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THEME

Global existence and exponential stability for some hyperbolic systems with damping terms

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Dedication

I dedicate this work,

To the person who always sacrificed herself for me, my eternal example, my source of inspiration, my life and my happiness, the one who helped me and encouraged me during my studies to you, my mom.

To the person who supported me during all these years, who always encouraged me to do my best, to you, my father.

To my brother Hocine and my sister Anya.

To my grandmother and my aunts for being supportive and caring, for bringing the joy and happiness to my days.

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Résumé

Dans ce travail nous étudions l'existence, l'unicité et la stabilité de la solution de quelques systèmes hyperboliques. Nous commençons par rappeler quelques notions de base, ensuite nous introduirons une équation non-linéaire unidimensionnelle de thermoélasticité avec deuxième son, nous étudions l'existence, la régularité et la stabilité de la solution. Ensuite nous allons voir le système de Timoshenko ou nous distinguerons deux cas : Celui ou la propagation est égale et celui ou elle ne l'est pas, nous construirons une fonctionnelle de Lyapunov pour étudier les différentes formes de stabilité.

Mots clés : Système hyperbolique, système de Timoshenko, stabilité, régularité, fonctionnelle de Lyapunov, thermoélasticité, deuxième son.

ملخص

في هذا العمل ندرس وجود، وحدانية واستقرار حل بعض الأنظمة الزائدية. نبدأ بتذكر بعض المفاهيم الأساسية، ثم سنقدم معادلة غير خطية أحادية البعد للمرونة الحرارية مع الصوت الثاني ندرس وجود الحل وانتظامه واستقراره. ثم سنرى نظام تيموشينكو « Timoshenko » حيث سنميز حالتين: الحالة التي يكون فيها سرعة انتشار الأمواج متساويَّة والحالة التي تكون فيها مختلفة، سنقوم ببناء مؤثر ليابونوف « Lyaponov » لدراسة الأشكال المختلفة للاستقرار.

الكلمات المفتاحية: النظام الزائدي، نظام تيموشينكو « Timoshenko »، الاستقرار، الانتظام، مؤثر ليابونوف« Lyaponov »، ا لمرونة الحرارية، الصوت الثاني.

Abstract

In this work we study the existence, the unicity and the stability of the solution of some hyperbolic systems. We start by recalling some basic notions, then we will introduce a one-dimensional nonlinear equation of thermoelasticity with second sound we study the existence, the regularity and the stability of the solution. Then we will see the Timoshenko system where we will distinguish two cases: The one where the propagation is equal and the one where it is not, we will build a Lyapunov functional to study the different forms of stability.

Keywords: Hyperbolic system, Timoshenko system, stability, regularity, Lyapunov functional, thermoelasticity, second sound.

Contents

General Introduction

As we all know Mathematic is one of the richest sciences. Among its different field of research we find PDEs (Partial Differential Equations) where most of physics, biologic, civil engineering and other science problems are modeled into a system of PDEs. Then the researcher will look for a solution and investigate for the stability of this one.
In our work we will study two famous types of this equation.
The first one is the equation of one-dimensional non-linear thermoelasticity with thermal memory and second sound. Thermoelastic model is based on the continuum physics and thermodynamics, considering the distribution of the thermoelastic body deformation and temperature.
The classical model for the heat propagation turns into the well-known Fourier law

$$q + k\nabla\theta = 0, \tag{1}$$

where θ is temperature (difference to a fixed constant reference temperature), q is the heat conduction vector and k is the coefficient of thermal conductivity. The model using classic Fourier law exhibits the physical paradox of infinite propagation speed of signals. To eliminate this paradox a generalized thermoelasticity theory has been developed subsequently. The development of this theory was accelerated by the advent of the second sound effects observed experimentally in materials at a very low temperature. In heat transfer problems involving very short time intervals and/or very high heat fluxes, it has been revealed that the inclusion of the second sound effects to the original theory yields results which are realistic and very much different from those obtained with the classical Fourier's law.
The first theory, developed by Lord and Shulman [15], replaces the equation (1) with the Cattaneo–Maxwell law

$$\tau_0 q_t + q + k\theta_x = 0. \tag{2}$$

The heat equation associated with equation (2) becomes hyperbolic and, hence, automatically eliminates the paradox of infinite speeds. The positive parameter τ_0 is the relaxation time describing the time lag in the response of the heat flux to a gradient in the temperature. In this work, we shall prove the global existence and exponential stability of solutions to nonlinear thermoelastic equations with second sound provided that the initial data are close to the equilibrium and the relaxation kernel is strongly positive definite and decays exponentially. In [2] Z.KHALILI,D.OUCHENANE and A. ELHAMIDI studied Stability results for laminated beam with thermo-visco-elastic effects and localized nonlinear damping

The other considered model is the Timoshenko-type system with a past history. The first one who introduced this model is Stephen TIMOSHENKO in [27], he modelled the transverse vibrations of a beam by the conservative system of hyperbolic coupled equation:

$$\begin{cases} \rho u_{tt} = (k(u_x - \varphi))_x, & in \quad (0,L) \times (0,+\infty), \\ \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + k(u_x - \varphi), & in \quad (0,L) \times (0,+\infty). \end{cases} \tag{3}$$

where t denotes the time variable and x is the space variable along the beam of length L; in its equilibrium conguration, u is the transverse displacement of the beam and φ is the rotation angle of the lament of the beam. The coeffcients ρ, I_ρ, E, I an k Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus. The model considered here is a modified Timoshenko system, that is, a Timoshenko system in classical thermoelasticity of type I. In the last three decades, the system (3) has been intensively studied for possible damping mechanisms. Muñoz Rivera and Racke [25] introduced a thermal damping by coupling system (3) with the classical heat equation. They proved that the system

$$\begin{cases} \rho_1 \varphi_{tt} = k(\varphi_x + \psi)_x, \\ \\ \rho_2 \psi_{tt} = b\psi_{xx} - k(\varphi_x + \psi) + \delta\theta_x, \\ \\ c\theta_t = \kappa\theta_{xx} - \gamma\psi_{xt}, \end{cases} \tag{4}$$

(of course with some boundary and initial conditions), is exponentially stable if and only if

$$\frac{\rho_1}{k} = \frac{\rho_2}{b} \tag{5}$$

if (5) does not hold Guesmia *et al.* [3] established a polynomial decay result provided that the initial data are regular enough. Almeida Junior *et al.* [4] considered the thermal coupling of the system (3) in shear force

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x + \sigma\theta_x = 0, & in \quad (0,L) \times (0,+\infty), \\ \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi)_x - \sigma\theta_x = 0, & in \quad (0,L) \times (0,+\infty), \\ \\ \rho_3 \theta_t - \gamma\theta_{xx} - \sigma(\varphi_x + \psi)_t = 0, & in \quad (0,L) \times (0,+\infty). \end{cases} \tag{6}$$

subjected to either the boundary conditions

$$\varphi(t,0) = \varphi(t,L) = \psi(t,0) = \psi(t,L) = \theta(t,0) = \theta(t,L) = 0, \tag{7}$$

or

$$\varphi(t,0) = \varphi(t,L) = \psi_x(t,0) = \psi_x(t,L) = \theta_x(t,0) = \theta_x(t,L) = 0. \tag{8}$$

and proved that the solution is exponentially stable if and only if

$$\chi = \frac{k}{\rho_1} - \frac{b}{\rho_2} = 0. \tag{9}$$

The model considered in our work is a modified Timoshenko system, that is, a Timoshenko system in classical thermoelasticity of type I.
We shall use multiplier techniques to prove the stability property for the system with a past history. For a kernel of polynomial decay, we prove the polynomial stability results for the equal wave-speed propagation, and establish a decay result for the nonequal wave-speed case under the assumption that g decays exponentially.
Moreover, the existence of the global attractor is achieved.
This memory is devided into three major chapters as follows:
Chapter 1: In this chapter we remind of some basic knowledge in modern analysis, some basic definitions and theorems, some useful inequalities and basic theory of semigroups.
Chapter 2: We investigate on global existence and exponential stability of one-dimensional nonlinear thermoelasticity with thermal memory and second sound, using the semigroups theory, the regularity theorem and the fundamental theorem of Hille-Yoshida.
Chapter 3: We introduce a Timoshenko-type system with a past history. We study the stability of the solution of this system for both equal and non-equal wave speed cases using the estimation of a Lyaponov functional equivalent to the energy function. We will see that it's exponentially stable for the first one and decays in the rate of $(1/t)$ (Polynomial Stability) for the other one.

Chapter 1

Preliminary

1.1 Some definitions and theorems in modern analysis

In this memory we denote by $\mathbb{R}^n$ the Euclid space, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $C^k(\mathbb{R}^n)$ (or $C^k(\Omega)$) is the kth differentiable continuous function space in $\mathbb{R}^n$ (or Ω)), $C^\infty(\mathbb{R}^n)$ (or $C^\infty(\Omega)$) is the ∞th differentiable continuous functions space in $\mathbb{R}^n$ (or Ω), $C_c^\infty(\mathbb{R}^n)$ (or $C_c^\infty(\Omega)$) is the ∞th differentiable continuous functions space with compact support in $\mathbb{R}^n$ (or Ω)).The norm for the space $C^\infty(\mathbb{R}^n)$ is definded as

$$C^k(\mathbb{R}^n) = \left\{ f | f \in C^k(\mathbb{R}^n), \| f \|_{C^k(\mathbb{R}^n)} = \sum_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |\partial^\alpha f(x)| < +\infty \right\}, \tag{1.1.1}$$

where $\partial^\alpha f = \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} ... \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$,$|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$.

The notation $\mathcal{S}(\mathbb{R}^n)$ denote the Schwartz space defined as:

$$\mathcal{S}(\mathbb{R}^n) = \left\{ \phi | \phi \in C^\infty(\mathbb{R}^n), ||\phi||_{(\alpha,\beta)} = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| < +\infty, \text{for all } \alpha, \beta \in \mathbb{N}^n \right\}. \tag{1.1.2}$$

The space $\mathcal{S}(\mathbb{R}^n)$ endowds the quasi norm $||.||_{(\alpha,\beta)}$ in (1.2) is a Frechet space and $C_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$, the space $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of $\mathcal{S}(\mathbb{R}^n)$. $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz class of rapidly decreasing functions.

Let (X, M, μ) be a non-atomic measurable space, for a complex or real-valued measurable function $f(x)$ defined on X, its distributional function is defined by:

$$f_*(\sigma) = \mu\left\{x \in X : f(x) > \sigma, \text{for all } \sigma > 0\right\} \tag{1.1.3}$$

which is non-increasing and continuous from the right. Furthermore, its non-increasing rearrangement $f*$ is defined by;

$$f*(t) = \inf\left\{s > 0 : f_*(s) \leq t\right\}, \text{ for all } t > 0, \tag{1.1.4}$$

which is also non increasing and continuous from the right and has the same distributional function as $f(x)$. $\xi'(\mathbb{R}^3)$ is the space of all tempred distributions on $\mathbb{R}^3$, $\mathcal{D}(\mathbb{R}^3)$ is the set of all scalar polynomials defined on $\mathbb{R}^3$.
$L^p(\mathbb{R}^3)$ $(1 \leq p < +\infty)$ denotes the lebesgue space of all $L^p(\mathbb{R}^3)$ integrable functions associated with the norms

$$||f||_p = \left(\int_{\mathbb{R}^3} |f(x)|^p dx\right)^{\frac{1}{p}}, 1 \leq p < +\infty. \tag{1.1.5}$$

$$||f||_\infty = ess\sup_{x\in\mathbb{R}^3} |f(x)|, p = +\infty. \tag{1.1.6}$$

Definition 1.1.1 *Let $\Omega \in \mathbb{R}^n$ be an open set, $\mathcal{D}(\Omega)$ of all ϕ in $C_0^\infty(\Omega)$ endowed with a topology so that ϕ_i converges to an element ϕ in $\mathcal{D}(\Omega)$ if and only if*

1. *there exists a compact set $K \in \Omega$ such that $\phi_i \in K$ for every i;*
2. $\lim_{x\to+\infty} D^\alpha \phi_i = D^\alpha \phi$ *uniformely on K for each multi-index α.*

Definition 1.1.2 *Let $d_{K,m}(\Omega) = \sup_{x\in K} \sum_{|\alpha|=m} |D^\alpha \phi(x)|$, where K is compact in Ω. $\mathcal{D}'(\Omega)$ is the dual space of $\mathcal{D}(\Omega)$, and*

$$\langle D^\alpha T, \phi\rangle = (-1)^\alpha \langle T, D^\alpha \phi\rangle, \text{ for all } \phi \in \mathcal{D}(\Omega), \tag{1.1.7}$$

where T is a linear functional.

$$D^\alpha T = \frac{\partial^{\alpha_1} ... \partial^{\alpha_n} T}{\partial {x_1}^{\alpha_1} ... \partial {x_n}^{\alpha_n}}, \tag{1.1.8}$$

with $|\alpha| = \sum_{i=1}^{n} \alpha_i$.

Definition 1.1.3 *$f \in L^1_{loc}(\Omega)$ defines a distribution $f \in \mathcal{D}'(\Omega)$ by*

$$\langle T_f, \phi \rangle = Re\left(\int_\Omega f(x)\overline{\phi(x)}dx \right) \text{ for all } \phi \in \mathcal{D}(\Omega). \tag{1.1.9}$$

Definition 1.1.4 *The Sobolev space $W^{m,p}(\Omega)$ is defined as*

$$W^{m,p}(\Omega) = \left\{ u \in L^p(\Omega) \middle| D^\alpha u \in L^p(\Omega), |\alpha| \le m \right\} \tag{1.1.10}$$

where $0 < m, p \le +\infty$, $p, m \in \mathbb{N}$. If $u \in W^{m,p}(\Omega)$, then the norm of u is defined as

$$||u||_{W^{m,p}(\Omega)} = \left(\sum_{0 \le |\alpha| \le m} ||D^\alpha u||^p_{L^p(\Omega)} \right)^{1/p} \tag{1.1.11}$$

Specially, when $m = 0$, $W^{0,p} = L^p$; when $p = 2$, $W^{m,2}(\Omega)$ is denoted as $H^m(\Omega)$, the norm of $u \in H^m(\Omega)$ is defined as

$$||u||_{H^m(\Omega)} = \left(\sum_{0 \le |\alpha| \le m} \int_\Omega |D^\alpha u(x)|^2 dx \right)^{1/2} \tag{1.1.12}$$

and the inner product is expressed as

$$(u, v)_{H^m(\Omega)} = \sum_{0 \le |\alpha| \le m} Re \int_\Omega D^\alpha u(x) \overline{D^\alpha v(x)} dx \tag{1.1.13}$$

We have

$$H^m_0(\Omega) = W^{m,2}_0$$

Theorem 1.1.1 *(Adams[1]) The continuous embeddingd in Sobolev space ar as:*

1. *When $1 \le p < n, \frac{1}{p} - \frac{m}{n} > 0$, we have $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ where $q \in [p, \frac{np}{n-p}]$.*
2. *If $p = n > 1$, then $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \in [p, +\infty]$.*
3. *When $p = n = 1$, there holds $W^{1,1} \hookrightarrow L^\infty(\Omega)$.*
4. *When $p = 1$ or $p = +\infty$, the Sobolev spaces $W^{k,1}(\Omega)$ and $W^{k,\infty}$ do not have the reflexible propruprety.*
5. *If $\frac{1}{p} - \frac{m}{n} < 0$, then $W^{m,p}(\Omega) \hookrightarrow C^{k,\alpha}(\Omega)$, here $k = [m - \frac{n}{p}], \alpha = (m - \frac{n}{p}) = m - \frac{n}{p} - k$.*

Here [.] and (.) denote the integral part and fractional part of numbers respectively.

Definition 1.1.5 *The Sobolev-Slobodeckiî space $W^{k+\beta,p}(Q)$(Q is a domain in $\mathbb{R}^L$, $1 \leq p < +\infty, 0 < \beta < 1, k = 0, 1, 2, ...$) is defined as*

$$W^{k+\beta,p}(Q) = \left\{ u \in L^p(\Omega) \middle| ||u||_{W^{k+\beta,p}(Q)} = \left(||u||_{W^{k+\beta,p}(Q)} + \sum_{|\alpha|=k} \int_Q \int_Q \frac{|\partial^\alpha v(y) - \partial^\alpha v(z)|^p}{|y-z|^{L+\beta p}} dydz \right)^{\frac{1}{p}} \right\} \tag{1.1.14}$$

Remark 1.1.1 *Here we introduce some examples of special Sobolev-Slobodechiî spaces. The topology of homogeneous space $W_2^l(\Omega)$ is equipped as*

$$||u||^2_{W_2^l}(\Omega) = \begin{cases} \sum\limits_{|\alpha|=l} ||\partial_x^\alpha u||^2_{L^2(\Omega)} & \text{if } l \in \mathbb{Z}, \\ \sum\limits_{|\alpha|=[l]} \int\limits_\Omega \int\limits_\Omega \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x-y|^{m+2(l-[l])}} dxdy & \text{if } l \in \mathbb{R}^+/\mathbb{Z}^+ \end{cases} \tag{1.1.15}$$

The topology of non-homogeneous space $W_2^l(\Omega)$ is equipped as

$$||u||_{W_2^l(\Omega)} = \left(\sum_{|\alpha|\leq l} ||\partial_x^\alpha u||^2_{L^2(\Omega)} + ||u||^2_{W_2^l(\Omega)} \right)^{\frac{1}{2}} \tag{1.1.16}$$

Denoting as $Q_T = \Omega \times (0, T)$, we can define the space $W_2^{l,\frac{1}{2}}(Q_T)$ as

$$W_2^{l,\frac{1}{2}}(Q_T) = L^2(0, T; W_2^l(Q)) \cap L^2(Q; W_2^{\frac{1}{2}}(0, T)) \tag{1.1.17}$$

with the norm

$$\begin{aligned} ||u||^2_{W_2^{l,\frac{1}{2}}}(Q_T) &= \left(||u||^2_{W_2^{l,0}(Q_T)} + ||u||^2_{W_2^{0,\frac{1}{2}}(Q_T)} \right)^{\frac{1}{2}} \\ &= \left(\int_0^T ||u||^2_{W_2^l(Q)} dt + \int\limits_\Omega ||u||^2_{W_2^{\frac{1}{2}}(0,T)} dx \right)^{\frac{1}{2}} \end{aligned} \tag{1.1.18}$$

where the topology equipped for the space $W_2^l(0, T)$ is defined as

$$||u||^2_{W_2^{l,\frac{1}{2}}(0,T)} = \begin{cases} \left(\sum_{j=0}^{i} ||D_t^j u||^2_{L^{2\cdot}(0,T)}\right)^{\frac{1}{2}} & \text{if } l \in \mathbb{Z} \\ \left(\sum_{j=0}^{i} ||D_t^j u||^2_{L^2(0,T)} + \int_0^T \int_0^t \frac{|D^{[l]} - tu(t) - D^{[l]} - tu(\tau)|^2}{|t-\tau|^{l+2(l-[l])}} dt d\tau\right)^{\frac{1}{2}} & \text{, if } l \in \mathbb{R}^+/\mathbb{Z}^+. \end{cases} \tag{1.1.19}$$

1.2 Some useful inequalities

In this section, we shall recall some inequalities which will be used in the subsequent chapters.

Theorem 1.2.1 *(Young [28]) (Young's Inequality) the following inequalities hold*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},\ q = \frac{p}{p-1}, 1 < p < +\infty,\ \textit{for all } a, b, \epsilon > 0, \tag{1.2.1}$$

$$ab \leq \frac{\epsilon}{p} a^p + \frac{1}{q\epsilon^{\frac{1}{p-1}}} b^q,\ q = \frac{p}{p-1}, 1 < p < +\infty,\ \textit{for all } a, b, \epsilon > 0. \tag{1.2.2}$$

Theorem 1.2.2 *(Adams [1]) (The Cauchy-Schwartz Inequality) There holds that*

$$|x.y| \leq |x||y|, \textit{for all } x, y \in \mathbb{R}^n \tag{1.2.3}$$

here $|x| = (x,x)^{(1/2)} = (\sum_{i=1}^n x_i^2)^{1/2}$ *for all* $x \in \mathbb{R}^n$

Theorem 1.2.3 *(Hölder [13])(Hölder's Inequality) Let* $\Omega \subseteq \mathbb{R}^n$ *be a domain, assume that* $u \in L^p(\Omega), v \in L^q(\Omega)$ *with* $1 \leq p, q < +\infty$ *and* $\frac{1}{p} + \frac{1}{q} = 1$*. Then*

$$\int_\Omega |uv| dx \leq ||u||_{L^p(\Omega)} ||v||_{L^q(\Omega)} \tag{1.2.4}$$

In 1986, Minkowski established the following Inequality.

Theorem 1.2.4 *(Hölder [13])(Minkowski's Inequality) Assume* $1 \leq p \leq +\infty$ *then for any* $u, v \in L^p(\Omega)$

$$||u+v||_{L^p(\Omega)} \leq ||u||_{L^p(\Omega)} + ||v||_{L^p(\Omega)} \tag{1.2.5}$$

Theorem 1.2.5 *(Cazenave [9])(Poincaré's Inequality) Assume that $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, then there holds*

$$||u|| \leq \lambda_1^{-\frac{1}{2}} ||u||_1, \textit{for all } u \in H_0^1(\Omega) \tag{1.2.6}$$

where λ_1 is the first eigenvalue of A under the homegeneous Dirichlet boundary condition, $||u||_1$ is the norm of u in $H_0^1, A = -\Delta$

Theorem 1.2.6 *(Gronwall [11])(Gronwall's Inequality) Let $a(t) \in L^1(0,T), a \geq 0$, $\beta(t) \in L^1(0,T), b_0 \in \mathbb{R}, b(\tau) = b_0 + \int_0^\tau \beta(t)dt$.*
Assume $\gamma(t) \in L^\infty(0,T)$ satisfies

$$\gamma(\tau) \leq b(\tau) + \int_0^\tau a(t)\gamma(t)dt \textit{ for almost all (a.a.) } \tau \in [0,T]. \tag{1.2.7}$$

Then for a.a. $\tau \in [0,T]$, we have

$$\gamma(t) \leq b_0 exp\left(\int_0^\tau a(s)ds\right) + \int_0^\tau \beta(t) exp\left(\int_t^\tau a(s)ds\right)dt. \tag{1.2.8}$$

Under assumptions of Theorem 1.2.6, the following theorem inequalities hold for dimendion $n = 3$.

Theorem 1.2.7 *(Martinez [16]) Let $\phi(t) : \mathbb{R}^+ \to \mathbb{R}^+$ be a non-increasing function and $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ be a strictly increasing C^1 function, with $\sigma(t) \to +\infty$ as $t \to +\infty$. Assume that there exist $p, q \geq 0$ and $c > 0$ such that*

$$\int_s^{+\infty} \sigma'(t)\phi 1 + p(t)dt \leq c\phi^{1+p}(s) + \frac{c\phi(s)}{\sigma^q(s)}, \ 1 \leq s < +\infty$$

Then there exist positive constants k and δ such that

$$\begin{cases} \phi(t) \leq ke^{-\delta\sigma(t)}, & \textit{for all } t \geq 1, \quad \textit{if } p = q = 0, \\ \phi(t) \leq \dfrac{k}{\sigma(t)^{(1+q)/p}}, & \textit{for all } t \geq 1, \quad \textit{if } p > 0. \end{cases}$$

1.3 Basic theory of semigroups

In this section, we recall some basic knowledge in semigroups, most of which will be used in the subsequent chapters. For the detailed proofs the reader can refer to the related documents, e.g., Batty and Duyckaerts [6], Belleni-Morante and McBriden [7], Borichev and Tomilov [8], Liu and Zheng [14], Prüs [20], Qin [21], Pazy [19].

Definition 1.3.1 *(Pazy [19], Zheng [29]) (Semigroups) Let X be a Banach space or a closed subset of a Banach space, the parameter family $S(t)$,*
$0 \leq t < +\infty$ from X to X is called a semigroup if

1. *$S(0) = Id$(Identity operator on X),*
2. *$S(t+s) = S(t)S(s)$.*

Definition 1.3.2 *(Pazy [19], Zheng [29]) (C_0-Semigroups) A semigroup $S(t), 0 \leq t \leq +\infty$ from X to X is called a strong continuous semigroup of bounded linear operators if*

$$\lim_{t\to 0^+} S(t)x = x \ \text{or} \ \lim_{t\to 0^+} ||S(t)x - x|| = 0, \ \text{for all } x \in X, \tag{1.3.1}$$

i.e., $S(t)$ is a C_0-semigroup.

Definition 1.3.3 *(Pazy [19], Zheng [29]) (Contraction Semigroups) The semigroup $S(t)$ is a contraction semigroup if there exists a constant $\alpha > 0$ $(0 < \alpha < 1)$ such that for all $t \geq 0$,*

$$||S(t)x - S(t)y|| \leq ||x - y||, \ \text{for all } x, y \in X. \tag{1.3.2}$$

Definition 1.3.4 *(Pazy [19], Zheng [29]) (Analytic Semigroups) Let*
$\Delta = \left\{ z | \phi_1 < argz < \phi_2, \phi_1 < 0 < \phi_2 \right\}$ and for all $z \in \Delta, S(z)$ be a bounded linear operator. The family $S(z)(z \in \Delta)$ is an analytic semigroup in Δ if

1. *$z \mapsto S(z)$ is analytic in Δ,*
2. *$S(0) = Id, i.e., \ \lim_{z\to 0} S(z)x = x$ for every $x \in X$,*
3. *$S(z_1 + z_2) = S(z_1)S(z_2)$ for all $z_1, z_2 \in \Delta$. A semigroup $S(t)$ is called analytic if it is analytic in some sector containing the nonnegative real axis.*

Definition 1.3.5 *(Pazy [19], Zheng [29]) The linear operator $\mathcal{A}$ defined by*

$$D(\mathcal{A}) = \{x \in X : \lim_{t\to 0^+} (S(t)x - x)/t \ \text{exists} \ \} \tag{1.3.3}$$

and

$$\mathcal{A}x = \lim_{t\to 0^+} (S(t)x - x)/t = \frac{d(S(t)x)}{dt}|_{t=0} \ \text{for all } x \in D(\mathcal{A}), \tag{1.3.4}$$

is called the infinitesimal generator of the semigroup $S(t)$, $D(\mathcal{A})$ is called the domain of $\mathcal{A}$.

In the following, for the sake of convenience, we introduce maximal accretive operator (see,e.g. Pazy [19] and Zheng [29])

Definition 1.3.6 *(Pazy [19] and Zheng [29]) Let $\mathcal{A}$ be a linear operator defined in a Banach space $X, \mathcal{A} : D(\mathcal{A}) \subset X \mapsto X$. If for any $x, y \in D(\mathcal{A})$ and any $\lambda > 0$,*

$$||x - y|| \leq ||x - y + \lambda(\mathcal{A}x - \mathcal{A}y)||, \tag{1.3.5}$$

then $\mathcal{A}$ is said to be an accretive operator. Moreover, if $\mathcal{A}$ is densely defined accretive operator, and $I + \mathcal{A}$ is surjective, i.e., $R(I + \mathcal{A}) = X$, then $\mathcal{A}$ is said to be a maximal accretive operator.

Theorem 1.3.1 *(Pazy [19] and Zheng [29]), Let $S(t), 0 \leq t < +\infty$ be a C_0semigroup on a Banach space X.Then there exist constants $M > 0$ and $\omega \geq 0$ such that*

$$||S(t)|| \leq Me^{\omega t} \text{ for all } t \geq 0. \tag{1.3.6}$$

Obviously, if $M = 1$ and $\omega = 0$ in (1.3.6), then we obtain a C_0-semigroup of nonexpansions or contractions.

Definition 1.3.7 *For real numbers $M > 0$ and $\omega \geq 0$, let $\mathcal{G}(M, \omega; X)$ denote the set of generators of C_0-semigroups $S(t), 0 \leq t < +\infty$ on a Banach space X satisfying (1.3.5).*

Theorem 1.3.2 *(Pazy [19] , Zheng [29]), for any $x \in D$, $S(t)x \in C^1([0, +\infty), X)$. Moreover, for all $t \geq 0$,*

$$x - S(t)x = \int_0^t \mathcal{A}S(\tau)x d\tau = \int_0^t S(\tau)\mathcal{A}x d\tau, \tag{1.3.7}$$

and

$$\frac{d(S(t)x)}{dt} + \mathcal{A}(S(t)x) = 0. \tag{1.3.8}$$

Theorem 1.3.3 *(Pazy [19], Zheng [29]) (Hille-Yoshida Theorem) $\mathcal{A} \in \mathcal{G}(M, \omega; X)$ if and only if*

1. *$\mathcal{A}$ is a closed linear operator whose domain $D(\mathcal{A})$ is dense in X; and*
2. *for all real numbers $\lambda > \omega, \lambda \in \rho(\mathcal{A})$(the resolvent set of $\mathcal{A}$),and*

$$||\ [R(\lambda, \mathcal{A})]^n|| \leq \frac{M}{(\lambda - \omega)^n} \text{ for n=1,2,...} \tag{1.3.9}$$

where $R(\lambda, \mathcal{A}) = (\mathcal{A} - \lambda I)^{-1}$

Theorem 1.3.4 *(Batty [5]) Let $S(t), t > 0$ be a bounded C_0-semigroup on Banach space X with generator $\mathcal{A}$. Suppose that $i\mathbb{R}$ is contained in the resolvent set $\rho(\mathcal{A})$ of $\mathcal{A}$.*
Then

$$||S(t)\mathcal{A}^{-1}|| \to 0, \ t \to +\infty. \tag{1.3.10}$$

Lemma 1.3.1 *(Pazy [19] and Zheng [29]) Let $\mathcal{A}$ be a linear operator defined in a Hilbert space $H, \mathcal{A} : D(\mathcal{A}) \subset H \to H$. Then the necessary and sufficient conditions for $\mathcal{A}$ being maximal accretive are:*

1. *$\Re(\mathcal{A}x, x) \geq 0$, for all $x \in D(\mathcal{A})$,*
2. *$R(I + \mathcal{A}) = H$.*

Proof We first proof the necessity. By (1.3.5),

$$(x, x) = ||x||^2 \leq ||x + \lambda\mathcal{A}x||^2 = (x, x) + 2\lambda Re(\mathcal{A}, x) + \lambda^2||\mathcal{A}x||^2. \tag{1.3.11}$$

Thus, for all $\lambda > 0$,

$$Re(\mathcal{A}x, x) \geq -\frac{\lambda}{2}||\mathcal{A}x||^2. \tag{1.3.12}$$

Letting $\lambda \to 0$, we get 1. Furthermore, 2. immediately follows from the fact that $\mathcal{A}$ is m-accretive.

We now prove the suffiency. It follows from 1. that for all $\lambda > 0$,

$$||x - y||^2 \leq Re(x - y, x - y + \lambda\mathcal{A}(x - y)) \leq ||x - y||||x - y + \lambda(\mathcal{A}x - \mathcal{A}y)||, \tag{1.3.13}$$

i.e.,(1.3.5) holds. Now it remains to prove that $\mathcal{A}$ is densly defined.We use a contradiction argument. Suppose that it is not true. Then there is a nontrivial element x_0 belonging to the supplement of $\overline{D(\mathcal{A})}$ such that for all $x \in D(\mathcal{A})$,

$$(x, x_0) = 0. \tag{1.3.14}$$

It follows from 2. that there is an $x^* \in D(\mathcal{A})$ such that

$$x^* + \mathcal{A}x^* = x_0. \tag{1.3.15}$$

Taking the inner product of (1.3.15) with x^*, we deduce that

$$(x^* + \mathcal{A}x^*, x^*) = 0. \tag{1.3.16}$$

Taking the real part of (1.3.15) we deduce that $x^* = 0$, and by (1.3.16), $x_0 = 0$, a contradiction. Thus the proof is complete.

□

Definition 1.3.8 *the opperator $\mathcal{A}$ is called dissipative if*

$$Re(\mathcal{A}x, x) \leq 0$$

Lemma 1.3.2 *Let A be a linear operator with dense domain $D(\mathcal{A})$ in a Hilbert space $\mathcal{H}$. IfA is dissipative and $0 \in \rho(A)$(the resolvent set of A), then A is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof See,e.g., Liu and Zheng [14] or Pazy [19].

□

For the abstract initial value problem

$$\begin{cases} \dfrac{du}{dt} + \mathcal{A}u = K, \\ \\ u(0) = u_0. \end{cases} \tag{1.3.17}$$

Where $\mathcal{A}$ is a maximal accretive operator defined in a dense subset $D(\mathcal{A})$ of a Banach space H.

Lemma 1.3.3 *(Pazy [19], Zheng [29]) Suppose that $\mathcal{A}$ is m-accretive in a Banach space H,$K = 0$ and $u_0 \in D(\mathcal{A})$. Then the problem (1.3.17) has a unique classical solution u such that*

$$u \in C^1([0, +\infty), H) \cap C([0, +\infty), D(\mathcal{A})),$$

Note that $K = 0$ in (1.3.17) corresponds to the Lumer-Philips theorem.

Proof By Theorem 1.3.2 and Hillie-Yoshida Theorem, $u(t) = S(t)u_0$ is a classical solution in the required class.
To prove the uniqueness, we use a contradiction argument. Suppose that there ar two solutions u_1 and u_2. Then

$$u = u_1 - u_2 \in C^1([0, +\infty), H) \cap C([0, +\infty), D(\mathcal{A})),$$

and it satisfies

$$\begin{cases} \dfrac{du}{dt} + \mathcal{A}u = 0, \\ \\ u(0) = 0, \end{cases} \tag{1.3.18}$$

For any $T > 0$, let

$$\phi(t) = S(t)u(T-t), \ \forall t \in [0, T]. \tag{1.3.19}$$

Then by Theorem 1.3.2, we have

$$\frac{d\phi}{dt} = -S(t)\frac{du(T-t)}{dt} - S(t)\mathcal{A}u(T-t) = S(t)\mathcal{A}u(T-t) - S(t)\mathcal{A}u(T-t) = 0. \tag{1.3.20}$$

Thus,

$$u(T) = \phi(0) = \phi(T) = S(T)u(0) = 0. \tag{1.3.21}$$

Hence, the uniqueness follows.

$\square$

Lemma 1.3.4 *(Pazy [19], Zheng [29]) Suppose that $K = K(t)$, and*

$$K(t) \in C^1([0, +\infty), H) \ , \ u_0 \in D(\mathcal{A}).$$

Then problem (1.3.17) admits a unique global classical solution u such that

$$u \in C^1([0, +\infty), H) \cap C([0, +\infty), D(\mathcal{A})), \tag{1.3.22}$$

which can be described as

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)K(\tau)d\tau. \tag{1.3.23}$$

Proof Since $S(t)u_0$ satisfies the homogeneous equation and nonhomogeneous initial condition, it suffices to verify that $w(t)$ given by

$$w(t) = \int_0^t S(t-\tau)f(\tau)d\tau, \tag{1.3.24}$$

belongs to

$$C^1([0,+\infty)\,,H)\cap C([0,+\infty)\,,D(\mathcal{A}))$$

and satisfies the nonhomogeneous equation.Consider the following quotient of difference

$$\begin{aligned}\frac{w(t+h)-w(t)}{h} &= \frac{1}{h}\left(\int_0^{t+h} S(t+h-\tau)f(\tau)d\tau - \int_0^t S(t-\tau)f(\tau)d\tau\right)\\ &= \left(\int_t^{t+h} S(t+h-\tau)f(\tau)d\tau + \frac{1}{h}\int_0^t (S(t+h-\tau)-S(t-\tau))f(\tau)d\tau\right)\\ &= \frac{1}{h}\int_t^{t+h} S(z)f(t+h-z)dz + \frac{1}{h}\int_0^t S(z)(f(t+h-z)-f(t-z)dz,\end{aligned} \tag{1.3.25}$$

when $h\to 0$, the terms in the last line of (1.3.25)have limits

$$S(t)f(0)+\int_0^t S(z)f'(t-z)dz \in C([0,+\infty)\,,H). \tag{1.3.26}$$

It turns out that $w \in C^1([0,+\infty), H)$ and the terms in the third line of (1.3.25) have limits too, which should be

$$S(0)f(t) - \mathcal{A}w(t) = f(t) - \mathcal{A}w(t). \tag{1.3.27}$$

Thus the proof is complete.

$\square$

From (1.3.24) we immediately have the following result.

Lemma 1.3.5 *(Pazy [19], Zheng [29]) Suppose that* $K = K(t)$*, and*

$$K(t) \in C([0,+\infty), D(\mathcal{A}))\ u_0 \in D(\mathcal{A}).$$

Then problem (1.3.17) admits a unique global classical solution $u(t)$*.*

Lemma 1.3.6 *(Pazy [19], Zheng [29]) Suppose that* $K = K(t)$*, and*

$$K(t) \in C([0,+\infty), H)\ u_0 \in D(\mathcal{A})$$

and for any $T > 0$*,*

$$K_t \in L^1([0,T], H).$$

Then problem (1.3.17) admits a unique global classical solution $u(t)$*.*

Proof We first prove that any $g \in L^1([0,T], H)$, the function w given by the following integral

$$w(t) = \int_0^t S(t-\tau)g(\tau)d\tau, \tag{1.3.28}$$

belongs to $C([0,T], H)$.Nothing that

$$\begin{aligned} w(t+h) - w(t) &= \int_0^{t+h} S(t+h-\tau)g(\tau)d\tau - \int_0^t S(t-\tau)g(\tau)d\tau \\ &= (S(h) - I)w(t) + \int_t^{t+h} S(t+h-\tau)g(\tau)d\tau, \end{aligned} \tag{1.3.29}$$

we get as $h \to 0$.

$$||w(t+h) - w(t)|| \le ||(S(h) - I)w(t)|| + \int_t^{t+h} ||g(\tau)|| \to 0, \tag{1.3.30}$$

where we have used the strong continuity of $S(t)$ and the absolute continuity of integral for $||g|| \in L^1[0,T]$.
Now it can be seen from the last line of (1.325) that for almost every $t \in [0,T]$, $\frac{dw}{dt}$ exists and it equals

$$S(t)f(0) + \int_0^t S(z)f'(t-z)dz = S(t-\tau)f'(\tau)d\tau \in C([0,T],H). \tag{1.3.31}$$

Thus, for almost every t,

$$\frac{dw}{dt} = -\mathcal{A}w + f. \tag{1.3.32}$$

Since w and f both belongs to $C([0,T],H)$, it follows from (1.3.32) that for almost every $t, \mathcal{A}w$ belongs to $C([0,T],H)$. Since $\mathcal{A}$ is a closed operator, we conclude that

$$w \in C([0,T], D(\mathcal{A})) \bigcap C^1([0,T],H)$$

and (1.3.32) holdfs for every t. Thus the proof is complete.

□

Lemma 1.3.7 *(Pazy [19], Zheng [29]) When $K = K(u)$ satisfies the global Lipschitz condition, i.e., there is a positive constant L such that for all $u, v \in H$,*

$$||K(u) - K(v)||_H \leq ||u - v||_H. \tag{1.3.33}$$

Furthermore, suppose that $u_0 \in H$. Then problem (1.3.18) admits a global mild solution $u(t)$ such that $u(t)$ belongs to $C([0,+\infty), H)$ and satisfies the following integral equation :

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)K(u(s))ds. \tag{1.3.34}$$

Moreover, let $u(t), \hat{u}(t)$ be the global mild solutions corresponding to u_0 and $\hat{u}_0$. Then for all $t \geq 0$, the following estimates holds:

$$||u(t) - \hat{u}(t)||_H \leq e^{Lt}||u_0 - \hat{u}_0||_H. \tag{1.3.35}$$

Proof We use the contraction mapping theorem to prove the present theorem. Two key steps for using the contraction mapping theorem are to figure out a closed set of considered Banach space, and an auxiliary problem so that the nonlinear operator defined by the auxilary problem maps from this closed set into itself, and turns out to be a contraction. In the following we proceed alog this line.
Let

$$\phi(v) = S(t)u_0 + \int_0^t S(t-\tau)K(v(t))d\tau \tag{1.3.36}$$

and

$$X = v \in C([0,+\infty)\,, H)\Big| \underset{t\geq 0}{sup}||v(t)||e^{-kt} < +\infty, \tag{1.3.37}$$

where k is a positive constant such that$k > L$. In X, we introduce the following norm:

$$||u||_X = \underset{t\geq 0}{sup}\; e^{-kt}||u(t)||. \tag{1.3.38}$$

Clearly, X is a Banach space. We now show that the nonlinear operator ϕ defined by (1.3.36) maps X into itself, and the mapping is a contraction. In deed, for $v \in X$, by (1.3.33),we have

$$\begin{aligned} ||\phi(v)|| \quad &\leq ||S(t)u_0 + \int_0^t ||S(t-\tau)||||K(v(t))||d\tau \\ &\leq ||u_0|| + C_0 t + L\; \underset{t\geq 0}{sup}\; e^{-kt}||v(t)||. \int_0^t e^{k\tau}d\tau \\ &\leq ||u_0|| + C_0 t + \frac{L}{k}e^{kt}||v(t)||_X. \end{aligned} \tag{1.3.39}$$

where $C_0 = K(0)$. Thus,

$$||\phi(v)||_X \leq \underset{t\geq 0}{sup}\Big((||u||_0 + C_0 t)e^{-kt}\Big) + \frac{L}{k}||v(t)||_X < +\infty. \tag{1.3.40}$$

i.e., $\phi(v) \in X$.
For $v_1, v_2 \in X$, we have

$$\begin{aligned} ||\phi(v_1) - \phi(v_2)||_X \quad & \underset{t\geq 0}{sup}\; e^{-kt}||\int_0^t S(t-\tau)K(v_1(\tau)) - K(v_2(\tau))d\tau|| \\ & \underset{t\geq 0}{sup}\; e^{-kt}L\int_0^t ||v_1 - v_2||d\tau \leq \frac{L}{k}||v_1 - v_2||_X. \end{aligned} \tag{1.3.41}$$

Therefore, by the contraction mapping theorem, (1.3.34) has a unique solution in X. To show that uniqueness also holds in $C([0,+\infty), H)$, let $u_1, u_2 \in C([0,+\infty), H)$ be two solutions of (1.3.34) and let $u = u_1 - u_2$. Then

$$u(t) = \int_0^t S(t-\tau)(K(u_1) - K(u_2))d\tau, \tag{1.3.42}$$

$$||u(t)|| \leq L \int_0^t ||u(\tau)||d\tau. \tag{1.3.43}$$

By the Gronwall inequality, we immediately conclude that $u(t) = 0$, i.e., the uniqueness in $C([0,+\infty), H)$ follows.
It remains to prove the stability result (1.3.35). In the same way as above we have

$$u(t) - \hat{u}(t) = S(t)(u_0 - \hat{u}_0) + \int_0^t S(t-\tau)(K(u) - K(\hat{u}))d\tau. \tag{1.3.44}$$

Therefore,

$$||u(t) - \hat{u}(t)|| \leq ||u_0 - \hat{u}|| + L \int_0^t ||u(\tau) - \hat{u}(\tau)||d\tau. \tag{1.3.45}$$

By the Gronwall inequality, (1.3.35) follows. Thus proof is complete.

□

For the abstract initial value problem

$$\begin{cases} \dfrac{du}{dt} = \mathcal{A}u, \\ \\ u(0) = u_0, \end{cases}$$

where $\mathcal{A}$ was defined as in Theorem 1.3.10, we define a continuous non-decreasing function

$$M(\eta) = \underset{t\in[-\eta,\eta]}{max} ||R(it, \mathcal{A})||, \ \ \eta > 0, \tag{1.3.46}$$

where $R(\lambda, \mathcal{A}) = (\lambda I - \mathcal{A})^{-1}, I$ is a unit operator on the Banach space H. then the associated function

$$M_{\log}(\eta) := M(\eta)(\log(1 + M(\eta)) + \log(1+\eta)), \ \ \eta > 0. \tag{1.3.47}$$

Let $M_{\log}^{-1}$ be the inverse of $M_{\log}$ defined on $[M_{\log}(0), +\infty)$. Then, we have the following lemmas.

Lemma 1.3.8 *(Batty and Duyckaerts[6]) Let $S(t), t \geq 0$ be a bounded C_0-semigroup on a Banach space X with generator $\mathcal{A}$, such that $i\mathbb{R} \subset \rho(\mathcal{A})$. Let the functions M and M_{log} be defined by (1.3.46) and (1.3.47). Then there exist constants $C, B > 0$ such that for all $t \geq B$,*

$$||S(t)\mathcal{A}^{-1}|| \leq \frac{C}{M_{log}^{-1}(t/C)}. \tag{1.3.48}$$

If $\alpha > 0, M(\eta) \leq C(1+\eta^{\alpha}), \eta \geq 0$, then for allt $\geq B$,

$$||S(t)\mathcal{A}^{-1}|| \leq C\left(\frac{logt}{t}\right)^{\frac{1}{\alpha}}. \tag{1.3.49}$$

Proof Let $t \geq 0$ and $R > 0$, and let γ be the contour consisting of the right-hand of the circle $|z| = R$ and any path γ' in $z \in \rho(-\mathcal{A}) : Rez < 0$ from $i\mathbb{R}$ to $i\mathbb{R}$.By Cauchy's Theorem,

$$e^{t\mathcal{A}}\mathcal{A}^{-1} = \frac{1}{2\pi i}\int_{\gamma}\left(1+\frac{z^2}{R^2}\right)(z+\mathcal{A})^{-1}e^{-t\mathcal{A}}\frac{dz}{z}.$$

We first show that

$$||e^{t\mathcal{A}}\mathcal{A}^{-1}|| \leq \frac{2\tilde{C}}{R} + \frac{1}{2\pi}||\int_{\gamma}\left(1+\frac{z^2}{R^2}\right)(z+\mathcal{A})^{-1}e^{tz}\frac{dz}{z}||. \tag{1.3.50}$$

If $z = Re^{i\theta}, -\pi/2 < \theta\pi/2$, we have

$$||(z+\mathcal{A})^{-1}e^{-t\mathcal{A}}|| = ||e^{tz}\int_{t}^{+\infty} e^{-(z+\mathcal{A})s}|| \leq \frac{\tilde{C}}{Rcos\theta}.$$

Noticing that $|1 + z^2/R^2 = 2|cos\theta$, we got the bound

$$||\int_{|z|=R}\left(1+\frac{z^2}{R^2}\right)(z+\mathcal{A})^{-1}e^{-t\mathcal{A}}\frac{dz}{z}|| \leq \frac{2\pi\tilde{C}}{R}. \tag{1.3.51}$$

Next we define the analytic function of z

$$h_t(z) = \int_0^t e^{(t-sz)z}e^{-s\mathcal{A}}ds = (z+\mathcal{A})e^{tz} - (z+\mathcal{A})^{-1}e^{-t\mathcal{A}}. \tag{1.3.52}$$

The Cauchy's Theorem and similar estimate as for (1.3.51) yield

$$||\int_{\gamma'}\left(1+\frac{z^2}{R^2}\right)h_t(z)\frac{dz}{z}|| = ||\int_{|z|=R}\left(1+\frac{z^2}{R^2}\right)h_t(z)\frac{dz}{dz}|| \leq \frac{2\pi\tilde{C}}{R}, \tag{1.3.53}$$

which together with (1.3.52) and (1.3.53), yield the estimate (1.3.50).
By means of standard Neumann series, we may take γ' to be the union of γ_0, γ_+ and γ_-, where

$$\gamma_0(\tau) = -\frac{1}{M(\tau)} + i\tau \ (-R \geq \tau \geq R),\ \gamma_{\pm}(s) = s \pm iR \ (-(2M(R))^{-1} \leq s < 0).$$

Although γ_0 may not be piecewise smooth, it can be approximated by smooth paths and (1.3.50) remains valid. Moreover,

$$||(\gamma_0(\tau) + \mathcal{A}^{-1})|| \leq 2M(|\tau|),\ ||(\gamma_{\pm}(s) + \mathcal{A}^{-1})|| \leq 2M(R).$$

Now assume that $R > 1$. On $\gamma_{\pm}, |1 + z^2/R^2| \leq C/R$, so we can estimate the norls of the integral in (1.3.50) over $\gamma_{\pm}$ by

$$C \int_0^{(2M(R))^{-1}} \frac{1}{R} M(R) \frac{e^{-ts}}{R} ds \leq \frac{CM(R)}{R^2 t},$$

where $C > 0$ is a constant depending only on M. On $\gamma_0, 1+z^2/R^2$ and $1/z$ are bounded independently of R, and $e^{tz} \leq e^{-t/2M(R)}$.
The length of γ_0 is at most $2(M(0)+R)$.Hence, we can estimate the norm of the integral over γ_0 by

$$CM(R)(1 + R)e^{-t/2M(R)},$$

where again $C > 0$ depends only on M.
Thus we have the estimate

$$||e^{-t\mathcal{A}}\mathcal{A}^{-1}|| \leq C\left(\frac{1}{R} + \frac{M(R)}{R^2 t} + \frac{(1 + M(R))^2(1 + R)^2}{R} e^{-t/2M(R)}\right).$$

Here, C depends on $\tilde{C}$ as well as M. Given $t > 4M_{log}(1)$, choose $R = M_{log}^{-1}(t/4)$. Then

$$\frac{M(R)}{R^2 t} = \frac{1}{4R^2 log((1 + M(R))(1 + R))} \leq \frac{C}{R},$$

$$(1 + M(R))^2(1 + R)^2 = 1.$$

Hence, the required result is proved.

□

Lemma 1.3.9 *(Borichev and Tomilov [8]) Given $Q > 0$ and $\varepsilon > 0$, there exist an integer $k > Q$ and a complex measure μ with compact support in $\mathbb{C}\backslash\Omega$ such that for some $B = B(\alpha, \beta)$, we have*

$$\left| \int_{\mathbb{C}\backslash\Omega} \frac{d\mu(\zeta)}{z - \zeta} \right| \leq B(1 + |(Imz)|)^{\beta}.\chi_{\{\xi:|\xi|>Q\}}(z) + \varepsilon,\ \ e \in \Omega, \tag{1.3.54}$$

$$\left| \int_{\mathbb{C}\backslash\Omega} e^{t\zeta} d\mu(\zeta) \right| \leq B\chi_{\{r:|r|>Q\}}(t) + \varepsilon,\ \ t \geq 0, \tag{1.3.55}$$

$$1/B \leq \left| \int_{\mathbb{C}\setminus\Omega} e^{t\zeta} d\mu(\zeta) \right| \leq B, \tag{1.3.56}$$

$$\left| \int_{\mathbb{C}\setminus\Omega} e^{t\zeta} \frac{d\mu(\zeta)}{\zeta} \right|^{\alpha} \leq B \frac{logt}{t} . \chi_{(Q,2k)}(t) + \frac{\varepsilon}{t+1}, \ t \geq 0, \tag{1.3.57}$$

and

$$\left| \int_{\mathbb{C}\setminus\Omega} e^{k\zeta} \frac{d\mu(\zeta)}{\zeta} \right|^{\alpha} \geq \frac{logk}{Bk}. \tag{1.3.58}$$

Proof Choose $H > Q$ large enough, and for integer $k, k \geq 2$, such that

$$H^{\alpha} \leq k \leq H^{3\alpha/2},$$

define

$$A = 2klogK, \ \tau = A^{k-1}/\sqrt{k}, \ q = e^{2\pi i/k}, \ w = iH - 1.$$

We define also a finite measure

$$\mu = \tau \sum_{1 \leq s \leq k} q^s (1 + q^s/(Aw)) \delta_{w+q^s/A},$$

where δ_x is the unit mass at x.

Observe that

$$\sum_{1 \leq s \leq k} \frac{q^s}{x - q^s} = \frac{k}{x^k - 1}, \ \sum_{1 \leq s \leq k} \frac{q^{2s}}{x - q^s} = \frac{kx}{x^k - 1}. \tag{1.3.59}$$

Indeed to prove the fist identity we use that

$$\sum_{1 \leq s \leq k} \frac{q^s}{x - q^s} = \frac{P(x)}{x^k - 1},$$

for some polynomial P with deg$P < k$ such that $P(x) = P(qx)$ so that $P(x)$=const., and then $P(x) = P(0) = k$. The second equality can be prouved by using that if

$$\sum_{1 \leq s \leq k} \frac{q^s}{x - q^s} = \frac{P(x)}{x^k - 1}$$

then $P(qx) = qP(x)$ which yields $P(x) = kx$.
Now using (1.3.59), we get

$$\begin{aligned}
\mathbb{C}_{\mu(z)} &= \int_{\mathbb{C}\setminus\Omega} \frac{d\mu(\zeta)}{\zeta} = \tau \sum_{1\le s\le k} \frac{q^s(1+q^s/(Aw))}{(z-w)-q^s/A} \\
&= \tau \sum_{1\le s\le k} \left[\frac{Aq^s}{A(z-w)-q^s} + \frac{q^{2s/w}}{A(z-w)-q^s} \right] \\
&= \tau \frac{Ak}{A^k(z-w)^k-1} + \frac{\tau}{w} \frac{kA(z-w)}{A^k(z-w)^k-1} \\
&= \frac{z}{w} \frac{\tau Ak}{A^k(z-w)^k-1}.
\end{aligned}$$

If $z \in \Omega, H > 1$, then

$$|z-w| \geq 1 - \frac{1}{H^\alpha}.$$

Now if $\frac{H}{2} \leq Imz \leq |z| \leq 2H, z \in \Omega$, then we use that $A|z-w| > 2$, $|A^k(z-w)^k - 1| > A^k e^{-k/H^\alpha}/2$, to obtain that

$$\left| \frac{z}{w} \frac{\tau Ak}{A^k(z-w)^k-1} \right| \leq c\tau k A^{1-k} e^{k/H^\alpha} = c\sqrt{k} e^{k/H^\alpha}.$$

From now on, we assume that k satisfies the condition

$$\sqrt{k} e^{k/H^\alpha} \leq H^\beta. \tag{1.3.60}$$

Then

$$|\mathcal{C}_{\mu(z)}| \leq cH^\beta, \ \frac{H}{2} \leq Imz \leq |z| \leq 2H, \ z \in \Omega. \tag{1.3.61}$$

If $z \in \Omega$ and $|Imz - H| + ||z| - H| > \frac{H}{2}$, then $|z-w| > c \max(|z|, H)$, and under condition (1.3.360), we have for large H,

$$|\mathcal{C}_{\mu(z)}| \leq \frac{c_1 |z| \sqrt{k}}{H(c\ max(|z|, H))^{\alpha+1}} \leq \varepsilon.$$

This together with (1.3.61), proves (1.3.54).
Next, define

$$\mathcal{L}_{\mu(t)} = \int_{\mathbb{C}\setminus\Omega} e^{t\zeta} d\mu(\zeta) = \tau \sum_{1\le s\le k} q^s(1+q^s/(Aw))e^{t(w+q^s/A)},$$

and

$$|\mathcal{L}_{\mu(t)}| = \tau e^{-t}\left|\sum_{1\le s\le k} q^s(1+q^s/(Aw))e^{t(q^s t/A)}\right|. \tag{1.3.62}$$

Furthmore, we have

$$\begin{aligned}
&\sum_{1\le s\le k} q^s(1+q^s/(Aw))e^{t(q^s t/A)} \\
&= \sum_{1\le s\le k}\sum_{n\ge 0}(q^s+q^{2s}/(Aw))(q^s t/A)^n \frac{1}{n!} \\
&= k\sum_{m\ge 1}\left[\frac{t^{km-1}}{A^{km-1}}.\frac{1}{(km-1)!}+\frac{t^{km-2}}{A^{km-2}}.\frac{1}{(km-2)!}.\frac{1}{Aw}\right] \\
&= \frac{kt^{k-1}}{A^{k-1}(k-1)!}\sum_{m\ge 1}\left(\frac{t^k}{A^k}\right)^{m-1}\frac{(k-1)!}{(km-1)!}\left(1+\frac{km-1}{tw}\right),
\end{aligned}$$

and

$$|\mathcal{L}_{\mu(t)}| = \frac{k^{3/2}t^{k-1}e^{-t}}{k!}\sum_{m\ge 1}\left(\frac{t^k}{A^k}\right)^{m-1}\frac{(k-1)!}{(km-1)!}\left(1+\frac{km-1}{tw}\right)$$

Thus, for some constants c, c_1, c_2, c_3, we have

$$\begin{aligned}
|\mathcal{L}_{\mu(t)}| &\le \frac{k^{3/2}t^{k-1}e^{-t}}{k!}\sum_{m\ge 1}\left(\frac{t^k}{HA}\right)^{k(m-1)}\frac{(k-1)!}{(km-1)!}\frac{km}{H} \\
&\le c_1 e^{-t}k^{5/2}t^{k-2}/(k!H),\ 0\le t\le k/H,
\end{aligned}$$

and

$$c_2 e^{-t}k^{3/2}t^{k-1}/(k!) \le |\mathcal{L}_{\mu(t)}| \le c_3 e^{-t}k^{3/2}t^{k-1}/(k!)\ k/H\le t\le A.$$

If $t\ge A$, then by (1.3.62),

$$|\mathcal{L}_{\mu(t)}| \le 2\tau e^{-t}ke^{t/A} = 2\sqrt{k}A^{k-1}e^{-t(1-1/A)}.$$

The function $t \mapsto e^{-t}t^{k-2}$ attains its maximum on $[0, k/H]$ at $t = k/H$;

The function $t \mapsto e^{-t}t^{k-1}$ attains its maximum on $[k/H, k]$ at $t = k-1$;

The function $t \mapsto e^{-t(1-1/A)}$ attains its maximum on $[A, +\infty)$ at $t = A$.

Using that $A = 2k \log k$, by Stirling's formula, we obtain that

$$0 < c_1 \leq |\mathcal{L}_{\mu(k)}| \leq c_2|\mathcal{L}_{\mu(k-1)}| \leq c_3,\ \max_{[k/H,k/2]}|\mathcal{L}_\mu| = o(1),\ H \to +\infty,$$
$$e^{-k/H}k^{5/2}(k/H)^{k-2}/(k!H) \to 0,\ \sqrt{k}A^{k-1}e^{-A} \to 0,\ H \to +\infty.$$

Hence,

$$0 < c_1 \leq |\mathcal{L}_{\mu(k)}| \leq c_2\max_{\mathbb{R}^+}|\mathcal{L}_\mu| \leq c_3,\ \max_{[0,k/2]\cup[A,+\infty)}|\mathcal{L}_\mu| = o(1),\ H \to +\infty,$$

and (1.3.55) and (1.3.56) follow for large H.
Finally, define

$$\mathcal{N}_{\mu(t)} = \int_{\mathbb{C}\setminus\Omega} e^{t\zeta}\frac{d\mu(\zeta)}{\zeta} = \frac{\tau e^{-t+iHt}}{w}\sum_{1\leq s\leq k} q^s e^{q^s t/A}.$$

Here we use the formula

$$\begin{aligned}\sum_{1\leq s\leq k} q^s e^{(q^s t/A)} &= \sum_{1\leq s\leq k}\sum_{n\geq 0} q^s(q^s t/A)^n\frac{1}{n!}\\ &= \frac{kt^{k-1}}{A^{k-1}(k-1)!}\sum_{m\geq 1}\left(\frac{t^k}{A^k}\right)^{m-1}\frac{(k-1)!}{(km-1)!}\\ &= (1+o(1))\frac{kt^{k-1}}{A^{k-1}(k-1)!},\ ,0 \leq t \leq A,\ ,H \to +\infty.\end{aligned}$$

Therefore, by Stirling's formula, we have for large H:

$$|\mathcal{N}_{\mu(k)}| \geq c_1\frac{\tau e^{-k}}{H}.\frac{k^k}{A^{k-1}(k-1)!} \geq \frac{c_2}{H}. \tag{1.3.63}$$

Moreover,

$$t^{1/\alpha}|\mathcal{N}_{\mu(t)}| \leq \varepsilon + c_1(k^{1/\alpha}/H).\chi_{(k/2,2k)},\ 0 \leq t \leq A, \tag{1.3.64}$$

and

$$t^{1/\alpha}|\mathcal{N}_{\mu(t)}| \leq \frac{c\tau e^{-t}}{H}.kt^{1/\alpha}e^{t/A} \leq c_1\frac{A^{k-1+1/\alpha}\sqrt{k}e^{-A}}{H} \leq \varepsilon,\ t \geq A. \tag{1.3.65}$$

Now fix $0 < \psi < \beta - \dfrac{\alpha}{2}$ and $k = \psi H^\alpha \log H$ in such a way that $k \in \mathbb{N}$.

Then (1.3.60) is satisfied for large H, (1.3.63) implies (1.3.58), and (1.3.64)-(1.3.65) imply (1.3.75).

□

Lemma 1.3.10 *(Borichev and Tomilev [8]) Given $\alpha > 0$, there exist a Bnach space X_α and a bounded C_0-semigroup $S(t)$ on X_α with generator $\mathcal{A}$ such that*

1. $||R(is,\mathcal{A})|| = O(|s|^\alpha),\ |s| \to +\infty$
2. $\lim\limits_{t\to+\infty} sup\left(\dfrac{t}{logt}\right)^{\frac{1}{\alpha}}||S(t)\mathcal{A}^{-1}|| > 0.$

Proof Let $T(t)_{t\geq 0}$ be the left shift semigroup on $B\cup C(\mathbb{R}^+,H)$ and $B\cup C(\mathbb{R}^+,H)$ is the space of bounded uniformly continuous H-valued functions .

Let $\Omega := \{\lambda \in \mathbb{C} : Re\lambda > -1(1+|Im\lambda|)^\alpha\}$, and let $\Omega_0 := \Omega \bigcap \{\lambda \in \mathbb{C} : |Re\lambda| < 1\}$.

Furthemore, let H_α be the space of functions $f \in B\cup C(\mathbb{R}^+)$ such that the Laplace transform $\hat{f}$ ewtends to an analytic function in Ω_0, and

$$|\hat{f}(\lambda)|(1+|Im\lambda|)^{-\alpha} \to 0,\ \lambda \to \infty,\ \lambda \in \Omega_0.$$

Then H_α equipped with the norm

$$||f||_{H_\alpha} =: ||f||_\infty + ||f||_\alpha = ||f||_\infty + \sup_{\lambda\in\Omega_0}|\hat{f}(\lambda)|(1+|Im\lambda|)^{-\alpha}$$

is Banach space. Moreover, $T(t)H_\alpha \subset H_\alpha, t \geq 0$, and the restriction $S(t)$ of $T(t)$ to H_α is also a C_0-semigroup. To prove this assertion, it suffices to observe that

$$\begin{aligned}\hat{f}(\lambda) - \widehat{S(t)f}(\lambda) &= \int_0^{+\infty} e^{-\lambda s}f(s)ds - \int_0^{+\infty} e^{-\lambda s}f(t+s)ds \\ &= (1-e^{\lambda t})\hat{f}(\lambda) + e^{\lambda t}\int_0^{+\infty} e^{-\lambda s}f(s)ds,\ \mathrm{Re}\lambda > 0,\end{aligned}$$

and the same equality holds on Ω_0. By the definition of H_α,

$$||S(t)f - f||_\alpha \to 0,\, t \to 0^+,$$

and then

$$||S(t)f - f||_{H_\alpha} \to 0,\, t \to 0^+.$$

Let $\mathcal{A}$ stand for the generator of semigroup $S(t)$. Next we prove that for every $f \in H_\alpha$, the local resolvent $R(\lambda, \mathcal{A})f$ satisfies the estimate

$$||R(\lambda, \mathcal{A})f||_{H_\alpha} \leq C(1 + |Im\lambda|)^\alpha ||f||_{H_\alpha},\ 0 < Re\lambda < 1. \tag{1.3.66}$$

We will estimate the quantities $||R(\lambda, \mathcal{A})f||_\infty$ and $||R(\lambda, \mathcal{A})f||_\alpha$ separately.

Observe first that for every $t \in \mathbb{R}^+$ and every $\lambda \in \mathbb{C}_+$, one has

$$(R(\lambda, \mathcal{A})f)(t) = \int_0^{+\infty} e^{-\lambda s} f(t+s)ds = e^{\lambda t}\hat{f}(\lambda) - \int_0^t e^{\lambda(t-s)} f(s)ds.$$

It follows that for every fixed $t \in \mathbb{R}$, the function $\lambda \mapsto (R(\lambda, \mathcal{A})f)(t)$ extends to an analytic function on Ω_0 and moreover

$$|(R(\lambda, \mathcal{A})f)(t)| \leq \frac{||f||_\infty}{|Re\lambda|},\ \text{if} Re\lambda > 0,$$

$$|(R(\lambda, \mathcal{A})f)(t)| \leq \frac{||f||_\infty}{|Re\lambda|} + |\hat{f}(\lambda)|,\ \text{if} Re\lambda < 0.$$

Applying Levinson's log-log theorem, we conclude that

$$||R(\lambda, \mathcal{A})||_\infty \leq C(1 + |Im\lambda|)^\alpha (||f||_\infty + ||f||_\alpha),\ 0 < Re\lambda < 1. \tag{1.3.67}$$

Fix $\lambda \in (0,1)$. To estimate $||R(\lambda, \mathcal{A})f||_\alpha$, note that

$$\begin{aligned}
(\widehat{R(\lambda, \mathcal{A})}f)(\mu) &= \int_0^{+\infty} e^{-\mu t} \int_0^{+\infty} e^{-\lambda s} f(t+s)dsdt \\
&= \int_0^{+\infty} e^{(\lambda-\mu)t} \int_0^{+\infty} e^{-\lambda s} f(s)dsdt \\
&= -\frac{1}{\lambda - \mu}\int_0^{+\infty} e^{-\lambda s} f(s)ds + \frac{1}{\lambda-\mu}\int_0^{+\infty} e^{-\mu t} f(t)dt \\
&= -\frac{\hat{f}(\lambda) - \hat{f}(\mu)}{\lambda - \mu},\ \text{Re}\ \mu > 1.
\end{aligned}$$

Therefore, $\widehat{R(\lambda, \mathcal{A})f}$ extends analytically to Ω_0, and

$$\widehat{R(\lambda, \mathcal{A})}f(\mu) = -\frac{\hat{f}(\lambda) - \hat{f}(\mu)}{\lambda - \mu}, \ \lambda \neq \mu, \ \mu \in \Omega_0,$$

$$\widehat{R(\lambda, \mathcal{A})}f(\mu) = -f'(\mu), \ \lambda = \mu.$$

Now, if $|\lambda - \mu| \geq 1, \mu \in \Omega_0$, then

$$|\widehat{R(\lambda, \mathcal{A})}f(\mu)| \leq |\hat{f}(\lambda)| + |\hat{f}(\mu)| \leq c(1 + |Im\mu|)^{\alpha}(1 + |Im\lambda|)^{\alpha}||f||_{\alpha}.$$

Furthermore, if

$$\frac{1}{2(1 + |Im\lambda|)^{\alpha}} \leq |\lambda - \mu| \leq 1, \ \mu \in \Omega_0,$$

then we have

$$\begin{aligned} |\widehat{R(\lambda, \mathcal{A})}f(\mu)| &\leq 2(1 + |Im\lambda|)^{\alpha}|\hat{f}(\lambda)| + |\hat{f}(\mu)| \\ &\leq c(1 + |Im\lambda|)^{\alpha}(1 + |Im\lambda|)^{\alpha}||f||_{\alpha}. \end{aligned}$$

Finally, if

$$|\lambda - \mu| \leq \frac{1}{2(1 + |Im\lambda|)^{\alpha}},$$

then applying Cauchy's formula on the circle

$$C_{\lambda} := \left\{ z \in \mathbb{C} : |z - \lambda| \frac{2}{3}(1 + |Im\lambda|)^{-\alpha} \right\},$$

we obtain that

$$|\widehat{R(\lambda, \mathcal{A})}f(\mu)| \leq c(1 + |Im\lambda|)^{\alpha}(1 + |Im\lambda|)^{\alpha}||f||_{\alpha}.$$

Thus,

$$||R(\lambda, \mathcal{A})f||_{\alpha} \leq C(1 + |\lambda|)^{\alpha}||f||_{\alpha}, \ 0 < Re\lambda < 1. \tag{1.3.68}$$

the estimate (1.3.67) and (1.3.68) together give us (1.3.66).

Since

$$||R(\lambda,\mathcal{A})f||_{H_\alpha} \geq \frac{1}{\mathrm{dist}(\lambda,\sigma(\mathcal{A}))},$$

the estimate (1.3.66) implies $i\mathbb{R} \subset \mathbb{C}\backslash\sigma(\mathcal{A})$, and that

$$||R(\lambda,\mathcal{A})f||_{H_\alpha} \leq C(1+|\lambda|)^\alpha,\ \lambda \in i\mathbb{R}. \tag{1.3.69}$$

Since $\sigma((A)) \cap i\mathbb{R} = \varnothing$, by Theorem(1.3.4), we obtain, for every $f \in H_\alpha$,

$$\mathcal{A}^{-1}f = \lim_{t\to+\infty}\Big(\mathcal{A}^{-1}f - S(t)\mathcal{A}^{-1}f\Big) = \int_0^{+\infty} S(t)f dt.$$

By lemma (1.3.9), there exist $f_n \in H_\alpha, f_n = \mathcal{L}_{\mu_n}$, and $k_n \to +\infty$ as $n \to +\infty$, such that

$$||f_n||_{H_\alpha} \leq 1,\ |\mathcal{N}_{\mu_n}(k_n)| \geq C(\alpha)\Big(\frac{\log\ k_n}{k_n}\Big)^{1/\alpha},\ n \geq 1.$$

Therefore,

$$\begin{aligned} ||S(k_n)\mathcal{A}^{-1}||_{H_\alpha} &\geq ||S(k_n)\mathcal{A}^{-1}f_n||_{H_\alpha} \geq \Big|\int_{k_n}^{+\infty} f_n(r)dr\Big| \\ &\geq \Big|\hat{f}_n(0) - \int_0^{k_n} f_n(s)ds\Big| = |\mathcal{N}_{\mu_n}(k_n)| \geq C(\alpha)\Big(\frac{\log\ k_n}{k_n}\Big)^{1/\alpha}. \end{aligned}$$

□

Lemma 1.3.11 *A semigroup of contractions $\{e^{t\mathcal{A}}\}_{t\geq 0}$ in a Hilbert space with norm $||.||$ is exponentialy stable if and only if*

1. *the resolvant set $\rho(\mathcal{A})$ of $\mathcal{A}$ contains the imaginary axis and*
2. $\sup\limits_{\lambda\to+\infty} ||(i\lambda Id - \mathcal{A})^{-1}|| < +\infty$

hold.

Proof The only if part is obvious. We shall prove the if part. Let $\omega = \tau + i\lambda$, $\tau \in [-1/2, 0]$, $M = sup\{||(i\lambda I - \mathcal{A})^{-1}||\}$, $M_0 = sup\{||(\omega I - \mathcal{A})^{-1}||\}$. Then, we have

$$\begin{aligned}||[I+\tau(i\lambda-\mathcal{A})^{-1}]x|| &\geq ||x||-\frac{1}{2M}||(i\lambda-\mathcal{A})^{-1}x|| \\ &\geq ||x||-\frac{1}{2M}||x||=\frac{||x||}{2},\end{aligned}$$

then we have $||[I+\tau(i\lambda-\mathcal{A})^{-1}]x|| \leq 2$. Thus, for $\tau \in (-1/2M, 0], -\infty < \lambda < +\infty$, we have $\omega \in \rho(\mathcal{A})$ and

$$\begin{aligned}||(\omega I-\mathcal{A})|| &= ||(i\lambda-\mathcal{A})^{-1}[I+\tau(i\lambda-\mathcal{A})^{-1}]^{-1}|| \\ &\leq ||(i\lambda-\mathcal{A})^{-1}||||[I+\tau(i\lambda-\mathcal{A})^{-1}]^{-1}|| \leq 2M.\end{aligned}$$

Hence, we have

$$\{\omega : -1 \leq 2M_0 \leq \text{Re }\omega \leq 1/2M_0\} \subset \rho(\mathcal{A}) \tag{1.3.70}$$

and

$$\sup\{||(\omega I-\mathcal{A})^{-1}|| : -1/2M_0 \leq \text{Re }\omega \leq 1/2M_0\} \leq 2M_0. \tag{1.3.71}$$

On the other hand, from $||e^{t\mathcal{A}}|| \leq M, \{\omega : \text{Re}\omega \subset \rho(\mathcal{A})\}$ and $\sup\{||(\omega I-\mathcal{A})^{-1}|| :$ $\text{Re }\omega \geq 1/2M_0\} \leq 2M_0M$, we have

$$\sup\{||(\omega I-\mathcal{A})^{-1}|| : \text{Re }\omega \geq 1/2M_0\} \leq 2M_0(1+M),$$

then, set $\mathcal{A}_1 = \mathcal{A}+\frac{1}{M_0}I$, we have there exists a positive constant number M such that $||e^{t\mathcal{A}_1}|| \leq M$, thus $||e^{t\mathcal{A}}|| \leq Me^{-\frac{1}{M_0}t}$. The theorem is proved.

□

Lemma 1.3.12 *If a bounded C_0-semigroup $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ satisfies*

$$i\mathbb{R} \subset \rho(\mathcal{A}),\ \underset{|\beta|\geq 1}{sup}\frac{1}{\beta^j}||(i\beta I-\mathcal{A})^{-1}||_{\mathcal{H}} < +\infty, \tag{1.3.72}$$

from some $j > 0$, then for any positive integer m there exists a constant $C_m > 0$ such that

$$||e^{t\mathcal{A}}z_0||_{\mathcal{H}} \leq C_m\left(\frac{ln\ t}{t}\right)^{\frac{m}{j}}(ln\ t)||z_0||_{D(\mathcal{A}^m)} \tag{1.3.73}$$

for all $z_0 \in D(\mathcal{A}^m)$.

For more detailled proof, we thus refer to Huang [12]

Chapter 2

Exponential stability for nonlinear thermoelastic equation with second sound

2.1 Introduction

This chapter is concerned with the global existence and asymptotic behavior of solutions to the equations of one-dimensional nonlinear thermoelasticity with thermal memory and second sound. We adopt the results in this chapter from [22].
The reference configuration under consideration is the unit interval $\Omega = (0, 1)$.The equations under consideration read as follows

$$u_{tt} - S(u_x, \theta)_x = 0, \tag{2.1.1}$$

$$\theta_t + \gamma q_x + k_1 * q_x + \beta u_{xt} = 0, \tag{2.1.2}$$

$$q_t + q + k\theta_x = 0, \tag{2.1.3}$$

subject to the boundary conditions

$$u(0, t) = u(1, t) = q(0, t) = q(1, t) = 0 \tag{2.1.4}$$

and initial conditions

$$\begin{aligned} &u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), u_{tt}(x, 0) = u_2(x), \theta(x, 0) = \theta_0(x), \theta_t(x, 0) = \theta_1(x), \\ &q(x, 0) = q_0(x), q_t(x, 0) = q_1(x). \end{aligned} \tag{2.1.5}$$

Here by $u = u(x,t)$, $\theta = \theta(x,t)$ and $q = q(x,t)$, we denote the displacement, absolute temparature and heat flux respectively, $S(u_x, \theta)$ is the Piola-Kirchnoff stress tensor, $k_1 = k_1(t)$ is the relaxation kernel. the sign $*$ denotes the convolution product, i.e., $k_1 * y(.,t) = \int_0^t k_1(t-\tau)y(.,\tau)d\tau$.
Finally, α, β, γ are postive constants.

When the problem (2.1.1)-(2.1.3) has no the relaxation kernel k_1 (i.e., $k_1 = 0$), Messaoudi and Said-Houari [17] considered a one-dimensional homogeneous body occupying, in its reference configuration, an interval I, the laws of balance of momentum, balance of energy, and growth of entropy with the forms

$$\begin{cases} \rho u_{tt} = \sigma_x + b, \\ \\ e_t + q_x = \sigma \varepsilon_t + r, \\ \\ \eta_t \geq \dfrac{r}{\theta} - \left(\dfrac{q}{\theta}\right)_x, \end{cases}$$

where the displacement u, the strain $\varepsilon = u_x$, the stress σ the 'absolute' temperature θ, the heat flux q, the internal energy e, the body force b, and the external heat supply r are all functions of $(x,t)(t \geq 0, x \in I = (0,1))$.
Moreover, the strain and the temperature are required to satisfy $\varepsilon > -1$ and $\theta > 0$, in the absence of the body force b and the external heat supply r,

$$\begin{cases} u_{tt} - a u_{xx} + b\theta_x = \alpha_1 q q_x, \\ \\ \theta_t + g q_x + d u_{tx} = \alpha_2 q q_t, \\ \\ \tau q_t + q_x + k\theta_x = 0, \end{cases}$$

where

$$a = a(u_x, \theta, q),\ b = b(u_x, \theta, q),\ g = (u_x, \theta, q),\ d = d(u_x, \theta, q)$$
$$\tau = \tau(u_x, \theta),\ \alpha_1 = \alpha_1(u_x, \theta),\ k = k(u_x, \theta),\ \alpha_2 = \alpha_2(u_x, \theta),$$

the authors established an exponential decay result for solutions with sufficiently small initial data and proved that the dissipation given by the heat conduction is strong enough to stabilize the system exponentially.
This work has extended the result of Racke [24] to a more general situation.
We assume that S are C^3-functions satisfying

$$\frac{\partial S}{\partial u_x}(0,0) = 1 > 0,\ \frac{\partial S}{\partial \theta}(0,0) \neq 0. \tag{2.1.6}$$

Concerning the kernel, we assume that $k_1(t) \in C^1(\mathbb{R}^+)$ and that $k_1(t)$ is a strongly positive definite kernel. Additionally, we assume that there exist positive constants $c_0 \leq c_1$, such that for all $t \geq 0$,

$$k_1(t) > 0, k_1'(t) + c_0 k_1(t) \leq 0 \leq k_1'(t) + c_1 k_1(t). \tag{2.1.7}$$

To simplify notations, we shall introduce $-\dfrac{\partial S}{\partial \theta}(0,0) = \alpha$ satisfying the product $\alpha\beta > 0$. For the initial data, we assume that

$$\begin{cases} (u_0, u_1, u_2) \in H^3(0,1) \times H^2(0,1) \times H^1(0,1), \\ (\theta_0, \theta_1) \in H^2(0,1) \times H^1(0,1), \\ (q_0, q_1) \in H^2(0,1) \times H^1(0,1), \end{cases} \tag{2.1.8}$$

with

$$\int_0^1 \theta_0(x)dx = 0, \ u_2 = u_{tt}|_{t=0} = \frac{d}{dx} S(u_0(x), \theta_0(x)). \tag{2.1.9}$$

We put $||.|| = ||.||_{L^2(0,1)}$, and we use C (sometimes C_1, C_2,...) to denote the generic positive constant independent of time $t > 0$.

Our main result in this chapter reads as follows

Theorem 2.1.1 *Under assumptions* (2.1.6)-(2.1.9), *there exists a small constant* $0 < \epsilon_0 < 1$ *such that for any* $\epsilon \in (0, \epsilon_0)$ *and for any initial data* $(u_0, u_1, \theta_0, q_0)$ *satisfying*

$$||u_0||_{H^3}^2 + ||u_1||_{H^2}^2 + ||\theta_0||_{H^2}^2 + ||q_0||_{H^2}^2 < \epsilon, \tag{2.1.10}$$

problem (2.1.1)-(2.1.5) *admits a unique global solution* $(u(t), \theta(t), q(t))$ *satisfying*

$$\begin{cases} u(t) \in \bigcap_{m=0}^{2} C^m([0,+\infty), H^{3-m}(0,1) \cap H_0^1), (\partial_t^3 u)(t) \in C(([0,+\infty), L^2(0,1)), \\ (k_1 * \theta)(t), \theta(t) \in \bigcap_{m=0}^{1} C^m(([0,+\infty), H^{2-m}(0,1)) \\ (k_1 * \theta)(t), \theta(t) \in C^2(([0,+\infty), L^2(0,1)), \\ q(t) \in C^1(([0,+\infty), L^2(0,1)) \cap C(([0,+\infty), H^1(0,1)), \\ (k_1 * \partial_t^i \theta_x)(t), (k_1 * \partial_t^j \theta_{xx})(t) \in L^2(([0,+\infty), L^2(0,1)), \ , i = 0,1,2; \ j = 0,1, \\ \partial_t^i q(t), \partial_t^j q_x(t) \in L^2(([0,+\infty), L^2(0,1)), \ , i = 0,1,2; \ j = 0,1, \end{cases} \tag{2.1.11}$$

Moreover, the solution $(u(t), \theta(t), q(t))$ decays exponentially as $t \to +\infty$, i.e., there exists a large time $t_0 > 0$ such that as $t \geq t_0$,

$$||u(t)||^2_{H^3} + ||u_t(t)||^2_{H^2} + ||\theta(t)||^2_{H^2} + ||q(t)||^2_{H^2} < Ce^{-C't}, \tag{2.1.12}$$

where C and C' are positive constants.

2.2 Global existence and stability

In this section, we shall prove Theorem 2.1.1. The main idea of the proof is that we first shall prove the local solutions, by a standard contraction mapping argument, we then can show that problem (2.1.1)-(2.1.5) admits a unique local solution $(u(t), \theta(t), q(t))$ such that

$$\begin{cases} u(t) \in \bigcap\limits_{m=0}^{2} C^m([0,T), H^{3-m}(0,1) \cap H_0^1), (\partial_t^3 u)(t) \in C([0,T), L^2(0,1)), \\ (k_1 * \theta)(t), \theta(t) \in \bigcap\limits_{m=0}^{1} C^m([0,T), H^{2-m}(0,1)), \\ (k_1 * \theta)(t), \theta(t) \in C^2([0,T), L^2(0,1)), \\ (k_1 * \partial_t^i \theta_x)(t), (k_1 * \partial_t^j \theta_{xx})(t) \in L^2(([0,T), L^2(0,1)), \ , i = 0,1,2; \ j = 0,1, \\ \partial_t^i q(t), \partial_t^j q_x(t) \in L^2(([0,T), L^2(0,1)), \ , i = 0,1,2; \ j = 0,1, \\ q(t) \in C^1(([0,T), L^2(0,1)) \cap C(([0,T), H^1(0,1)), \end{cases} \tag{2.2.1}$$

where the constant $T > 0$ is the maximal existence interval of solutions, and second we shall use the contradiction argument to continue the local solutions in time. The next lemma concerns the property of a strongly positive definite kernel.

Lemma 2.2.1 *Assume that $\hat{k}(t) \in L^1(\mathbb{R}^+)$ is a strongly positive definite kernel satisfying $\hat{k}'(t) \in L^1(\mathbb{R}^+)$, thenn for any $y(e) \in L^1_{loc}(\mathbb{R}^+)$, it follows that*

$$\int_0^t \left| \hat{k} * y(\tau) \right|^2 d\tau \leq \beta_0 k_2 \int_0^t y(\tau) \ \hat{k} * y(\tau) d\tau \tag{2.2.2}$$

where $k_2 = \left(\int_0^{+\infty} |\hat{k}(t)| dt \right)^2 + 4 \left(\int_0^{+\infty} |\hat{k}'(t)| dt \right)^2$ and $\beta_0 > 0$, is a constant such that the function $\hat{k}(t) - \beta_0 e^{-t}$ is a positively definite kernel.

Proof Define

$$y_t(\tau) = \begin{cases} y(\tau), & 0 \leq \tau \leq t, \\ 0, & \text{otherwise.} \end{cases}$$

By the Plancherel identity and the fact that convolution is mapped into pointwise multiplication by the Fourier transform,

$$\begin{aligned}\int_0^t \left|\hat{k} * y(\tau)\right|^2 d\tau \quad &\leq \int_0^{+\infty} \left|\int_0^{\tau} \hat{k}(\tau - s) y_t(s) ds\right|^2 d\tau \\ &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|\tilde{\hat{k}}(w)\right|^2 |\tilde{y}_t(w)| dw,\end{aligned} \tag{2.2.3}$$

where

$$\left|\tilde{\hat{k}}(w)\right| = \left|\int_0^{+\infty} e^{-iwt} \hat{k}(t) dt\right| \leq \int_0^{+\infty} \left|\hat{k}(t)\right| dt, \tag{2.2.4}$$

$$\left|w\tilde{\hat{k}}(w)\right| = \left|\int_0^{+\infty} (e^{-iwt} - 1) \hat{k}'(t) dt\right| \leq 2 \int_0^{+\infty} \left|\hat{k}'(t)\right| dt. \tag{2.2.5}$$

and $\tilde{f}$ denotes the Fourrier transform of f.
Square these two inequatilites (2.2.4)-(2.2.5), and add (2.2.5) to (2.2.4) to get

$$\left|\tilde{\hat{k}}(w)\right|^2 \leq \frac{k_2}{1+w^2},$$

which, combined with (2.2.3), yields

$$\begin{aligned}\int_0^t \left|\int_0^{\tau} \hat{k}(\tau - s) y(s) ds\right|^2 d\tau \quad &\leq \frac{k_2}{2\pi} \int_{-\infty}^{+\infty} \frac{|\tilde{y}_t|^2}{1+w^2} dw \\ &= \frac{k_2}{2} \int_{-\infty}^{+\infty} y_t(\tau) \int_{-\infty}^{+\infty} e^{-|\tau - s|} y_t(s) ds d\tau \\ &= k_2 \int_0^t y(\tau) \int_0^{\tau} e^{-(\tau - s)} y(s) ds d\tau \\ &= k_2 \int_0^t y(\tau)(e^{-\tau} * y)(\tau) d\tau \leq k_2 \beta_0 k_2 \int_0^t y(\tau)(\hat{k} * y)(\tau) d\tau.\end{aligned}$$

Thus (2.2.2) follows.

□

From (2.1.3), we can deduce that

$$q = q_0 e^{-t} - \int_0^t e^{s-t} \theta_x ds. \tag{2.2.6}$$

By inserting (2.2.6) and (2.1.6) into (2.1.1)-(2.1.3), system (2.1.1)-(2.1.3) reduces to the system

$$\begin{cases} u_{tt} - u_{xx} + \alpha\theta_x = f, \\ \theta_t - \tilde{k}_1 * \theta_{xx} + \beta u_{xt} = g, \\ q_t + q + k\theta_x = 0, \end{cases} \tag{2.2.7}$$

where

$$f = \left[\frac{\partial S}{\partial u_x}(u_x, \theta) - 1\right] u_{xx} + \left[\frac{\partial S}{\partial \theta}(u_x, \theta) + \alpha\right] \theta_x, \tag{2.2.8}$$

and

$$g = -\gamma q_{0x} k_2 - k_1 * k_2 q_{0x},\ k_2 = e^{-t},\ \tilde{k}_1 = \gamma k k_2 + k k_2 * k_2. \tag{2.2.9}$$

For simplicity, we put

$$\eta_1 = \frac{\partial S}{\partial u_x}(u_x, \theta) - 1,\ \eta_2 = \frac{\partial S}{\partial \theta}(u_x, \theta) + \alpha. \tag{2.2.10}$$

From (2.2.10), we can derive that, if $\gamma > k_1(0)$, then there exists positive constants $c_2 \le c_3$, such that for all $t \ge 0$,

$$\tilde{k}_1(t) > 0,\ \tilde{k}_1'(t) + c_2\tilde{k}_1(t) \le 0 \le \tilde{k}_1'(t) + c_3\tilde{k}_1(t). \tag{2.2.11}$$

If $\gamma > k_1(0)$, from (2.2.10), there exists some time $t_0 > 0$, as $t \ge t_0$, we have $\gamma > k_1(t_0)$, so we can define $\hat{t} = t - t_0$, then we have for all $\hat{t} \ge 0$

$$\tilde{k}_1(\hat{t}) > 0,\ \tilde{k}_1'(\hat{t}) + c_2\tilde{k}_1(\hat{t}) \le 0 \le \tilde{k}_1'(\hat{t}) + c_3\tilde{k}_1(\hat{t}).$$

Thus without loss of generality, we may assume $\gamma > k_1(0)$.
It follows from (2.2.11) that kernel $\tilde{k}_1(t)$ decays exponentially as time goes to infinity and satisfies

$$\tilde{k}_1(0)e^{-c_3 t} \le \tilde{k}_1(t) \le \tilde{k}_1(0)e^{-c_2 t}.$$

Thus we can choose $\delta \in \delta_0 \equiv (0, \min(1, c_3/2, c_2/2))$ so small that fo any $t \ge 0$,

$$\tilde{k}_1(0)e^{-(c_3/2)t} \le \hat{k}(t) := e^{\delta t}\tilde{k}_1(t) \le \tilde{k}_1(0)e^{-(c_2/2)t} \tag{2.2.12}$$

and for all $t \ge 0$,

$$\hat{k}(t) > 0,\ \hat{k}'(t) + \frac{c_2}{2}\hat{k}(t) \le 0 \le \hat{k}'(t) + c_3\hat{k}(t). \tag{2.2.13}$$

Let us denote

$$v(x,t) = e^{\delta t}u(x,t),\ \phi(x,t) = e^{\delta t}\theta(x,t),\ p(x,t) = e^{\delta t}q(x,t). \tag{2.2.14}$$

Then the system (2.2.7) can be rewritten as

$$\begin{cases} v_{tt} - v_{xx} + \alpha\phi_x = F, \\ \phi_t - \hat{k} * \phi_{xx} + \beta v_{xt} = G, \\ p_t + (1-\delta)p + k\phi_x = 0, \\ v(0,t) = v(1,t) = p(0,t) = p(1,t) = 0, \\ v(x,0) = v_0,\ v_t(x,0) = v_1,\ ,\phi(x,0) = \phi_0, p(x,0) = p_0, \end{cases} \tag{2.2.15}$$

where

$$F(t) = fe^{\delta t} + 2\delta v_t - \delta^2 v,\ G(t) = ge^{\delta t} + \delta\phi + \delta\beta v_x, \tag{2.2.16}$$

$$\int_0^1 \phi_0(x)dx = \int_0^1 \theta_0(x)dx = 0. \tag{2.2.17}$$

We easily derive from 2.1.2, 2.1.10 and (2.2.17) that

$$\int_0^1 \phi(x,t)dx = \int_0^1 \theta(x,t)dx = 0. \tag{2.2.18}$$

To facilitate our analysis, let us introduce the linear problem

$$\begin{cases} V_{tt} - V_{xx} + \alpha\Phi_x = \mathcal{F}, \\ \Phi_t - \hat{k} * \Phi_{xx} + \beta V_{xt} = \mathcal{G}, \\ P_t + (1-\delta)P + k\Phi_x = 0, \\ V(0,t) = V(1,t) = P(0,t) = P(1,t) = 0, \\ V(x,0) = v_0,\ V_t(x,0) = V_1,\ ,\Phi(x,0) = \Phi_0, P(x,0) = P_0, \end{cases} \tag{2.2.19}$$

with

$$\int_0^1 \Phi_0(x)dx = 0. \tag{2.2.20}$$

It follows from (2.2.18) and (2.2.20), that when $(V,\Phi) = (v,\phi), (\mathcal{F},\mathcal{G}) = (F,G)$ or $(V,\Phi) = (v_t,\phi_t), (\mathcal{F},\mathcal{G}) = (F_t, G_t + \hat{k}(t)\phi_{0xx})$,

$$\int_0^1 \Phi(x,t)dx = 0,\ \text{for all} t > 0. \tag{2.2.21}$$

In the sequel, we shall study the linearized system (2.2.19). To this end, we define the following energy fuctions

$$\begin{cases} E_1(t,V,\Phi) = \dfrac{1}{2}\displaystyle\int_0^1 (V_t^2 + V_x^2 + \alpha\beta^{-1}\Phi^2)dx, \\ E_2(t,V,\Phi) = \dfrac{1}{2}\displaystyle\int_0^1 (V_{tt}^2 + V_{tx}^2 + \alpha\beta^{-1}\Phi_t^2)dx, \\ E_3(t,V,\Phi) = \dfrac{1}{2}\displaystyle\int_0^1 (V_{tx}^2 + V_{xx}^2 + \alpha\beta^{-1}\Phi_x^2)dx. \end{cases} \tag{2.2.22}$$

Multiplying the first and the second equation in(2.2.19) by V_t and $\alpha\beta^{-1}\Phi$, respectively, and suming the results, we have

$$\frac{d}{dt}E_1(t,V,\Phi) = -\alpha\beta^{-1}\int_0^1 \Phi_x + \hat{k} * \Phi_x dx + \int_0^1 (\mathcal{F}V_t + \alpha\beta^{-1}\mathcal{G}\Phi)dx. \tag{2.2.23}$$

Assuming regular initial data and noting that V_t and Φ_t satisfy the same boundary conditions, we get

$$\begin{aligned}
\frac{d}{dt}E_2(t,V,\Phi) &= -\alpha\beta^{-1}\int_0^1 \Phi_{tx}\hat{k} * \Phi_{tx}dx - \alpha\beta^{-1}\hat{k}(t)\int_0^1 \Phi_{0x}\Phi_{tx}dx \\
&+ \int_0^1 (\mathcal{F}_t V_{tt} + \alpha\beta^{-1}\mathcal{G}_t\Phi_t)dx \\
&= -\alpha\beta^{-1}\int_0^1 \Phi_{tx}\hat{k} * \Phi_{tx}dx - \alpha\beta^{-1}\frac{d}{dt}\left(\hat{k}(t)\int_0^1 \Phi_{0x}\Phi_{tx}dx\right) \\
&+\alpha\beta^{-1}\frac{d}{dt}\hat{k}'(t)\int_0^1 \Phi_{0x}\Phi_{tx}dx + \int_0^1 (\mathcal{F}_t V_{tt} + \alpha\beta^{-1}\mathcal{G}_t\Phi_t)dx.
\end{aligned} \tag{2.2.24}$$

Similarly, multiplying the first and the second equation of (2.2.19) by V_{xxt} and $\alpha\beta^{-1}\Phi_{xx}$, respectively, and summing the results, we have

$$\begin{aligned}
\frac{d}{dt}E_3(t,V,\Phi) &= -\alpha\beta^{-1}\int_0^1 \Phi_{xx}\hat{k} * \Phi_{xx}dx + \int_0^1 (\mathcal{F}V_{xxt} + \alpha\beta^{-1}\mathcal{G}\Phi_{xx})dx \\
&= -\alpha\beta^{-1}\int_0^1 \Phi_{xx}\hat{k} * \Phi_{xx}dx - \frac{d}{dt}\int_0^1 \mathcal{F}V_{xx}dx \\
&+ \int_0^1 (\mathcal{F}_t V_{xt} + \alpha\beta^{-1}\mathcal{G}\Phi_{xx})dx.
\end{aligned} \tag{2.2.25}$$

Now we introduce the following funtions:

$$\begin{cases}
E_4(t,V,\Phi) = -\int_0^1\int_0^x \Phi_t dy V_{tt} dx, \\
E_5(t,V,\Phi) = \int_0^1 V_{tx}\Phi dx, \\
E_6(t,V,\Phi) = \int_0^1 V_{tx}V_x dx, \\
E_7(t,V,\Phi) = -\int_0^1 \Phi_t\hat{k}\Phi_t dx, \\
E_8(t,V,\Phi) = -\int_0^1 \Phi_x\hat{k}\Phi_x dx,
\end{cases}$$

Thus integrating the second equation in(2.2.19) over $(0, x)$, using the boundary conditions and the third equation in(2.2.19), we derive

$$\int_0^x \Phi_t dy - \hat{k} * \Phi_x + \beta V_t = \int_0^x \mathcal{G} dy. \tag{2.2.26}$$

By (2.2.19) and (2.2.26), we easily get

$$\begin{aligned} E_4(t,V,\Phi) \quad &\le -\frac{\beta}{2}||V_{tt}||^2 + \frac{\beta}{8}||\frac{\beta}{2}_{tx}||^2 + \frac{1}{\beta}\Big(k^2(0)||\Phi_x||^2 + ||\hat{k}' * \Phi_x||^2\Big) \\ &+(\alpha + \frac{2}{\beta})||\Phi_t||^2 + \int_0^1 \Big(\int_0^x \mathcal{G}_t dy V_{tt} + \int_0^x \Phi_t dy \mathcal{F}_t\Big) dx. \end{aligned} \tag{2.2.27}$$

Now we define

$$n(t,V,\Phi) = \int_0^1 \Big(V_{tt}^2 + V_{tx}^2 + V_{xx}^2 + \Phi_t^2 + \Phi_x^2\Big)(t) dx$$

and

$$L(t,V,\Phi) = N\Big(E_1(t,V,\Phi) + E_2(t,V,\Phi) + E_3(t,V,\Phi) + \alpha\beta^{-1}\hat{k}\int_0^1 \Phi_{0x}\Phi_x dx\Big)$$

$$+E_4(t,V,\Phi) + E_5(t,V,\Phi) + \frac{\beta}{4}E_6(t,V,\Phi) + a_1 E_7(t,V,\Phi) + a_2 E_8(t,V,\Phi),$$

When $N > 0$ is a parameter sufficiently large and

$$a_1 = \frac{4}{\tilde{k}_1(0)}\Big(\alpha + \frac{2}{\beta}\Big),\ a_2 = \frac{4}{\tilde{k}_1(0)}\Big(\alpha + \frac{4 + \tilde{k^2}_1(0)}{\beta} + \frac{\alpha^2\beta}{8} + a_1\Big).$$

Under the above notations, we can derive the following lemma.

Lemma 2.2.2 *There exist positive constant $\beta_1, \beta_2, \beta_3, C_3, C_4$ and sufficiently large constant N such that $L(t,V,\Phi)$ satisfies the following inequalities:*

$$\begin{aligned} \frac{d}{dt}L(t,V,\Phi) \le \quad & -C_3 n(t,V,\Phi) + C_4\Big(||\hat{k} * \Phi_x||^2 + ||\hat{k} * \Phi_{xt}||^2 + ||\hat{k} * \Phi_{xx}||^2\Big) \\ & -\alpha N\beta^{-1}\int_0^1 (\Phi_x \hat{k} * \Phi_x + \Phi_{tx}\hat{k} * \Phi_{tx} + \Phi_{xx}\hat{k} * \Phi_{xx}) dx \\ & +R(t,V,\Phi), \end{aligned} \tag{2.2.28}$$

$$L(t,V,\Phi) \leq \beta_2\Big(n(t,V,\Phi) + ||\hat{k}*\Phi_x||^2 + ||\hat{k}*\Phi_t||^2 + \hat{k}^2(t)||\Phi_{0x}||^2\Big), \tag{2.2.29}$$

$$L(t,V,\Phi) \geq \beta_1 n(t,Vn\Phi) - \beta_3\Big(||\hat{k}*\Phi_x||^2 + ||\hat{k}*\Phi_t||^2 + \hat{k}^2(t)||\Phi_{0x}||^2\Big), \tag{2.2.30}$$

where

$$\begin{aligned} R(t,V,\Phi) = \ & N\int_0^1 \Big(\mathcal{F}V_t + \alpha\beta^{-1}\mathcal{G}\Phi + \mathcal{F}_t V_{tt} + \alpha\beta^{-1}\mathcal{G}_t\Phi_t + \mathcal{F}_t V_{xx} - \alpha\beta^{-1}\mathcal{G}\Phi_{xx}\Big)dx \\ & -N\frac{d}{dt}\int_0^1 \mathcal{F}V_{xx}dx + \alpha N\beta^{-1}\hat{k}'(t)\int_0^1 \Phi_{0x}\Phi_x dx - a_1\int_0^1 \mathcal{G}_t\hat{k}*\Phi_t dx \\ & +\int_0^1\Big(\int_0^x \mathcal{G}_t dy V_{tt} + \int_0^x \Phi_t dy \mathcal{F}_t\Big)dx + a_2\int_0^x \mathcal{G}\hat{k}*\Phi_{xx}dx \\ & +\int_0^1\Big(V_{tx}\mathcal{G} - \mathcal{F}\Phi_x - \frac{\beta}{4}\mathcal{F}V_{xx}\Big)dx. \end{aligned} \tag{2.2.31}$$

Proof By (2.2.19)and integration by parts, we get

$$\begin{aligned} \frac{d}{dt}E_5(t,V,\Phi) \ & = -\beta||V_{tx}||^2 + \alpha||\Phi_x||^2 - \int_0^1 (V_{xx}+\mathcal{F})\Phi_x dx \\ & +\int_0^1 V_{tx}(\hat{k}*\Phi_{xx} + \mathcal{G})dx \\ & \leq -\frac{\beta}{2}||V_{tx}||^2 + \frac{\beta}{16}||V_{xx}||^2 + \Big(\alpha + \frac{4}{\beta}\Big)||\Phi_x||^2 \\ & +\frac{1}{2\beta}||\hat{k}*\Phi_{xx}||^2 + \int_0^1 (V_{tx}\mathcal{G} - \mathcal{F}\Phi_x)dx. \end{aligned} \tag{2.2.32}$$

and

$$\begin{aligned} \frac{d}{dt}E_6(t,V,\Phi) \ & = ||V_{tx}||^2 - ||V_{xx}||^2 + \alpha\int_0^1 \Phi_x V_{xx} + \mathcal{F}dx - \int_0^1 \mathcal{F}V_{xx}dx \\ & \leq -\frac{1}{2}||V_{xx}||^2 + \frac{\alpha^2}{2}||\Phi_x||^2 + ||V_{tx}||^2 - \int_0^1 \mathcal{F}V_{xx}dx. \end{aligned} \tag{2.2.33}$$

Thus it follows from (2.2.27) and (2.2.32)-(2.2.33) that

$$
\begin{aligned}
&\frac{d}{dt}\Bigg(E_4(t,V,\Phi)+E_5(t,V,\Phi)+\frac{\beta}{4}E_6(t,V,\Phi)\Bigg)\\
&\le -\frac{\beta}{2}||V_{tt}||^2-\frac{\beta}{16}||V_{xx}||^2-\frac{\beta}{8}||V_{tx}||^2+\left(\alpha+\frac{2}{\beta}\right)||\Phi_t||^2\\
&+\left(\alpha+\frac{4}{\beta}+\frac{\alpha^2\beta}{8}+\frac{\tilde{k^2}_1(0)}{\beta}\right)||\Phi_x||^2+\frac{1}{\beta}||\hat{k}'*\Phi_x||^2+\frac{1}{2\beta}||\hat{k}*\Phi_{xx}||^2 \qquad (2.2.34)\\
&+\int_0^1\left(\int_0^x\mathcal{G}_tdyV_{tt}+\int_0^x\Phi_tdy\mathcal{F}_t\right)dx\\
&+\int_0^1\left(V_{tx}\mathcal{G}-\mathcal{F}\Phi_x-\frac{\beta}{4}\mathcal{F}V_{xx}\right)dx.
\end{aligned}
$$

On the other hand, differentiating the second equation in (2.2.19) with respect to t, multiplying the resulting equation by $\hat{k}*\Phi_t$ and integrating it by parts, we deduce

$$
\begin{aligned}
\frac{d}{dt}E_7(t,V,\Phi)= \;& -\tilde{k}_1(0)||\Phi_t||^2+\int_0^1\left(\Phi_t\hat{k}'*\Phi+\tilde{k}_1(0)\Phi_x\hat{k}*\Phi_{tx}+\hat{k}'*\Phi_x\hat{k}*\Phi_{tx}\right)dx\\
&-\int_0^1(\beta V_{tt}\hat{k}*\Phi_{tx}-\mathcal{G}_t\hat{k}*\Phi_t)dx\\
&\le -\frac{\tilde{k}_1(0)}{2}||\Phi_t||^2+\frac{\beta}{4a_1}||V_{tt}||^2+||\Phi_x||^2+\frac{1}{2\tilde{k}_1(0)}||\hat{k}'*\Phi_t||^2\\
&+\frac{\tilde{k}_1^2(0)}{4}||\hat{k}*\Phi_{tx}||^2+\frac{1}{2}\left(||\hat{k}'*\Phi_x||^2+||\hat{k}*\Phi_{tx}||^2\right)\\
&+\beta a_1||\hat{k}*\Phi_{tx}||^2-\int_0^1\mathcal{G}_t\hat{k}*\Phi_tdx.
\end{aligned}
\qquad (2.2.35)
$$

Similary, differentiating the second equation in(2.2.19) with respect to x, multiplying the resulting equation by $\hat{k} * \Phi_x$ and integrating it by parts, we infer

$$
\begin{aligned}
\frac{d}{dt}E_8(t,V,\Phi) = \quad & -\tilde{k}_1(0)||\Phi_x||^2 + ||\hat{k} * \Phi_{xx}||^2 + \int_0^1 \mathcal{G}\hat{k} * \Phi_{xx}dx \\
& - \int_0^1 \big(\beta V_{tx}\hat{k} * \Phi_{xx} + \Phi_x \hat{k}' * \Phi_x\big)dx \\
& \le -\frac{\tilde{k}_1(0)}{2}||\Phi_x||^2 + \frac{\beta}{4a_2}||V_{tx}||^2 + \frac{1}{2\tilde{k}_1(0)}||\hat{k}' * \Phi_x||^2 \\
& +(1+\beta a_2)||\hat{k} * \Phi_{xx}||^2 - \int_0^1 \mathcal{G}\hat{k} * \Phi_{xx}dx.
\end{aligned}
\tag{2.2.36}
$$

Combining (2.2.35) and (2.2.36) with (2.2.34) gives

$$
\begin{aligned}
& \frac{d}{dt}\left(E_4(t,V,\Phi) + E_5(t,V,\Phi) + \frac{\beta}{4}E_6(t,V,\Phi) + a_1E_7(t,V,\Phi) + a_2E_8(t,V,\Phi)\right) \\
& \le -C_3 n(t,V,\Phi) + \Big(\frac{1}{\beta} + \frac{a_1}{2} + \frac{a_2}{2\tilde{k}_1(0)}\Big)||\hat{k}' * \Phi_x||^2 \\
& +\Big[\frac{1}{\beta} + (1+\beta a_2)a_2\Big]||\hat{k} * \Phi_{xx}||^2 + \frac{a_1}{2\tilde{k}_1(0)}||\hat{k}' * \Phi_t||^2 \\
& +\Big(\tilde{k^2}_1(0) + \frac{a_1}{4} + \frac{a_1}{2} + \beta a_1^2\Big)||\hat{k} * \Phi_{tx}||^2 + R_1(t,V,\Phi),
\end{aligned}
\tag{2.2.37}
$$

where $C_3 = \min\Big(\beta/16, \tilde{k}_1(0)a_1/4, \tilde{k}_1(0)a_2/4\Big)$. In view of (2.2.13), (2.2.21) and Poincare's inequality, we have

$$
||\hat{k}' * \Phi_t|| \le ||\hat{k}' * \Phi_{tx}|| \le C||\hat{k} * \Phi_{tx}||. \tag{2.2.38}
$$

Thus it follows from (2.2.23)-(2.2.25) and (2.2.37)-(2.2.38) that (2.2.28) holds.

From the definition of $L(t,V,\Phi)$, we easily know that there exist constants $\beta_1, \beta_2, \beta_3 > 0$ and sufficiently large constant N such that (2.2.29) and (2.2.30) hold. The proof is complete.

□

Now we define

$$
\mathcal{M}(t,V,\Phi) = n(t,v,\phi) + n(t,v_t,\phi_t) + ||\phi_{xx}(t)||^2.
$$

Differentiating the second equation in(2.2.15) with respect to t, we arrive at

$$\phi_{tt} - \tilde{k}_1(0)\phi_{xx} - \hat{k} * \phi_{xx} + \beta v_{ttx} = G_t,$$

which, combined with (2.2.13),the second equation in(2.2.15) and (2.2.16), yields

$$\begin{aligned}||\phi_{xx}||^2 &\leq C\Big(||\phi_{tt}||^2 + ||v_{ttx}||^2 + ||\phi_t||^2 + ||v_{tx}||^2 + ||G||^2 + ||G_t||^2\Big)\\ &\leq C_4\Big(n(t,v,\phi) + n(t,v_t,\phi_t)\Big) + C_5\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2\Big)||p_{0x}||^2.\end{aligned} \tag{2.2.39}$$

Thus,

$$\begin{aligned}n(t,v,\phi) + n(t,v_t,\phi_t) &\leq \mathcal{M}(t,v,\phi) \leq C_6\Big(n(t,v,\phi) + n(t,v_t,\phi_t)\Big)\\ &+C_5\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2\Big)||p_{0x}||^2.\end{aligned} \tag{2.2.40}$$

By smallness condition the last formula of (2.1.11) of initial data, there is a constant $\alpha_1 > 1$, independent of δ, such that

$$\mathcal{M}(0,u,\theta,q) < \alpha_1\epsilon^2 \tag{2.2.41}$$

Using equations (2.2.11)-(2.2.14), there exixts a constant $\alpha_2 > 1$, independent of δ, such that

$$n(t,v,\phi) + n(t,v_t,\phi_t) \leq \mathcal{M}(0,v,\phi) \leq \alpha_2\mathcal{M}(0,u,\theta) < \alpha_1\alpha_2\epsilon^2. \tag{2.2.42}$$

We derive from (2.1.7), (2.2.9), (2.2.12)-(2.2.13) and (2.2.42) that there exists a constant $\eta_0 > 0$, independent of δ, such that

$$\begin{aligned}&\int_0^{+\infty}\Big\{\lambda_3(\hat{k}'(t))^2\Big(||\phi_{0x}||^2 + ||\Phi_{1x}||^2\Big) + \Big[\lambda_1(\hat{k}'(t))^2 + \lambda_2(\hat{k}(t))^2\Big]||\phi_{0xx}||^2\\ &+\lambda_4\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2 + ((k_1 * k_2)'(t))^2\Big)||p_{0x}||^2\Big\}dt < \eta_0\epsilon^2,\end{aligned} \tag{2.2.43}$$

where

$$\lambda_1 = 1 + \frac{2C_6}{C_3} + \frac{4N^2\alpha^2C_6}{C_3\beta^2},\ \lambda_2 = \frac{2C_6}{C_3} + \frac{2N^2\alpha^2C_6}{C_3\beta^2},\ \lambda_3 = \frac{2N^2\alpha^2C_6}{C_3\beta^2},\ \lambda_4 = \lambda_2 + NC_5. \tag{2.2.44}$$

Using the continuity of the solutions, it follows that there exist constants $\alpha_0 > 0$ and $t_0 \in [0, T)$ such that

$$\mathcal{M}(t, v, \phi) \le \alpha_0 \epsilon^2, \text{ for all } t \in [0, t_0)\,. \tag{2.2.45}$$

Now we define

$$t_1 = \sup\Big\{\tau_1 > 0;\ \mathcal{M}(t, v, \phi) \le \alpha_0 \epsilon^2 \text{ in } [0, \tau_1)\Big\}. \tag{2.2.46}$$

By Soblev's embedding theorem and (2.2.46), we obtain that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$|v_x(x, t)| + |\phi(x, t)| + |\phi_x(x, t)| + |\phi_t(x, t)| \le C_7 \epsilon, \tag{2.2.47}$$

which implies that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$|u_x(x, t)| + |\theta(x, t)| + |\theta_x(x, t)| + |\theta_t(x, t)| \le C_8 \epsilon e^{-\delta t}. \tag{2.2.48}$$

Thus, if ϵ is small enough, we have that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$|u_x(x, t)| < \rho_0. \tag{2.2.49}$$

Define

$$v = \sup_{|x|+|y| \le \rho_0} \{|\partial^\rho \eta_i|; i = 1, 2; 0 \le |\rho| \le 3\},$$

where ∂^ρ denotes the partial derivatives of order $|\rho|$. Recalling the definitions of $\eta_i (i = 1, 2)$ and using the above inequalities, we deduce

$$|\eta_i| \le C_9 \epsilon,\ i = 1, 2, \tag{2.2.50}$$

with $C_9 = C_9(v) > 0$, being a constant. By (2.2.46)-(2.2.50), we easily derive that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$|v_t(x, t)| + |v_{tx}(x, t)| + |v_{tt}(x, t)| \le C_{10} \epsilon,$$

which, together with (2.2.12),(2.2.14), implies that for any $(x, t) \in [0, 1] \times [0, t_1)$,

$$|u_t(x, t)| + |u_{tx}(x, t)| + |u_{tt}(x, t)| \le C_{11} e^{-\delta t} \epsilon, \tag{2.2.51}$$

By (2.2.15), (2.2.47), and (2.2.50)-(2.2.51), we get

$$|v_{xx}(x,t)|+ \leq C\epsilon + C\epsilon|v_{xx}(x,t)|,$$

which gives

$$|v_{xx}(x,t)| \leq C_{12}\epsilon,\ |u_{xx}(x,t)| \leq C_{12}\epsilon e^{-\delta t},\ \text{for all } (x,t) \in [0,1]\times[0,t_1) \qquad (2.2.52)$$

Similarly, differentiating (2.2.15) with respect to x, we conclude

$$||v_{xxx}(t)||^2 \leq C\Big[\mathcal{M}(t,v,\phi) + ||F_x||^2\Big] \leq C\mathcal{M}(t,v,\phi) + C\epsilon^2||v_{xxx}(t)||^2,$$

which gives that for any $t \in [0,t_1)$,

$$||v_{xxx}(t)||^2 \leq C\mathcal{M}(t,v,\phi) \leq C_{13}\epsilon^2,\ ||u_{xxx}(t)||^2 \leq C\mathcal{M}(t,v,\phi) \leq C_{13}\epsilon^2 e^{-\delta t} \quad (2.2.53)$$

provided that ϵ is small enough.

Lemma 2.2.3 *Under the same assumptions as in Theorem 2.1.1, the following inequalities holds for any $t \in [0,t_1)$:*

$$\int_0^1 \mathcal{F}_t V_{tt} dx \leq C(\delta+\epsilon)\mathcal{M}(t,v,\phi) - \frac{1}{2}\frac{d}{dt}\int_0^1 \eta_1 V_{tx}^2 dx, \qquad (2.2.54)$$

$$\int_0^1 \mathcal{F}_t V_{xx} dx \leq C(\delta+\epsilon)\mathcal{M}(t,v,\phi) + \frac{1}{2}\frac{d}{dt}\int_0^1 \eta_1 V_{xx}^2 dx, \qquad (2.2.55)$$

$$\int_0^1 \mathcal{F} V_t dx \leq C(\delta+\epsilon)\mathcal{M}(t,v,\phi), \qquad (2.2.56)$$

$$\int_0^1 \mathcal{F} V_{xx} dx \leq C(\delta+\epsilon)\mathcal{M}(t,v,\phi), \qquad (2.2.57)$$

$$\int_0^1 \mathcal{F}^2 dx \leq C(\delta+\epsilon)\mathcal{M}(t,v,\phi). \qquad (2.2.58)$$

Proof We only consider the case of $(V,\Phi) = (v_t,\phi_t)$ and $\mathcal{F} = F_t$ to prove (2.2.54). The case of $(V,\Phi) = (v,\phi)$ and $\mathcal{F} = F$ is simple. By (2.2.8) and noting that

$$\begin{cases} F_{tt} = f_{tt}e^{\delta t} + 2\delta f_t e^{\delta t} + \delta^2 f e^{\delta t} + 2\delta v_{ttt} - \delta^2 v_t, \\ f_{tt} = \eta_{1tt} + 2\eta_{1t}u_{xxt} + \eta_1 u_{xxtt} + \eta_{2tt}\theta_x + 2\eta_{2t}\theta_{xt} + \eta_2\theta_{xtt}, \\ \eta_{1tt} = (u_{xt},\theta_t)\mathcal{H}_{\eta_1}(u_{xt},\theta_t)^\tau + \nabla\eta_1.(u_{xtt},\theta_{tt}),\ \eta_{1t} = \nabla\eta_1.(u_{xt},\theta_t), \end{cases}$$

equalities hold for any $t \in [0, t_1)$*:*

$$R(t,v,\phi) \le C(\epsilon+\delta)\mathcal{M}(t,v,\phi) + (C\delta + \frac{a_1}{4})||\hat{k} * \phi_{tx}||^2 + C\delta||\hat{k} * \phi_{txx}||^2$$
$$+\frac{C_3}{8C_6}\Big(||v_{xx}|| + ||v_{txx}|| + ||\phi_x|| + ||\phi_{tx}||\Big)$$
$$+\Big(\lambda_1(\hat{k}'(t))^2 + \lambda_2((\hat{k}(t))^2\Big)||\phi_{0xx}||^2 + \lambda_3((\hat{k}'(t))^2||\phi_{1x}|^2|$$
$$+\lambda_4\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2 + ((k_1 * k_2)'(t))^2\Big)||p_{0x}||^2$$
$$+\frac{d}{dt}\int_0^1 \Big[\frac{N}{2}(\eta_1 v_{txx}^2 - v_{txx}^2) - N\alpha\beta^{-1}\hat{k}(t)\phi_{0xx}\phi_{xx} - NF_t v_{txx}\Big]dx, \tag{2.2.59}$$

$$\begin{aligned} N\int_0^1 F_t dx &\le C(\epsilon+\delta)\mathcal{M}(t,v,\phi) - N\alpha\beta^{-1}\hat{k}(t)\int_0^1 \phi_{0xx}\phi_{xx}dx \\ &\le \frac{\beta_1}{2}\Big(n(t,v,\phi) + n(t,v_t,\phi_t)\Big) + \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}\hat{k}^2(t)||\phi_{0xx}||^2. \end{aligned} \tag{2.2.60}$$

Let us introduce the following functions

$$\Upsilon_1(t) = L(t,v,\phi) + L(t,v_t,\phi_t) + N\int_0^1 (Fv_{xx} + F_t v_{txx})dx$$
$$+\frac{N}{2}\int_0^1 \eta_1(v_{ttx}^2 - v_{txx}^2)dx + N\alpha\beta^{-1}\hat{k}(t)\int_0^1 \phi_{0xx}\phi_{xx}dx.$$

Then it follows from (2.2.40),(2.2.48)-(2.2.53) and lemmas 2.2.2-2.2.5. that if $\epsilon+\delta$ is small enough

$$\Upsilon_1(t) \le \Big(\beta_2 + \frac{\beta_1}{2}\Big)\Big(n(t,v,\phi) + n(t,v_t,\phi_t)\Big) + C(\epsilon+\delta)\mathcal{M}(t,v,\phi)$$
$$+\frac{N^2\alpha^2C_4}{2\beta_1\beta_2}\hat{k}^2(t)||\phi_{0xx}||^2 + \beta_2\Big(||\hat{k} * \phi_t||^2 + ||\hat{k} * \phi_{tt}||^2 + ||\hat{k} * \phi_x||^2 \tag{2.2.61}$$
$$+||\hat{k} * \phi_{tx}||^2 + \hat{k}^2(t)||\phi_{0x}||^2 + \hat{k}^2(t)||\phi_{1x}||^2\Big)$$

$$\begin{aligned}\Upsilon_1(t) \leq\ & \Big(\beta_2 + \frac{\beta_1}{2}\Big)[n(t,v,\phi) + n(t,v_t,\phi_t)] \\ & + C_5 C(\epsilon+\delta)\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2\Big)||p_{0x}||^2 \\ & + \Big(\beta_2||\phi_{0x}||^2 + \beta_2||\phi_{1x}||^2 + \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}||\phi_{0xx}||^2\Big)\hat{k}^2(t) \\ & + \beta_2\Big(||\hat{k} * \phi_t||^2 + ||\hat{k} * \phi_{tt}||^2 + ||\hat{k} * \phi_x||^2 + ||\hat{k} * \phi_{tx}||^2 Big)\end{aligned} \tag{2.2.62}$$

and

$$\begin{aligned}\Upsilon_1(t) \geq\ & \frac{\beta_1}{2}[n(t,v,\phi) + n(t,v_t,\phi_t)] - C(\epsilon+\delta)\mathcal{M}(t,v,\phi) \\ & - \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}\hat{k}^2(t)||\phi_{0xx}||^2 - \beta_3\Big(||\hat{k} * \phi_t||^2 + ||\hat{k} * \phi_{tt}||^2 + ||\hat{k} * \phi_x||^2 \\ & + ||\hat{k} * \phi_{tx}||^2 + \hat{k}^2(t)||\phi_{0x}||^2 + \hat{k}^2(t)||\phi_{1x}||^2\Big) \\ \geq\ & \frac{\beta_1}{4}[n(t,v,\phi) + n(t,v_t,\phi_t)] \\ & - C_5 C(\epsilon+\delta)\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2\Big)||p_{0x}||^2 \\ & - \Big(\beta_3||\phi_{0x}||^2 + \beta_3||\phi_{1x}||^2 + \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}||\phi_{0xx}||^2\Big)\hat{k}^2(t) \\ & + \beta_3\Big(||\hat{k} * \phi_t||^2 + ||\hat{k} * \phi_{tt}||^2 + ||\hat{k} * \phi_x||^2 + ||\hat{k} * \phi_{tx}||^2 Big)\end{aligned} \tag{2.2.63}$$

Define

$$\begin{aligned}\Upsilon(t) =\ & \Upsilon_1(t) C_5 C(\epsilon+\delta)\Big((k_2(t))^2 + ((k_1 * k_2)(t))^2\Big)||p_{0x}||^2 \\ & + \Big(\beta_3||\phi_{0x}||^2 + \beta_3||\phi_{1x}||^2 + \frac{N^2\alpha^2 C_4}{2\beta_1\beta_2}||\phi_{0xx}||^2\Big)\hat{k}^2(t) \\ & + \beta_3\Big(||\hat{k} * \phi_t||^2 + ||\hat{k} * \phi_{tt}||^2 + ||\hat{k} * \phi_x||^2 + ||\hat{k} * \phi_{tx}||^2 Big).\end{aligned} \tag{2.2.64}$$

Then it follows from (2.2.63),(2.2.13) and (2.2.40) that if $\epsilon+\delta$ is small enough, we have

$$\Upsilon(t) \geq \frac{\beta_1(n(t,v,\phi) + n(t,v_t,\phi_t))}{4} \geq \frac{\beta_1}{4C_6}\mathcal{M}(t,v,\phi), \tag{2.2.65}$$

$$\begin{aligned}\frac{d}{dt}\Upsilon(t) \le \ & \frac{d}{dt}\Upsilon_1(t) + \frac{C_3}{8}(n(t,v,\phi) + n(t,v_t,\phi_t)) \\ & + C_{14}\beta_3\Big(||\hat{k} * \phi_{tx}||^2 + ||\hat{k} * \phi_{ttx}||^2 + ||\hat{k} * \phi_x||^2\Big).\end{aligned} \tag{2.2.66}$$

Proof of Theorem 2.1.1 We shall assume that the initial data belong to $H^4(0,1)$. Our result will follow the standard density. By virtue of Lemmas 2.2.4 and 2.2.5, we easily obtain

$$\begin{aligned}\frac{d}{dt}\Upsilon_1(t) \le \ & -N\alpha\beta^{-1}\int_0^1 \Big(\phi_x\hat{k} * \phi_x + 2\phi_{tx}\hat{k} * \phi_{tx} + \phi_{ttx}\hat{k} * \phi_{ttx} \\ & + \phi_{xx}\hat{k} * \phi_{xx} + \phi_{txx}\hat{k} * \phi_{txx}\Big)dx \\ & - C_3(n(t,v,\phi), n(t,v_t,\phi_t)) + \frac{C_3}{8C_6}\mathcal{M}(t,v,\phi) \\ & + C_{15}(\epsilon+\delta)\mathcal{M}(t,v,\phi) + C_4\Big(||\hat{k} * \phi_x||^2 + ||\hat{k} * \phi_{tx}||^2 \\ & + ||\hat{k} * \phi_{ttx}||^2 + ||\hat{k} * \phi_{xx}||^2 + ||\hat{k} * \phi_{txx}||^2\Big) + \frac{a_1^2}{4}||\hat{k} * \phi_{tx}||^2 \\ & + C_{16}\delta\Big(||\hat{k} * \phi_{tx}||^2 + ||\hat{k} * \phi_{xx}||^2 + ||\hat{k} * \phi_{txx}||^2\Big) \\ & + \Big[\lambda_1(\hat{k}'(t))^2 + \lambda_2(\hat{k}(t))^2\Big]||\phi_{0xx}||^2 + \lambda_3(\hat{k}'(t))^2(||\phi_{1x}||^2 + ||\phi_{0x}||^2) \\ & + \lambda_4\Big[(k_2(t))^2 + ((k_1 * k_2)(t))^2 + ((k_1 * k_2)'(t))^2\Big]||p_{0x}||^2,\end{aligned} \tag{2.2.67}$$

which, together with (2.2.65)-(2.2.66), yields that if $\epsilon + \delta$ is small enough

$$
\begin{aligned}
\frac{d}{dt}\Upsilon_1(t) \leq \;& -N\alpha\beta^{-1}\int_0^1 \Big(\phi_x\hat{k}*\phi_x + 2\phi_{tx}\hat{k}*\phi_{tx} + \phi_{ttx}\hat{k}*\phi_{ttx} \\
& +\phi_{xx}\hat{k}*\phi_{xx} + \phi_{txx}\hat{k}*\phi_{txx}\Big)dx \\
& -\frac{C_3}{2}\Big((n(t,v,\phi), n(t,v_t,\phi_t))\Big) \\
& +\Big(2C_4 + \frac{a_1^2}{4}\Big)\Big(||\hat{k}*\phi_x||^2 + ||\hat{k}*\phi_{tx}||^2 \\
& +||\hat{k}*\phi_{ttx}||^2 + ||\hat{k}*\phi_{xx}||^2 + ||\hat{k}*\phi_{txx}||^2\Big) + \\
& +\Big[\lambda_1(\hat{k}'(t))^2 + \lambda_2(\hat{k}(t))^2\Big]||\phi_{0xx}||^2 + \lambda_3(\hat{k}'(t))^2(||\phi_{1x}||^2 + ||\phi_{0x}||^2) \\
& +\lambda_4\Big[(k_2(t))^2 + ((k_1*k_2)(t))^2 + ((k_1*k_2)'(t))^2\Big]||p_{0x}||^2,
\end{aligned}
\tag{2.2.68}
$$

Integrating (2.2.67) with respect to t, using Lemma 2.2.2, (2.2.62)-(2.2.65) and (2.2.42), taking ϵ and δ small enough, we deduce

$$
\begin{aligned}
\Upsilon(t) \;& +\frac{C_3}{2}\int_0^t \Big(n(\tau,v,\phi) + n(\tau,v_t,\phi_t)\Big)d\tau \\
& +C_{17}\int_0^t \Bigg[\sum_{i=0}^{2}||\hat{k}*\partial_t^i\phi_x||^2 + \sum_{i=0}^{1}||\hat{k}*\partial_t^i\phi_x||^2\Bigg]d\tau \qquad (2.2.69) \\
\leq \;& (\beta_1+\beta_2)\frac{\alpha_1\alpha_2\epsilon^2}{2} + \tilde{k}_1(0)\Big[2(\beta_1+\beta_2) + \frac{N^2\alpha^2C_4}{\beta_1\beta_2}\Big]\alpha_1\alpha_2\epsilon^2 + \eta_0\epsilon^2 \\
= \;& \alpha_3\epsilon^2.
\end{aligned}
$$

Thus it follows from (2.2.40), (2.2.43), (2.2.65) and (2.2.69) that for any $t \in [0, t_1)$,

$$
\begin{aligned}
\mathcal{M}(t,v,\phi) \;& +\frac{2C_3}{\beta_1}\int_0^t \mathcal{M}(\tau,v,\phi)d\tau \\
& +\frac{4C_6C_{17}}{\beta_1}\int_0^t \Bigg(\sum_{i=0}^{2}||\hat{k}*\partial_t^i\phi_x||^2 + \sum_{i=0}^{1}||\hat{k}*\partial_t^i\phi_x||^2\Bigg)d\tau \qquad (2.2.70) \\
\leq \;& \frac{4C_6\alpha_3\epsilon^2}{\beta_1} = (\alpha_0 - \alpha_1\alpha_2)\epsilon^2.
\end{aligned}
$$

Letting $t \to t_1$ in (2.2.70), we have

$$\mathcal{M}(t_1, v, \phi) \leq (\alpha_0 - \alpha_1\alpha_2)\epsilon^2 \leq \alpha_0\epsilon^2,$$

which contradicts the definition of t_1, By repeating the same procedure, taking ϵ even smaller if necessary, and using the continuity of $\mathcal{M}(t, v, \phi)$, (2.2.70) is established for all $t > 0$. On the other hand, it is easy to verify that for all $t > 0$,

$$C_{18}^{-1}\mathcal{M}(t, \theta, v)e^{2\delta t} \leq \mathcal{M}(t, v, \phi) \leq \mathcal{M}(t, \theta, v)e^{2\delta t}. \tag{2.2.71}$$

By (2.2.11), we deduce

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=0}^{1}||(\hat{k} * \partial_t^i\phi_x)(t)||^2 + ||(\hat{k} * \partial_t^i\phi_{xx})(t)||^2\right) \\
&\leq C\left(\sum_{i=0}^{1}(||(\hat{k} * \partial_t^i\phi_x)(t)||^2 + ||(\partial_t^i\phi_x)(t)||^2) + ||(\hat{k} * \partial_t^i\phi_{xx})(t)||^2 + ||\phi_{xx}(t)||^2\right) \\
&\leq C\left(\sum_{i=0}^{1}||(\hat{k} * \partial_t^i\phi_x)(t)||^2 + ||(\hat{k} * \partial_t^i\phi_{xx})(t)||^2 + \mathcal{M}(t, v, \phi)\right).
\end{aligned} \tag{2.2.72}$$

Integrating (2.2.72) with respect to t, and exploiting (2.2.70), we finally obtain

$$\sum_{i=0}^{1}\left(||(\hat{k} * \partial_t^i\phi_x)(t)||^2 + ||(\hat{k} * \partial_t^i\phi_{xx})(t)||^2\right) \leq C. \tag{2.2.73}$$

Then by (2.2.6), (2.2.14), (2.2.70), (2.2.71) and (2.2.73), the proof of Theorem 2.1.1 is now complete

$\square$

Chapter 3

Energy decay for Timoshenko-type system with a past history

3.1 Introduction

In this chapter, we shall consider the following Timoshenko-type system

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \int_0^{+\infty} g(s)\psi_{xx}(t-s)ds + k(\varphi_x + \psi) + \delta\theta_x = 0, \\ \rho_3\theta_t - \beta\theta_{xx} + \delta\psi_{xt} = 0, \end{cases} \tag{3.1.1}$$

with positive constants ρ_1, ρ_2, ρ_3, k, b, β, δ together with initial conditions

$$\varphi(.,0) = \varphi_0,\ \varphi_t(.,0) = \varphi_1,\ \psi(.,0) = \psi_0,\ \psi_t(.,0) = \psi_1,\ \theta(.,0) = \theta_0 \tag{3.1.2}$$

and boundary conditions

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = 0, \tag{3.1.3}$$

where the functions φ, ψ and θ depend on $(x,t) \in [0,1] \times [0,+\infty)$ and model the transverse displacement of a beam with reference configuration $(0,1) \subset \mathbb{R}$, the rotation angle of a filament and the temperature difference respectively.

In this chapter, we have extended and improved those results in [26] with weaker conditions on g, that is, we shall use a different method (the multiplier techniques, which was also used in [18]) from that in [26] to prove the exponential stability result which was also obtained in [26] only for the case of equal wave speeds. Furthermore, we also prove a polynomial stability result which has been never studied before.We adopt the results in this chapter from [23].

3.2 Preliminaries

In order to state our main result, we make the following hypotheses:
(H1). $g : \mathbb{R}^+ \to \mathbb{R}^+$ is differentiable function such that

$$g(0) > 0,\ b - \int_0^{+\infty} g(s)ds = \overline{b} > 0.$$

(H2). There exist constants $\gamma > 0$ and $1 \le p < 3/2$ such that

$$g'(s) \le -\gamma g^p(s),\ \forall s \ge 0.$$

Following the idea of [10], we introduce

$$\eta^t(x,s) = \psi(x,t) - \psi(x,t-s),\ \ s \ge 0 \tag{3.2.1}$$

and obtain the following initial and boundary conditions

$$\eta^t(x,0) = 0,\ \eta^t(0,s) = \eta^t(1,s) = 0,\ \eta^0(x,s) = \eta_0(x,s),\ \text{for all } s,t \ge 0. \tag{3.2.2}$$

Clearly, (3.2.1) gives

$$\eta_t^t(x,s) + \eta_s^t(x,s) = \psi_t(x,t). \tag{3.2.3}$$

Remark 3.2.1 Under the hypothesis (H1)-(H2), it is easy to verify that

$$G_0 = \int_0^{+\infty} g^{1/2}(s)ds < +\infty,\ G_p = \int_0^{+\infty} g^{2-p}(s)ds < +\infty,\ 1 \le < 3/2 \tag{3.2.4}$$

From semigroup theory (see, e.g., [14, 29]) we have the following existence and regularity result, for the explicit proofs, we refer the reader to [26].

Lemma 3.2.1 *Let $(\varphi_0,\varphi_1), (\psi_0,\psi_1) \in H_0^1(0,1) \times L^2(0,1), \theta_0 \in L^2(0,1)$. Assume that (H1) is satisfied, then problem* (3.1.1) *has a unique global weak solution $(\varphi(t),\psi(t),\theta(t))$ verifying*

$$\varphi(t), \psi(t) \in C(\mathbb{R}^+, H_0^1(0,1)) \cap C(\mathbb{R}^+, L^2(0,1)),\ \theta \in C(\mathbb{R}^+, L^2(0,1)$$

Moreover, if

$$(\varphi_0,\varphi_1), (\psi_0,\psi_1) \in (H_0^2(0,1) \times H_0^1(0,1)) \times H_0^1(0,1),\ \theta_0 \in H_0^1(0,1)$$

then the solution $(\varphi(t),\psi(t),\theta(t))$ satisfies

$$\begin{cases} \varphi(t), \psi(t) \in C(\mathbb{R}^+, H_0^2(0,1) \times H_0^1(0,1)) \cap C^1(\mathbb{R}^+, H_0^1(0,1)) \cap C^2(\mathbb{R}^+, L^2(0,1)), \\ \theta(t) \in C(\mathbb{R}^+, H_0^2(0,1) \times H_0^1(0,1)) \cap C^1(\mathbb{R}^+, H_0^1(0,1)). \end{cases}$$

3.3 Case of equal wave-speeds

In this section, we shall state and prove a decay result in the case of equal wave speeds propagation. First, combining (3.1.1)–(3.1.3) and (3.2.1)–(3.2.2), we give a reformulation system as

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - \bar{b}\psi_{xx} - \int_0^{+\infty} g(s)\eta^t_{xx}(t-s)ds + k(\varphi_x + \psi) + \delta\theta_x = 0, \\ \rho_3\theta_t - \beta\theta_{xx} + \delta\psi_{xt} = 0, \\ \eta^t_t(x,s) + \eta^t_s(x,s) - \psi_t(x,s) = 0, \end{cases} \tag{3.3.1}$$

in $(0,1)\times(0,+\infty)$ together with initial conditions and boundary conditions

$$\varphi(.,0) = \varphi_0,\ \varphi_t(.,0) = \varphi_1,\ \psi(.,0) = \psi_0,\ \psi_t(.,0) = \psi_1,\ \theta(.,0) = \theta_0,\ \eta^t(x,0) = 0 \tag{3.3.2}$$

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = \theta(0,t) = \theta(1,t) = \eta^t(0,s) = \eta^t(1,s) = 0. \tag{3.3.3}$$

The associated energy of system (3.3.1) is given by

$$\begin{aligned} E(t) = &\ \frac{1}{2}\int_0^1 (\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_3\theta^2 + k|\varphi_x + \psi|^2 + \bar{b}\psi_x^2)dx \\ &+\frac{1}{2}\int_0^1\int_0^{+\infty} g(s)|\eta^t_x(x,s)|^2 dsdx. \end{aligned} \tag{3.3.4}$$

Let $L^2_g(\mathbb{R}^+, H^1_0)$ be the Hilbert space H^1_0-valued functions on $\mathbb{R}^+$, endowed with the norm

$$||w||_{L^2_g(\mathbb{R}^+,H^1_0)} = \left(\int_0^1\int_0^{+\infty} g(s)|\eta^t_x(x,s)|^2 dsdx\right)^{1/2}.$$

We are now ready to state our main stability result.

Theorem 3.3.1 *Let* $(\varphi_0,\varphi_1),(\psi_0,\psi_1) \in H^1_0(0,1)\times L^2(0,1), \eta^t_0 \in L^2_g\Big(\mathbb{R}^+, H^1_0(0,1)\Big)$ *and* $\theta_0 \in L^2(0,1)$. *Assume that (H1)-(H2) and* $\dfrac{\rho_1}{k} = \dfrac{\rho_2}{b}$ *are satisfied, then there exist two positive constants* C *and* ω, *such that*

$$E(t) \le Ce^{-\omega t},\ p = 1,\ \textit{for all}\ t \ge 0, \tag{3.3.5}$$

$$E(t) \leq \frac{C}{(t+1)^{1/(p-1)}},\ p > 1,\ for\ all\ t \geq 0, \tag{3.3.6}$$

The proof of our result will be established through the following several lemmas.

Lemma 3.3.1 *Let $(\varphi, \psi, \theta, \eta^t)$ be the solution of problem (3.3.1), then we have*

$$E'(t) = -\beta \int_0^1 \theta_x^2 dx + \frac{1}{2} \int_0^1 \int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 ds dx \leq 0. \tag{3.3.7}$$

Proof of the lemma and the enegy function
Multiplying the first equation of (3.3.1) by φ_t, and integrating their sum over (0,1), we can obtain the following formulation

$$\int_0^1 [\rho_1 \varphi_{tt} - k(\varphi_x + \psi)_x]\varphi_t dx = \int_0^1 \frac{1}{2}\frac{d}{dt}\varphi_t^2 dx \underbrace{-k(\varphi_x+\psi)|_{x=0}^{x=1}}_{=0} + k\int_0^1 (\varphi_x + \psi)\varphi_{xt} dx = 0 \tag{3.3.8}$$

Multiplying the second equation of (3.3.1) by ψ_t, and integrating their sum over (0,1), we can obtain the following formulation

$$\int_0^1 [\rho_2 \psi_{tt} - \bar{b}\psi_{xx} - \int_0^{+\infty} g(s)\eta_{xx}^t(t-s)ds + k(\varphi_x + \psi) + \delta\theta_x]\psi_t dx =$$
$$\int_0^1 \frac{1}{2}\frac{d}{dt}\psi_t^2 dx \underbrace{-(\bar{b}\psi_x\psi_t)|_{x=0}^{x=1}}_{=0} + \bar{b}\int_0^1 (\psi_x\psi_{xt})dx - \underbrace{-\int_0^{+\infty}(g(s)\eta_x^t)|_{x=0}^{x=1} ds}_{=0} + \int_0^1\int_0^{+\infty} g(s)\eta_x^t\psi_{xt} ds dx$$
$$+k\int_0^1 (\varphi_x+\psi)\psi_t dx \underbrace{+(\delta\theta\psi_t)|_{x=0}^{x=1}}_{=0} - \int_0^1 \theta\psi_{xt} dx$$
$$= \int_0^1 \frac{1}{2}\frac{d}{dt}\psi_t^2 dx + \bar{b}\int_0^1 \frac{1}{2}\frac{d}{dt}\psi_x^2 dx + \int_0^1\int_0^{+\infty} g(s)\eta_x^t\psi_{xt} ds dx + k\int_0^1 (\varphi_x+\psi)\psi_t dx - \int_0^1 \theta\psi_{xt} dx \tag{3.3.9}$$

Now using the value of ψ_t in the fourth equation of the system (3.3.1) we get the following result

$$\begin{aligned}
\int_0^1\int_0^{+\infty} g(s)\eta_x^t\psi_{xt} ds dx &= \int_0^1\int_0^{+\infty} g(s)\eta_x^t(\eta_{tx}+\eta_{sx}) ds dx \\
&= \int_0^1\int_0^{+\infty} \frac{1}{2}\frac{d}{dt} g(s)(\eta_x^t)^2 ds dx + \int_0^1\int_0^{+\infty} g(s)\frac{1}{2}\frac{d}{ds}(\eta_x^t)^2 ds dx \\
&= \int_0^1\int_0^{+\infty} \frac{1}{2}\frac{d}{dt} g(s)(\eta_x^t)^2 ds dx + \lim_{r\to+\infty} (g(s)(\eta_x^t)^2)|_{s=0}^{s=r} \\
&- \int_0^1\int_0^{+\infty} g'(s)(\eta_x^t)^2 ds dx
\end{aligned} \tag{3.3.10}$$

Substituting the result of (3.3.10) in (3.3.9) results

$$\int_0^1 \frac{1}{2}\frac{d}{dt}\psi_t^2 dx + \bar{b}\int_0^1 \frac{1}{2}\frac{d}{dt}\psi_x^2 dx + \int_0^1\int_0^{+\infty} g(s)\eta_x^t\psi_{xt} dsdx + k\int_0^1 (\varphi_x+\psi)\psi_t dx - \int_0^1 \theta\psi_{xt} dx =$$
$$\int_0^1 \frac{1}{2}\frac{d}{dt}\psi_t^2 dx + \bar{b}\int_0^1 \frac{1}{2}\frac{d}{dt}\psi_x^2 dx + k\int_0^1 (\varphi_x+\psi)\psi_t dx - \int_0^1 \theta\psi_{xt} dx + \int_0^1\int_0^{+\infty} \frac{1}{2}\frac{d}{dt} g(s)(\eta_x^t)^2 dsdx$$
$$+ \lim_{r\to+\infty}(\frac{1}{2}g(s)(\eta_x^t)^2)|_{s=0}^{s=r} - \int_0^1\int_0^{+\infty} g'(s)(\eta_x^t)^2 dsdx \tag{3.3.11}$$

Multiplying the third equation of (3.3.1) by φ_t, and integrating their sum over (0,1), we can obtain the following formulation

$$\int_0^1 [\rho_3\theta_t - \beta\theta_{xx} + \delta\psi_{xt}]\theta dx = \int_0^1 \frac{1}{2}\frac{d}{dt}\theta^2 dx \underbrace{-(\beta\theta_x\theta)|_{x=0}^{x=1}}_{=0} + \beta\int_0^1 \theta_x^2 dx + \delta\int_0^1 \psi_{xt}\theta dx = 0 \tag{3.3.12}$$

From (3.3.8), (3.3.11) and (3.3.12) the following results

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [\varphi_t^2 + \psi_t^2 + \bar{b}\psi_x^2] dx + k(\varphi_x+\psi)(\varphi_{xt}+\psi_t)dx + \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^{+\infty} g(s)(\eta_x^t)^2 dsdx =$$
$$-\lim_{r\to+\infty}(\frac{1}{2}g(s)(\eta_x^t)^2)|_{s=0}^{s=r} + \int_0^1\int_0^{+\infty} g'(s)(\eta_x^t)^2 dsdx - \beta\int_0^1 \theta_x^2 dx \tag{3.3.13}$$

we conclude that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 [\varphi_t^2 + \psi_t^2 + \bar{b}\psi_x^2] dx + k\int_0^1 \frac{1}{2}\frac{d}{dt}(\varphi_x+\psi)^2 dx + \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^{+\infty} g(s)(\eta_x^t)^2 dsdx =$$
$$-\lim_{r\to+\infty}(\frac{1}{2}g(s)(\eta_x^t)^2)|_{s=0}^{s=r} + \int_0^1\int_0^{+\infty} g'(s)(\eta_x^t)^2 dsdx - \beta\int_0^1 \theta_x^2 dx \tag{3.3.14}$$

$\square$

Lemma 3.3.2 *Let* $(\varphi, \psi, \theta, \eta^t)$ *be the solution of problem* (3.3.1), *then we have for* $1 < p < 3/2$,

$$\Big(\int_0^1\int_0^{+\infty} g(s)|\eta_t^x(x,s)|^2 dsdx\Big)^{2p-1} \le C\int_0^1\int_0^{+\infty} g^p(s)|\eta_t^x(x,s)|^2 dsdx \tag{3.3.15}$$

for a constant $C > 0$.

Proof Using Hölder's inequality, it is straightforward to see that, for any $r > 1$, we have

$$\int_0^1\int_0^{+\infty} g(s)|\eta_t^x(x,s)|^2 dsdx = \int_0^1\int_0^{+\infty} g^{\frac{1}{2r}}(s)|\eta_t^x(x,s)|^{\frac{2r-2}{r}} dsdx$$

$$\le \Big(\int_0^1\int_0^{+\infty} g^{\frac{1}{2}}(s)|\eta_t^x(x,s)|^2 dsdx\Big)^{\frac{1}{r}}\Big(\int_0^1\int_0^{+\infty} g^{\frac{2r-1}{2r-2}}(s)|\eta_t^x(x,s)|^2 dsdx\Big)^{\frac{r-1}{r}}.$$

Then from (3.2.1), (3.2.4), (3.3.4) and (3.3.7) it follows

$$\int_0^1 \int_0^{+\infty} g^{\frac{1}{2}}(s)|\eta_t^x(x,s)|^2 dsdx \leq 2E(0) \int_0^{+\infty} g^{\frac{1}{2}}(s)ds = 2G_0 E(0).$$

Taking $r = (2p-1)(2p-2)$, we can derive (3.3.15).

Lemma 3.3.3 *Let* $(\varphi, \psi, \theta, \eta^t)$ *be the solution of problem* (3.3.1)-(3.3.3)*, then we have for* $1 \leq p \leq 3/2$,

$$\int_0^1 \Big(\int_0^{+\infty} g(s)\eta_t^x(x,s)ds \Big)^2 dx \leq G_p \int_0^1 \int_0^{+\infty} g^p(s)|\eta_t^x(x,s)|^2 dsdx. \tag{3.3.16}$$

Proof Using the Cauchy-Schwartz's inequality, we get

$$\int_0^{+\infty} g(s)\eta_t^x(x,s)ds = \int_0^{+\infty} g^{1-p/2}(s) g^{p/2}(s)\eta_t^x(x,s)ds \leq \Big(\int_0^{+\infty} g^{2-p}(s)ds \Big)^{1/2} \times \Big(\int_0^{+\infty} g^p(s)|\eta_t^x(x,s)|^2 ds \Big)^{1/2}$$

Therefore (3.3.16) follows from (3.2.4).

□

Now we are going to construct a Lyapunov functional $\mathcal{F}(t)$ equivalent to $E(t)$. To this end, we define several functionals which allow us to obtain the needed estimates.
Let

$$I_1 := \int_0^1 (\rho_1 \varphi_t w + \rho_2 \psi_t \psi)dx, \tag{3.3.17}$$

where w is the solution of

$$-w_{xx} = \psi_x, \ w_0 = w_1 = 0.$$

Lemma 3.3.4 *Let* $(\varphi, \psi, \theta, \eta^t)$ *be the solution of problem* (3.3.1)-(3.3.3)*, then we have for any* $\varepsilon_1, c_1 > 0$,

$$\begin{aligned} \frac{dI_1(t)}{dt} \leq \ & (-\bar{b} + 2c_1) \int_0^1 \psi_x^2 dx + \varepsilon_1 \rho_1 \int_0^1 \varphi_t^2 dx + \Big(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\Big) \int_0^1 \psi_t^2 dx \\ & + \frac{1}{4c_1} \int_0^1 \theta_x^2 dx + \frac{G_p}{4c_1} \int_0^1 \int_0^{+\infty} g^p(s)|\eta_t^x(x,s)|^2 dsdx. \end{aligned} \tag{3.3.18}$$

Proof Taking the derivative in t of (3.3.17) and using (3.3.1), we obtain

$$\begin{aligned}\frac{dI_1(t)}{dt} = & -\bar{b}\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx - k\int_0^1 \psi^2 dx \\ & +k\int_0^1 w_x^2 dx + \rho_1 \int_0^1 \varphi_t w_t dx - \int_0^1 \psi_x \int_0^{+\infty} g(s)\eta_t^x(x,s)dsdx.\end{aligned}$$

Using Young's inequality and the following inequalities

$$\int_0^1 w_x^2 dx \le C\int_0^1 \psi^2 dx \le C\int_0^1 \psi_x^2 dx$$
$$\int_0^1 w_t^2 dx \le C\int_0^1 w_{tx}^2 dx \le C\int_0^1 \psi_t^2 dx$$

we find that

$$\begin{aligned}\frac{dI_1(t)}{dt} \le & \; (-\bar{b}+c_1)\int_0^1 \psi_x^2 dx + \varepsilon_1\rho_1 \int_0^1 \varphi_t^2 dx + \Big(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\Big)\int_0^1 \psi_t^2 dx \\ & + \frac{1}{4c_1}\int_0^1 \theta_x^2 dx - \int_0^1 \psi_x \int_0^{+\infty} g(s)\eta_t^x(x,s)dsdx.\end{aligned} \tag{3.3.19}$$

By using (3.3.16), the last term on right-hand side of (3.3.19) can be estimated as follows

$$\Big| -\int_0^1 \psi_x \int_0^{+\infty} g(s)\eta_t^x(x,s)dsdx \Big| \le c_1 \int_0^1 \psi_x^2 dx + \frac{G_p}{4c_1}\int_0^1\int_0^{+\infty} g^p(s)|\eta_t^x(x,s)|^2 dsdx. \tag{3.3.20}$$

Inserting (3.3.20) into (3.3.19), we can obtain the desired result.

□

Lemma 3.3.5 *Let* $(\varphi,\psi,\theta,\eta^t)$ *be the solution of problem* (3.3.1)-(3.3.3)*, then the functional* $I_2(t)$ *defined by*

$$I_2 = -\rho_2 \int_0^1 \psi_t(x,t)\int_0^{+\infty} g(s)\eta_t(x,s)dsdx$$

satsifies the estimate

$$\begin{aligned}\frac{dI_2(t)}{dt} \le & -\frac{\rho_2 g_0}{2}\int_0^1 \psi_t^2 dx + \varepsilon_2 \bar{b}^2 \int_0^1 \psi_x^2 dx + \varepsilon_2 k^2 \int_0^1 (\varphi_x+\psi)^2 dx \\ & + \varepsilon_2\delta^2 \int_0^1 \theta_x^2 dx + G_p\Big(1 + \frac{1}{4\varepsilon_2} + \frac{C^*}{4\varepsilon_2} + \frac{C^*}{2}\Big)\int_0^1\int_0^{+\infty} g^p(s)\eta_x^t(x,s)dsdx \\ & - \frac{C^* g(0)}{2\rho_2}\int_0^1\int_0^{+\infty} g'(s)\eta_x^t(x,s)dsdx,\end{aligned} \tag{3.3.21}$$

for any $\varepsilon_2 > 0$, *where* $g_0 = \int_0^{+\infty} g(s)ds$.

Proof Using (3.3.1) and Young's inequality, similarly to (3.3.18), we can easily obtain the result.

□

Next we introduce the functional

$$I_3(t) := \rho_2 \int_0^1 \psi_t(\varphi_x + \psi)dx + \frac{\rho_1 \bar{b}}{k} \int_0^1 \psi_x \varphi_t dx + \frac{\rho_1}{k} \int_0^1 \varphi_t \int_0^{+\infty} g(s)\eta_t^x(x,s)dsdx. \tag{3.3.22}$$

Lemma 3.3.6 *Let* $(\varphi, \psi, \theta, \eta^t)$ *be the solution of problem* (3.3.1)-(3.3.3), *then for any* $\varepsilon_2 > 0$, *we conclude*

$$\begin{aligned} \frac{dI_3(t)}{dt} \leq\ & \Big(\varphi_x(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_t^x(x,s)ds)\Big)\Big|_{x=0}^{x=1} - k\int_0^1 (\varphi_x + \psi)^2 dx + \rho_2 \int_0^1 \psi_t dx \\ & + 2\varepsilon_3 \int_0^1 \varphi_t^2 dx + \frac{\delta^2}{4\varepsilon_3} \int_0^1 \theta_x^2 dx - g(0)C(\varepsilon_3) \int_0^1 \int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 dsdx. \end{aligned} \tag{3.3.23}$$

Proof Differentiating $I_3(t)$ and using (3.3.1), we obtain

$$\begin{aligned} \frac{dI_3(t)}{dt} =\ & \int_0^1 (\varphi_x + \psi)dx\Big(\bar{b}\psi_{xx} + \int_0^{+\infty} g(s)\eta_t^{xx}(x, t-s)ds - k(\varphi_x + \psi) - \delta\theta_x\Big)dx \\ & + \rho_2 \int_0^1 \psi_t^2 dx + \bar{b}\int_0^1 \psi_x(\varphi_x + \psi)_x dx + \Big(\frac{\rho_1 b}{k} - \rho_2\Big)\int_0^1 \psi_{tx}\varphi_t dx \\ & + \frac{\rho_1}{k}\int_0^1 \varphi_t \int_0^{+\infty} g(s)\eta_{sx}^t(x,s)dsdx + \int_0^1 (\varphi_x + \psi)_x \int_0^{+\infty} g(s)\eta_x^t(x,s)dsdx. \end{aligned}$$

Using $\dfrac{k}{\rho_1} = \dfrac{b}{\rho_2}$ and Young's inequality, we conclude (3.3.23),

□

We now define the following function to handle the boundary terms appearing in (3.3.23)

$$q(x) = 2 - 4x,\ x \in [0,1].$$

Consequently, we have the following result.

Lemma 3.3.7 *Let $(\varphi, \psi, \theta, \eta^t)$ be the solution of problem* (3.3.1)-(3.3.3), *then we have*

$$
\begin{aligned}
&\left(\varphi_x\Big(\overline{b}\psi_x+\int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)\right)\Big|_{x=0}^{x=1}\\
&\leq -\frac{\varepsilon_3}{k}\frac{d}{dt}\int_0^1 \rho_1 q(x)\varphi_t\varphi_x dx+3\varepsilon_3\int_0^1 \varphi_x^2 dx+\frac{2\rho_1\varepsilon_3}{k}\int_0^1 \varphi_t^2 dx\\
&-\frac{1}{4\varepsilon_3}\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\Big(\overline{b}\psi_x+\int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)+4\delta^2 c_3\int_0^1 \theta_x^2 dx\\
&+\frac{1}{\varepsilon_3}\Big(\overline{b}^2+\frac{\overline{b}^2}{4c_3}+\varepsilon_3^2\Big)\int_0^1 \psi_x^2 dx+\frac{G_0}{4\varepsilon_3}\Big(4+\frac{1}{c_3}\Big)\int_0^1\int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2 dsdx\\
&+\frac{\rho_2}{4\varepsilon_3}(2\overline{b}+\varepsilon_3)\int_0^1 \psi_t^2 dx+\frac{c_3k^2}{\varepsilon_3}\int_0^1 (\varphi_x+\psi)^2 dx\\
&-\rho_2 g(0)c(\varepsilon_3)\int_0^1\int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 dsdx,
\end{aligned}
\tag{3.3.24}
$$

for any $c_3,\ \varepsilon_3 > 0$.

Proof Using Young's inequality, we easily see that for $\varepsilon_3 > 0$

$$
\begin{aligned}
&\left(\varphi_x\Big(\overline{b}\psi_x+\int_0^{+\infty} g(s)\eta_x^t(x,s)\Big)\right)\Big|_{x=0}^{x=1}\\
&\leq \varepsilon_3+\Big(\varphi_x^2(1,t)+\varphi_x^2(0,t)\Big)+\frac{1}{4\varepsilon_3}\Big(\overline{b}\psi_x(0,t)+\int_0^{+\infty} g(s)\eta_x^t(0,s)ds\Big)^2 \qquad (3.3.25)\\
&+\frac{1}{4\varepsilon_3}\Big(\overline{b}\psi_x(1,t)+\int_0^{+\infty} g(s)\eta_x^t(1,s)ds\Big)^2.
\end{aligned}
$$

Exploiting

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\Big(\overline{b}\psi_x+\int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx\\
&=\int_0^1 \rho_2 q(x)\psi_{tt}\Big(\overline{b}\psi_x+\int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx\\
&+\int_0^1 \rho_2 q(x)\psi_t\Big(\overline{b}\psi_{tx}+\int_0^{+\infty} g(s)\eta_{tx}^t(x,s)ds\Big)dx
\end{aligned}
$$

and (3.3.1), we get

$$
\begin{aligned}
&\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx \\
&= \int_0^1 q(x)\Big(\bar{b}\psi_{xx} + \int_0^{+\infty} g(s)\eta_{xx}^t(x,s)ds - k(\varphi_x+\psi) - \delta\theta_x\Big) \\
&\times\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx \\
&+ \int_0^1 \rho_2 q(x)\psi_t\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx.
\end{aligned}
\tag{3.3.26}
$$

A direct calculation shows

$$
\begin{aligned}
&\int_0^1 q(x)\Big(\bar{b}\psi_{xx} + \int_0^{+\infty} g(s)\eta_{xx}^t(x,s)ds\Big)\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx \\
&= -\frac{1}{2}\int_0^1 q'(x)\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)^2 dx \\
&+\Big(\frac{q(x)}{2}\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)^2\Big)\Big|_{x=0}^{x=1}.
\end{aligned}
\tag{3.3.27}
$$

By using (3.3.1), the las term in (3.3.26) can be treated as follows

$$\begin{aligned}
&\int_0^1 \rho_2 q(x)\psi_t\bigg(\overline{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\bigg)dx\\
&= \rho_2\overline{b}\int_0^1 q(x)\psi_t\psi_{tx}dx + \rho_2\int_0^1 q(x)\psi_t\int_0^{+\infty} g(s)\eta_{tx}^t(x,s)dsdx\\
&= -\frac{\rho_2\overline{b}}{2}\int_0^1 q'(x)\psi_t^2dx + \rho_2\int_0^1 q(x)\psi_t\int_0^{+\infty} g(s)\eta_{tx}^t(x,s)dsdx\\
&= -\frac{\rho_2\overline{b}}{2}\int_0^1 q'(x)\psi_t^2dx + \rho_2\int_0^1 q(x)\psi_t\int_0^{+\infty} g(s)(\psi_{tx}(t)-\eta_{sx}^t(x,s))dsdx\\
&= -\frac{\rho_2\overline{b}}{2}\int_0^1 q'(x)\psi_t^2dx + g_0\rho_2\int_0^1 q(x)\psi_t\psi_{tx} - \rho_2\int_0^1 q(x)\psi_t\int_0^{+\infty} g(s)\eta_{sx}^t(x,s)dsdx\\
&= -\frac{\rho_2(\overline{b}+g_0)}{2}\int_0^1 q'(x)\psi_t^2dx + \rho_2\int_0^1 q(x)\psi_t\int_0^{+\infty} g'(s)\eta_x^t(x,s)dsdx.
\end{aligned} \tag{3.3.28}$$

Inserting (3.3.27) and (3.3.28) into (3.3.26), we arrive at

$$\begin{aligned}
&\bigg(\overline{b}\psi_x(0,t) + \int_0^{+\infty} g(s)\eta_x^t(0,s)ds\bigg)^2 + \bigg(\overline{b}\psi_x(1,t) + \int_0^{+\infty} g(s)\eta_x^t(1,s)ds\bigg)^2\\
&= -\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\bigg(\overline{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\bigg)dx\\
&+2\int_0^1\bigg(\overline{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\bigg)^2dx\\
&-k\int_0^1 q(x)(\varphi_x+\psi)\bigg(\overline{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\bigg)dx\\
&-\delta\int_0^1 q(x)\theta_x\bigg(\overline{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\bigg)dx\\
&+2\rho_2(\overline{b}+g_0)\int_0^1 \psi_t^2dx + \rho_2\int_0^1 q(x)\psi_t\int_0^{+\infty} g'(s)\eta_x^t(x,s)dsdx.
\end{aligned} \tag{3.3.29}$$

We now estimate each term on the right-hand side of (3.3.29), by exploiting Young's inequality and (3.3.16) as follows: The second, third and fourth terms can be estimated as

$$2\int_0^1 \left(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\right)^2 dx \quad \le 4\bar{b}^2 \int_0^1 \psi_x^2 dx$$
$$+4G_p \int_0^1 \int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2 dsdx, \tag{3.3.30}$$

$$\begin{aligned}
&\Big|k\int_0^1 q(x)(\varphi_x+\psi)\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx\Big| \\
&\le 2k\Big|\int_0^1 q(x)(\varphi_x+\psi)\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx\Big| \\
&\le 4k^2c_3\int_0^1 (\varphi_x+\psi)^2 + \frac{1}{4c_3}\int_0^1 \left(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\right)dx \\
&\le 4k^2c_3\int_0^1 (\varphi_x+\psi)^2 dx + \frac{\bar{b}^2}{2c_3}\int_0^1 \psi_x^2 dx + \frac{G_p}{2c_3}\int_0^1\int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2 dsdx,
\end{aligned} \tag{3.3.31}$$

and

$$\begin{aligned}
&\Big|\delta\int_0^1 q(x)\theta_x\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx\Big| \\
&\le 2\delta\Big|\int_0^1 \theta_x\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx\Big| \\
&\le 4\delta^2c_3\int_0^1 \theta_x^2 + \frac{1}{4c_3}\int_0^1 \left(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\right)dx \\
&\le 4\delta^2c_3\int_0^1 \theta_x^2 + \frac{\bar{b}^2}{2c_3}\int_0^1 \psi_x^2 dx + \frac{G_p}{2c_3}\int_0^1\int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2 dsdx,
\end{aligned} \tag{3.3.32}$$

The last term can be handled similarly

$$\begin{aligned}
&\Big|\rho_2\int_0^1 q(x)\psi_t\int_0^{+\infty} g'(s)\eta_x^t(x,s)dsdx\Big| \\
&\le \rho_2\varepsilon_3\int_0^1 \psi_t^2 dx - \rho_2 g(0)C(\varepsilon_3)\int_0^1\int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 dsdx.
\end{aligned} \tag{3.3.33}$$

Combining (3.3.29)-(3.3.33), we obtain

$$\begin{aligned}
&\Big(\bar{b}\psi_x(0,t)+\int_0^{+\infty} g(s)\eta_x^t(0,s)ds\Big)^2+\Big(\bar{b}\psi_x(1,t)+\int_0^{+\infty} g(s)\eta_x^t(1,s)ds\Big)^2\\
&\le -\frac{d}{dt}\int_0^1 \rho_2 q(x)\psi_t\Big(\bar{b}\psi_x+\int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx\\
&+\bar{b}^2\left(4+\frac{1}{c_3}\right)\int_0^1 \psi_x^2dx+\frac{G_p}{2c_3}+G_p\left(4+\frac{1}{c_3}\right)\int_0^1\int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2dsdx \quad (3.3.34)\\
&+2\rho_2\left(\bar{b}+g_0+\frac{\varepsilon_3}{2}\right)\int_0^1 \psi_t^2dx+4k^2c_3\int_0^1(\varphi_x+\psi)^2dx\\
&+4\delta^2c_3\int_0^1 \theta_x^2-\rho_2 g(0)C(\varepsilon_3)\int_0^1\int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2dsdx.
\end{aligned}$$

Similary, using (3.3.1), we arrive at

$$\frac{d}{dt}\int_0^1 \rho_1 q\varphi_t\varphi_x dx\le -k\Big(\varphi_x^2(1)+\varphi_x^2(0)\Big)+3k\int_0^1\varphi_x^2dx+k\int_0^1\psi_x^2dx+2\rho_1\int_0^1\varphi_t^2dx. \quad (3.3.35)$$

Hence the assertion of lemma follows from (3.3.25), (3.3.34) and (3.3.35).

Lemma 3.3.8 *Let $(\varphi,\psi,\theta,\eta^t)$ be the solution of problem* (3.3.1)-(3.3.3)*, the functional I_4 defined by*

$$I_4 := -\rho_1\int_0^1 \varphi_t\varphi dx-\rho_2\int_0^1\psi_t\psi dx$$

satisfies the estimate

$$\begin{aligned}
\frac{dI_4}{dt}\le\ & -\rho_1\int_0^1\varphi_t^2dx-\rho_2\int_0^1\psi_t^2dx+(\bar{b}+2\varepsilon_3)\int_0^1\psi_x^2dx+\frac{\delta^2}{4\varepsilon_3}\int_0^1\theta_x^2dx\\
&+k\int_0^1(\varphi_x+\psi)^2dx+\frac{G_p}{4\varepsilon_3}\int_0^1\int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2dsdx,
\end{aligned} \quad (3.3.36)$$

for any $\varepsilon_3>0$.

Proof It is easy to check that

$$\begin{aligned}
\frac{dI_4}{dt}=\ & -\rho_1\int_0^1\varphi_t^2dx-\rho_2\int_0^1\psi_t^2dx+\bar{b}\int_0^1\psi_x^2dx+\delta\int_0^1\theta_x\psi dx\\
&+k\int_0^1(\varphi_x+\psi)^2dx+\int_0^1\psi_x\int_0^{+\infty} g(s)\eta_x^t(x,s)dsdx,
\end{aligned} \quad (3.3.37)$$

and for Lemma 3.3.3 and Young's inequality, (3.3.36) follows.
Now for N, N_1, N_2, $\lambda > 0$, we define a Lyapunov functional $\mathcal{F}(t)$ as follows

$$\begin{aligned}\mathcal{F}(t) = \; & NE(t) + N_1 I_1(t) + N_2 I_2(t) + I_3(t) + \lambda I_4(t) + \frac{\varepsilon_3}{k}\int_0^1 \rho_1 q \varphi_t \varphi_x dx \\ & + \frac{1}{4\varepsilon_3}\int_0^1 \rho_2 q(x)\psi_t\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)\eta_x^t(x,s)ds\Big)dx.\end{aligned} \tag{3.3.38}$$

Using (3.3.7), (3.3.18), (3.3.21), (3.3.23), (3.3.24) and (3.3.36), we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) \leq \; & \Upsilon_1 \int_0^1 \psi_x^2 dx + \Upsilon_2 \int_0^1 \varphi_t^2 dx + \Upsilon_3 \int_0^1 \psi_t^2 dx + \Upsilon_4 \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \Upsilon_5 \int_0^1 \theta_x^2 dx + C_2 \int_0^1 \int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2 ds dx \\ & + (N - C_1)\int_0^1 \int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 ds dx,\end{aligned} \tag{3.3.39}$$

where C_1, C_2 are positive constants independent of N and

$$\begin{cases} \Upsilon_1 = -N_1(\bar{b} - 2c_1) + N_2\varepsilon_2\bar{b} + C(c_3, \lambda, \varepsilon_3), \\ \Upsilon_2 = N_1\varepsilon_1\rho_1 + \varepsilon_3 + \dfrac{2\rho_1\varepsilon_3}{k} - \lambda\rho_1, \\ \Upsilon_3 = N_1(\rho_2 + \dfrac{\rho_1}{4\varepsilon_1}) - N_2\dfrac{g_0\rho_2}{2} + \rho_2 + \dfrac{1}{4\varepsilon_3} - \lambda\rho_2, \\ \Upsilon_4 = -k + \dfrac{k^2 c_3}{\varepsilon_3} + N_2\varepsilon_2 k^2 + 2(3\varepsilon_3 + \lambda k), \\ \Upsilon_5 = -N\beta + \dfrac{1}{4c_1} + \varepsilon_2\delta^2 c_3. \end{cases}$$

Now we choose our constants very carefully and properly so that there exists a constant $\mu > 0$, such that the following form holds

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) \leq \; & -\mu\Big(\int_0^1 \Big(\psi_x^2 + \psi_t^2 + \varphi_t^2 + \theta_x^2 + (\varphi_x + \psi)^2\Big) \\ & + \int_0^1 \int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2 ds dx\Big).\end{aligned} \tag{3.3.40}$$

On the other hand, we know that $\mathcal{F}(t)$ is equivalent to $E(t)$, i.e.

$$\mathcal{F}(t) \sim E(t) \tag{3.3.41}$$

We are now ready to prove Theorem 3.3.1.
Proof of Theorem 3.3.1 We distinguish two cases.
Case 1 $p = 1$. From (3.3.40) and (3.3.41), it follows

$$\frac{d}{dt}\mathcal{F}(t) \leq -\omega \mathcal{F}(t). \tag{3.3.42}$$

Integrating (3.3.42) over $(0, t)$, and using (3.3.41), (3.3.5) follows immediately.
Case 2 $p > 1$. Firstly, from (3.3.4) and (3.3.7), we derive

$$\begin{aligned} E^{2p-1} \leq\ & C(E(0))^{2p-2} \int_0^1 \Big(\psi_x^2 + \psi_t^2 + \varphi_t^2 + \theta_x^2 + (\varphi_x + \psi)^2\Big) dx \\ & + C \int_0^1 \left(\int_0^{+\infty} g(s) |\eta_x^t(x, s)|^2 ds \right)^{2p-1} dx \\ \leq\ & C(E(0))^{2p-2} \int_0^1 \Big(\psi_x^2 + \psi_t^2 + \varphi_t^2 + \theta_x^2 + (\varphi_x + \psi)^2\Big) dx \\ & + C \int_0^1 \int_0^{+\infty} g^p(s) |\eta_x^t(x, s)|^2 ds dx. \end{aligned} \tag{3.3.43}$$

Combining (3.3.40), (3.3.41) and (3.3.43), we have

$$\mathcal{F}'(t) \leq -CE^{2p-1}(t) \leq -C\mathcal{F}^{2p-1}(t),$$

which, integrated over $(0, t)$, leads to

$$\mathcal{F}(t) \leq \frac{C}{(t+1)^{1/(2p-2)}}. \tag{3.3.44}$$

We observe that

$$\begin{aligned} & \int_0^1 \int_0^{+\infty} g(s) |\eta_x^t(x, s)|^2 ds dx \\ \leq & \left(\int_0^1 \int_0^{+\infty} |\eta_x^t(x, s)|^2 ds dx \right)^{(p-1)/p} \times \left(\int_0^1 \int_0^{+\infty} g^p(s) |\eta_x^t(x, s)|^2 ds dx \right)^{1/p} \end{aligned} \tag{3.3.45}$$

and use (3.3.4) and (3.3.44) to conclude, for $p < 3/2$,

$$\begin{aligned}\int_0^1\int_0^{+\infty}|\eta_x^t(x,s)|^2dsdx = {} & 2\int_0^1\int_0^{+\infty}|\psi_{x,t}(x,s)|^2dsdx \\ & +2\int_0^1\int_0^{+\infty}|\psi_x(x,t-s)|^2dsdx \\ \leq {} & 2t\int_0^1|\psi_x(x,t)|^2dx+\frac{4}{\bar{b}}\int_0^t E(t-s)ds \\ \leq {} & \frac{Ct}{(t+1)^{1/(2p-2)}}+C\int_0^t\frac{ds}{(t-s+1)^{1/(2p-2)}} \\ \leq {} & \Lambda,\end{aligned}$$

where $\Lambda > 0$ is a constant independent of t. Hence, we get

$$\int_0^1\int_0^{+\infty}|\eta_x^t(x,s)|^2dsdx \leq \Lambda. \tag{3.3.46}$$

Consequently, a combination of (3.3.45) and (3.3.46) gives

$$\int_0^1\int_0^{+\infty}|g(s)\eta_x^t(x,s)|^2dsdx \leq \Lambda^{(p-1)/p}\Big(\int_0^1\int_0^{+\infty}g^p(s)|\eta_x^t(x,s)|^2dsdx\Big)^{1/p}$$

or

$$\Big(\int_0^1\int_0^{+\infty}|g(s)\eta_x^t(x,s)|^2dsdx\Big)^p \leq C\int_0^1\int_0^{+\infty}g^p(s)|\eta_x^t(x,s)|^2dsdx.$$

Similarly to (3.3.43), we obtain

$$\begin{aligned}E^p(t) \leq {} & C\int_0^1(\psi_x^2+\psi_t^2+\varphi_t^2+\theta_x^2+(\varphi_x+\psi)^2)dx \\ & +C\int_0^1\int_0^{+\infty}g^p(s)|\eta_x^t(s)|^2dsdx,\end{aligned} \tag{3.3.47}$$

which, together with (3.3.40), (3.3.41) and (3.3.47), yields

$$\mathcal{F}'(t) \leq -C\mathcal{F}^p(t). \tag{3.3.48}$$

Integrating (3.3.48) gives

$$\mathcal{F}(t) \leq \frac{C}{(t+1)^{1/(p-1)}}, \tag{3.3.49}$$

which, along with (3.3.41), yields (3.3.6). This complete the proof of Theorem 3.3.1.

3.4 Case of nonequal wave-speeds

In this section, we treat the case of different wave speeds propagation $(\frac{\rho_1}{k} \neq \frac{\rho_2}{b})$, we shall show that if the initial data are regular enough, the solution energy $E(t)$decays in the rate of $1/t$ when the relaxation function g decays exponentially.
Define the second-order energy as

$$E_1(t) = E(\varphi_t, \psi_t, \theta_t, \eta_t^t),$$

where E is given in (3.3.4).

Theorem 3.4.1 *Let* (φ_0, φ_1), $(\psi_0, \psi_1) \in (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$, $\theta_0 \in H_0^1(0,1)$ *and* $\eta_0^t \in L_g^2(\mathbb{R}^+, H^2(0,1) \cap H_0^1(0,1))$.
Then there exists a constant $C > 0$ *such that for all* $t > 0$,

$$E(t) \leq Ct^{-1/(2p-1)}, \; p \geq 1. \tag{3.4.1}$$

In order to prove the above theorem, we need the following two lemmas.

Lemma 3.4.1 *Let* $(\varphi, \psi, \theta, \eta^t)$ *be the solution of problem* (3.3.1)-(3.3.3), *then we have*

$$E'(t) = -\beta \int_0^1 \theta_{tx}^2 dx + \frac{1}{2} \int_0^1 \int_0^{+\infty} g'(s) |\eta_{tx}^t(x,s)|^2 ds dx \leq 0. \tag{3.4.2}$$

Proof: Similary to Lemma 3.3.1 we can prove the lemma.

Lemma 3.4.2 *Let* $(\varphi, \psi, \theta, \eta^t)$ *be the solution of problem* (3.3.1)-(3.3.3), *then for any* $\varepsilon_3 > 0$, *we conclude*

$$\begin{aligned}\frac{dI_3(t)}{dt} \leq\ & \left(\varphi_x\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)|\eta_x^t(x,s)\Big)\right)\Bigg|_{x=0}^{x=1} - k\int_0^1 (\varphi_x+\psi)^2 dx \\ & +\rho_2\int_0^1 \psi_t^2 dx + \varepsilon_3\int_0^1 \varphi_t^2 dx + \frac{1}{4\varepsilon_3}\int_0^1 \theta_x^2 dx \\ & -g(0)C(\varepsilon_3)\int_0^1\int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 ds dx \\ & +C(\varepsilon_3)\int_0^1\int_0^{+\infty} g^p(s)|\eta_{tx}^t(x,s)|^2 ds dx.\end{aligned} \tag{3.4.3}$$

Proof Similarly to the proof of Lemma 3.3.6, we obtain

$$\begin{aligned}\frac{dI_3(t)}{dt} =\ & \int_0^1 (\varphi_x+\psi)\Big(\bar{b}\psi_{xx} + \int_0^{+\infty} g(s)\eta_{xx}^t(x,t-s)ds - k(\varphi_x+\psi) + \delta\theta_x\Big)dx \\ & +\rho_2\int_0^1 \psi_t^2 dx + \bar{b}\int_0^1 \psi_x(\varphi_x+\psi)_x dx + \Big(\frac{\rho_1 b}{k} - \rho_2\Big)\int_0^1 \psi_{tx}\varphi_t dx \\ & -\frac{\rho_1}{k}\int_0^1 \varphi_t \int_0^{+\infty} g(s)\eta_{sx}^t(x,s) ds dx + \int_0^1 (\varphi_x+\psi)_x \int_0^{+\infty} g(s)\eta_x^t(x,s) ds dx.\end{aligned}$$

Since $\frac{\rho_1}{k} \neq \frac{\rho_2}{b}$, we have to handle the term $\int_0^1 \psi_{tx}\varphi_t dx$ on the right-hand side of the above equation. To this end, we use the idea of [26] to obtain

$$\int_0^1 \psi_{tx}\varphi_t dx = \frac{1}{g_0}\int_0^1 \Big(\int_0^{+\infty} g(s)\eta_{tx}^t(x,s)ds\Big)\varphi_t dx - \frac{1}{g_0}\int_0^1 \Big(\int_0^{+\infty} g'(s)\eta_x^t(x,s)ds\Big)\varphi_t dx. \tag{3.4.4}$$

Therefore, using Young's inequality and (3.3.16), we obtain the following two estimates

$$\begin{aligned}& \frac{1}{g_0}\Big(\frac{\rho_1 b}{k} - \rho_2\Big)\int_0^1 \Big(\int_0^{+\infty} g(s)\eta_{tx}^t(x,s)ds\Big)\varphi_t dx \\ & \leq \frac{\varepsilon_3}{2}\int_0^1 \varphi_t^2 dx + C(\varepsilon_3)\int_0^1\int_0^{+\infty} g^p(s)|\eta_{tx}^t(x,s)|^2 ds dx\end{aligned} \tag{3.4.5}$$

and

$$\begin{aligned}& \frac{1}{g_0}\Big(\frac{\rho_1 b}{k} - \rho_2\Big)\int_0^1 \Big(\int_0^{+\infty} g'(s)\eta_{tx}^t(x,s)ds\Big)\varphi_t dx \\ & \leq \frac{\varepsilon_3}{2}\int_0^1 \varphi_t^2 dx - g(0)C(\varepsilon_3)\int_0^1\int_0^{+\infty} g'(s)|\eta_{tx}^t(x,s)|^2 ds dx.\end{aligned} \tag{3.4.6}$$

Inserting (3.4.5) and (3.4.6) into (3.4.4), using the estimates of Lemma 3.3.7, we obtain the desired result (3.4.3) .

□

Proof of the Theorem 3.4.1 We define a Lyapunov functional $\mathcal{F}$ as follows

$$\begin{aligned}\mathcal{F}(t) = \; & N\Big(E(t) + E_1(t)\Big) + N_1 I_1 + N_2 I_2 + I_3 + \lambda I_4 + \frac{\varepsilon_3}{k}\int_0^1 \rho_1 q \varphi_t \varphi_x dx \\ & + \frac{1}{4\varepsilon_3}\int_0^1 \rho_2 q(x)\psi_t\Big(\bar{b}\psi_x + \int_0^{+\infty} g(s)|\eta_x^t(x,s)|^2 ds\Big)dx.\end{aligned} \tag{3.4.7}$$

Consequetly, we get

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) \le \; & \Upsilon_1 \int_0^1 \psi_x^2 dx + \hat{\Upsilon}_2 \int_0^1 \varphi_t^2 dx + \Upsilon_3 \int_0^1 \psi_t^2 dx + \Upsilon_4 \int_0^1 (\varphi_x + \psi)^2 dx \\ & + \Upsilon_5 \int_0^1 \theta_x^2 dx + C_2 \int_0^1 \int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2 ds dx \\ & + (\frac{N}{2} - C_1) \int_0^1 \int_0^{+\infty} g'(s)|\eta_x^t(x,s)|^2 ds dx \\ & + \frac{N}{2} \int_0^1 \int_0^{+\infty} g'(s)|\eta_{tx}^t(x,s)|^2 ds dx \\ & + C(\varepsilon_3) \int_0^1 \int_0^{+\infty} g^p(s)|\eta_{tx}^t(x,s)|^2 ds dx,\end{aligned}$$

where

$$\hat{\Upsilon}_2 = \Upsilon_2 + \varepsilon_3.$$

Similarly to the proof of (3.3.40), there is a constant $\mu_1 > 0$ such that

$$\begin{aligned}\frac{d}{dt}\mathcal{F}(t) \le \; & -\mu_1\Big(\int_0^1 (\psi_x^2 + \psi_t^2 + \varphi_t^2 + \theta_x^2 + (\varphi_x + \psi)^2) dx \\ & + \int_0^1 \int_0^{+\infty} g^p(s)|\eta_x^t(x,s)|^2 ds dx + \int_0^1 \int_0^{+\infty} g^p(s)|\eta_{tx}^t(x,s)|^2 ds dx\Big).\end{aligned} \tag{3.4.8}$$

Case 1 p=1: For p=1, it is not hard to see

$$\frac{d}{dt}\mathcal{F}(t) \le -cE(t). \tag{3.4.9}$$

Integrating (3.4.9) and noticing (3.4.7), we get

$$\int_0^t E(s)ds \leq C\Big(\mathcal{F}(0) - \mathcal{F}(t)\Big) \leq C\mathcal{F}(0) \leq C\Big(E(0) + E_1(t)\Big), \text{ for all } t > 0. \quad (3.4.10)$$

Noting that

$$\frac{d}{dt}\Big(tE(t)\Big) = E(t) + t\frac{d}{dt}E(t) \leq E(t),$$

we have

$$tE(t) = \int_0^t E(s)ds \leq C\Big(E(0) + E_1(t)\Big), \text{ for all } t > 0,$$

Consequently, we get

$$E(t) \leq \frac{C}{t}, \text{ for all } t > 0. \quad (3.4.11)$$

Case 2 $p > 1$: We saw previously that

$$\frac{d\mathcal{F}}{dt} \leq -CE^{2p-1}$$

which implies

$$\int_0^t E^{2p-1}(s)ds \leq C\Big(\mathcal{F}(0) - \mathcal{F}(t)\Big) \leq C\mathcal{F}(0), \text{ for all } t > 0. \quad (3.4.12)$$

On other hand we have

$$\frac{d}{dt}\Big(tE^{2p-1}(t)\Big) = E^{2p-1}(t) + (2p-1)tE^{2p-2}\frac{d}{dt}E^{2p-1}(t) \leq E^{2p-1}(t), \quad (3.4.13)$$

Similar calculations, using (3.4.12) and (3.4.13), yield

$$E(t) \leq Ct^{-1/(2p-1)}, \text{ for all } t > 0. \quad (3.4.14)$$

□

Conclusion

In this memory we studied a one-dimensional nonlinear thermoelasticity with thermal memory and second sound we proved the global existence using a regularity theorem and that this solution exponentialy decays. Then we considered a Timoshenko-type system with a past history we mentionned that the existence of the solution was already studied using the same method as in Chapter 2 and we focused on the energy decays using Lyaponov functional where we find that the solution decays exponentially in the case of equal wave speeds but it decays in the rate of $(1/t)$ (Polynomial Stability) for the case of the non-equal one.

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