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{sp(x)} dx \leq 1\right\} \\
&= \left[\inf\left\{\mu > 0 \ : \ \int_\Omega \left|\frac{u(x)}{\mu}\right|^{sp(x)} dx \leq 1\right\}\right]^s \\
&= \left[\inf\left\{\mu > 0 \ : \ \varrho_{sp(.)}\left(\frac{u}{\mu}\right) \leq 1\right\}\right]^s \\
&= \|u\|^s_{sp(.)}
\end{aligned}$$

*which is the desired result.*

**Lemma 1.2.13** *Let $p \in \mathcal{P}(A,\mu)$. Then the set of simple functions $S(A,\mu)$ is contained in $L^{p(.)}(A,\mu)$ and*

$$\min\{1,\mu(E)\} \leq \|\chi_E\|_{\bar{\varphi}_{p(.)}} \leq \max\{1,\mu(E)\},$$

*for every measurable set $E \subset A$.*

**Proof.** *Let $E \subset A$ be measurable with $\mu(E) < \infty$. Then*

$$\begin{aligned}
\bar{\varrho}_{p(.)}\left(\frac{\chi_E}{\max\{1,\mu(E)\}}\right) &= \int_E \frac{1}{(\max\{1,\mu(E)\})^{p(x)}}\, dx \\
&\leq \int_E \frac{1}{\max\{1,\mu(E)\}}\, dx \leq 1.
\end{aligned}$$

*Hence, by the unit ball property $\|\chi_E\|_{p(.)} \leq \max\{1,\mu(E)\}$. Since simple functions are finite linear combinations of characteristic functions, we get $S(A,\mu) \subset L^{p(.)}(A,\mu)$. Now, let $\lambda > 1$, then*

$$\begin{aligned}
\bar{\varrho}_{p(.)}\left(\frac{\lambda\chi_E}{\min\{1,\mu(E)\}}\right) &= \int_E \frac{\lambda^{p(x)}}{(\min\{1,\mu(E)\})^{p(x)}}\, dx \\
&\geq \int_E \frac{\lambda}{\min\{1,\mu(E)\}}\, dx \geq \lambda > 1.
\end{aligned}$$

*Hence,* $\|\lambda\chi_E\|_{\bar{\varphi}_{p(.)}} > \min\{1, \mu(E)\}$, *for every* $\lambda > 1$ *(by the unit ball property), which proves* $\|\chi_E\|_{\bar{\varphi}_{p(.)}} \geq \min\{1, \mu(E)\}$.

The following lemma is especially useful if $\frac{1}{s^-} - \frac{1}{s^+}$ is small.

**Lemma 1.2.14** *Let* $s \in \mathcal{P}(A, \mu)$. *Then*

$$\frac{1}{2} \min\left\{\mu(A)^{\frac{1}{s^+}}, \mu(A)^{\frac{1}{s^-}}\right\} \leq \|1\|_{L^{s(.)}(A,\mu)} \leq 2 \max\left\{\mu(A)^{\frac{1}{s^+}}, \mu(A)^{\frac{1}{s^-}}\right\}$$

*for every measurable set A with* $\mu(A) > 0$. *If* $\varphi_p = \bar{\varphi}_p$, *then we can omit the factors* $\frac{1}{2}$ *and* 2.

**Proof.** *The case* $\varphi_{p(.)} = \bar{\varphi}_{p(.)}$ *follows from Lemma* 1.2.12. *The case* $\varphi_{p(.)} = \widetilde{\varphi}_{p(.)}$ *then follows by (*1.9*).*

As we see in the *Definition* 1.2.11 they define the variable exponent Lebesgue space $L^{p(.)}(A, \mu)$ as the Musielak-Orlicz space $L^{\varphi_{p(.)}}(A, \mu)$, So the properties of the continuity and lower semicontinuity in $L^{p(.)}(A, \mu)$ is a results from the properties in the space $L^{\varphi_{p(.)}}(A, \mu)$. Recall that $E^{p(.)}(A, \mu)$ denotes the set of finite elements of $L^{p(.)}(A, \mu)$, see [7, *Definition* 2.3.11]

**Lemma 1.2.15** *Every* $\Phi$*-function is lower semicontinuous.*

From this Lemma we deduce lemma 1.2.17:

**Lemma 1.2.16** *Let* $\varphi \in \Phi(A, \mu)$ *and* $f_k, f, g \in L^0(A, \mu)$.
*(a) If* $f_k \to f$ $\mu$*-almost everywhere, then* $\varrho_\varphi(f) \leq \liminf_{k\to\infty} \varrho_\varphi(f_k)$.
*(b) If* $|f_k| \nearrow |f|$ $\mu$*-almost everywhere, then* $\varrho_\varphi(f) = \lim_{k\to\infty} \varrho_\varphi(f_k)$.
*(c) If* $f_k \to f$ $\mu$*-almost everywhere and* $|f_k| \leq |g|$ $\mu$*-almost everywhere, and* $\varrho_\varphi(\lambda g) < \infty$ *for every* $\lambda > 0$, *then* $f_k \to f$ *in* $L^\varphi$.

**Proof.** *By Lemma 1.2.15 the mappings* $\varphi(y, .)$ *are lower semicontinuous. Thus Fatou's lemma implies*

$$\begin{aligned}
\varrho_\varphi(f) &= \int_A \varphi(y, \lim_{k\to\infty} |f_k(y)|)\, d\mu(y) \\
&\leq \int_A \liminf_{k\to\infty} \varphi(y, |f_k(y)|)\, d\mu(y) \\
&\leq \liminf_{k\to\infty} \int_A \varphi(y, |f_k(y)|)\, d\mu(y) \\
&= \liminf_{k\to\infty} \varrho_\varphi(f_k).
\end{aligned}$$

*This proves (a).*

*To prove (b) let* $|f_k| \nearrow |f|$. *Then by the left-continuity and monotonicity of* $\varphi(y,.)$, *we have* $0 \le \varphi(.,|f_k(.)|) \nearrow \varphi(.,|f(.)|)$ *almost everywhere. So, the theorem of monotone convergence gives*

$$\begin{aligned}\varrho_\varphi(f) &= \int_A \varphi(y, \lim_{k\to\infty} |f_k(y)|)\, d\mu(y) \\ &= \int_A \lim_{k\to\infty} \varphi(y, |f_k(y)|)\, d\mu(y) \\ &= \lim_{k\to\infty} \int_A \varphi(y, |f_k(y)|)\, d\mu(y) \\ &= \lim_{k\to\infty} \varrho_\varphi(f_k).\end{aligned}$$

*To prove (c) assume that* $f_k \to f$ *almost everywhere,* $|f_k| \le |g|$, *and* $\varrho(\lambda g) < \infty$ *for every* $\lambda > 0$. *Then* $|f_k - f| \to 0$ *almost everywhere,* $|f| \le |g|$ *and* $|f_k - f| \le 2|g|$. *Since* $\varrho_\varphi(2\lambda g) < \infty$, *we can use the theorem of dominated convergence to conclude that*

$$\lim_{k\to\infty} \varrho_\varphi(\lambda|f - f_k|) = \int_A \varphi\left(y, \lim_{k\to\infty} \lambda|f(y) - f_k(y)|\right) d\mu(y) = 0.$$

*Since* $\lambda > 0$ *was arbitrary. Lemma 1.2.3 implies that* $f_k \to f$ *in* $L^\varphi$.

**Lemma 1.2.17** *Let* $p \in \mathcal{P}(A,\mu)$ *and* $f_k, f, g \in L^0(A,\mu)$.

*(a) If* $f_k \to f$ $\mu$*-almost everywhere, then* $\varrho_{p(.)}(f) \le \liminf_{k\to\infty} \varrho_{p(.)}(f_k)$.

*(b) If* $|f| \nearrow |f|$ $\mu$*-almost everywhere, then* $\varrho_{p(.)}(f) = \lim_{k\to\infty} \varrho_{p(.)}(f_k)$.

*(c) If* $f_k \to f$ $\mu$*-almost everywhere,* $|f_k| \le |g|$ $\mu$*-almost everywhere and* $g \in E^{p(.)}$, *then* $f_k \to f$ *in* $L^{p(.)}$.

In analogy with the properties for the integral, the claims of the previous lemma will be called Fatou's lemma (for the modular), monotone convergence and dominated convergence, respectively.

**Theorem 1.2.10** *Let* $\varrho$ *be a semimodular on X. Then* $\varrho^*$ *is a semimodular on* $X^*_\varrho$.

**Proof.** *It is easily seen that* $\varrho^*(0) = 0$, $\varrho^*(\lambda x^*) = \varrho^*(x^*)$ *for* $|\lambda| = 1$, *and* $\varrho^*(x^*) \ge 0$ *for every* $x^* \in X^*_\varrho$. *Let* $x^*_0,\ x^*_1 \in X^*_\varrho$ *and* $\theta \in (0,1)$. *Then*

$$\begin{aligned}\varrho^*((1-\theta)x^*_0 + \theta x^*_1) &= \sup_{x\in X}(|\langle (1-\theta)x^*_0 + \theta x^*_1, x\rangle| - \varrho(x)) \\ &\le (1-\theta)\sup_{x\in X}(|\langle x^*_0, x\rangle| - \varrho(x)) \\ &\quad +\theta \sup_{x\in X}(|\langle x^*_1, x\rangle| - \varrho(x)) \\ &= (1-\theta)\varrho^*(x^*_0) + \theta\varrho^*(x^*_1).\end{aligned}$$

*Finally, let $\varrho^*(\lambda x^*) = 0$ for every $\lambda > 0$. For $x \in X_\varrho$ choose $\eta > 0$ such that $\varrho(\eta x) < \infty$. Then by (1.6)*

$$\lambda\eta|\langle x^*, x\rangle| \le \varrho(\eta x) + \varrho^*(\lambda x^*) = \varrho(\eta x).$$

*Taking $\lambda \to \infty$ we obtain $|\langle x^*, x\rangle| = 0$. Hence $x^* = 0$. It remains to show that $\varrho^*$ is left-continuous. For $\lambda \to 1^-$ and $x^* \in X_\varrho^*$ we have*

$$\begin{aligned}\lim_{\lambda\to 1^-} \varrho^*(\lambda x^*) &= \lim_{\lambda\to 1^-} \sup_{x\in X}(|\langle \lambda x^*, x\rangle| - \varrho(x)) \\ &= \sup_{0<\lambda<1} \sup_{x\in X}(|\lambda||\langle x^*, x\rangle| - \varrho(x)) \\ &= \sup_{x\in X}(|\langle x^*, x\rangle| - \varrho(x)) = \varrho^*(x).\end{aligned}$$

*Thus $\varrho^*$ is left-continuous.*

**Theorem 1.2.11** *Let $\varrho$ be a semimodular on $X$. Then $\varrho^{**} = \varrho$ on $X_\varrho$.*

**Proof.** *see [7].*

**Theorem 1.2.12** *Let $\varrho$ be a semimodular on $X$, then $\varrho$ is lower semicontinuous on $X_\varrho$, i.e.*

$$\varrho(x) \le \liminf_{k\to\infty} \varrho(x_k)$$

*for all $x_k, x \in X_\varrho$ with $x_k \to x$ (in norm) for $k \to \infty$.*

**Theorem 1.2.13** *Let $\varrho$ be a semimodular on $X$, then the semimodular $\varrho$ is weakly (sequentially) lower semicontinuous, i.e. if $f_k \to f$ weakly in $X_\varrho$, then $\varrho(f) \le \liminf_{k\to\infty} \varrho(f_k)$.*

**Proof.** *Let $f_k,\ f \in X_\varrho$ with $f_k \to f$. Then, by Theorem 1.2.11, $\varrho = \varrho^{**}$, which*

$$\begin{aligned}\varrho(f) = \varrho^{**}(f) &= \sup_{g^*\in X_\varrho^*} (|\langle g^*, f\rangle| - \varrho^*(g^*)) \\ &= \sup_{g^*\in X_\varrho^*} \left(\lim_{k\to\infty} |\langle g^*, f_k\rangle| - \varrho^*(g^*)\right) \\ &\le \liminf_{k\to\infty} \Big( \sup_{g^*\in X_\varrho^*} (|\langle g^*, f_k\rangle| - \varrho^*(g^*)) \Big) \\ &= \liminf_{k\to\infty} \varrho^{**}(f_k) \\ &= \liminf_{k\to\infty} \varrho(f_k).\end{aligned}$$

From theorem 1.2.13 we obtain:

**Theorem 1.2.14** *If $p \in \mathcal{P}(A,\mu)$, then the modular is weakly (sequentially) lower semicontinuous, i.e. $\varrho_{p(.)}(f) \leq \liminf_{k\to\infty} \varrho_{p(.)}(f_k)$ if $f_k \to f$ weakly in $L^{p(.)}(A,\mu)$.*

Since strong convergence implies weak convergence, the conclusion of the previous theorem holds also if $f_k \to f$ in $L^{p(.)}(A,\mu)$.

**Theorem 1.2.15** *Let $\varphi \in \Phi(A,\mu)$. Then the following hold.*
*(a) $\|f\|_\varphi = \| \, |f| \, \|_\varphi$ for all $f \in L^\varphi$.*
*(b) If $f \in L^\varphi$, $g \in L^0(A,\mu)$, and $0 \leq |g| \leq |f|$ $\mu$-almost everywhere, then $g \in L^\varphi$ and $\|g\|_\varphi \leq \|f\|_\varphi$.*
*(c) If $f_k \to f$ almost everywhere, then $\|f\|_\varphi \leq \liminf_{k\to\infty} \|f_k\|_\varphi$.*
*(d) I $|f_k| \nearrow |f|$ $\mu$-almost everywhere with $f_k \in L^\varphi(A,\mu)$ and $\sup_k \|f_k\|_\varphi < \infty$, then $f \in L^\varphi(A,\mu)$ and $\|f_k\|_\varphi \nearrow \|f\|_\varphi$.*

**Proof.** *The properties (a) and (b) are obvious. Let us now prove (c). So let $f_k \to f$ $\mu$-almost everywhere. There is nothing to prove for $\liminf_{k\to\infty} \|f_k\|_\varphi = \infty$. Let $\lambda > \liminf_{k\to\infty} \|f_k\|_\varphi$. Then $\|f_k\|_\varphi < \lambda$ for large k. Thus by the unit ball property $\varrho_\varphi(f_k/\lambda) \leq 1$ for large k. Now Fatou's lemma for the modular (lemma 1.2.16) implies $\varrho_\varphi(f/\lambda) \leq 1$. So $\|f\|_\varphi \leq \lambda$ again by the unit ball property which implies $\|f\|_\varphi \leq \liminf_{k\to\infty} \|f_k\|_\varphi$.*
*It remains to prove (d). So let $|f_k| \nearrow |f|$ $\mu$-almost everywhere with $\sup_k \|f_k\|_\varphi < \infty$. From (a) and (c) follows $\|f\|_\varphi \leq \liminf_{k\to\infty} \|f_k\|_\varphi \leq \sup_k \|f_k\|_\varphi < \infty$, which proves $f \in L^\varphi$. On the other hand $|f_k| \nearrow |f|$ and (b) implies that $\|f_k\|_\varphi \nearrow \limsup_{k\to\infty} \|f_k\|_\varphi \leq \|f\|_\varphi$. Thus $\lim_{k\to\infty} \|f_k\|_\varphi = \|f\|_\varphi$ and $\|f_k\|_\varphi \nearrow \|f\|_\varphi$.*

From Lemma 1.2.5 and lemma 1.2.6 we deduce:

**Lemma 1.2.18** *Let $p \in \mathcal{P}(A,\mu)$ and let $f_k \in L^{p(.)}(A,\mu)$.*
*(a) If $f_k$ is a Cauchy sequence, then there exists a subsequence of $f_k$ which converges $\mu$-almost everywhere to a measurable function f.*
*(b) If $\mu(A) < \infty$ and $\|f_k\|_{p(.)} \to 0$, then $f_k \to 0$ in measure.*

*Theorem* 1.2.15 implies that $L^{p(.)}(A,\mu)$ is circular, solid, satisfies Fatou's lemma (for the norm) and has the Fatou property, i.e.

- $\|f\|_{p(.)} = \| \, |f| \, \|_{p(.)}$ for all $f \in L^{p(.)}(A,\mu)$.
- if $f \in L^{p(.)}(A,\mu)$, $g \in L^0(A,\mu)$ and $0 \leq |g| \leq |f|$ $\mu$- almost everywhere, then $g \in L^{p(.)}(A,\mu)$ and $\|g\|_{p(.)} \leq \|f\|_{p(.)}$.

- if $f_k \to f$ $\mu$-almost everywhere, then $\|f\|_{p(.)} \leq \liminf_{k\to\infty} \|f_k\|_{p(.)}$.
- if $|f_k| \nearrow |f|$ $\mu$-almost everywhere with $f_k \in L^{p(.)}(A,\mu)$ and $\sup_k \|f_k\|_{p(.)} < \infty$. Then $f \in L^{p(.)}(A,\mu)$ and $\|f_k\|_{p(.)} \nearrow \|f\|_{p(.)}$, respectively.

**Definition 1.2.13** *Let $\varphi \in \Phi(A,\mu)$. Then*

$$(L^{\varphi}(A,\mu))' := \left\{ g \in L^0(A,\mu) \ : \ \|g\|_{(L^{\varphi}(A,\mu))'} < \infty \right\}$$

*with norm*

$$\|g\|_{(L^{\varphi}(A,\mu))'} := \sup_{f\in L^{\varphi} \,:\, \|f\|_{\varphi}\leq 1} \int_A |f||g|\, d\mu,$$

*will be called the associate space of $L^{\varphi}(A,\mu)$ or $(L^{\varphi})'$ for short.*

**Definition 1.2.14** *A normed space $(Y, \|.\|_Y)$ with $Y \subset L^0(A,\mu)$ is called a Banach function space, if*

*(a) $(Y, \|.\|_Y)$ is circular, solid and satisfies the Fatou property.*

*(b) If $\mu(E) < \infty$, then $\chi_E \in Y$.*

*(c) If $\mu(E) < \infty$, then $\chi_E \in Y'$, i.e. $\int_E |f|\, d\mu \leq c(E)\|f\|_Y$ for all $f \in Y$.*

*From Theorem* 1.2.15 *we know that $L^{\varphi}$ satisfies (a) for every $\varphi \in \Phi(A,\mu)$ so one need only check (b) and (c). These properties are equivalent to $S \subset L^{\varphi}$ and $S \subset (L^{\varphi})'$, where $S$ is the set of simple functions. These inclusions may or may not hold, depending on the function $\varphi$.*

**Lemma 1.2.19** *Let $\varphi \in \Phi(A,\mu)$. Then*

$$\|g\|_{(L^{\varphi})'} = \sup_{f\in S\cap L^{\varphi} \,:\, \|f\|_{\varphi}\leq 1} \int_A |f||g|\, d\mu$$

*for all $g \in (L^{\varphi}(A,\mu))'$.*

**Theorem 1.2.16** *If $\varrho$ be a semimodular on $X$, then for every $x^* \in X^*_{\varrho}$*

$$\|x^*\|_{\varrho^*} \leq \|x^*\|_{X^*_{\varrho}} \leq 2\|x^*\|_{\varrho^*}.$$

**Theorem 1.2.17** *Let $\varphi \in \Phi(A,\mu)$ be such that $S(A,\mu) \subset L^{\varphi}(A,\mu)$. Then $L^{\varphi^*}(A,\mu) = (L^{\varphi}(A,\mu))'$, $\varrho_{\varphi^*}(g) = (\varrho_{\varphi})^*(J_g)$ and*

$$\|g\|_{\varphi^*} \leq \|g\|_{(L^{\varphi})'} = \|J_g\|_{(L^{\varphi})^*} \leq 2\|g\|_{\varphi^*}$$

*for every $g \in L^{\varphi^*}(A,\mu)$, where $J_g : f \mapsto \int_A fg\, d\mu$. (or complex- valued functions, the constant 2 should be replaced by 4.)*

**Proof.** *For the sake of simplicity we assume $\mathbb{K} = \mathbb{R}$. In the case $\mathbb{K} = \mathbb{C}$ we can proceed analogously by splitting $g$ into its real and imaginary part.*

*We already know that $L^{\varphi^*} \subset (L^\varphi)'$, $\|g\|_{(L^\varphi)'} = \|J_g\|_{(L^\varphi)^*} \leq 2\|g\|_{\varphi^*}$, and $(\varrho_\varphi)^*(J_g) \leq \varrho_{\varphi^*}(g)$ for every $g \in L^{\varphi^*}$. Fix $g \in (L^\varphi)'$. We claim that $g \in L^{\varphi^*}$ and $\varrho_{\varphi^*}(g) \leq (\varrho_\varphi)^*(J_g)$.*

*Since $\mu$ is $\sigma$-finite, we find measurable sets $A_k \subset A$ with $\mu(A_k) < \infty$ and $A_1 \subset A_2 \subset \ldots$ such that $A = \bigcup_{k=1}^\infty A_k$. Let $\{q_1, q_2, \ldots\}$ be a countable, dense subset of $[0, \infty)$ with $q_j \neq q_k$ for $j \neq k$ and $q_1 = 0$. For $k \in \mathbb{N}$ and $y \in A$ define*

$$r_k(y) := \chi_{A_k}(y) \max_{j=1,\ldots,k} \left( q_j |g(y)| - \varphi(y, q_j) \right).$$

*The special choice $q_1 = 0$ implies $r_y(y) \geq 0$ for all $y \geq 0$. Since $\{q_1, q_2, \ldots\}$ is dense in $[0, \infty)$ and $\varphi(y, .)$ is left-continuous, $r_k(y) \nearrow \varphi^*(y, |g(y)|)$ for any $y \in A$ as $k \to \infty$. For every $k \in \mathbb{N}$ there exists a simple function $f_k$ with $f_k(A) \subset \{q_1, \ldots, q_k\}$ and $f_k(y) = 0$ for all $y \in A \backslash A_k$ such that*

$$r_k(y) = f_k(y)|g(y)| - \varphi(y, f_k(y))$$

*for all $y \in A$. As a simple function, $f_k$ belongs by assumption to $L^\varphi(A, \mu)$. Define $h_k(y) := f_k(y) sgn(g(y))$ for $y \in A$, where $sgn(a)$ denotes the sign of $a$. Then also $h_k$ is a simple function (here we use $\mathbb{K} = \mathbb{R}$) and therefore*

$$(\varrho_\varphi)^*(J_g) \geq J_g(h_k) - \varrho_\varphi(h_k) = \int_A g(y)h_k(y) - \varphi(y, |h_k(y)|)\, d\mu(y).$$

*By the definition of $h_k$ it follows that*

$$(\varrho_\varphi)^*(J_g) \geq \int_A |g(y)| f_k(y) - \varphi(y, |f_k(y)|)\, d\mu(y) = \int_A r_k(y)\, d\mu(y).$$

*Since $r_k \geq 0$ and $r_k(y) \nearrow \varphi^*(y, |g(y)|)$, we get by the theorem of monotone convergence that*

$$(\varrho_\varphi)^*(J_g) \geq \limsup_{k\to\infty} \int_A r_k(y)\, d\mu(y) = \int_A \varphi^*(y, |g(y)|)\, d\mu(y) = \varrho_{\varphi^*}(g).$$

*Together with $(\varrho_\varphi)^*(J_g) \leq \varrho_{\varphi^*}(g)$ we g et $(\varrho_\varphi)^*(J_g) = \varrho_{\varphi^*}(g)$.*

*Since $g \mapsto J_g$ is linear, it follows that $(\varrho_\varphi)^*(\lambda J_g) = \varrho_{\varphi^*}(\lambda g)$ for every $\lambda > 0$ and therefore $\|g\|_{\varphi^*} = \|J_g\|_{(\varrho_\varphi)^*} \leq \|J_g\|_{(L^\varphi)^*} = \|g\|_{(L^\varphi)'}$ using in the second step Theorem 1.2.16.*

**Corollary 1.2.2 (Norm conjugate formula)** *Let $\varphi \in \Phi(A, \mu)$. If $S(A, \mu) \subset L^{\varphi^*}(A, \mu)$, then*

$$\|f\|_\varphi \leq \sup_{g \in L^{\varphi^*} \,:\, \|g\|_{\varphi^*} \leq 1} \int |f||g|\, d\mu \leq 2\|f\|_\varphi.$$

*for every $f \in L^0(A, \mu)$. the supremum is unchanged if we replace the condition $g \in L^{\varphi^*}$ by $g \in S(A, \mu)$.*

**Proof.** *Applying Theorem 1.2.17 to $\varphi^*$ and taking into account that $\varphi^{**} = \varphi$, we have*

$$\|f\|_\varphi \leq \|f\|_{(L^{\varphi^*})'} \leq 2\|f\|_\varphi$$

*for $f \in L^\varphi$. That the supremum does not change for $g \in S(A,\mu)$ follows by Lemma 1.2.19. The claim also follows in the case $f \in L^0 \backslash L^{\varphi^*} = L^0 \backslash (L^\varphi)'$, since both sides of the formula are infinite.*

**Definition 1.2.15** *A generalized $\Phi$-function $\varphi \in \Phi(A,\mu)$ is called proper if the set of simple functions $S(A,\mu)$ satisfies $S(A,\mu) \subset L^\varphi(A,\mu) \cap (L^\varphi(A,\mu))'$.*
*So $\varphi$ is proper if and only if $L^\varphi$ is a Banach function space. Moreover, if $\varphi$ is proper then the norm conjugate formula for $f$ $L^\varphi$ and $L^{\varphi^*}$ holds (Corollary 1.2.2) and $L^{\varphi^*} = (L^\varphi)'$.*

**Corollary 1.2.3** *Let $\varphi \in \Phi(A,\mu)$. Then the following are equivalent :*
*(a) $\varphi$ is proper.*
*(b) $\varphi^*$ is proper.*
*(c) $S(A,\mu) \subset L^\varphi(A,\mu) \cap L^{\varphi^*}(A,\mu)$.*

**Proof.** *If (a) or (c) holds, then $S \subset L^\varphi$. Hence $(L^\varphi)' = L^{\varphi^*}$ by theorem 1.2.17, which obviously implies the equivalence of (a) and (c).*
*Applying this equivalence for the function $\varphi^*$, and taking into account that $\varphi^{**} = \varphi$, yields the equivalence of (b) and (c).*

**Theorem 1.2.18** *Let $p \in \mathcal{P}(A,\mu)$. Then $\varphi_{p(.)}$ is proper and $L^{p(.)}(A,\mu)$ is a Banach function space. Its associate space satisfies $(L^{p(.)}(A,\mu))' = L^{p'(.)}(A,\mu)$ and*

$$\begin{aligned} \|g\|_{p'(.)} \leq \|g\|_{(L^{p(.)})'} &\leq 2\|g\|_{p'(.)} \quad \text{if } \varphi_{p(.)} = \widetilde{\varphi}_{p(.)}, \\ \frac{1}{2}\|g\|_{p'(.)} \leq \|g\|_{(L^{p(.)})'} &\leq 2\|g\|_{p'(.)} \quad \text{if } \varphi_{p(.)} = \bar{\varphi}_{p(.)} \end{aligned}$$

*for every $g \in L^0(A,\mu)$.*

**Proof.** *If follows from Lemma 1.2.13 that simple functions are contained in $L^{p(.)}$ and $L^{p'(.)}$. Thus $\varphi_{p(.)}$ is proper by Corollary 1.2.3 and therefore $L^{p(.)}$ is a Banach function space, see also [7, Section 2.7]. We can apply Theorem 1.2.17 to $\varphi_{p(.)}$ to get $(L^{p(.)}(A,\mu))' = L^{p'(.)}(A,\mu)$ and*

$$\|g\|_{(\varphi_{p(.)})^*} \leq \|g\|_{(L^{p(.)})'} \leq 2\|g\|_{(\varphi_{p(.)})^*},$$

*which is the first estimate of clain if $\varphi_{p(.)} = \widetilde{\varphi}_{p(.)}$, since $(\widetilde{\varphi}_{p(.)})^* = \widetilde{\varphi}_{p'(.)}$. From Lemma 1.1.2 we deduce*

$$\|g\|_{(\bar{\varphi}_{p(.)})^*} \leq \|g\|_{\bar{\varphi}_{p'(.)}} \leq 2\|g\|_{(\bar{\varphi}_{p(.)})^*},$$

*which in combination with the previous estimate proves the second estimate of the claim.*

**Corollary 1.2.4 (Norm conjugate formula)** *Let $p \in \mathcal{P}(A,\mu)$. Then*

$$\frac{1}{2}\|f\|_{p(.)} \leq \sup_{g \in L^{p'(.)} \,:\, \|g\|_{p'(.)} \leq 1} \int |fg|\, d\mu \leq 2\|f\|_{p(.)}$$

*for all $f \in L^0(A,\mu)$. The factor $\frac{1}{2}$ can be omitted if $\varphi_{p(.)} = \widetilde{\varphi}_{p(.)}$.*
*The supremum is unchanged if we replace the condition $g \in L^{p'(.)}(A,\mu)$ by $g \in S(A,\mu)$ or even $g \in S_c(\Omega)$ when $p \in \mathcal{P}(\Omega)$, where $S_c(\Omega)$ is the set of simple functions with compact support in $\Omega$.*

**Proof.** *The proof of the formula is exactly the same as in Corollary* 1.2.2 *if we additionally use the estimates of Theorem* 1.2.18. *That the supermum does not change for $g \in S(A,\mu)$ follows by Lemma* 1.2.19. *The case $g \in S_c(\Omega)$ requires a simple straightforward modification of Lemma* 1.2.19

**Lemma 1.2.20 (Young's inequality)** *Let $p, q, s \in [1,\infty]$ with*

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q}.$$

*Then for all $a, b \geq 0$*

$$\varphi_s(ab) \leq \varphi_p(a) + \varphi_q(b), \tag{1.12}$$

$$\bar{\varphi}_s(ab) \leq \frac{s}{p}\bar{\varphi}_p(a) + \frac{s}{q}\bar{\varphi}_q(b), \tag{1.13}$$

*where we use the convention $\frac{s}{p} = \frac{s}{q} = 1$ for $s = p = q = \infty$. Moreover, if $1 \leq s < \infty$ then for all $a \geq 0$*

$$\widetilde{\varphi}_p(a) = \sup_{b \geq 0} \left( \widetilde{\varphi}_s(ab) - \widetilde{\varphi}_q(b) \right). \tag{1.14}$$

**Proof.** *Assume first that $s = \infty$. Then necessarily $p = q = \infty$. There is nothing to show for $a, b \in [0,1]$, since in this case $\varphi_\infty(ab) = 0$. If $a > 1$ or $b > 1$, then $\varphi_p(a) = \infty$ or $\varphi_q(b) = \infty$, respectively. Thus the claim holds in this case also.*
*Assume then that $1 \leq s < \infty$. In order to prove* (1.12) *for $\widetilde{\varphi}$ it suffices to prove* (1.14). *If $s = 1$, then $p = q'$ and $\widetilde{\varphi}_p = (\widetilde{\varphi}_q)^*$ by Lemma* 1.1.2. *Thus,*

$$\widetilde{\varphi}_p(a) = (\widetilde{\varphi}_q)^*(a) = \sup_{b \geq 0} \left( ab - \widetilde{\varphi}_q(b) \right) = \sup_{b \geq 0} \left( \widetilde{\varphi}_1(ab) - \widetilde{\varphi}_q(b) \right)$$

*for all $a, b \geq 0$. If $1 < s < \infty$, then*

$$1 = \frac{1}{p/s} + \frac{1}{q/s},$$

*so by Lemma* 1.1.2 $(\widetilde{\varphi}_{p/s})^* = \widetilde{\varphi}_{q/s}$. *Using the case* $s = 1$ *we deduce*

$$\widetilde{\varphi}_p(a) = \frac{1}{s}\widetilde{\varphi}_{p/s}(a^s) = \frac{1}{s}\sup_{b\geq 0}\left(a^s b^s - \widetilde{\varphi}_{q/s}(b^s)\right) = \sup_{b\geq 0}\left(\widetilde{\varphi}_s(ab) - \widetilde{\varphi}_q(b)\right)$$

*for all* $a, b \geq 0$.

*It remains to prove* (1.13)*, since this inequality is stronger than* (1.12) *for* $\varphi = \bar{\varphi}$. *If* $s = \infty$ *, then* $s = p = q = \infty$ *and* (1.13) *follows from* (1.12)*, since* $\bar{\varphi}_\infty = \widetilde{\varphi}_\infty$. *So in the following let* $1 \leq s < \infty$. *Now* $s \leq p$ *and* $s \leq q$ *and we obtain using the previous case that*

$$\bar{\varphi}_s(ab) \leq s\widetilde{\varphi}_s(ab) \leq s\left(\widetilde{\varphi}_p(a) + \widetilde{\varphi}_q(b)\right) = \frac{s}{p}\bar{\varphi}_p(a) + \frac{s}{q}\bar{\varphi}_q(b) \leq \bar{\varphi}_p(a) + \bar{\varphi}_q(b)$$

*for all* $a, b \geq 0$. *It remains to prove* (1.12) *with* $\varphi = \bar{\varphi}$ *for* $1 \leq s < \infty$. *Now* $s \leq p$ *and* $s \leq q$ *and we obtain using the previous case that*

$$\bar{\varphi}_s(ab) \leq s\widetilde{\varphi}_s(ab) \leq s\left(\widetilde{\varphi}_p(a) + \widetilde{\varphi}_q(b)\right) = \frac{s}{p}\bar{\varphi}_p(a) + \frac{s}{q}\bar{\varphi}_q(b)$$

*for all* $a, b \geq 0$.

**Remark 1.2.3** *Note that* (1.12) *holds for both* $\widetilde{\varphi}$ *and* $\bar{\varphi}$*, but it is sharp only for* $\widetilde{\varphi}$ *as is shown by* (1.14) *and this example : if* $s = 1$ *and* $p = q = 2$*, then* $\sup_{b\geq 0}(\bar{\varphi}_1(ab) - \bar{\varphi}_2(b)) = \frac{1}{4}a^2 \neq a^2 = \bar{\varphi}_2(a)$ *for* $a > 0$.

**Lemma 1.2.21 (Hölder's inequality)** *Let* $p, q, s \in \mathcal{P}(A, \mu)$ *be such that*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}$$

*for* $\mu$*-almost every* $y \in A$. *Then*

$$\varrho_{s(.)}(fg) \leq \varrho_{p(.)}(f) + \varrho_{q(.)}(g), \tag{1.15}$$

$$\|fg\|_{s(.)} \leq 2\|f\|_{p(.)}\|g\|_{q(.)}, \tag{1.16}$$

$$\|fg\|_{\bar{\varphi}_{s(.)}} \leq \left(\left(\frac{s}{p}\right)^+ + \left(\frac{s}{q}\right)^+\right)\|f\|_{\bar{\varphi}_{p(.)}}\|g\|_{\bar{\varphi}_{q(.)}}, \tag{1.17}$$

*for all* $f \in L^{p(.)}(A, \mu)$ *and* $g \in L^{q(.)}(A, \mu)$*, where in the case* $s = p = q = \infty$ *we use the convention* $\frac{s}{p} = \frac{s}{q} = 1$.

*In particular,* $fg \in L^{s(.)}(A, \mu)$. *If additionally* $f \in E^{p(.)}(A, \mu)$ *or* $g \in E^{q(.)}(A, \mu)$*, then* $fg \in E^{s(.)}(A, \mu)$.

**Proof.** *Let $f \in L^{p(.)}$ and $g \in L^{q(.)}$. Since $f$ and $g$ are measurable, also $fg$ is measurable. Then* (1.15) *follows from* (1.12) *by integration over $y \in A$.*

*The following argument applies to both $\widetilde{\varphi}_{p(.)}$ and $\bar{\varphi}_{p(.)}$. If $\|f\|_{p(.)} \leq 1$ and $\|g\|_{q(.)} \leq 1$, then $\varrho_{p(.)}(f) \leq 1$ and $\varrho_{q(.)}(g) \leq 1$ by the unit ball property. Using* (1.15) *we estimate*

$$\varrho_{s(.)}\left(\frac{1}{2}fg\right) \leq \frac{1}{2}\varrho_{s(.)}(fg) \leq \frac{1}{2}\left(\varrho_{p(.)}(f) + \varrho_{q(.)}(g)\right) \leq 1.$$

*This implies $\|fg\|_{s(.)} \leq 2$ by the unit ball property. The scaling argument proves* (1.16).

*Now let $\|f\|_{\bar{\varphi}_{p(.)}} \leq 1$ and $\|g\|_{\bar{\varphi}_{q(.)}} \leq 1$, then by the unit ball property $\bar{\varrho}_{p(.)}(f) \leq 1$ and $\bar{\varrho}_{q(.)}(g) \leq 1$. Using* (1.13) *integrated over $y \in A$ we get*

$$\bar{\varrho}_{s(.)}(ab) \leq \left(\frac{s}{p}\right)^{+} \bar{\varrho}_{p(.)}(f) + \left(\frac{s}{q}\right)^{+} \bar{\varrho}_{q(.)}(g) \leq \left(\frac{s}{p}\right)^{+} + \left(\frac{s}{q}\right)^{+}.$$

*This implies $\|fg\|_{\bar{\varphi}_{s(.)}} \leq ess\sup \frac{s}{p} + ess\sup \frac{s}{q}$ by the unit ball property. The scaling argument proves* (1.17).

*Assume now that additionally $f \in E^{p(.)}$, i.e. $\varrho_{p(.)}(\lambda f) < \infty$ for every $\lambda > 0$. Let $\gamma > 0$ be such that $\varrho_{q(.)}(g/\gamma) < \infty$. Then for every $\lambda > 0$*

$$\varrho_{r(.)}(\lambda fg) \leq \varrho_{p(.)}(\lambda\gamma f) + \varrho_{q(.)}(g/\gamma) < \infty.$$

*Since $\lambda > 0$ was arbitrary, this proves $fg \in E^{r(.)}$. The case $g \in E^{q(.)}$ follows by symmetry.*

The case $s = 1$ in *Lemma* 1.2.21 is of special interest:

$$\int_A |f||g|\,d\mu \leq \varrho_{p(.)}(f) + \varrho_{p'(.)}(g),$$

$$\int_A |f||g|\,d\mu \leq 2\|f\|_{p(.)}\|g\|_{p'(.)},$$

$$\int_A |f||g|\,d\mu \leq \left(1 + \frac{1}{p^-} - \frac{1}{p^+}\right)\|f\|_{\bar{\varphi}_{p(.)}}\|g\|_{\bar{\varphi}_{p'(.)}}$$

for all $f \in L^{p(.)}(A,\mu)$ and $g \in L^{p'(.)}(A,\mu)$.

## 1.3 Embeddings

**Definition 1.3.1** *Let $X$ and $Y$ be two normed spaces and $c$ be a constant not depending on $u$. Then*

*i. if all elements of $X$ are in $Y$, $(X \subset Y)$ and*

*ii. for all $u \in X$, the inequality*

$$\|u\|_Y \leq c\|u\|_X$$

*holds, then X is said to be embedded in Y and denoted by* $X \hookrightarrow Y$. *The notation* $\hookrightarrow$ *replaces by* $\hookrightarrow\hookrightarrow$ *if the embedding is compact.*

**Theorem 1.3.1** *Let* $\varphi, \psi \in \Phi(A, \mu)$ *and let the measure* $\mu$ *be atom-less.*[1] *Then* $L^{\varphi}(A, \mu) \hookrightarrow L^{\psi}(A, \mu)$ *if and only if there exists* $c' > 0$ *and* $h \in L^1(A, \mu)$ *with* $\|h\|_1 \leq 1$ *such that*

$$\psi\left(y, \frac{t}{c'}\right) \leq \varphi(y, t) + h(y)$$

*for almost all* $y \in A$ *and all* $t \geq 0$.

*Moreover,* $c'$ *is bounded by the embedding constant, whereas the embedding constant is bounded by* $2c'$.

**Proof.** *Let us start by showing that the inequality implies the embedding. Let* $\|f\|_{\varphi} \leq 1$, *which yields by the unit ball property that* $\varrho_{\varphi}(f) \leq 1$. *Then*

$$\varrho_{\psi}\left(\frac{f}{2c'}\right) \leq \frac{1}{2}\varrho_{\psi}\left(\frac{f}{c'}\right) \leq \frac{1}{2}\varrho_{\varphi}(f) + \frac{1}{2}\int_A h(y)\,dy \leq 1.$$

*This and the unit ball property yield* $\|f/(2c')\|_{\psi} \leq 1$. *Then the embedding follows by the scaling argument.*

*Assume next that the embedding holds with embedding constant* $c_1$. *For* $y \in A$ *and* $t \geq 0$ *define*

$$\alpha(y, t) := \begin{cases} \psi\left(y, \frac{t}{c_1}\right) - \varphi(y, t) & \text{if } \varphi(y, t) < \infty, \\ 0 & \text{if } \varphi(y, t) = \infty. \end{cases}$$

*Since* $\varphi(y, .)$ *and* $\psi(y, .)$ *are left-continuous for all* $y \in A$, *also* $\alpha(y, .)$ *is left-continuous for all* $y \in A$. *Let* $(r_k)$ *be a sequence of distinct numbers with* $\{r_k : k \in \mathbb{N}\} = Q \cap [0, \infty)$ *and* $r_1 = 0$. *Then*

$$\psi\left(y, \frac{r_k}{c_1}\right) \leq \varphi(y, r_k) + \alpha(y, r_k)$$

*for all* $k \in \mathbb{N}$ *and* $y \in A$. *Define*

$$b_k(y) := \max_{1 \leq j \leq k} \alpha(y, r_j).$$

*Since* $r_1 = 0$ *and* $\alpha(y, 0) = 0$, *we have* $b_k \geq 0$. *Moreover, the functions* $b_k$ *are measurable and nondecreasing in k. The function* $b := \sup_k b_k$ *is measurable, non-negative, and satisfies*

$$b(y) = \sup_{t \geq 0} \alpha(y, t),$$

$$\psi\left(y, \frac{t}{c_1}\right) \leq \varphi(y, t) + b(y)$$

[1] A measure $\mu$ is called atom-less if for any measurable set A with $\mu(A) > 0$ there exists a measurable subset A' of A such that $\mu(A) > \mu(A')$.

*for all $y \in A$ and all $t \geq 0$, where we have used that $\alpha(y,.)$ is left-continuous and the density of $\{r_k \ : \ k \in \mathbb{N}\}$ in $[0,\infty)$.*

*We now show that $b \in L^1(A,\mu)$ with $\|b\|_1 \leq 1$. We consider first the case $|b| < \infty$ a.e., and assume to the contrary that there exists $\varepsilon > 0$ such that*

$$\int_A b(y)\, d\mu(y) \geq 1 + 2\varepsilon.$$

*Define*

$$V_k := \left\{ y \in A \ : \ \alpha(y, r_k) > \frac{1}{1+\varepsilon} b(y) \right\},$$
$$W_{k+1} := V_{k+1} \backslash (V_1 \cup \cdots \cup V_k)$$

*for all $k \in \mathbb{N}$. Note that $V_1 = \emptyset$ due to the special choice $r_1 = 0$. Since $\{r_k \ : \ k \in \mathbb{N}\}$ is dense in $[0,\infty)$ and $\alpha(y,.)$ is left-continuous for every $y \in A$, we have $\bigcup_{k=1}^{\infty} V_k = \bigcup_{k=2}^{\infty} W_k = \{y \in A \ : \ b(y) > 0\}$.*

*Let $f := \sum_{k=2}^{\infty} r_k \chi_{W_k}$. For every $y \in W_k$ we have $\alpha(y, r_k) > 0$ and therefore $\varphi(y, r_k) < \infty$. If $y$ is outside of $\bigcup_{k=2}^{\infty} W_k$, then $\varphi(y, |f(y)|) = 0$. This implies that $\varphi(y, |f(y)|)$ is everywhere finite. Moreover, by the definition of $W_k$ and $\alpha$ we get*

$$\psi\left(y, \frac{|f(y)|}{c_1}\right) \geq \varphi(y, |f(y)|) + \frac{1}{1+\varepsilon} b(y) \tag{1.18}$$

*for all $y \in A$.*

*If $\varrho_\varphi(f) \leq 1$, then $\varrho_\psi\left(\frac{f}{c_1}\right) \leq 1$ by the unit ball property since $c_1$ is the embedding constant. However, this contradicts*

$$\varrho_\psi\left(\frac{f}{c_1}\right) \geq \varrho_\varphi(f) + \frac{1}{1+\varepsilon} \int_A b(y)\, d\mu(y) \geq \frac{1+2\varepsilon}{1+\varepsilon} > 1,$$

*where we have used (1.18) and $\bigcup_{k=2}^{\infty} W_k = \{y \in A \ : \ b(y) > 0\}$. So we can assume that $\varrho_\varphi(f) > 1$. Since $\mu$ is atom-less and $\varphi(y, |f(y)|)$ is almost everywhere finite, there exists $U \subset A$ with $\varrho_\varphi(f\chi_U) = 1$. Thus*

$$\begin{aligned} \varrho_\psi\left(\frac{f}{c_1}\chi_U\right) &\geq \varrho_\varphi(f\chi_U) + \frac{1}{1+\varepsilon} \int_U b(y)\, d\mu(y) \\ &= 1 + \frac{1}{1+\varepsilon} \int_U b(y)\, d\mu(y). \end{aligned} \tag{1.19}$$

*Now, $\varrho_\varphi(f\chi_U) = 1$ implies that $\mu(U \cap \{f \neq 0\}) > 0$. Since $\{f \neq 0\} = \bigcup_{k=2}^{\infty} W_k = \{y \in A \ : \ b(y) > 0\}$ we get $\mu(U \cap \{y \in A \ : \ b(y) > 0\}) > 0$ and*

$$\int_U b(y)\, d\mu(y) > 0.$$

*This and (1.19) imply that*

$$\varrho_\psi(f/c_1\chi_U) > 1.$$

*Which contradicts $\varrho_\psi(f/c_1) \leq 1$. Thus the case where $|b| < \infty$ a.e. is complete.*
*If we assume that there exists $E \subset A$ with $b|_E = \infty$ and $\mu(E) > 0$, then a similar argument with $V_k := \left\{y \in E \,:\, \alpha(y, r_k) \geq \frac{2}{\mu(E)}\right\}$ yields a contradiction. Hence this case cannot occur, and the proof is complete by what was shown previously.*

Recall that the norm of the embedding $L^{p(.)}(A) \hookrightarrow L^{q(.)}(A)$ is the smallest constant $K > 0$ for which $\|f\|_{q(.)} \leq K\|f\|_{p(.)}$.

**Theorem 1.3.2** *Let $p, q \in \mathcal{P}(A, \mu)$. Define the exponent $r \in \mathcal{P}(A, \mu)$ by $\frac{1}{r(y)} := \max\left\{\frac{1}{q(y)} - \frac{1}{p(y)}, 0\right\}$ for all $y \in A$.*
*(a) If $q \leq p$ $\mu$-almost everywhere and $1 \in L^{r(.)}(A, \mu)$, then $L^{p(.)}(A, \mu) \hookrightarrow L^{q(.)}(A, \mu)$ with norm at most $2\|1\|_{L^{r(.)}(A)}$.*
*(b) If the measure $\mu$ is atom-less and $L^{p(.)}(A, \mu) \hookrightarrow L^{q(.)}(A, \mu)$ with norm $K > 0$, then $q \leq p$ $\mu$-almost everywhere and $\|1\|_{L^{r(.)}(A)} \leq 4K$.*

**Proof.** *We begin with the proof of (a). Since $q \leq p$ almost everywhere and $\frac{1}{r} + \frac{1}{p} = \frac{1}{q}$, we can apply Hölder's inequality, Lemma* 1.2.21*, to get*

$$\|f\|_{q(.)} \leq 2\|1\|_{r(.)}\|f\|_{p(.)}.$$

*Let us now prove (b). We begin with the case $\varphi_{p(.)} = \widetilde{\varphi}_{p(.)}$. Assume that $L^{p(.)}(A) \hookrightarrow L^{q(.)}(A)$. Then by Theorem* 1.3.1 *there exists $h \in L^1(A, \mu)$ with $h \geq 0$ and $\|h\|_1 \leq 1$ such that*

$$\widetilde{\varphi}_{q(y)}(t/K) \leq \widetilde{\varphi}_{p(y)}(t) + h(y) \tag{1.20}$$

*for almost all $y \in A$ and all $t \geq 0$. The limit $t \to \infty$ implies that $q \leq p$ almost everywhere. If $q(y) < \infty$, then* (1.14) *and* (1.20) *imply that*

$$\begin{aligned}\widetilde{\varphi}_{r(y)}(1/K) &= \sup_{t\geq 0}\left(\widetilde{\varphi}_{q(y)}(t/K) - \widetilde{\varphi}_{p(y)}(t)\right)\\ &\leq \sup_{t\geq 0}\left(\widetilde{\varphi}_{p(y)}(t) - \widetilde{\varphi}_{p(y)}(t) + h(y)\right)\\ &= h(y).\end{aligned} \tag{1.21}$$

*If the set $E := \{q = \infty\}$ has measure zero, then we can integrate this inequality over $y \in A$ and get $\widetilde{\varrho}_{r(.)}(1/K) \leq \|h\|_1 \leq 1$, so $1 \in L^{r(.)}(\Omega)$ and $\|1\|_{r(.)} \leq K$.*

*If $\mu(E) > 0$, then it fellows from* (1.20) *with $t = 1$ that $\widetilde{\varphi}_\infty(1/K) \le h(y)$ for almost all $y \in E$. Since $h$ is a.e. finite on $E$ and $\mu(E) > 0$, this implies $K \ge 1$. Since $r = \infty$ on the set $E$, we get*

$$\widetilde{\varphi}_{r(y)}(1/K) = \varphi_\infty(1/K) = 0 \le h(y)$$

*for almost every $y \in E$. So* (1.21) *also holds on the set $E$. Thus we can proceed exactly as in the previous case to conclude $\|1\|_{r(.)} \le K$.*
*The case $\varphi_{p(.)} = \bar{\varphi}_{p(.)}$ follows from this using* (1.9).

If $\mu(A) < \infty$, then by *Lemma* 1.2.13 and /or *Lemma* 1.2.14 the condition $1 \in L^{r(.)}(A)$ of the last theorem is always satisfied. Hence,

**Corollary 1.3.1** *Let $p, q \in \mathcal{P}(A, \mu)$ and let the measure $\mu$ be atom-less with $\mu(A) < \infty$. Then $L^{p(.)}(A, \mu) \hookrightarrow L^{q(.)}(A, \mu)$ if and only if $q \le p$ $\mu$-almost everywhere in $A$. The embedding constant is less or equal to $2(1 + \mu(A))$ and $2 \max\left\{\mu(A)^{\left(\frac{1}{q}-\frac{1}{p}\right)^+}, \left(\frac{1}{q}-\frac{1}{P}\right)^-\right\}$.*

**Lemma 1.3.1** *Let $p \in \mathcal{P}(\mathbb{R}^n)$ and $p_\infty \in [1, \infty]$. Define $s \in \mathcal{P}(\mathbb{R}^n)$ by*

$$\frac{1}{s(x)} := \left|\frac{1}{p(x)} - \frac{1}{p_\infty}\right|.$$

*Then $1 \in L^{s(.)}(\mathbb{R}^n)$ if and only if*

$$L^{\max\{p(.), p_\infty\}}(\mathbb{R}^n) \hookrightarrow L^{p(.)}(\mathbb{R}^n) \hookrightarrow L^{\min\{p(.), p_\infty\}}(\mathbb{R}^n).$$

**Proof.** *If the embeddings hold, then Theorem* 1.3.2 *implies that $1 \in L^{s(.)}$.*
*Assume now that $1 \in L^{s(.)}(\mathbb{R}^n)$. Let $\gamma \in (0, 1)$ such that $\bar{\varrho}_{s(.)}(\gamma) < \infty$. Define $r_1, r_2 \in \mathcal{P}(\mathbb{R}^n)$ by*

$$\frac{1}{r_1(x)} := \min\left\{0, \frac{1}{p_\infty} - \frac{1}{p(x)}\right\} = \frac{1}{\max\{p(x), p_\infty\}} - \frac{1}{p(x)},$$
$$\frac{1}{r_2(x)} := \min\left\{0, \frac{1}{p(x)} - \frac{1}{p_\infty}\right\} = \frac{1}{p(x)} - \frac{1}{\max\{p(x), p_\infty\}}$$

*for all $x \in \mathbb{R}^n$, Then $s \le r_1$ and $s \le r_2$ almost everywhere. Thus it follows from the definition that $\bar{\varrho}_{r_1(.)}(\gamma) \le \bar{\varrho}_{s(.)}(\gamma) < \infty$ and $\bar{\varrho}_{r_2(.)}(\gamma) \le \bar{\varrho}_{s(.)}(\gamma) < \infty$ for $\gamma \in (0, 1)$. In particular, $1 \in L^{r_1(.)}$ and $1 \in L^{r_2(.)}$. This and Theorem* 1.3.2 *prove the embeddings.*

**Theorem 1.3.3** *Let $p.q, r \in \mathcal{P}(A, \mu)$ with $p \le q \le r$ $\mu$-almost everywhere in $A$. Then*

$$L^{p(.)}(A, \mu) \cap L^{r(.)}(A, \mu) \hookrightarrow L^{q(.)}(A, \mu) \hookrightarrow L^{p(.)}(A, \mu) + L^{r(.)}(A, \mu).$$

*The embedding constants are at most 2. More precisely, for* $g \in L^{q(.)}(A,\mu)$ *the functions* $g_0 := \operatorname{sgn} g \ \max\{|g|-1,0\}$ *and* $g_1 := \operatorname{sgn} g \min\{|g|,1\}$ *satisfy* $g = g_0+g_1$, $|g_0|,\ |g_1| \le |g|,\ \|g_0\|_{p(.)} \le 1$ *and* $\|g_1\|_{r(.)} \le 1$.

**Proof.** *see[7]*

**Lemma 1.3.2** *Let* $p \in \mathcal{P}(\mathbb{R}^n)$ *and* $p_\infty \in [1,\infty]$. *Assume that* $1 \in L^{s(.)}(\mathbb{R}^n)$, *where* $s \in \mathcal{P}(\mathbb{R}^n)$ *is defined by* $\frac{1}{s(x)} := |\frac{1}{p(x)} - \frac{1}{p_\infty}|$. *Then*

$$L^{p(.)}(\mathbb{R}^n) \cap L^{p^+}(\mathbb{R}^n) \cong L^{p_\infty}(\mathbb{R}^n) \cap L^{p^+}(\mathbb{R}^n),$$
$$L^{p(.)}(\mathbb{R}^n) \hookrightarrow L^{p_\infty}(\mathbb{R}^n) + L^{p^-}(\mathbb{R}^n).$$

**Proof.** *Using Lemma* 1.3.1, *Theorem* 1.3.3 *twice, and then Lemma* 1.3.1 *again we deduce*

$$\begin{aligned} L^{p(.)} \cap L^{p^+} &\hookrightarrow L^{\min\{p(.),p_\infty\}} \cap L^{p^+} \\ &\hookrightarrow L^{p_\infty} \cap L^{p^+} \\ &\hookrightarrow L^{\max\{p(.),p_\infty\}} \cap L^{p^+} \\ &\hookrightarrow L^{p(.)} \cap L^{p^+}. \end{aligned}$$

*This proves the first assertion. Analogously, by Lemma* 1.3.1 *and Theorem* 1.3.3

$$L^{p(.)} \hookrightarrow L^{\min\{p(.),p_\infty\}} \hookrightarrow L^{p_\infty} + L^{p^-}.$$

*This proves the second assertion.*

# Chapter 2

# Introduction to Sobolev spaces with variables exponents

In this chapter we start our study of Sobolev functions. It is not possible to study this functions before study its properties in the previous space $L^{p(\cdot)}$. We need this space in the partial differential equations because the solutions belong naturally to Sobolev Space. In the first section we will talk about the basic properties (functional analysis-type properties) in this space, the results are from [8] by Diening, [9] by Fan and Zhao, and [17] by Kováčik and Rákosník, then we will turn to an important inequality, it's called "Poincaré inequality".

## 2.1 Basic properties

Let $\Omega \subset \mathbb{R}^n$ be an open set.

**Definition 2.1.1** *Given $\Omega$, a function $f \in L^1_{loc}(\Omega)$ and a multi-index $\alpha$, $f$ has a weak derivative of order $\alpha$ if there exists $\alpha$ function $g_\alpha \in L^1_{loc}(\Omega)$ such that for every function $\phi \in C_c^\infty(\Omega)$,*

$$\int_\Omega f(x) D^\alpha \phi(x)\, dx = (-1)^{|\alpha|} \int_\Omega g_\alpha(x)\phi(x)\, dx.$$

*It can be shown that $g_\alpha$ is unique, and we write $D^\alpha f = g_\alpha$.*
*If a function $f$ is smooth, then by integration by parts we have that the classical derivatives of $f$ are also weak derivatives.*

**Definition 2.1.2** *Let $\Omega \subset \mathbb{R}^n$, $m$ be a positive integer, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be multi-index and*

*$p : \Omega \to [1, \infty)$ be a measurable function. Then the space defined by*

$$W^{m,p(.)}(\Omega) = \left\{ u \in L^{p(.)}(\Omega) \ : \ D^{\alpha} u \in L^{p(.)}(\Omega),\ 0 \leq |\alpha| \leq m \right\}$$

*is called the variable exponent Sobolev space. This space is a Banach space with respect to the norm*

$$\|u\|_{W^{m,p(.)}(\Omega)} = \|u\|_{m,p(.)} = \sum_{0 \leq |\alpha| \leq m} \|D^{\alpha} u\|_{p(.)}.$$

*If $1 < p^{-} \leq p^{+} < \infty$ is satisfied, then the space $W^{m,p(.)}(\Omega)$ is reflexive. Moreover the closure of $C_0^{\infty}(\Omega)$ space on $W^{m,p(.)}(\Omega)$ space is $W_0^{m,p(.)}(\Omega)$.*

the previous definition is one of the other definitions which is about $W^{m,p(.)}$ in another definition [7, *Definition* 8.1.2] they define a semimodular on $W^{m,p(.)}(\Omega)$ by

$$\varrho_{W^{m,p(.)}(\Omega)}(u) \coloneqq \sum_{0 \leq |\alpha| \leq m} \varrho_{L^{p(.)}(\Omega)}(\partial_{\alpha} u)$$

which induces a norm by

$$\|u\|_{W^{m,p(.)}(\Omega)} \coloneqq \inf \left\{ \lambda > 0 \ : \ \varrho_{W^{m,p(.)}(\Omega)} \left( \frac{u}{\lambda} \right) \leq 1 \right\}.$$

**Remark 2.1.1** *It is also possible to define the semimodular $\varrho_{W^{k,p(.)}(\Omega)}$ on the larger set $W^{k,1}_{loc}(\Omega)$ or even $L^{1}_{loc}(\Omega)$. Then $W^{k,p(.)}(\Omega)$ is just the corresponding semimodular space.*

Sobolev functions, as Lebesgue functions, are defined only up to measure zero and thus we identify functions that are equal almost everywhere. If the set $\Omega$ is clear from the content, we abbreviate $\|u\|_{W^{k,p(.)}(\Omega)}$ to $\|u\|_{k,p(.)}$ and $\varrho_{W^{k,p(.)}(\Omega)}$ to $\varrho_{k,p(.)}$.

**Remark 2.1.2** *(i) note that in $W^{k,p(.)}(\Omega)$*

$$\sum_{m=0}^{k} \varrho_{L^{p(.)}(\Omega)}(|\nabla^{m} u|) \quad \textit{and} \quad \sum_{m=0}^{k} \| \, |\nabla^{m} u| \, \|_{L^{p(.)}(\Omega)}$$

*define a semimodular and a norm equivalent to the Sobolev semimodular and the Sobolev norm, respectively. We abbreviate $\| \, |\nabla^{m} u| \, \|_{L^{p(.)}(\Omega)}$ as $\|\nabla^{m} u\|_{L^{p(.)}(\Omega)}$, $m \in \mathbb{N}$.*
*(ii) One easily proves, using Lemma* 1.2.10*, that for each 1-finite partition $(\Omega_i)_{i \in \mathbb{N}}$ of $\Omega$ we have*

$$\|u\|_{W^{k,p(.)}(\Omega)} \leq \sum_{i=1}^{\infty} \|u\|_{W^{k,p(.)}(\Omega_i)}$$

*for all $u \in W^{k,p(.)}(\Omega)$.*

**Definition 2.1.3** *Let $p \in \mathcal{P}(\Omega)$ and $k \in \mathbb{N}$. The Sobolev space $W_0^{k,p(.)}(\Omega)$ with zero boundary values is the closure of the set of $W^{k,p(.)}(\Omega)$-functions with compact support, i.e.*

$$\left\{u \in W^{k,p(.)}(\Omega) : \ u = u_{\chi_K} \text{ for a compact } K \subset \Omega\right\}$$

*in $W^{k,p(.)}(\Omega)$.*

In [5, lamma 2.2], for $v \in W_0^{1,q(.)}(\Omega)$, they give the equivalent norm

$$\|v\|_{1,q(.)} = \|\nabla v\|_{q(.)}.$$

$W_0^{-1,q'(.)}(\Omega)$ sign to the dual of $W_0^{1,q(.)}(\Omega)$ in which $\frac{1}{q(.)} + \frac{1}{q'(.)} = 1$.

**Warning 2.1.1** *The closure of $C_0^\infty(\Omega)$ in the space $W^{k,p(.)}(\Omega)$ is denoted by $H_0^{k,p(.)}(\Omega)$.*

**Remark 2.1.3** *In contrast to $H_0^{1,p(.)}(\Omega)$, the space $W_0^{1,p(.)}(\Omega)$ has the following fundamental property: if $u \in W^{1,p(.)}(\Omega)$ and $v$ is a Lipschitz continuous function with compact support in $\Omega$, then $uv \in W_0^{1,p(.)}(\Omega)$. In [7, Section 11.5] we will see that for certain exponents $p$ the product $uv$ need not to be in $H_0^{1,p(.)}(\Omega)$ and thus it may hold that $H_0^{1,p(.)}(\Omega) \subsetneq W_0^{1,p(.)}(\Omega)$.*

**Definition 2.1.4** *A function $u$ belongs to $W_{loc}^{k,p(.)}(\Omega)$ if $u \in W^{k,p(.)}(U)$ for every open $U \subset\subset \Omega$. We equip $W_{loc}^{k,p(.)}(\Omega)$ with the initial topology induced by the embeddings $W_{loc}^{k,p(.)}(\Omega) \hookrightarrow W^{k,p(.)}(U)$ for all open $U \subset\subset \Omega$.*

**Proposition 2.1.1** *Let $X$ be a Banach space and let $Y$ denote either a closed subset of $X$ or a Cartesian product $X^N$. Then:*
*(i) $Y$ is a Banach space.*
*(ii) If $X$ is reflexive, then $Y$ is reflexive.*
*(iii) If $X$ is separable, then $Y$ is separable.*
*(iv) If $X$ is uniformly convex, then $X$ is reflexive.*
*(v) If $X$ is uniformly convex, then $Y$ is uniformly convex.*

**Theorem 2.1.1** *Let $p \in \mathcal{P}(\Omega)$. The space $W^{k,p(.)}(\Omega)$ is a Banach space, which is separable if $p$ is bounded, and reflexive and uniformly convex if $1 < p^- \leq p^+ < \infty$.*

**Proof.** *We proof only the case $k = 1$, the proof for the general case is similar. We first show that the Sobolev spaces is a Banach space; for that let $(u_i)$ be a Cauchy sequence in $W^{1,p(.)}(\Omega)$. We*

*have to show that there exists* $u \in W^{1,p(.)}(\Omega)$ *such that* $u_i \to u$ *in* $W^{1,p(.)}(\Omega)$ *as* $i \to \infty$. *Since the Lebesgue space* $L^{p(.)}(\Omega)$ *is a Banach space* (*Theorem* 1.2.7), *there exist* $u, g_1, \dots, g_n \in L^{p(.)}(\Omega)$ *such that* $u_i \to u$ *and* $\partial_j u_i \to g_i$ *in* $L^{p(.)}(\Omega)$ *for every* $j = 1, \dots, n$. *Let* $\psi \in C_0^\infty(\Omega)$. *Since* $u_i$ *is in* $W^{1,p(.)}(\Omega)$ *we have*

$$\int_\Omega u_i \partial_j \psi \, dx = -\int_\Omega \psi \partial_j u_i \, dx.$$

*Strong convergence in* $L^{p(.)}(\Omega)$ *implies weak convergences and hence we have*

$$\int_\Omega u_i \partial_j \psi \, dx \to \int_\Omega u \partial_j \psi \, dx \quad \text{and} \quad \int_\Omega \psi \partial_j u_i \, dx \to \int_\Omega \psi g_j \, dx$$

*as* $i \to \infty$. *Thus the right-hand sides on the previous line yield that* $(g_1, \dots, g_n)$ *is the weak gradient of u. It follows that* $u \in W^{1,p(.)}$ *and* $u_j \to u$ *in* $W^{1,p(.)}$.

*By Theorem* 1.2.11, $L^{p(.)}(\Omega)$ *is separable if* $p^+ < \infty$ *and by Theorem* 1.2.7, $L^{p(.)}(\Omega)$ *is reflexive if* $1 < p^- \le p^+ < \infty$. *By the mapping* $u \mapsto (u, \nabla u)$, *the space* $W^{1,p(.)}(\Omega)$ *is a closed subspace of* $L^{p(.)}(\Omega) \times (L^{p(.)}(\Omega))^n$. *Thus* $W^{1,p(.)}(\Omega)$ *is separable if* $p^+ < \infty$, *and reflexive if* $1 < p^- \le p^+ < \infty$ *by Proposition* 2.1.1.

*For the uniform convexity we note that* $W^{1,p(.)}(\Omega)$ *satisfies the* $\Delta_2$*-condition provided that* $p^+ < \infty$. *The* $L^{p(.)}$*-modular is uniformly convex for* $p^- > 1$ *by Theorem* 1.2.4 *and the proof of Theorem* 1.2.6. *Thus* $\varrho_{W^{k,p(.)}}$ *is uniform convex as a sum of uniform convex modulars* (*Lemma* 1.2.9). *Thus* $W^{1,p(.)}(\Omega)$ *is uniform convex with its own norm by Theorem* 1.2.5.

**Theorem 2.1.2** *Let* $p \in \mathcal{P}(\mathbb{R}^n)$. *The space* $W_0^{k,p(.)}(\Omega)$ *is a Banach space, which is separable if p is bounded, and reflexive and uniformly convex if* $1 < p^- \le p^+ < \infty$.

**Proof.** *Since* $W_0^{k,p(.)}(\Omega)$ *is a closed subspace of* $W^{k,p(.)}(\Omega)$, *the properties follow by Proposition 2.1.1 and Theorem* 2.1.1.

A normed space X has the Banach-Saks property if $\frac{1}{m}\sum_{i=1}^m u_i \to u$ whenever $u_i \to u$. By [12] every uniformly convex space has the Banach-Saks property. This together with the previous theorem implies the following corollary.

**Corollary 2.1.1** *Let* $p \in \mathcal{P}(\Omega)$ *with* $1 < p^- \le p^+ < \infty$. *Then the Sobolev space* $W^{k,p(.)}(\Omega)$ *has the Banach-Saks property.*

In the next proposition, the notation " $\subset$ " is denote to the embedding.

**Proposition 2.1.2** *Given* $\Omega$, $p(.) \in \mathcal{P}(\Omega)$ *and* $k \geq 1$, $W^{k,p(.)}(\Omega) \subset W^{k,p_-}_{loc}(\Omega)$. *If* $|\Omega| < \infty$, *then* $W^{k,p(.)}(\Omega) \subset W^{k,p_-}(\Omega)$.

**Proposition 2.1.3** *Let* $p \in \mathcal{P}(\Omega)$. *If* $u, v \in W^{1,p(.)}(\Omega)$, *then* $\max\{u, v\}$ *and* $\min\{u, v\}$ *are in* $W^{1,p(.)}(\Omega)$ *with*

$$\nabla \max(u.v)(x) = \begin{cases} \nabla u(x), & \text{for almost every } x \in \{u \geq v\}; \\ \nabla v(x), & \text{for almost every } x \in \{v \geq u\}; \end{cases}$$

*and*

$$\nabla \min(u, v)(x) = \begin{cases} \nabla u(x), & \text{for almost every } x \in \{u \leq v\}; \\ \nabla v(x), & \text{for almost every } x \in \{v \leq u\}. \end{cases}$$

*In particular,* $|u|$ *belongs to* $W^{1,p(.)}(\Omega)$ *and* $|\nabla |u|| = |\nabla u|$ *almost everywhere in* $\Omega$.

**Proof.** *It suffices to prove the assertions for* $\max\{u, v\}$ *since* $\min\{u, v\} = -\max\{-u, -v\}$. *By Lemma* 2.1.2 *we know that* $W^{1,p(.)}(\Omega) \hookrightarrow W^{1,1}_{loc}(\Omega)$ *and so the formulas for* $\nabla \max(u, v)$ *and* $\nabla \min(u, v)$ *follow from [16, Theorem 1.20]. We next note that* $\|\max\{u, v\}\|_{p(.)} \leq \|u\|_{p(.)} + \|v\|_{p(.)}$ *and* $\|\nabla \max\{u, v\}\|_{p(.)} \leq \|\nabla u\|_{p(.)} + \|\nabla v\|_{p(.)}$. *Thus it follows that* $\max\{u, v\} \in W^{1,p(.)}(\Omega)$. *Analogously, we get* $\min\{u, v\} \in W^{1,p(.)}(\Omega)$. *The claims for* $|u|$ *follow by noting that* $|u| = \max\{u, 0\} - \min\{u, 0\}$.

The previous proposition yields that in a set where u is constant $\nabla u = 0$ almost everywhere.

**Lemma 2.1.1** *Let* $p \in \mathcal{P}(\mathbb{R}^n)$ *and* $u \in W^{k,p(.)}_0(\Omega)$. *Then u extended by zero to* $\mathbb{R}^n \backslash \Omega$ *belongs to* $W^{k,p(.)}(\mathbb{R}^n)$.

**Proof.** *Let* $u \in W^{k,p(.)}(\Omega)$ *with compact support, i.e. there exists a compact set* $K \subset \Omega$ *such that* $u = \chi_K u$ *almost everywhere. We define* $\mathcal{E}u$ *to be the extension of u (as a measurable function) by zero outside of* $\Omega$. *We claim that* $\mathcal{E}u \in W^{k,p(.)}(\mathbb{R}^n)$ *and* $\partial_\alpha \mathcal{E}(u) = \mathcal{E}(\partial_\alpha u)$ *almost everywhere for* $|\alpha| \leq k$. *Choose* $\eta \in C^\infty_0(\Omega)$ *such that* $\chi_K \leq \eta\eta\chi_\Omega$. *Then for all* $\psi \in C^\infty_0(\mathbb{R}^n)$ *and* $|\alpha| \leq k$ *we have*

$$\begin{aligned} \int_{\mathbb{R}^n} \mathcal{E}u \partial_\alpha \psi \, dx &= \int_\Omega u \partial_\alpha(\psi\eta) \, dx = (-1)^{|\alpha|} \int_\Omega (\partial_\alpha u)\psi\eta \, dx \\ &= (-1)^{|\alpha|} \int_{\mathbb{R}^n} (\mathcal{E}(\partial_\alpha u))\psi \, dx, \end{aligned}$$

*where we used that* $u = 0$ *and* $\partial_\alpha u = 0$ *outside of K and* $\eta = 1$ *on K. This proves* $\partial_\alpha \mathcal{E}(u) = \mathcal{E}(\partial_\alpha u)$. *Since* $\mathcal{E}(\partial_\alpha u) \in L^{p(.)}(\mathbb{R}^n)$, *it follows that* $\mathcal{E}u \in W^{k,p(.)}(\mathbb{R}^n)$. *Moreover,* $\|u\|_{W^{k,p(.)}(\Omega)} = \|\mathcal{E}u\|_{W^{k,p(.)}(\mathbb{R}^n)}$,

*so $\mathcal{E}$ is a isometry on the set of compactly supported $W^{k,p(.)}(\Omega)$ functions. Since those functions are by definition dense in $W_0^{k,p(.)}(\Omega)$, the extension operator is also an isometry from $W_0^{k,p(.)}(\Omega)$ to $W^{k,p(.)}(\mathbb{R}^n)$. In particular, $u \in W_0^{k,p(.)}(\Omega)$ implies $\mathcal{E}u \in W^{k,p(.)}(\mathbb{R}^n)$.*

## 2.2 Poincaré inequalities

In this section we will study the poincaré inequalities in two form "norm version and modular version", In the constant exponent case the poincaré inequality is known to hold for more irregular domains but the inequality is mostly used in John domains. Recall that $I_1$ denotes the Riesz potential operator (cf. [7, Definition 6.1.1]); note also the convention that $I_1 f$ denotes $I_1(\chi_\Omega f)$ if the function f is defined only in $\Omega$. By a cube we always mean a non-degenerate cube with faces parallel to the coordinate axes. The average integral of f over the set A is defined by ${}_A f\, d\mu := \frac{1}{\mu(A)} \int_A f\, d\mu,\ \mu(A) > 0$ and the diameter of $\Omega$ by $diam(\Omega) = \max_{u,v\in V} d(u,v)$ where V is the finite set of the vertices in $\Omega$.

**Definition 2.2.1** *A bounded domain $\Omega \subset \mathbb{R}^n$ is called an $\alpha$-John domain, $\alpha > 0$, if there exists $x_0 \in \Omega$ (the John center) such that each point in $\Omega$ can be joined to $x_0$ by a rectifiable path $\gamma$ (the John path) parametrized by its arc-length such that*

$$B\left(\gamma(t), \frac{1}{\alpha}t\right) \subset \Omega$$

*for all $t \in [0, l(\gamma)]$, where $l(\gamma) \leq \alpha$ is the length of $\gamma$. The ball $B\left(x_0, \frac{diam(\Omega)}{2\alpha}\right)$ is called the John ball.*

**Definition 2.2.2** *A family $\mathcal{Q}$ of measurable sets $E \subset \mathbb{R}^n$ is called locally N-finite, $N \in \mathbb{N}$, if $\sum_{E\in\mathcal{Q}} \chi_E \leq N$ almost everywhere in $\mathbb{R}^n$. We simply say that $\mathcal{Q}$ is locally finite if it is locally N-finite for some $N \in \mathbb{N}$.*
*Note that a family $\mathcal{Q}$ of open, bounded sets $Q \subset \mathbb{R}^n$ is locally 1-finite if and only if the sets $Q \in \mathcal{Q}$ are pairwise disjoint.*

**Definition 2.2.3** *Let $\Omega \subset \mathbb{R}^n$ and let $\sigma_2 > \sigma_1 > 1$. Then we say that $\Omega$ satisfies the emanating chain condition with constants $\sigma_1$ and $\sigma_2$ if there exists a countable covering $\mathcal{Q}$ of $\Omega$ consisting of open cubes (or balls) such that :*
*(B1) We have $\sigma_1 Q \subset \Omega$ for all $Q \in \mathcal{Q}$ and $\sum_{Q\in\mathcal{Q}} \chi_{\sigma_1 Q} \leq \sigma_2 \chi_\Omega$ on $\mathbb{R}^n$.*

*(B2) If $\Omega$ is bounded :*

*there exists a central cube $Q_\infty \in \mathcal{Q}$ with $\Omega \subset \sigma_2 Q_\infty$. For every $Q \in \mathcal{Q}$ there exists a chain $Q_1, \dots, Q_m$ emanating from $Q$ ending in $Q_\infty$ : $Q_1, \dots, Q_m$ are pairwise distinct such that $Q_1 = Q$, $Q_m = Q_\infty$, and $Q_i \subset \sigma_2 Q_j$ for $1 \le i \le j \le m$. Moreover, $Q_i \cap Q_{i+1}$ contains a ball $B_i$ such that $Q_i \cup Q_{i+1} \subset \sigma_2 B_i$ for $i = 1, \dots, m-1$.*

*If $\Omega$ is unbounded :*

*For every $Q \in \mathcal{Q}$ then exists an unending chain $Q_1, Q_2, \dots$ emanating from $Q$ : $Q_1, Q_2, \dots$ are pairwise distinct such that $Q_1 = Q$ and $Q_i \subset \sigma_2 Q_j$ for $1 \le i \le j$. Moreover, $Q_i \cap Q_{i+1}$ contains a ball $B_i$ such that $Q_i \cup Q_{i+1} \subset \sigma_2 B_i$ for $i \ge 1$.*

*(B3) The set $\{Q \in \mathcal{Q} \ : \ Q \cap K \neq \emptyset\}$ if finite for every compact subset $K \subset \Omega$.*

*The family $\mathcal{Q}$ is called the chain-covering of $\Omega$.*

**Definition 2.2.4** *For a function $f \in L^0(\mathbb{R}^n)$ and an open, bounded set $U \subset \mathbb{R}^n$ (usually a cube or a ball) we define*

$$M_U f := \fint_U |f(y)|\,dy = \frac{1}{|U|}\int_U |f(y)|\,dy \tag{2.1}$$

*The (norm-centered) maximal function $Mf$ of $f$ is defined by*

$$Mf(x) := \sup_{Q \ni x} M_Q f = \sup_{Q \ni x} \fint_Q |f(y)|\,dy$$

*for all $x \in \mathbb{R}^n$, where the supremum is taken over all cubes (or balls) $Q \subset \mathbb{R}^n$ which contain $x$. The operator $M : f \mapsto Mf$ is called the Hardy-Littlewood maximal operator or just maximal operator. Furthermore, for $f \in L^s_{loc}(\mathbb{R}^n)$, $s \in [1, \infty)$, and an open, bounded set $U \subset \mathbb{R}^n$ we define*

$$M_{s,U} f := (M_U(|f|^s))^{\frac{1}{s}} = \left( \fint_U |f(y)|^s\,dy \right)^{\frac{1}{s}},$$

$$M_s f := (M(|f|^s))^{\frac{1}{s}}.$$

Note that $\langle u \rangle_B = \frac{1}{|B|} \fint_B u(y)\,dy$.

**Lemma 2.2.1** *(a) For every $u \in W^{1,1}_0(\Omega)$, the inequality*

$$|u| \le c I_1 |\nabla u|$$

*holds a.e. in $\Omega$ with the constant $c$ depending only on the dimension $n$.*

*(b) If $\Omega \subset \mathbb{R}^n$ is a bounded $\alpha$-John domain, then there exists a ball $B \subset \Omega$ and a constant $c$ such that*

$$|u(x) - \langle u \rangle_B| \le c I_1 |\nabla u|(x)$$

*holds a.e. in $\Omega$ for every $u \in W^{1,1}(\Omega)$. The ball $B$ satisfies $|B| \leq |\Omega| \leq c'|B|$ and the constants $c$ and $c'$ depend only on the dimension $n$ and $\alpha$.*

**Proof.** *We prove only $(b)$. The proof of $(a)$ is similar, see for example [11, Lemma 7.14, p. 154]. We consider first claim $(b)$ when $\Omega$ is a ball. Assume that $u \in C^\infty(\Omega) \cap W^{1,1}(\Omega)$. We have*

$$u(x) - u(y) = -\int_0^{|x-y|} \nabla u\left(x + r\frac{y-x}{|y-x|}\right).\frac{y-x}{|y-x|}\, dr.$$

*Integrating with respect to $y$ over $\Omega$ and dividing the result by $|\Omega|$. we obtain*

$$u(x) - \langle u\rangle_\Omega = -\frac{1}{|\Omega|}\int_\Omega \int_0^{|x-y|} \nabla u\left(x + r\frac{y-x}{|y-x|}\right).\frac{y-x}{|y-x|}\, dr\, dy.$$

*Using the notation*

$$D(z) = \begin{cases} \nabla u(z) & \text{if } z \in \Omega; \\ 0 & \text{if } z \notin \Omega; \end{cases}$$

*we obtain*

$$\begin{aligned} |u(x) - \langle u\rangle_\Omega| &\leq \frac{1}{|\Omega|}\int_{\{|x-y|\leq diam(\Omega)\}} \int_0^\infty \left|D\left(x + r\frac{y-x}{|y-x|}\right)\right| dr\, dy \\ &= \frac{1}{|\Omega|}\int_0^\infty \int_{\partial B(0,1)} \int_0^{diam(\Omega)} |D(x+rw)|\varrho^{n-1}\, d\varrho\, dw\, dr \\ &\leq \int_0^\infty \int_{\partial B(0,1)} |D(x+rw)|\, dw\, dr \\ &= \int_\Omega \frac{|D(y)|}{|x-y|^{n-1}}\, dy = I_1|\nabla u|(x). \end{aligned}$$

*This concludes the proof in the ball when $u$ is smooth. For $u \in W^{1,1}(\Omega)$, we take smooth approximations $\psi_i$ such that $\psi_i \to u$ in $W^{1,1}(\Omega)$ and almost everywhere. Then $\langle\psi_i\rangle_\Omega \to \langle u\rangle_\Omega$ and $I_1|\nabla\psi_i|(x) \to I_1|\nabla u|(x)$, where we also used the continuity of the Riesz potential in $L^1(\Omega)$ (cf. [21, Theorem 1.1.31]). This yields the claim for $u$.*

*Suppose now that $\Omega$ is a bounded $\alpha$-John domain. Then $\Omega$ satisfies the emanating chain condition with constants depending only on $\alpha$. Let $Q_0$ be the central emanating ball. If $x \in Q_0$, then the claim follows by what was just proved. Otherwise, let $(Q_0)_{j=0}^m$ be the emanating chain connecting $x$ and $Q_0$. Let $B_j$ be the balls in the intersection $Q_j \cap Q_{j+1}$ as in the Definition 2.2.1. Then*

$$\begin{aligned} |u(x) - \langle u\rangle_{Q_0}| &\leq |u(x) - \langle u\rangle_{Q_m}| + \sum_{j=0}^{m-1} |\langle u\rangle_{Q_{j+1}} - \langle u\rangle_{B_j}| + |\langle u\rangle_{B_j} - \langle u\rangle_{Q_j}| \\ &\leq I|\nabla u|(x) + 2\sum_{j=0}^{m-1} |\langle u\rangle_{B_j} - \langle u\rangle_{Q_j}| \end{aligned}$$

*Let us estimate the second term:*

$$\begin{aligned}|\langle u\rangle_{B_j} - \langle u\rangle_{Q_j}| &\leq \fint_{B_j} |u - \langle u\rangle_{Q_j}|\, dy \\ &\leq \sigma_2^n \fint_{Q_j} |u - \langle u\rangle_{Q_j}|\, dy \\ &\leq c\sigma_2^n diam(Q_j) \fint_{Q_j} |\nabla u|\, dy.\end{aligned}$$

*Since $|x - y| \leq \sigma_2 diam(Q_j)$ and $(Q_j)$ has overlap less than or equal to $\sigma_1$, we obtain*

$$\sum_{j=0}^{m-1} diam(Q_j) \fint_{Q_j} |\nabla u|\, dy \leq c\sigma_2^{n-1} \sum_{j=0}^{m-1} \fint_{Q_j} \frac{|\nabla u|}{|x-y|^{n-1}}\, dy \leq c\sigma_1\sigma_2^{n-1} I|\nabla u|(x).$$

*The assertion follows when we combine the previous three estimates.*

**Remark 2.2.1** *One easily checks that the assertions in Lemma* 2.2.1 *(b) also holds for $u \in L^1_{loc}(\Omega)$ with $|\nabla u| \in L^{p(.)}(\Omega)$.*

**Definition 2.2.5** *We say that a function $\alpha : \Omega \to \mathbb{R}$ is locally log-Hölder continuous on $\Omega$ if there exists $c_1 > 0$ such that*

$$|\alpha(x) - \alpha(y)| \leq \frac{c_1}{\log(e + 1/|x-y|)}$$

*for all $x, y \in \Omega$. We say that $\alpha$ satisfies the log-Hölder decay condition if there exist $\alpha_\infty \in \mathbb{R}$ and a constant $c_2 > 0$ such that*

$$|\alpha(x) - \alpha_\infty| \leq \frac{c_2}{\log(e + |x|)}$$

*for all $x \in \Omega$.We say that $\alpha$ is globally log-Hölder continuous in $\Omega$ if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants $c_1$ and $c_2$ are called the local log-Hölder constant and the log-Hölder decay constant, respectively. The maximum $\max\{c_1, c_2\}$ is just called the log-Hölder constant of $\alpha$.*

**Definition 2.2.6** *We define the following class of variable exponents*

$$\mathcal{P}^{log}(\Omega) := \left\{ p \in \mathcal{P}(\Omega) :\ \frac{1}{p} \ \text{ is globally log-Hölder continuous} \right\}$$

*By $c_{log}(p)$ or $c_{log}$ we denote the log-Hölder constant of $\frac{1}{p}$. If $\Omega$ is unbounded, then we define $p_\infty$ by $\frac{1}{p_\infty} := \lim_{|x|\to\infty} \frac{1}{p(x)}$.*
*Note that although $\frac{1}{p}$ is bounded, the variable exponent $p$ itself can be unbounded. We would also like to remark that the definition of $p_\infty$ "commutes with duality", i.e. $p \in \mathcal{P}^{log}(\Omega)$ if and only if $p' \in \mathcal{P}^{log}(\Omega)$ and $(p_\infty)' = (p')_\infty$. Hence we do not have to distinguish between $(p_\infty)'$ and $(p')_\infty$, and write $p'_\infty$ for short.*

**Remark 2.2.2** *If $p \in \mathcal{P}(\Omega)$ with $p^+ < \infty$, then $p \in \mathcal{P}^{log}(\Omega)$ if and only if $p$ is globally log-Hölder continuous. This is due to the fact that $p \mapsto \frac{1}{p}$ is a bilipschitz mapping from $[p^-, p^+]$ to $[\frac{1}{p^+}, \frac{1}{p^-}]$.*

**Theorem 2.2.1 (Jensen's inequality)** *Let $\varphi : [0, \infty) \to \mathbb{R}$ be a convex function and $f \in L^1(a, b)$. Then*

$$\varphi\left( \fint_a^b |f(x)|\, dx \right) \leq \fint_a^b \varphi\left(|f(x)|\right)\, dx.$$

**Definition 2.2.7** *For a family $\mathcal{Q}$ of open, bounded sets $U \subset \mathbb{R}^n$ we define $T_{\mathcal{Q}} : L^1_{loc}(\mathbb{R}^n) \to L^0(\mathbb{R}^n)$ and $T_{s,\mathcal{Q}} : L^s_{loc}(\mathbb{R}^n) \to L^0(\mathbb{R}^n)$ with $s \in [1, \infty)$ by*

$$T_{\mathcal{Q}} f := \sum_{U \in \mathcal{Q}} \chi_U M_U f = \sum_{U \in \mathcal{Q}} \chi_U \fint_U |f(y)|\, dy,$$

$$T_{s,\mathcal{Q}} f := \sum_{U \in \mathcal{Q}} \chi_U M_{s,U} f = \sum_{U \in \mathcal{Q}} \chi_U \left( \fint_U |f(y)|^s\, dy \right)^{\frac{1}{s}}.$$

*The operators $T_{\mathcal{Q}}$ and $T_{s,\mathcal{Q}}$ are called averaging operator and s-averaging operator, respectively. Note that $T_{\mathcal{Q}} = T_{1,\mathcal{Q}}$. The functions $T_{\mathcal{Q}} f$ and $T_{s,\mathcal{Q}} f$ are well defined in $L^0(\mathbb{R}^n)$, since $M_{\mathcal{Q}} f \geq 0$, but might be infinite at many points or even everywhere. However, if $\mathcal{Q}$ is locally finite and $f \in L^1_{loc}(\mathbb{R}^n)$ and $g \in L^s_{loc}(\mathbb{R}^n)$, then $T_{\mathcal{Q}} f \in L^1_{loc}(\mathbb{R}^n)$ and $T_{s,\mathcal{Q}} g \in L^s_{loc}(\mathbb{R}^n)$. By Jensen's inequality we have $T_{\mathcal{Q}} f \leq T_{s,\mathcal{Q}}$ for $s \geq 1$.*

**Definition 2.2.8** *By $\mathcal{A}$ we denote the set of all generalized $\Phi$-functions $\varphi$ on $\mathbb{R}^n$ which have the property that the averaging operators $T_{\mathcal{Q}}$ are bounded from $L^\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$ uniformly for all locally 1-finite families $\mathcal{Q}$ of cubes in $\mathbb{R}^n$. The smallest constant $k$ for which*

$$\|T_{\mathcal{Q}} f\|_\varphi \leq k \|f\|_\varphi,$$

*for all locally 1-finite families of cubes $\mathcal{Q}$ in $\mathbb{R}^n$ and all $f \in L^\varphi(\mathbb{R}^n)$, will be called the $\mathcal{A}$-constant of $\varphi$. If $\varphi \in \mathcal{A}$, then we say that $\varphi$ is of class $\mathcal{A}$. In the case $\varphi_{p(.)}$, $p \in \mathcal{P}(\mathbb{R}^n)$, we denote $\varphi_{p(.)} \in \mathcal{A}$ simply by $p \in \mathcal{A}$ and call the $\mathcal{A}$-constant of $\varphi_{p(.)}$ also the $\mathcal{A}$-constant of $p$.*

*By $\mathcal{A}_{loc}$, we denote the set of all generalized $\Phi$-functions $\varphi$ on $\mathbb{R}^n$ which have the property that the averaging operators $T_{\{Q\}}$ over single cubes $Q$ are uniformly bounded from $L^\varphi(\mathbb{R}^n)$ to $L^\varphi(\mathbb{R}^n)$, ie $\sup_Q \|T_{\{Q\}} f\|_\varphi \leq k_2 \|f\|_\varphi$ for all $f \in L^\varphi(\mathbb{R}^n)$, where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. The smallest constant $k_2$ will be called the $\mathcal{A}_{loc}$-constant of $\varphi$. If $\varphi \in \mathcal{A}_{loc}$, then we say that $\varphi$ is of class $\mathcal{A}_{loc}$. In the case $\varphi_{p(.)}$, $p \in \mathcal{P}(\mathbb{R}^n)$ we denote $\varphi_{p(.)} \in \mathcal{A}_{loc}$ simply by $p \in \mathcal{A}_{loc}$.*

**Theorem 2.2.2** *Let $p \in \mathcal{P}(\mathbb{R}^n)$. Then the following are equivalent:*

*(a) $p \in \mathcal{A}_{loc}$.*

*(b) $p' \in \mathcal{A}_{loc}$.*

*(c) $\|\chi_Q\|_{p(.)}\|\chi_Q\|_{p'(.)} \approx |Q|$ uniformly for all cubes $Q \subset \mathbb{R}^n$*

*(d) $\|\chi_Q\|_{p(.)} \approx |Q|^{\frac{1}{p_Q}}$ and $\|\chi_Q\|_{p'(.)} \approx |Q|^{\frac{1}{p'_Q}}$ uniformly for all cubes $Q \subset \mathbb{R}^n$.*

*The statement remains true if we replace cubes by balls.*

**Corollary 2.2.1** *Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$. Then for every $m > 0$ there exists $\beta \in (0,1)$ only depending on $m$ and $c_{log}(p)$ such that*

$$\begin{aligned} \int_Q \varphi_{p(x)}\left(\beta \int_Q |f(y)|\,dy\right) dx &\leq \int_Q \varphi_{p(y)}(|f(y)|)\,dy + \int_Q (e+|y|)^{-m}\,dy, \\ \int_Q \varphi_{p(x)}\left(\beta \int_Q |f(y)|\,dy\right) dx &\leq \int_Q \varphi_{p(.)}(|f(y)|)\,dy + |\{y \in Q \,:\, 0 < |f(y)| \leq 1\}| \end{aligned}$$

*for every cube (or ball) $Q \subset \mathbb{R}^n$ and all $f \in L^{p(.)}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ with $\|f\|_{L^{p(.)}(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} \leq 1$.*

**Theorem 2.2.3** *If $p \in \mathcal{P}^{log}(\mathbb{R}^n)$, then $p \in \mathcal{A}$ with $\mathcal{A}$-constant depending only on $c_{log}(p)$ and $n$. Moreover,*

$$\|T_{\mathcal{Q}} f\|_{p(.)} \leq cN\|f\|_{p(.)}$$

*for every locally $N$-finite family $\mathcal{Q}$ of cubes (or balls) and all $f \in L^{p(.)}(\mathbb{R}^n)$. The constant $c$ depends only on $c_{log}(p)$.*

**Proof.** *Let $f \in L^{p(.)}(\mathbb{R}^n)$ with $\|f\|_{p(.)} \leq 1$ and let $\mathcal{Q}$ be a locally $N$-finite family of cubes (or balls). Then by the unit ball property $\varrho_{p(.)}(f) \leq 1$. Let $m > n$ be such that $\int_{\mathbb{R}^n}(e+|y|)^{-m}\,dy \leq 1$. Choose $\beta \in (0,1)$ as in Corollary 2.2.1. Then*

$$\begin{aligned} \varrho_{p(.)}\left(\frac{1}{2N}\beta T_{\mathcal{Q}} f\right) &\leq \frac{1}{2N}\sum_{Q\in\mathcal{Q}} \int_Q \varphi_{p(x)}\left(\beta\chi_Q M_Q f\right) dx \\ &\leq \frac{1}{2N}\sum_{Q\in\mathcal{Q}}\left(\int_Q \varphi_{p(y)}(|f(y)|)\,dy + \int_Q (e+|y|)^{-m}\,dy\right) \\ &\leq \frac{1}{2}\left(\int_{\mathbb{R}^n} \varphi_{p(x)}(f)\,dx + \int_{\mathbb{R}^n}(e+|x|)^{-m}\,dx\right) \\ &\leq \frac{1}{2}(\varrho_{p(.)}(f)+1) \leq 1. \end{aligned}$$

*Where we have used that $\mathcal{Q}$ is locally $N$-finite. This implies $\|T_{\mathcal{Q}} f\|_{p(.)} \leq \frac{2N}{\beta}$. A scaling argument yields the $\|T_{\mathcal{Q}} f\|_{p(.)} \leq \frac{2N}{\beta}\|f\|_\varphi$. The case $N = 1$ with cubes implies $p \in \mathcal{A}$.*

**Proposition 2.2.1** *If $p \in \mathcal{P}^{log}(\Omega)$, then it has an extension $q \in \mathcal{P}^{log}(\mathbb{R}^n)$ with $c_{log}(q) = c_{log}(p)$, $q^- = p^-$, and $q^+ = p^+$. If $\Omega$ is unbounded, then additionally $q_\infty = p_\infty$.*

**Proof.** *Let $c_1 > 0$ and $p_\infty \geq 1$ be such that*

$$\left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \frac{c_1}{\log(e + 1/|x-y|)} \text{ and } |p(x) - p_\infty| \leq \frac{c_1}{\log(e+|x|)},$$

*for all point $x, y \in \Omega$. Since $t \mapsto 1/\log(e + 1/t)$ is a modulus of continuity, we can use the extension of McShane-type [20] to extend $\frac{1}{p}$ to $\mathbb{R}^n$ with the same modulus of continuity and lower and upper bound. More precisely, we define $a \in C(\mathbb{R}^n)$ by*

$$\frac{1}{a(y)} := \sup_{x\in\Omega}\left(\frac{1}{p(z)} - \frac{c_1}{\log(e + 1/|z-y|)}\right)$$

*for $y \in \mathbb{R}^n$. In particular, a is locally log-Hölder continuous with local log-Hölder constant less or equal to $c_1$, and $a(y) = 1/p(y)$ for all $y \in \Omega$.*

*In order to ensure that our extension satisfies also the log-Hölder decay condition and has the same lower and upper bound as p we define q by truncation*

$$\frac{1}{q(y)} := \min\left\{\max\left\{\frac{1}{a(y)}, \frac{1}{p_\infty} - \frac{c_2}{\log(e+|x|)}, \frac{1}{p^+_\Omega}\right\}, \frac{1}{p_\infty} + \frac{c_1}{\log(e+|x|)}, \frac{1}{p^-_\Omega}\right\}$$

*for all $y \in \mathbb{R}^n$. Since $x \mapsto \frac{c_2}{\log(e+|x|)}$ is globally log-Hölder continuous with constant $c_2$, we see the log-Hölder constant of $\frac{1}{q}$ does not exceed $\max\{c_1, c_2\} = c_{log}(p)$. The decay condition of $\frac{1}{p}$ ensures that $\frac{1}{q(y)} = a(y) = \frac{1}{p(y)}$ for all $y \in \Omega$. Therefore q is the variable exponent we are looking for.*

**Lemma 2.2.2** *Let $\Omega$ be a bounded $\alpha$-John domain and let $p \in \mathcal{P}^{log}(\Omega)$. Then*

$$\|1\|_{L^{p(\cdot)}(\Omega)}\|1\|_{L^{p'(\cdot)}(\Omega)} \approx |\Omega|,$$

*where the constant only depends on $c_{log}(p), \alpha$ and n.*

**Proof.** *Extend p by Proposition 2.2.1 to $\mathbb{R}^n$ preserving $c_{log}(p)$. Let B be the John ball of $\Omega$, then $B \subset \Omega \subset 2\alpha B$. By Theorem 2.2.2 we find that*

$$\|1\|_{L^{p(\cdot)}(\Omega)}\|1\|_{L^{p'(\cdot)}(\Omega)} \leq \|1\|_{L^{p(\cdot)}(2\alpha B)}\|1\|_{L^{p'(\cdot)}(2\alpha B)} \leq c|2\alpha B| \leq c(2\alpha)^n|\Omega|.$$

*On the other hand by Hölder's inequality $|\Omega| \leq 2\|1\|_{L^{p(\cdot)}(\Omega)}\|1\|_{L^{p'(\cdot)}(\Omega)}$.*

**Lemma 2.2.3** *Let $\Omega$ be a bounded $\alpha$-John domain and let $p \in \mathcal{P}^{log}(\Omega)$. If $A \subset \Omega$ has positive finite measure, then*

$$c\frac{|A|}{|\Omega|}\|u - \langle u\rangle_A\|_{L^{p(\cdot)}(\Omega)} \leq \|u - \langle u\rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq c\|u - \langle u\rangle_A\|_{L^{p(\cdot)}(\Omega)}$$

*for $u \in L^{p(\cdot)}(\Omega)$, where $c$ depends on the dimension $n$, $c_{log}(p)$ and $\alpha$.*

**Proof.** *By the triangle inequality, $\|u - \langle u\rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq \|u - \langle u\rangle_A\|_{L^{p(\cdot)}(\Omega)} + \|\langle u\rangle_A - \langle u\rangle_\Omega\|_{L^{p(\cdot)}(\Omega)}$. We estimate the second term by Hölder's inequality:*

$$\begin{aligned}\|\langle u\rangle_A - \langle u\rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} &= |\langle u\rangle_A - \langle u\rangle_\Omega|\,\|1\|_{L^{p(\cdot)}(\Omega)}\\ &= |\Omega|^{-1}\|u - \langle u\rangle_A\|_{L^1(\Omega)}\|1\|_{L^{p(\cdot)}(\Omega)}\\ &\leq c\frac{\|1\|_{L^{p'(\cdot)}(\Omega)}\|1\|_{L^{p(\cdot)}(\Omega)}}{|\Omega|}\|u - \langle u\rangle_A\|_{L^{p(\cdot)}(\Omega)}.\end{aligned}$$

*Since $p \in \mathcal{P}^{log}(\Omega)$, the fraction in the last estimate is bounded by a constant according to Lemma 2.2.2. The lower bound is proved similarly.*

By $Q(x, r)$ we denote a cube with side length r and center at $x \in \mathbb{R}^n$.The cube is dyadic if it is of the form $Q(x, 2^k)$ for some $x \in 2^k\mathbb{Z}^n$ and $k \in \mathbb{Z}$.

**Definition 2.2.9** *For $k \in \mathbb{Z}$ we define the averaging operator $T_k$ by*

$$T_k f := \sum_{\substack{Q \text{ dyadic}\\ diam(Q)=2^{-k}}} \chi_Q M_{2Q} f$$

*for all $f \in L^1_{loc}(\mathbb{R}^n)$.*

**Remark 2.2.3** *Note that $T_k$ can be written as the sum of $2^n$-averaging operators $T_{Q_j}$ with locally 1-finite families of shifted dyadic cubes, namely, each $2Q$ is split into $2^n$ disjoint cubes of the same size as $Q$. This property ensures the boundedness of $T_k$ on $L^\varphi(\mathbb{R}^n)$ if $\varphi \in \mathcal{A}$.*

**Lemma 2.2.4** *Let $x \in \mathbb{R}^n$, $\delta > 0, 0 < \alpha < n$, and $f \in L^1_{loc}(\mathbb{R}^n)$. Then*

$$\int_{B(x,\delta)} \frac{|f(y)|}{|x - y|^{n-\alpha}}\,dy \leq c(\alpha)\delta^\alpha \sum_{k=0}^{\infty} 2^{-\alpha k} T_{k+k_0} f(x) \leq c(\alpha)\delta^\alpha Mf(x),$$

*where $k_0 \in \mathbb{Z}$ is chosen such that $2^{-k_0-1} \leq \delta \leq 2^{-k_0}$.*

**Proof.** *We split the integration domain into annuli and use the definition of $T_k$:*

$$\begin{aligned}\int_{B(x,\delta)} \frac{|f(y)|}{|x-y|^{n-\alpha}}\,dy &\le \sum_{k=1}^{\infty}(\delta 2^{-k})^{\alpha-n} \int_{2^{-k}\delta\le|x-y|\le 2^{-k+1}\delta} |f(y)|\,dy \\ &\le c\sum_{k=1}^{\infty}(\delta 2^{-k})^{\alpha} \int_{|x-y|<2^{-k+1}\delta} |f(y)|\,dy \\ &\le c\delta^{\alpha}\sum_{k=0}^{\infty} 2^{-\alpha k} T_{k+k_0} f(x).\end{aligned}$$

*This is the first inequality. Since $T_{k+k_0} f \le Mf$, the second inequality follows by convergence of the geometric series.*

**Definition 2.2.10** *By $\mathcal{G}$ we denote the set of all generalized $\Phi$-functions $\varphi$ on $\mathbb{R}^n$ which have the property that*

$$\sum_{Q\in\mathcal{Q}} \|\chi_Q f\|_{\varphi}\|\chi_Q g\|_{\varphi^*} \le K\|f\|_{\varphi}\|g\|_{\varphi^*}$$

*for all $f \in L^{\varphi}(\mathbb{R}^n)$, $g \in L^{\varphi^*}(\mathbb{R}^n)$, and all locally 1-finite families $\mathcal{Q}$ of cubes. The smallest constant $K$, is called the $\mathcal{G}$-constant of $\varphi$. If $\varphi \in \mathcal{G}$, then we say that $\varphi$ is of class $\mathcal{G}$. In the case $\varphi_{p(.)}$, we denote $\varphi_{p(.)} \in \mathcal{G}$ simply by $p \in \mathcal{G}$.*
*The name "class $\mathcal{G}$ is derived from the works of Berzhnoi for ideal Banach spaces [3]. In the notation of Berezhnoi $\varphi \in \mathcal{G}$ is just $(L^{\varphi}, L^{\varphi^*}) \in G(\mathcal{X}^n)$, where $\mathcal{X}^n$ is the set of all cubes in $\mathbb{R}^n$.*

**Corollary 2.2.2** *Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ and $\mathcal{Q}$ be a locally N-finite family of cubes or balls $Q \subset \mathbb{R}^n$. Then*

$$\|T_{p(.),\mathcal{Q}} f\|_{p(.)} \approx \left\|\sum_{Q\in\mathcal{Q}} \chi_Q g\right\|_{p(.)}$$

*for all $f \in L^{p(.)}_{loc}(\mathbb{R}^n)$. The implicit constants only depend on $c_{log}(p)$, $n$ and $N$. The dependence on $N$ is linear.*

**Theorem 2.2.4** *If $p \in \mathcal{P}^{log}(\mathbb{R}^n)$, then $p \in \mathcal{G}$ and the $\mathcal{G}$-constant only depends on $c_{log}(p)$ and $n$.*

**Proof.** *Remark 2.2.2 implies that $p' \in \mathcal{P}^{log}(\mathbb{R}^n)$. Thus Corollary 2.2.2 yields $\|T_{p(.),\mathcal{Q}} f\|_{p(.)} \approx \|\sum_{Q\in\mathcal{Q}} \chi_Q f\|_{p(.)}$ and $\|T_{p'(.),\mathcal{Q}} g\|_{p'(.)} \approx \|\sum_{Q\in\mathcal{Q}} \chi_Q g\|_{p'(.)}$ for all $f \in L^{p(.)}_{loc}(\mathbb{R}^n)$ and $g \in L^{p'(.)}_{loc}(\mathbb{R}^n)$. Moreover, by Theorem 2.2.3, $p \in \mathcal{A}$ and therefore Theorem 2.2.2 implies that $|Q| \approx \|\chi_Q\|_{p(.)}\|\chi_Q\|_{p'(.)}$.*

*Thus*

$$\begin{aligned}\sum_{Q\in\mathcal{Q}} \|\chi_Q f\|_{p(.)}\|\chi_Q g\|_{p'(.)} &\approx \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{Q}} \chi_Q \frac{\|\chi_Q f\|_{p(.)}}{\|\chi_Q\|_{p(.)}} \frac{\|\chi_Q g\|_{p'(.)}}{\|\chi_Q\|_{p'(.)}}\, dx \\ &\leq \int_{\mathbb{R}^n} T_{p(.),\mathcal{Q}} f T_{p'(.),\mathcal{Q}} g\, dx \\ &\leq c\|T_{p(.),\mathcal{Q}} f\|_{p(.)}\|T_{p(.),\mathcal{Q}} g\|_{p'(.)} \\ &\approx \|f\|_{p(.)}\|g\|_{p'(.)}.\end{aligned}$$

For domain with $|\Omega| < \infty$ we denote the subspace of $L^{p(.)}$ consisting of functions with vanishing integral by

$$L_0^{p(.)} := \left\{ f \in L^{p(.)}(\Omega) : \int_\Omega f(x)\, dx = 0 \right\}$$

In the case that $|\Omega| = \infty$ we set $L_0^{p(.)}(\Omega) := L^{p(.)}(\Omega)$.The space of compactly supported smooth functions with vanishing integral we denote by $C^\infty_{0,0}(\Omega)$.

**Theorem 2.2.5 (Decomposition on theorem)** *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the emanating chain condition with constants $\sigma_1$ and $\sigma_2$ and with chain covering $\mathcal{Q}$. Then there exists a family of linear operators $S_Q : L_0^\infty(\Omega) \to L_0^\infty(Q)$ with $Q \in \mathcal{Q}$ which also maps $C^\infty_{0,0}(\Omega) \to C^\infty_{0,0}(Q)$, such that for every $p \in \mathcal{P}^{log}(\Omega)$ with $p^- > 1$ the following holds :*

*(a) For each $Q \in \mathcal{Q}$, the operator norm of $S_Q : L_0^{p(.)}(\Omega) \hookrightarrow L_0^{p(.)}(Q)$ depends only on $\alpha,\ p^-,\ c_{log}(p)$ and $n$.*

*(b) For every $f \in L_0^{p(.)}(\Omega),\ |S_Q f| \leq c\sigma_2 \chi_Q M f$ a.e.*

*(c) The family $S_Q f$ is a decomposition of $f$, i.e.*

$$f = \sum_{Q\in\mathcal{Q}} S_Q f \quad \text{and in } L^1_{loc}(\Omega)$$

*for all $f \in L_0^{p(.)}(\Omega)$. If $p^+ < \infty$, then the sum converges unconditionally in $L_0^{p(.)}(\Omega)$, i.e. every permutation of the series converges in $L_0^{p(.)}(\Omega)$.*

*(d) For all $f \in L_0^{p(.)}(\Omega)$*

$$\left\| \sum_{Q\in\mathcal{Q}} \chi_Q \frac{\|S_Q f\|_{L_0^{p(.)}(Q)}}{\|\chi_Q\|_{p(.)}} \right\|_{p(.)} \approx \|f\|_{L_0^{p(.)}(\Omega)}$$

*with constant only depending on $\alpha,\ p^-,\ c_{log}(p)$ and $n$.*

*(e) For all $f \in L_0^{p(.)}(\Omega)$ and $g \in L^{p'(.)}(\Omega)$*

$$\int_\Omega fg\, dx = \sum_{Q\in\mathcal{Q}} \int_Q S_Q f g\, dx = \sum_{Q\in\mathcal{Q}} \int_Q S_Q f (g - \langle g\rangle_Q)\, dx.$$

*(f) If $\Omega$ is bounded and $f \in C^\infty_{0,0}(\Omega)$, then $\{Q \in \mathcal{Q} \ :\ S_Q f \neq 0\}$ is finite.*

**Proposition 2.2.2** *Let $\Omega \subset \mathbb{R}^n$ be a domain satisfying the emanating chain condition with constants $\sigma_1$ and $\sigma_2$ and with chain covering $\mathcal{Q}$. Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ with $p^+ < \infty$. Then*

$$\|f - \langle f\rangle_\Omega\|_{L_0^{p(.)}(\Omega)} \approx \left\| \sum_{Q\in\mathcal{Q}} \chi_Q \frac{\|f - \langle f\rangle_Q\|_{L^{p(.)}(Q)}}{\|\chi_Q\|_{p(.)}} \right\|_{L^{p(.)}(\Omega)}$$

*for all $f \in L^{p(.)}(\Omega)$. The constant only depends on $p^+$, $c_{log}(p)$ and the chain condition constants.*

**Proof.** *Let $f \in L^{p(.)}(\Omega)$ and $g \in L_0^{p'(.)}(\Omega)$ we use (e) of the decomposition theorem for the space $L_0^{p'(.)}(\Omega)$. Hölder's inequality, and Theorem 2.2.4 to conclude that*

$$\begin{aligned}
\int_\Omega (f - \langle f\rangle_\Omega) g\,dx &= \sum_{Q\in\mathcal{Q}} \int_Q (f - \langle f\rangle_Q) S_Q g\,dx \\
&\le 2 \sum_{Q\in\mathcal{Q}} \|f - \langle f\rangle_Q\|_{L^{p(.)}(Q)} \|\chi_Q S_Q g\|_{L_0^{p'(.)}(\Omega)} \\
&\le c \left\| \sum_{Q\mathcal{Q}} \chi_Q \frac{\|f - \langle f\rangle_Q\|_{L^{p(.)}(Q)}}{\|\chi_Q\|_{p(.)}} \right\|_{p(.)} \left\| \sum_{q\in\mathcal{Q}} \chi_Q S_Q g \right\|_{L_0^{p'(.)}(\Omega)}
\end{aligned}$$

*Note that $\sum_{Q\in\mathcal{Q}} \chi_Q S_Q g = g$, so the right-hand side simplifies. Then we take the supremum over $g$ with $\|g\|_{L_0^{p'(.)}(\Omega)} \le 1$ and use the norm conjugate formula (Corollary 1.2.4)*

$$\begin{aligned}
\|f - \langle f\rangle_\Omega\|_{L_0^{p(.)}(\Omega)} &\approx \sup_{\|g\|_{L_0^{p'(.)}(\Omega)} \le 1} \int_\Omega (f - \langle f\rangle_\Omega) g\,dx \\
&\le c \left\| \sum_{Q\in\mathcal{Q}} \chi_Q \frac{\|f - \langle f\rangle_Q\|_{L^{p(.)}(Q)}}{\|\chi_Q\|_{p(.)}} \right\|_{p(.)}.
\end{aligned}$$

*This proves the first direction of the claim. Since $p \in \mathcal{P}^{log}(\Omega)$ we have $p \in \mathcal{A}$ and therefore*

$$\begin{aligned}
\|f - \langle f\rangle_Q\|_{L^{p(.)}(Q)} &\le \|f - \langle f\rangle_\Omega\|_{L^{p(.)}(Q)} + \|\chi_Q M_Q f - \langle f\rangle_\Omega\|_{L^{p(.)}(Q)} \\
&\le c\|f - \langle f\rangle_\Omega\|_{L^{p(.)}(Q)}
\end{aligned}$$

*This and Theorem 2.2.4 prove the other direction of the claim.*

**Theorem 2.2.6 (Poincaré inequality)** . *Let $p \in \mathcal{P}^{log}(\Omega)$ or $p \in \mathcal{A}$.*
*(i) For every $u \in W_0^{1,p(.)}(\Omega)$, the inequality*

$$\|u\|_{L^{p(.)}(\Omega)} \le c\ diam(\Omega) \|\nabla u\|_{L^{p(.)}(\Omega)}$$

*holds with a constant c depending only on the dimension n and $c_{log}(p)$.*
*(ii) If $\Omega$ is a bounded $\alpha$-John domain, then*

$$\|u - \langle u\rangle_\Omega\|_{L^{p(.)}(\Omega)} \le c\ diam(\Omega) \|\nabla u\|_{L^{p(.)}(\Omega)}$$

*for $u \in W^{1,p(.)}(\Omega)$. The constant c depends only on the dimension n, $\alpha$ and $c_{log}(p)$.*

**Proof.** *We prove only the latter case. The proof for the first case is similar; the only difference is to use in Lemma* 2.2.1 *case* $(a)$ *instead of case* $(b)$*. We note that* $p \in \mathcal{P}^{log}(\Omega)$ *can be extended to* $\mathbb{R}^n$ *so that* $p \in \mathcal{A}$ (*Proposition* 2.2.1 *and Theorem* 2.2.3).

*By Lemma* 2.2.1 $(b)$ *and Lemma* 2.2.4 *we obtain*

$$|u(x) - \langle u\rangle_B| \leq I_1|\nabla u|(x) \leq c\ diam(\Omega) \sum_{k=0}^{\infty} 2^{-k} T_{k+k_0}|\nabla u|(x)$$

*for every* $u \in W^{1,p(.)}(\Omega)$ *and almost every* $x \in \Omega$*, where* $k_0 \in \mathbb{Z}$ *is chosen such that* $2^{-k_0-1} \leq diam(\Omega) \leq 2^{-k_0}$*. Since* $p \in \mathcal{A}$*, the averaging operator* $T_{k+k_0}$ *is bounded on* $L^{p(.)}(\Omega)$ *(cf. Remark 2.2.3). Using also the triangle inequality, we obtain*

$$\begin{aligned} \|u - \langle u\rangle_B\|_{L^{p(.)}(\Omega)} &\leq c\ diam(\Omega) \sum_{k=0}^{\infty} 2^{-k} \|T_{k+k_0}|\nabla u|\ \|_{L^{p(.)}} \\ &\leq c\ diam(\Omega) \|\nabla u\|_{L^{p(.)}(\Omega)}. \end{aligned}$$

*The estimate for* $\|u - \langle u\rangle_\Omega\|_{L^{p(.)}(\Omega)}$ *follows from this and Lemma* 2.2.3.

In the case that $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ with $p^+ < \infty$ we give an alternative proof for *Theorem* 2.2.6 $(b)$ based on the decomposition *Theorem* 2.2.5. The proof is not self-contained since it uses the variable exponent Poincaré inequality in cubes.

*Proof of Theorem* 2.2.6 $(b)$ *for* $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ *with* $p^+ < \infty$. Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ with $p^+ < \infty$. Let $Q$ be the chain covering of $\Omega$. Let $u \in W^{1,p(.)}(\Omega)$ with $\fint_\Omega u\,dx = 0$. Then *Proposition* 2.2.2, the Poincaré inequality in cubes, $diam(Q) \leq diam(\Omega)$ for $Q \in \mathcal{Q}$, and *Theorem* 2.2.4 yield

$$\begin{aligned} \|u\|_{L_0^{p(.)}(\Omega)} &\leq c \left\| \sum_{Q\in\mathcal{Q}} \chi_Q \frac{\|u - \langle u\rangle_Q\|_{L^{p(.)}(Q)}}{\|\chi_Q\|_{p(.)}} \right\|_{L^{p(.)}(\Omega)} \\ &\leq c \left\| \sum_{Q\in\mathcal{Q}} \chi_Q \frac{diam(Q)\|\nabla u\|_{L^{p(.)}(Q)}}{\|\chi_Q\|_{p(.)}} \right\|_{L^{p(.)}(\Omega)} \\ &\leq c\ diam(\Omega) \left\| \sum_{Q\in\mathcal{Q}} \chi_Q \frac{\|\nabla u\|_{L^{p(.)}(Q)}}{\|\chi_Q\|_{p(.)}} \right\|_{L^{p(.)}(\Omega)} \\ &\leq c\ diam(\Omega)\, \|\nabla u\|_{L^{p(.)}(\Omega)}\,. \end{aligned}$$

*Theorem* 2.2.6 $(a)$ immediately yields.

**Corollary 2.2.3** *Let* $\Omega$ *be bounded and* $p \in \mathcal{P}^{log}(\Omega)$ *or* $p \in \mathcal{A}$*. For every* $u \in W_0^{1,p(.)}(\Omega)$ *the inequality*

$$\|\nabla u\|_{L^{p(.)}(\Omega)} \leq \|u\|_{W^{1,p(.)}(\Omega)} \leq (1 + c\ diam(\Omega))\|\nabla u\|_{L^{p(.)}(\Omega)}$$

*holds with constant c depending only on the dimension n, and* $c_{log}(p)$*.*

*Theorem* 2.2.6 (*b*) and *Remark* 2.2.1 yields the following corollary:

**Corollary 2.2.4** *Let $\Omega$ be a bounded $\alpha$-John domain and let $p \in \mathcal{P}^{log}(\Omega)$ or $p \in \mathcal{A}$. Furthermore, let $A \subset \Omega$ be such that $|A| \approx |\Omega|$. Then*

$$\|u - \langle u \rangle_A\|_{L^{p(.)}(\Omega)} \leq c\, diam(\Omega) \|\nabla u\|_{L^{p(.)}(\Omega)}$$

*for $u \in L^1_{loc}(\Omega)$ with $|\nabla u| \in L^{p(.)}(\Omega)$. The constant $c$ depends only on the dimension $n$, $\alpha$ and $c_{log}(p)$.*

Next we will talk about the modular versions of the poincaré inequality. The following example shows that in the case of the variable exponent in general, the poincaré inequality can not hold in a modular form. Although, in the constant exponent case, there is an obvious connection between modular and norm version. For instance, A. Lerner showed that

$$\int_{\mathbb{R}^n} |Mf|^{p(x)}\, dx \leq c \int_{\mathbb{R}^n} |f|^{p(x)}\, dx$$

if and only if $p \in (1, \infty)$ is constant.

**Example 2.2.1** *Let $p : (-2, 2) \to [2, 3]$ be a Lipschitz continuous exponent that equals 3 in $(-2, -1) \cup (1, 2)$, 2 in $(-\frac{1}{2}, \frac{1}{2})$ and is linear elsewhere. Let $u_\lambda$ be a Lipschitz function such that $u_\lambda(\pm 2) = 0$, $u_\lambda = \lambda$ in $(-1, 1)$ and $|u'_\lambda| = \lambda$ in $(-2, -1) \cup (1, 2)$. Then*

$$\frac{\bar{\varrho}_{p(.)}(u_\lambda)}{\bar{\varrho}_{p(.)}(u'_\lambda)} = \frac{\int_{-2}^{2} |u_\lambda|^{p(x)}\, dx}{\int_{-2}^{2} |u'_\lambda|^{p(x)}\, dx} \geq \frac{\int_{-1/2}^{1/2} \lambda^2\, dx}{2 \int_{-2}^{-1} \lambda^3\, dx} = \frac{1}{2\lambda} \to \infty$$

*as $\lambda \to 0^+$.*

If p is continuous and has a minimum or maximum then the modular poincaré inequality $\varrho_{p(.)}(u) \leq c\varrho_{p(.)}(\nabla u)$ does not hold, this is according to Fan,Zhao and Zhang in [10]. In another way, the inequality holds if there exists a function $\xi \in W^{1,1}(\Omega)$ such that $\nabla p . \nabla \xi \geq 0$ and $\nabla \xi \not\equiv 0$, like Allegretto shows in [1]. this holds if p is suitably monotone.

**Proposition 2.2.3** *Let $p \in \mathcal{P}^{log}(\Omega)$ be a bounded exponent.*
*(a) Let $\Omega$ be bounded. For $m > 0$ there exist a constant $c$ depending on the dimension $n$, $c_{log}(p)$, $m$, and $p^+$ such that*

$$\int_\Omega \left( \frac{|u|}{diam(\Omega)} \right)^{p(x)} dx \leq c \int_\Omega |\nabla u|^{p(x)}\, dx + c \int_{B(x, diam(\Omega))} (e + |x|)^{-m}\, dx$$

*for all* $u \in W_0^{1,p(.)}(\Omega)$ *with* $\|\nabla u\|_{L^{p(.)}(\Omega)} \leq 1$ *and all* $z \in \Omega$.

*(b) Let* $\Omega$ *be a bounded* $\alpha$*-John domain. For* $m > 0$ *there exist a constant* $c$ *depending on the dimension* $n$, $c_{log}(p)$, $m$, $p^+$, $\alpha$, $|\Omega|$ *and* $diam(\Omega)$ *such that*

$$\int_\Omega \left(\frac{|v - \langle v\rangle_B|}{diam(\Omega)}\right)^{p(x)} dx \leq c \int_\Omega |\nabla v|^{p(x)}\, dx + c \int_{B(z,diam(\Omega)} (e + |x|)^{-m}\, dx$$

*for all* $v \in W^{1,p(.)}(\Omega)$ *with* $\|\nabla v\|_{L^{p(.)}(\Omega)} \leq 1$ *and all* $z \in \Omega$. *The ball* $B$ *is from Lemma* 2.2.1.

**Proof.** *Assume that* $u \in W_0^{1,p(.)}(\Omega)$ *is extended by zero outside* $\Omega$ (*Lemma* 2.1.1). *By Lemma 2.2.1 (a) and* 2.2.4 *we obtain*

$$|u(x)| \leq c \int_\Omega \frac{|\nabla u(y)|}{|x - y|^{n-1}}\, dy \leq c\, diam(\Omega) \sum_{k=0}^{\infty} 2^{-k} T_{k+k_0} |\nabla u|(x),$$

*where* $k_0 \in \mathbb{Z}$ *is chosen such that* $2^{-k_0-1} \leq diam(\Omega) \leq 2^{-k_0}$. *Exactly the same estimate holds with* $u$ *replaced by* $v - \langle v\rangle_B$ *using Lemma* 2.2.1 (*b*), *so it suffices to derive the estimate of the first claim involving* $u$.

*We divide by* $diam(\Omega)$ *and raise both sides of this inequality to the power* $p(x)$, *integrate over* $\Omega$, *and use* $p^+ < \infty$ *to obtain*

$$\begin{aligned}\int_\Omega \left(\frac{|u|}{diam(\Omega)}\right)^{p(x)} dx &\leq c \int_\Omega \Big(\sum_{k=0}^{\infty} 2^{-k} T_{k+k_0} |\nabla u|\Big)^{p(x)} dx \\ &\leq c \sum_{k=0}^{\infty} 2^{-k} \int_\Omega (T_{k+k_0} |\nabla u|)^{p(x)}\, dx,\end{aligned}$$

*where we used convexity in the second step. Since* $\varrho_{L^{p(\cdot)}(\Omega)}(\nabla u) \leq 1$ *by the unit ball property, we may use Corollary* 2.2.1 *for* $|\nabla u|_{\chi_\Omega}$ *on the ball* $B(z, diam(\Omega))$ *and get*

$$\int_\Omega (T_{k+k_0} |\nabla u|)^{p(x)}\, dx \leq c \int_\Omega |\nabla u|^{p(x)}\, dx + c \int_{B(z,diam(\Omega)} (e + |x|)^{-m}\, dx,$$

*where we also used that the dyadic cubes* $2Q$ *are locally N-finite. Combining the last two inequality proves the claim.*

**Remark 2.2.4** *If* $p \in \mathcal{P}^{log}(\Omega)$ *with no restriction on* $p^+$, *then the first estimate in Proposition* 2.2.3 *reads*

$$\int_\Omega \varphi_{p(x)} \left(\beta \frac{|u|}{diam(\Omega)}\right) dx \leq \int_\Omega \varphi_{p(x)}(|\nabla u|)\, dx + \int_{B(z,diam(\Omega)} (e + |x|)^{-m}\, dx$$

*for some* $\beta \in (0, 1)$ *and all* $u \in W_0^{1,p(.)}(\Omega)$ *with* $\|\nabla u\|_{L^{p(.)}(\Omega)} \leq 1$. *The constant* $\beta$ *depends only on* $c_{log}(p)$, $m$ *and* $n$. *The second estimate in Proposition* 2.2.3 *has to be changed accordingly. The proof is the same.*

**Remark 2.2.5** *It is possible to replace* $\langle u\rangle_B$ *in Lemma* 2.2.1 $(b)$ *and Proposition* 2.2.3 *by* $\langle u\rangle_\Omega$ *or even* $\langle u\rangle_A$*, where* $A \subset \Omega$ *is a set with* $|A| \approx |\Omega|$*. Indeed, it follows from Jensen's inequality and Corollary* 2.2.4 *for* $p = 1$ *that*

$$|\langle u\rangle_A - \langle u\rangle_B| \leq \fint_A |u - \langle u\rangle_B|\,dy \leq c \fint_\Omega |u - \langle u\rangle_B|\,dy \leq c\, diam(\Omega) \fint_\Omega |\nabla u|\,dy$$

*for all* $u \in W^{1,1}(\Omega)$*. In particular, this implies*

$$|\langle u\rangle_A - \langle u\rangle_B| \leq c \fint_\Omega \frac{|\nabla u(y)|}{(diam(\Omega))^{n-1}}\,dy \leq cI_1(\nabla u)(x).$$

*for any* $x \in \Omega$*. This proves the modified version of Lemma* 2.2.1 $(b)$*. With this new estimate we get the modified version of Proposition* 2.2.3 *with no change in the proof.*

**Theorem 2.2.7 (Key estimate)** *Let* $p \in \mathcal{P}^{log}(\mathbb{R}^n)$*. Then for every* $m > 0$ *there exists* $\beta \in (0,1)$ *only depending on m and* $c_{log}(p)$ *such that*

$$\varphi_{p(x)}\left(\beta \fint_Q |f(y)|\,dy\right)$$
$$\leq \begin{cases} \fint_Q \varphi_{p(y)}(|f(y)|)\,dy + \frac{1}{2}\left(\fint_Q ((e+|x|)^{-m} + (e+|y|)^{-m}\chi_{\{0<|f(y)|\leq 1\}}\,dy\right)^{p^-} \\ \fint_Q \varphi_{p(y)}(|f(y)|)\,dy + \frac{1}{2}\fint_Q ((e+|x|)^{-m} + (e+|y|)^{-m})\chi_{\{0<|f(y)|\leq 1\}}\,dy \end{cases}$$

*for every cube (or ball)* $Q \subset \mathbb{R}^n$*, all* $x \in Q$*, and all* $f \in L^{p(.)}(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ *with* $\|f\|_{L^{p(.)}(\mathbb{R}^n)+L^\infty(\mathbb{R}^n)} \leq 1$.

Note that if $p^+ < \infty$ then in the previous Lemmas [7, Lemma 4.2.3, Lemma 4.2.2, Lemma 4.2.1] and theorem we can take the constant $\beta$ out from $\varphi_{p(.)}$. For example, in the later case of the previous theorem we obtain

$$\varphi_{p(x)}\left(\fint_Q |f(y)|\,dy\right)$$
$$\leq c \fint_Q \varphi_{p(y)}(|f(y)|)\,dy + c \fint_Q ((e+|x|)^{-m} + (e+|y|)^{-m})\chi_{\{0<|f(y)|\leq 1\}}\,dy,$$

where the constant c depends only on m, $c_{log}(p)$ and $p^+$.

**Lemma 2.2.5 (Jensen inequality with singular measure)** *Let* $p \in \mathcal{P}^{log}(\mathbb{R}^n)$*. For every* $m > 0$ *there exists* $\beta \in (0,1)$ *only depending on m and* $c_{log}(p)$ *such that*

$$\varphi_{p(x)}\left(\beta \fint_B \frac{|f(y)|}{r|x-y|^{n-1}}\,dy\right) \leq \fint_B \frac{\varphi_{p(y)}(|f(y)|)}{r|x-y|^{n-1}}\,dy + M\left((e-|.|)^{-m}\right)(x)$$

*for every* $R > 0$*,* $B := B(z,r)$*,* $x \in B$ *and every* $f \in L^{p(.)}(B) \bigcap L^\infty(B)$ *with* $\|f\|_{L^{p(.)}(B)+L^\infty(B)} \leq 1$.

**Proof.** *Fix $r > 0$ and let $B$ be a ball with radius $r$. Define annuli $A_k := \{y \in B : 2^{-k} \le |x-y|R^{-1} \le 2^{1-k}\}$ for $k \ge 0$. As in the proof of Lemma 2.2.4, we split the $B$ into the annuli $A_k$ and obtain*

$$\int_B \frac{|f(y)|}{r|x-y|^{n-1}}\,dy \le c_1 \sum_{k=0}^{\infty} 2^{-k} \int_{B(x,\,2^{1-k})} \chi_{A_k}|f|\,dy.$$

*Let $\beta_2 > 0$ be the $\beta$ from the Key estimate Theorem 2.2.7. Since $\sum_{k=0}^{\infty} 2^{-k} \le 1$, it follows by convexity that*

$$(I) := \varphi_{p(x)}\left(\frac{\beta_2}{c_1} \int_B \frac{|f(y)|}{r|x-y|^{n-1}}\,dy\right) \le \sum_{k=1}^{\infty} 2^{-k}\varphi_{p(x)}\left(\beta_2 \int_{B(x,\,2^{1-k})} \chi_{A_k}|f|\,dy\right).$$

*So the Key estimate, Theorem 2.2.7, yields*

$$\begin{aligned}(I) &\le \sum_{k=1}^{\infty} 2^{-k}\Big( \int_{B(x,\,2^{1-k})} \chi_{A_k}\varphi_{p(y)}(|f|)\,dy \\ &\qquad + \frac{1}{2}(e+|x|)^{-m} + \frac{1}{2}\int_{B(x,\,2^{1-k})} (e+|y|)^{-m}\,dy\Big) \\ &\le \int_B \frac{\varphi_{p(.)}(|f(y)|)}{r|x-y|^{n-1}}\,dy + M((e+|.|)^{-m})(x).\end{aligned}$$

The following improvement of *Proposition* 2.2.3 is useful in the study of $p(.)$-minimizers and is the starting point for reverse Hölder estimates. The result is from Schwarzacher [28].

**Proposition 2.2.4** *Let $p \in \mathcal{P}^{log}(\mathbb{R}^n)$ satisfy $1 < p^- \le p^+ < \infty$ and let $s \le p^-$ satisfy $s \in [1, \frac{n}{n-1})$. Then for every $m > 0$ there exist a constant $c$ depending on $n$, $c_{log}(p)$, $m$, and $s$ such that*

$$\begin{aligned}\int_{B_R}\left(\frac{|u|}{R}\right)^{p(x)} dx &\le c\left(\int_{B_R} |\nabla u|^{\frac{p(.)}{s}}\,dx\right)^s + c\int_{B_R}(e+|x|)^{-ms}\,dx \\ \int_{B_R}\left(\frac{|v-\langle v\rangle_{B_R}|}{R}\right)^{p(x)} dx &\le c\left(\int_{B_R} |\nabla v|^{\frac{p(.)}{s}}\,dx\right)^s + c\int_{B_R}(e+|x|)^{-ms}\,dx\end{aligned}$$

*for every ball $B_R$ with radius $R$, and every $u \in W_0^{1,\frac{p(.)}{s}}(B_R)$, $v \in W^{1,\frac{p(.)}{s}}(B_R)$ with $\|\nabla u\|_{L^{p(.)/s}+L^\infty}$, $\|\nabla v\|_{L^{q(.)/s}+L^\infty} \le 1$.*

**Proof.** *By Jensen's inequality the case $s > 1$ implies the case $s = 1$ and thus we may assume that $s > 1$. By Lemma* 2.2.1 *we have, for $x \in B_R$,*

$$|u(x)| \le c\int_{B_R} \frac{|\nabla u(y)|}{|x-y|^{n-1}}\,dy, \qquad |v(x) - \langle v\rangle_{B_R}| \le c\int_{B_R} \frac{|\nabla v(y)|}{|x-y|^{n-1}}\,dy.$$

*Starting from here the proofs for u and v are the same, so we just present the estimate for u. With the previous estimate, the help of Lemma 2.2.5 applied to $p(.)/s$ and $p^+ < \infty$ we get*

$$\begin{aligned}(I) &:= \int_{B_R} \left(\frac{|u(x)|}{R}\right)^{p(x)} dx \leq c(p^+) \int_{B_R} \left( \int_{B_R} \frac{|\nabla u(y)|}{R|x-y|^{n-1}}\, dy \right)^{p(x)} dx \\ &\leq c \int_{B_R} \left( \int_{B_R} \frac{|\nabla u(y)|^{\frac{p(y)}{s}}}{R|x-y|^{n-1}}\, dy \right)^{s} dx + c \int_{B_R} \left(M\left((e+|.|)^{-m}\right)(x)\right)^{s}\, dx \\ &=: (II) + (III).\end{aligned}$$

*In order to estimate (II) we set $J := \int_{B_R} |\nabla u|^{p(.)/s}\, dx$. We can assume $J > 0$ in the following, since otherwise $\nabla u = 0$ and there is nothing to estimate. We apply Jensen's inequality for the probability measure $\mu := |\nabla u|^{p(.)/s}/J$ and the convex function $t \mapsto t^s$, then we use Fubini's theorem to change the integration order and $\int_{B_R} \frac{dx}{|x-y|^{s(n-1)}} \approx R^{-s(n-1)+n}$ for a.e. $y \in B_R$ using $s < \frac{n}{n-1}$:*

$$\begin{aligned}(II) &\leq c \int_{B_R} J^{s-1} \int_{B_R} \frac{|\nabla u(y)|^{\frac{p(y)}{s}}}{R^s |x-y|^{s(n-1)}}\, dy\, dx \\ &\leq c J^{s-1} R^{n-sn} \int_{B_R} |\nabla u(y)|^{\frac{p(y)}{s}}\, dy \leq c \left( \int_{B_R} |\nabla u(y)|^{\frac{p(y)}{s}}\, dy \right)^{s}.\end{aligned}$$

*Since $s > 1$, we can use the boundedness of M on $L^s(B_R)$ to conclude*

$$(III) \leq c \int_{B_R} (e+|x|)^{-ms}\, dx$$

*Combining the estimates for $(I) - (III)$ we obtain the result.*

**Remark 2.2.6** *Exactly as in the Remarks* 2.2.4 *and* 2.2.5 *it is possible to modify Proposition* 2.2.4 *to include the case $p^+ = \infty$: we have to replace the constant c on the right-hand side by a constant $\beta$ on the left-hand side. As in Proposition 2.2.3 and Remark* 2.2.5 *we can replace the integration domain $B_R$ by a bounded John domain $\Omega$ and the mean value $\langle v \rangle_{B_R}$ by $\langle v \rangle_A$ for any $A \subset \Omega$ with $|A| \approx |\Omega|$.*

In the following Lemma, the case $r < n$ and $q = r^*$ is due to B.Bojarski [2, (6,6)] and in the case $q < r^*$ follows from it by Hölder's inequality. By $r^*$ we denote the Sobolev conjugate exponent of $r < n$, $r^* := nr/(n-r)$.

**Lemma 2.2.6** *Let $\Omega \subset \mathbb{R}^n$ be a bounded $\alpha$-John domain. If $1 \leq r < n$ and $r \leq q \leq r^*$ or if $r \geq n$ and $q < \infty$, then*

$$\|u - \langle u \rangle_\Omega\|_{L^q(\Omega)} \leq c|\Omega|^{\frac{1}{n}+\frac{1}{q}-\frac{1}{r}} \|\nabla u\|_{L^r(\Omega)}$$

*for all $u \in W^{1,r}(\Omega)$. In the first case the constant c depends only on n, r and $\alpha$, while in the second case it depends also on q.*

We can prove the Poincaré inequality in bounded John domains for variable exponents by using the previous constant exponent Sobolev-Poincaré inequality.

**Lemma 2.2.7** *Let $\Omega \subset \mathbb{R}^n$ be bounded $\alpha$-John domain. If $p \in \mathcal{P}(\Omega)$ is bounded with $p^+ \leq (p^-)^*$ or $p^- \geq n$, then there exists a constant $c$ depending on the dimension $n$, $p^-$, $p^+$, and $\alpha$ such that for every $u \in W^{1,p(\cdot)}(\Omega)$ we have*

$$\|u - \langle u \rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq c(1 + |\Omega|)^2 |\Omega|^{\frac{1}{n} + \frac{1}{p^+} - \frac{1}{p^-}} \|\nabla u\|_{L^{p(\cdot)}(\Omega)}.$$

**Proof.** *Assume first that $p^+ \leq (p^-)^*$. Since $p(x) \leq p^+ \leq (p^-)^*$ we deduce from Corollary* 1.3.1 *and Lemma* 2.2.6 *that*

$$\begin{aligned}
\|u - \langle u \rangle_\Omega\|_{p(\cdot)} &\leq 2(1 + |\Omega|)\|u - \langle u \rangle_\Omega\|_{p^+} \\
&\leq c(n, p^-, \alpha)(1 + |\Omega|)|\Omega|^{\frac{1}{n} + \frac{1}{p^+} - \frac{1}{p^-}} \|\nabla u\|_{p^-} \\
&\leq c(n, p^-, \alpha)(1 + |\Omega|)^2 |\Omega|^{\frac{1}{n} + \frac{1}{p^+} - \frac{1}{p^-}} \|\nabla u\|_{p(\cdot)}.
\end{aligned}$$

*Let $p^- \geq n$. We choose a constant $q \in [1, n)$ such that $p^+ = q^*$. We obtain the result by using similar arguments than in the previous case. The only difference is that the constant in the second inequality in the above chain of inequalities is $c(n, p^+, \alpha)$.*

**Lemma 2.2.8** *For a constant $q \in (1, n)$, arbitrary $x \in \mathbb{R}^n$ and $R > r > 0$ we have*

$$\inf_{B(x,R)} |\nabla u|^q \, dx = c \left| \frac{q - n}{q - 1} \right|^{q-1} \left| R^{\frac{q-n}{q-1}} - r^{\frac{q-n}{q-1}} \right|^{1-q},$$

*where the infimum is taken over all $u \in C_0^\infty(B(x, R))$ with $u|_{B(x,r)} = 1$. Here the constant $c$ depends only on the dimension $n$.*

The next proposition shows that the poincaré inequality does not hold for general non-constant exponents,

**Proposition 2.2.5** *Let $B$ be a unit ball in the plane. For every $q_1 \in [1, 2)$ and $q_2 \in (2, \infty)$ there exists $p \in \mathcal{P}(B)$ with $p^- = q_1$ and $p^+ = q_2$ such that the norm-version of the Poincaré inequality,*

$$\|u - \langle u \rangle_\Omega\|_{L^{p(\cdot)}(B)} \leq c \|\nabla u\|_{L^{p(\cdot)}(B)},$$

*does not hold.*

**Proof.** *Our aim is to construct a sequence of functions in $B \subset \mathbb{R}^2$ for which the constant in the Poincaré inequality goes to infty. Let $e_1 = (1,0)$, $B_i := B\left(2^{-i}e_1, \frac{1}{4}2^{-i}\right) \subset B$ and $B_i' := B\left(2^{-i}e_1, \frac{1}{8}2^{-i^2}\right) \subset B$ for every $i \in \mathbb{N}$. Let $u_i \in C_0^\infty(B_i)$ with $u_i|_{\overline{B'}_i} = 1$. Define $p := q_2$ in every $B_i'$ and $p := q_1$ otherwise in $B$ with positive first coordinate. Since $\nabla u_i = 0$ in $B_i'$ we obtain*

$$\|\nabla u_i\|_{L^{p(\cdot)}(B_i)} = \|\nabla u_i\|_{L^{q_1}(B_i)}.$$

*Let $\tilde{B}_i := B\left(-2^{-i}e_1, \frac{1}{4}2^{-i}\right)$. We extend $u_i$ to $B$ as an odd function of the first coordinate in $\tilde{B}_i$ and by zero elsewhere. We extend $p$ to $B$ as an even function of the first coordinate. We denote these extensions by $\tilde{u}_i$ and $\frac{p^*}{n'}$.*

*By Lemma 2.2.8 we may choose the functions $u_i$ such that*

$$\|\nabla \tilde{u}_i\|_{L^{\frac{p^*(\cdot)}{n'}}(B)} \leq c(q_1)\left|\left(\frac{1}{4}2^{-i}\right)^{\frac{q_1-2}{q_1-1}} - \left(\frac{1}{8}2^{-i^2}\right)^{\frac{q_1-2}{q_1-1}}\right|^{\frac{1-q_1}{q_1}}.$$

*For large $i$, the right-hand side is approximately equal to $c(q_1)2^{-i^2\frac{2-q_1}{q_1}}$. Since $\langle \tilde{u}_i\rangle_B = 0$, we obtain*

$$\|\tilde{u}_i - \langle \tilde{u}_i\rangle_B\|_{L^{\frac{p^*(\cdot)}{n'}}(B)} = \|\tilde{u}_i\|_{L^{\frac{p^*(\cdot)}{n'}}(B)} \geq |B_i'|^{\frac{1}{q_2}} \geq c2^{-i^2\frac{2}{q_2}}.$$

*Combining the previous two inequalities, we find that*

$$\frac{\|\tilde{u}_i - \langle \tilde{u}_i\rangle_B\|_{L^{\frac{p^*(\cdot)}{n'}}(B)}}{\|\nabla \tilde{u}_i\|_{L^{\frac{p^*(\cdot)}{n'}}(B)}} \geq c(q_1)2^{i^2(\frac{2}{q_1}-1-\frac{2}{q_2})} \to \infty$$

*as $i \to \infty$ if $\frac{2}{q_1} - 1 - \frac{2}{q_2} > 0$, that is, if $q_2 > \frac{2q_1}{2-q_1} > 2$.*

**Theorem 2.2.8** *Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain and $p \in \mathcal{P}(\Omega)$ be bounded. Assume that there exist John domains $D_i$, $i = 1, \dots, j$, such that $\Omega = \cup_{i=1}^j D_i$ and either $p^+_{D_i} \leq (p^-_{D_i})^*$ or $p^-_{D_i} \geq n$ for every $i$. Then there exists a constant $c$ such that*

$$\|u - \langle u\rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} \leq c\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

*for every $u \in W^{1,p(\cdot)}(\Omega)$. The constant $c$ depends on $n$, $diam(\Omega)$, $|D_i|$, $p$ and the John constants of $\Omega$ and $D_i$, $i = 1, \dots, j$.*

**Proof.** *Using the triangle inequality we obtain*

$$\begin{aligned}\|u - \langle u\rangle_\Omega\|_{L^{p(\cdot)}(\Omega)} &\leq \sum_{i=1}^j \|u - \langle u\rangle_\Omega\|_{L^{p(\cdot)}(D_i)} \\ &\leq \sum_{i=1}^j \|u - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)} + \sum_{i=1}^j \|\langle u\rangle_\Omega - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)}.\end{aligned}$$

*We estimate the first part of the sum using Lemma* 2.2.7. *This yields for every* $i = 1, \dots, j$

$$\|u - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)} \leq c\|\nabla u\|_{L^{p(\cdot)}(D_i)} \leq c\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

*with constants depending on* $n, p^+_{D_i}, p^-_{D_i}, |D_i|, \alpha_i$, *where* $\alpha_i$ *is the John constant of* $D_i$. *Therefore it remains only to estimate the sum of the terms* $\|\langle u\rangle_\Omega - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)}$. *We use the constant exponent Poincaré inequality* (*in the third inequality* ):

$$\begin{aligned}
\|\langle u\rangle_\Omega - \langle u\rangle_{D_i}\|_{L^{p(\cdot)}(D_i)} &\leq \|1\|_{L^{p(\cdot)}(D_i)} \int_{D_i} |u(x) - \langle u\rangle_\Omega|\, dx \\
&\leq \|1\|_{L^{p(\cdot)}(D_i)}|D_i|^{-1} \int_{\Omega} |u(x) - \langle u\rangle_\Omega|\, dx \\
&\leq c(n, diam(\Omega), \alpha)|D_i|^{-1}\|1\|_{L^{p(\cdot)}(D_i)}\|\nabla u\|_{L^1(\Omega)} \\
&\leq c(n, diam(\Omega), \alpha)|D_i|^{-1}\|1\|_{L^{p(\cdot)}(D_i)}\|\nabla u\|_{L^{p(\cdot)}(\Omega)}
\end{aligned}$$

*for every* $i = 1, \dots, j$. *Here* $\alpha$ *is the John constant of* $\Omega$. *By Corollary* 1.3.1 $\|1\|_{L^{p(\cdot)}(D_i)}$ *depends only on* $p$ *and* $|D_i|$, *which completes the proof.*

The previous theorem shows that we can replace the condition $p^+ \leq (p^-)^*$ in lemma 2.2.7by a set of local conditions. Next, by using Lemma 2.2.7 we prove the poincaré inequality for Sobolev functions with zero boundary values.

**Theorem 2.2.9** *Let* $\Omega$ *be bounded. Assume that* $p \in \mathcal{P}(\Omega)$ *and there exists* $\delta > 0$ *such that for every* $x \in \Omega$ *either*

$$P^+_{B(x,\delta)\cap\Omega} \leq \frac{np^-_{B(x,\delta)\cap\Omega}}{n - p^-_{B(x,\delta)\cap\Omega}} \quad or \quad p^-_{B(x,\delta)\cap\Omega} \geq n. \tag{2.2}$$

*Alternatively, assume that* $p$ *is uniformly continuous in* $\Omega$. *Then the inequality*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq c\|\nabla u\|_{L^{p(\cdot)}(\Omega)},$$

*holds for every* $u \in W^{1,p(\cdot)}_0(\Omega)$. *Here the constant* $c$ *depends on* $p$, $\Omega$, $\delta$ *and the dimension* $n$.

**Proof.** *Note that if* $p$ *is continuous in* $\overline{\Omega}$ *or uniformly continuous in* $\Omega$, *then* $p$ *satisfies the first set of conditions of the theorem for some* $\delta > 0$.
*By the assumptions there exist* $x_1, \dots, x_j$ *and* $\delta > 0$ *such that*

$$\Omega \subset \bigcup_{i=1}^{j} B(x_i, \delta)$$

*and each ball* $B(x_i, \delta)$ *satisfies either of the two inequalities in* (2.2). *We write* $B_i := B(x_i, \delta)$ *and denote by* $\chi_i$ *the characteristic function of* $B_i \cap \Omega$. *In each* $B_i$ *we define* $p_i(x) := p(x)\chi_i + p^-_{B_i\cap\Omega}(1 -$

$\chi_i$). *Then in each* $B_i$ *either* $p_i^+ \le (p_i^-)^*$ *or* $p_i^- \ge n$. *Let* $\tilde{u}$ *be the zero extension of* $u \in W_0^{1,p(\cdot)}(\Omega)$ *to* $\mathbb{R}^n \backslash \Omega$ (*Lemma* 2.1.1). *By the triangle inequality we obtain*

$$\begin{aligned}\|u\|_{L^{p(\cdot)}(\Omega)} &\le \Big\|\tilde{u}\sum_{i=1}^{j}\chi_i\Big\|_{L^{p(\cdot)}(\Omega)} \le \sum_{i=1}^{j}\|\tilde{u}\|_{L^{p_i(\cdot)}(B_i)} \\ &\le \sum_{i=1}^{j}\|\tilde{u}-\langle\tilde{u}\rangle_{B_i}\|_{L^{p_i(\cdot)}(B_i)} + \sum_{i=1}^{j}|\langle\tilde{u}\rangle_{B_i}|\,\|1\|_{L^{p_i(\cdot)}(B_i)}.\end{aligned}$$

*We estimate the first sum on the right-hand side of the previous inequality. By Lemma* 2.2.7 *we obtain*

$$\begin{aligned}\|\tilde{u}-\langle\tilde{u}\rangle_{B_i}\|_{L^{p_i(\cdot)}(B_i)} &\le c(1+|B_i|)^2|B_i|^{\frac{1}{n}+\frac{1}{p_i^+}-\frac{1}{p_i^-}}\|\nabla\tilde{u}\|_{L^{p_i(\cdot)}(B_i)} \\ &\le c(1+|B_i|)^2|B_i|^{\frac{1}{n}+\frac{1}{p_i^+}-\frac{1}{p_i^-}}\|\nabla u\|_{L^{p(\cdot)}(\Omega)}.\end{aligned}$$

*for every* $i = 1, \dots, j$ *To estimate the second sum,* $\sum_{i=1}^{j}|\langle\tilde{u}\rangle_{B_i}|\|1\|_{L^{p_i(\cdot)}(B_i)}$, *in the above inequality we use Lemma* 1.2.14 *to estimate* $\|1\|_{L^{p_i(\cdot)}(B_i)}$ *by a constant depending only on* $p$ *and* $\delta$. *Further, the constant exponent Poincaré inequality implies that*

$$\begin{aligned}|\langle\tilde{u}\rangle_{B_i}| &\le \frac{c(n)}{\delta^n}\fint_{\Omega}|u|\,dx \le \frac{c}{\delta^n}diam(\Omega)\fint_{\Omega}|\nabla u|\,dx \\ &\le \frac{c(n)}{\delta^n}diam(\Omega)(1+|\Omega|)\|\nabla u\|_{L^{p(\cdot)}(\Omega)},\end{aligned}$$

*again for every* $i = 1, \dots, j$. *Combining the last three estimates yields the assertion.*

**Remark 2.2.7** *Assume that* $\Omega$ *is convex and* $p \in \mathcal{P}(\Omega)$ *is uniform continuous* (*or* $p \in C(\overline{\Omega})$). *As in the proof of Theorem* 2.2.9 *we may cover* $\Omega$ *by finitely many balls* $B(x_i, \delta)$ *so that* (2.2) *holds. Since* $\Omega$ *is convex so is* $B(x_i, \delta) \cap \Omega$ *and thus it is a John domain. Hence Theorem* 2.2.8 *yields that the Poincaré inequality*

$$\|u - \langle u\rangle_{\Omega}\|_{L^{p(\cdot)}(\Omega)} \le c\|\nabla u\|_{L^{p(\cdot)}(\Omega)}$$

*holds for every* $u \in W^{1,p(\cdot)}(\Omega)$.

# Conclusion

In Conclusion, the space $L^{p(.)}$ is a special case of generalized Orlicz space or (Musielak-Orlicz) space with $\varphi_{p(.)} = \widetilde{\varphi}_{p(.)}$ or $\varphi_{p(.)} = \bar{\varphi}_{p(.)}$ , for instance, we considered some proprieties in this space as a results from the properties in the Musielak-Orlicz space, and there are also a semimodular space. The study of this space was very import for begin our study of Sobolev functions, as we see in the definition 2.1.2 $W^{m,p(.)}(\Omega) = \left\{u \in L^{p(.)},\ D^{\alpha}u \in L^{p(.)}(\Omega),\ |\alpha| \leq m\right\}$. One motivation of study this functions is that solutions of partial differential equation belong to Sobolev space.
in the future, if we continue we will try to study the variational integrals with non-standard growth and coercivity conditions(non-standard variational problems).

# Bibliography


[1] W. Allegretto. *Form estimates for the $p(x)$-Laplacian*. Proc. Amer. Math. Soc. 135:2177-2185, 2007.

[2] B. Bojarski. *Remarks on Sobolev imbedding inequalities*. In Complex analysis, Joensua 1987, volume 1351 of Lecture Notes in Math, page 52-68. Springer, Berlin, 1988.

[3] E. I. Berezhnoi. Two-weighted estimations for the Hardy-Littlewood maximal function in ideal Banach spaces. Proc. Amer. Math. Soc, 127:79-87, 1999.

[4] H. Brezis. Analyse Fonctionnelle théorie et applications. Masson. Paris. 1987.

[5] S. Boulaaras, A. Choucha, D. Ouchenane and R. Jan. *Blow up, growth, and decay of solution for a class of coupled nonlinear viscoelastic Kirchhoff equations with distributed delay and variable exponents*. Springer. 2024.

[6] D. V. Cruz-Uribe, A. Fiorenza, Variable Lebesgue Spaces, Foundations and Harmonic Analysis, BirKhäuser/Springer, New York, 2013.

[7] L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and Sobolev Spaces with variable Exponents, Springer, 2017.

[8] L. Diening. *Maximal function on generalized Lebesgue spaces $L^{p(.)}$*. Math. Inequal. Appl., 7:245-253, 2004.

[9] X. Fan and D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$*, journal of Mathematical Analysis and Application, volume 263, pages 424-446, 2001.

[10] X. L. Fan, Q. H.Zhang, and D.Zhao. *Eigenvalues of $p(x)$-Laplacian Dirichlet problem*. J. Math. Anal. Appl., 302:306-317, 2005.

[11] D. Gilbarg and N. Trudinger. Elliptic partial differential equations of second order. Classics in Mathematics. Springer-Verlag. Berlin. 2001. Reprint of the 1998 edition.

[12] A. E. Hamidi. *Existence results to elliptic nonstandard growth conditions*. J. Math. Anal. Appl., 300(1):30-42, 2004.

[13] J. Heinonen, T. Kilpeläin, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*,(Dover Publications Inc, Mineola,NY, 2006) Unabridged republication of the 1993 original.

[14] P. Harjulehto, P. Hästö, Orlicz Spaces and Generalized Orlicz Spaces, Springer, Nature Switzerland, 2019.

[15] P. Halmos. Measure Theory. D.Van Nostrand Company, Inc, New York, N. Y., 1950.

[16] J. Heinonen, T. Kilpeläimen, and O. Martio. *Non linear potential theory of degenerate elliptic equations*. Dover Publications Inc, Mineola, NY, 2006. Unabridged republication of the 1993 original.

[17] O. Kováčik and J. Rákosnik, *On the spaces Lp(x) and Wk,p(x)*, czechoslovak math. j., 41(116):592-618, 1991.

[18] S. Kakutani, *Weak convergence in uniformly convex spaces*, Tohoku Math, J.,45:188-193, 1938.

[19] O. Méndez, J. Lang, Analysis on Function Spaces of Musielak-Orlicz Type, CRC Press Taylor&Francis Group,Boca Raton London New York, 2019.

[20] P. Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of cambridge studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1995.

[21] J. Malý and W. P. Ziemer. *Fine regularity of solutions of elliptic partial differential equations, volume 51 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.*

[22] C. P. Nicalescu, L. person,Convex Function and Their Application Springer, 2004.

[23] D. Ouchenane, S. Boulaaras, A. choucha & M. Alegga. *Blow-up and general decay of solutions for a Kirchhoff-type equation with distributed delay and variable-exponents.* Quaestions Mathematica. 2023.

[24] W. Orlicz. *Über konjugierte Exponentenfolgen*. Studia Math., 3:200-211, 1931.

[25] E. Picskin, B. Okutmuştur,An Introduction to Sobolev Space, Benthan Books, 2021.

[26] L. Pick, A. Kufuer, O. John, S. Fučik, Function Spaces, volume 1, De Gruyter, 2013.

[27] I. Sharapudinov. *On the topology of the space $L^{p(t)}$*([0; 1]). Math. Notes, 26(3-4):796-806, 1979.

[28] S. Schwarzacher. *Higher integrability of elliptic differential equations with variable growth*. Master's thesis, University of Freiburg, Germany, February 2010.