People`s Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

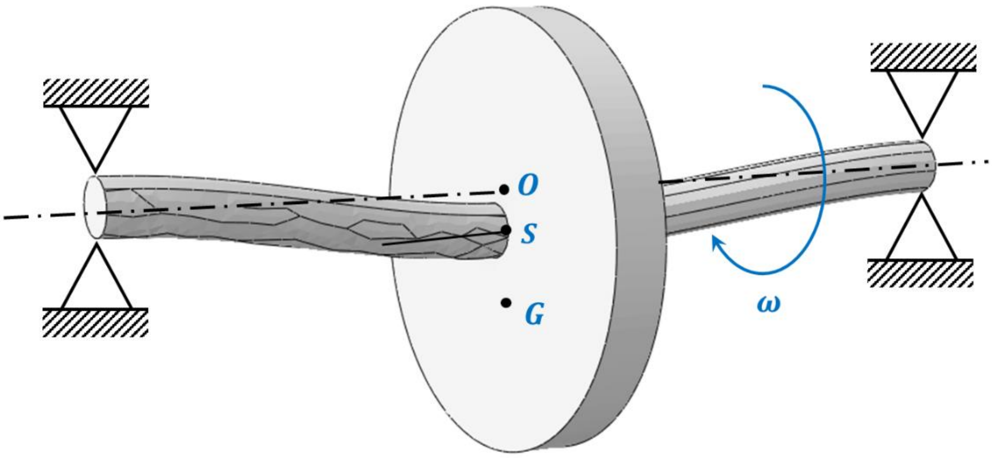
Ammar Telidji University – Laghouat

Faculty of Technology

Department of Mechanical Engineering

Course support

Vibration of rotating machines



For Master 1 Industrial Maintenance students

By Dr. Abdelkader Djilali HAMMOU

MCA, Department of Mechanical Engineering

Edition 1. 2024

Summary

Preamble

Students in the second cycle of LMD (Master) training generally find themselves faced with a problem of understanding the subjects studied and can get lost in the large number of books which cover the modules offered in their training. This course material aims to provide additional educational tools that allow these students to follow and assimilate the learning module.

This course is intended for "Master 1 industrial maintenance" students following the program established by the training teams at the Mechanical Engineering department of Ammar Telidji University in Laghouat. It is offered as a continuation of the “Dynamics of structures” module of semester six (L3) with a total theoretical hourly volume of 45 hours (15 weeks including a 1.5 hour course session and another weekly tutorial).

The objectives of this teaching are:

- Understand the physical principles of vibrational dynamics.

- Focus on vibration modeling techniques for rotating machines.

- Learn digital resolution methods and choose the appropriate modeling.

- Allow better control of the installation and use of rotating machines.

- Understand applications on industrial machines that are particularly sensitive to vibrational alterations of their components: turbo-alternators, pumping groups, motors, centrifuges, etc.

- Apply these methods to practical industrial cases.

Students must have recommended prior knowledge: Notions on: Oscillators with one degree of freedom; Dynamics of the rigid solid; Linear second order differential equations with constant coefficients; Basics of linear algebra.

This course has six chapters which can be theoretically divided as follows:

- Introduction to rotor dynamics.
- Hamilton principle.
- Energy formulation.
- Rotor modeling and equations of motion.
- Jeffcott rotor.
- Balancing techniques for rotating machines.

So that the student can assimilate the phenomena and theories covered in this course, I have proposed solved exercises. Of course the student is invited to solve the exercises proposed during the tutorials.

M. HAMMOU Abdelkader Djilali
Docteur en Mécanique
Maitre de Conférences A
Département de Génie Mécanique

1. Introduction to rotor dynamics

Rotating machines represent the largest and most important category of machines. They are used for fluid transportation (figure 01), metal working and forming (figure 02), power generation (figure 03), aircraft propulsion (figure 04), and other purposes.

Figure 01. Centrifugal pump (https://www.pump.co.uk/pumps-c280/surface-mounted-c221/cm-monobloc-range-c223/cm80-160-d-centrifugal-pump-p5473)

Figure 02. Milling machine (Laghouat University).

Figure 03. Electrical generator (https://www.electricalindia.in/electrical-generators-areas-of-application/).

Figure 04. Aircraft propulsion (https://en.wikipedia.org/wiki/Propeller_%28aeronautics%29).

Rotor dynamics is an extremely important part of the discipline of dynamics that concerns the operation and behavior of a large group of rotating machines (figure 05).

This machine behavior includes a wide variety of physical phenomena, which can hinder the proper functioning of machines and can even lead to catastrophic failures if not correctly identified and corrected.

1.1. Rotor:

The rotor is the rotating part of a machine, mechanical or electrical, which interacts with the fixed (static) part called the stator.

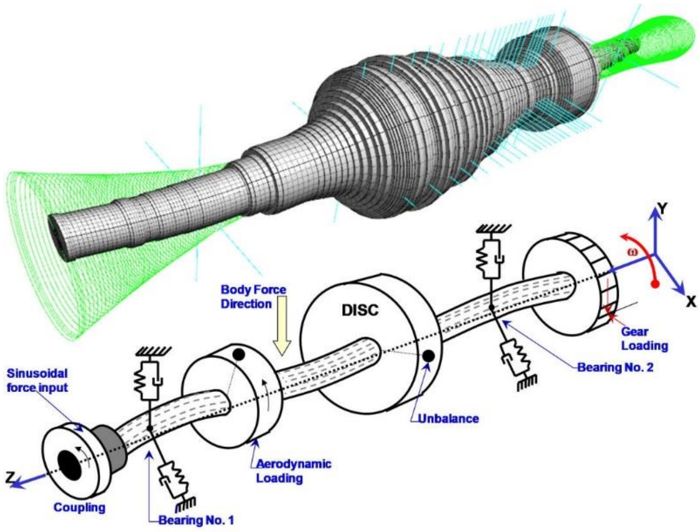


Figure 05. Dynamic model of rotor (https://www.rbts.com/rotanalysis.html).

The word "rotor" is used to describe the assembly of rotating parts in a rotating machine, including the shaft, bladed disks, wheels, bearings, gears, couplings and all other elements that are attached to the shaft (figures 06, 07 and 08).

Figure 06. Electric motor rotor

(https://upload.wikimedia.org/wikipedia/commons/8/80/Electric_Motor_Rotor.jpg).

Figure 07. Brake rotor (https://www.buybrakes.com/images/product/dba-smooth-plain-coated-street-brake-rotor.jpg).

Figure 08. Gas turbine rotor (https://empoweringpumps.com/sulzer-quick-turnaround-turbines/).

Since the invention of the wheel, rotors have been the most used parts of machines and mechanisms.

The rotational movement is used to obtain:

• The transmission, from the wheel to the axle (figure 09);

• To store energy, as in the old slingshot or flywheels (figures 10 and 11);

• Power transmission from one point to another using belts, toothed wheels or gear trains (figure 12);

• To obtain kinetic energy from other types of energy, such as thermal, chemical, nuclear or wind energy (figure 13) ...etc.

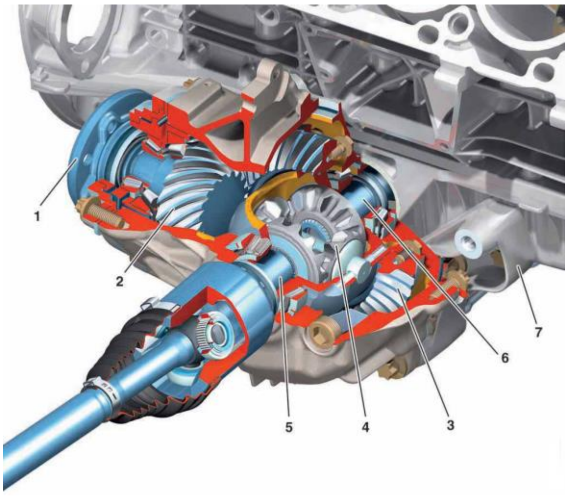


Figure 09. Front differential (https://www.forum-mercedes.com/topic-321-transmission-4matic-etude-de-la-chaine-cinematique.htmlhttp://www.forum-mercedes.com/forum-3-actros.html).

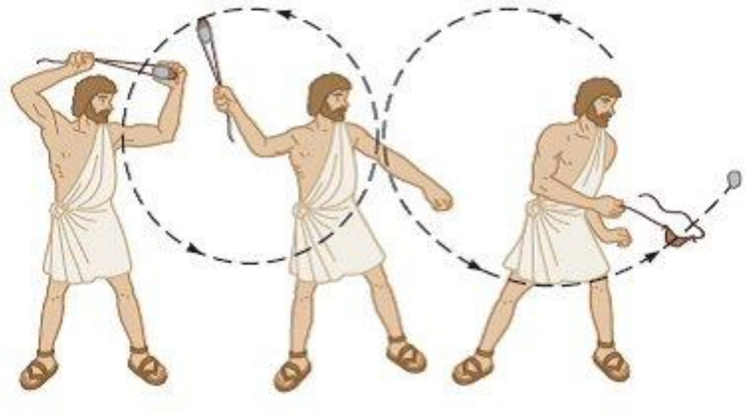


Figure 10. Old slingshot.

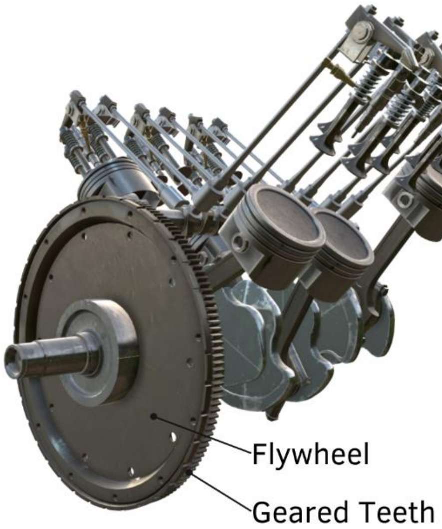


Figure 11. Flywheel (https://savree.com/en/encyclopedia/engine-flywheel).

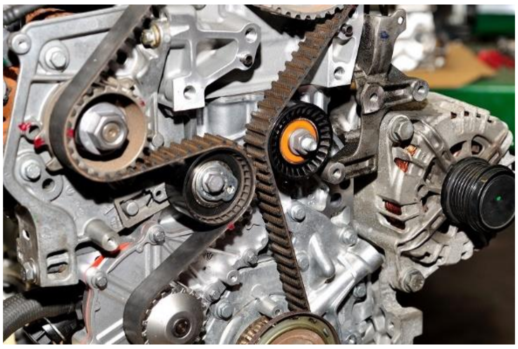


Figure 12. Timing belt (https://www.autotechwest.ca/what-is-a-timing-belt/).

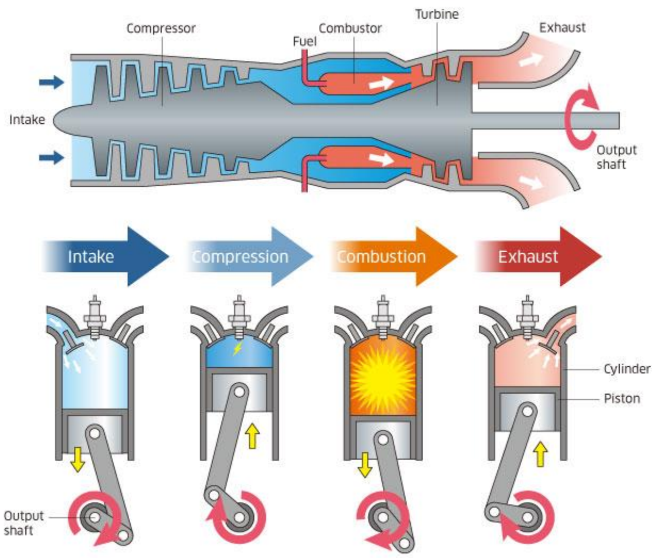


Figure 13. Thermal energy to mechanical energy (https://global.kawasaki.com/en/energy/equipment/gas_turbines/outline.html).

While fulfilling very important roles in machines, rotors are, at the same time, the main cause of concern in the normal operation of machines. Rotational motion around an appropriate axis, at the nominal rotational speed, imposed by the design, represents the crucial dynamic state for the rotors (figure 14).

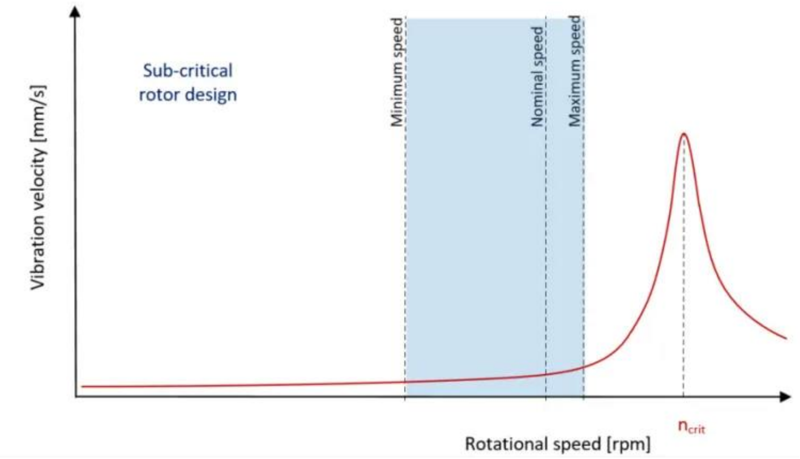


Figure 14. Rotor Vibration velocity vs nominal speed (https://mb-drive-serIVces.com/variable-speed-motors-and-critical-speeds/).

Due to several factors that contribute to energy transfer - from rotation to other forms of motion - rotor rotation can be accompanied by different Vibration modes.

All three main modes of rotor Vibration – lateral, torsional and axial modes – can be present during rotor operation (figure 15). Among these modes, the lateral modes (transverse Vibration) of the rotor are of most concern.

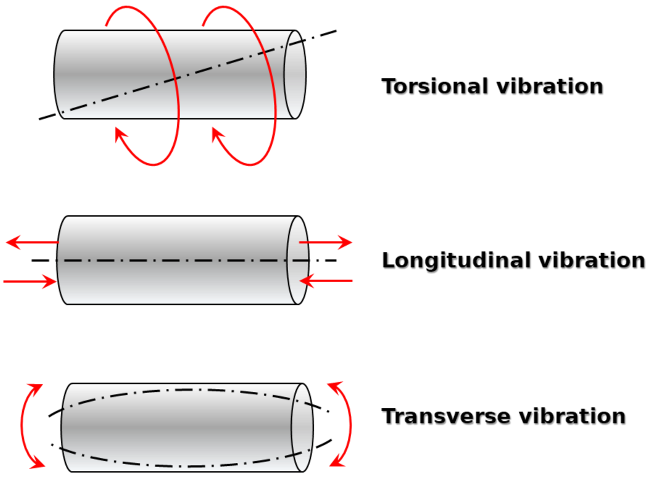


Figure 15. Vibration modes.

Lateral (transverse) Vibrations of the rotor, through the support bearings and through the fluid surrounding the rotor, are transmitted to the non-rotating (static) parts of the machine. Then, the Vibrations propagate to the machine foundation, adjacent equipment, building walls, and the surrounding air in the form of acoustic waves (for example Sayano-Shushenskaya power station accident on 17 August 2009, figure 16).

Figure 16. Sayano-Shushenskaya power station accident before and after.

There are several causes that can cause Vibrations. The first and most well-known among them is rotor imbalance (figure 17). As a result, the rotor responds with lateral Vibrations with a frequency, matching the rotation speed.

Since rotor unbalance is an almost unavoidable element, it is important to ensure that during operating conditions the Vibration amplitudes related to unbalance are acceptable and that during starts and stops a high-speed machine should smoothly pass several lateral resonance speeds "critical speeds".

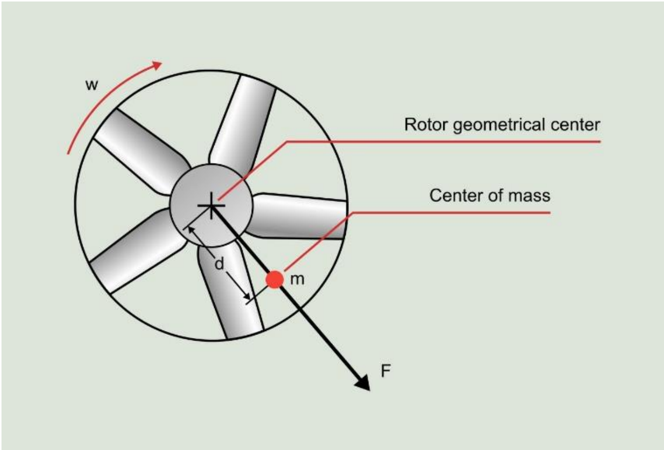


Figure 17. Rotor unbalance (https://power-mi.com/content/causes-imbalance).

Since rotor unbalance is not the only force that would excite rotor Vibrations, other periodic forces that excite rotor Vibrations (e.g., periodic excitations of blade passing frequency) must be recognized and kept under control (figure 18).

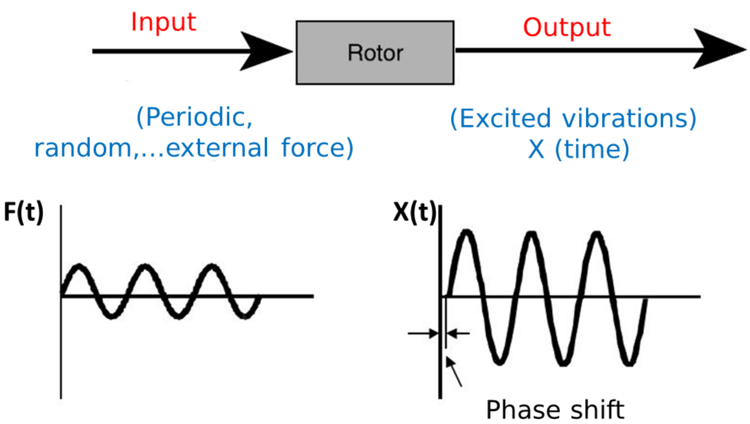


Figure 18. Vibration excitation and response.

There is another category of Vibrations in mechanical systems, called "transient Vibrations", which occur when the system is excited by a short impact, causing instantaneous changes in the system's acceleration, velocity and/or displacement (figure 19). The system responds to impact with free Vibrations, with "natural" frequencies, characteristic of the system.

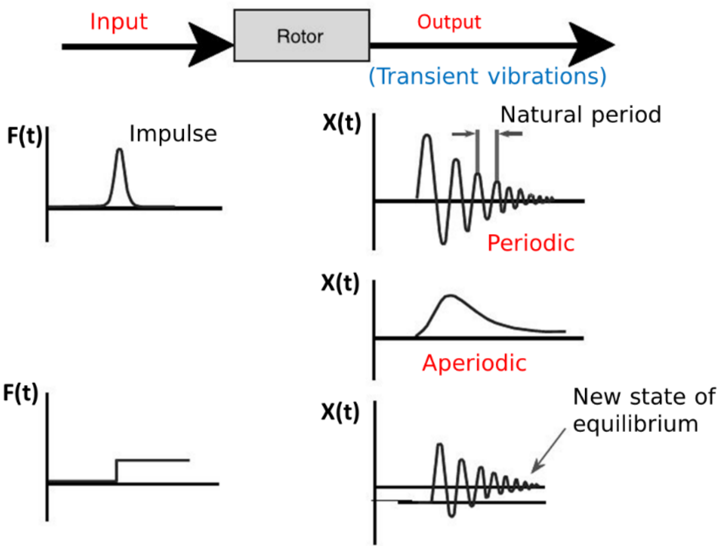


Figure 19. Transient excitation and response.

1.2. Components of a rotor:

The basic elements of a rotor are (figure 20): shaft, disk and bearings. The unbalance which cannot be completely avoided must also be taken into account.

Expressions for kinetic energy are necessary to characterize the disk, the shaft and the unbalance. The strain energy is necessary to characterize the shaft.

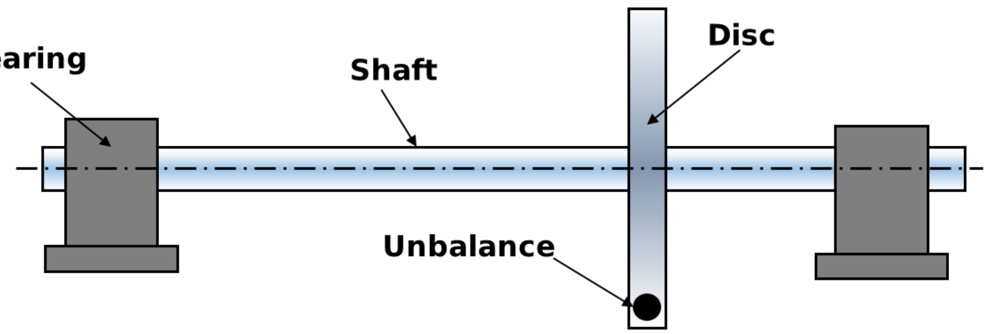


Figure 20. Rotor dynamic model.

2. Hamilton's principle:

Hamilton's principle is an approach to derive the equation of motion using the potential and kinetic energies of the mechanical system:

$$\delta\int_{t_1}^{t_2}\left(T-V\right)dt+\int_{t_1}^{t_2}\delta W_{nc}dt=0 \tag{1}$$

where T and V are respectively the kinetic and potential energy of the system, δW_{nc} is the virtual work of the non-conservative forces. Hamilton's principle is sometimes written in the following form:

$$\delta\int_{t_1}^{t_2}Ldt+\int_{t_1}^{t_2}\delta W_{nc}dt=0 \tag{2}$$

with L is the Lagrangian defined as:

$$L=T-V \tag{3}$$

Hamilton's principle states that the change in kinetic and potential energy plus the integral of the virtual work done by the non-conservative forces during the time interval between t_1 and t_2 must equal zero.

The potential energy V is defined as the strain energy minus the work of the conservative forces:

$$V=U-W_c \tag{4}$$

with W_c is the work of the conservative forces. We can define the virtual work of all forces, conservative and non-conservative, as:

$$\delta W=\delta W_{nc}+\delta W_c \tag{5}$$

Substituting equations (4) and (5) into (1) leads to the following form of Hamilton's principle:

$$\delta\int_{t_1}^{t_2}\left(T-U\right)dt+\int_{t_1}^{t_2}\delta Wdt=0 \tag{6}$$

When applying Hamilton's principle, it is assumed that the coordinates of the system are specified at the two end points t_1 and t_2, i.e.:

$$\delta q(t_1)=\delta q(t_2)=0 \tag{7}$$

Using Hamilton's principle is equivalent to applying the Lagrange equation. In fact, Hamilton's principle can be used to derive Lagrange's equation of motion. For this, we write:

$$\delta\int_{t_1}^{t_2} Tdt = \int_{t_1}^{t_2} \delta T dt = \int_{t_1}^{t_2}\left(\frac{\partial T}{\partial q}\delta q + \frac{\partial T}{\partial \dot{q}}\delta\dot{q}\right)dt \tag{8}$$

Let’s use integration by part:

$$\int_{t_1}^{t_2}\left(\frac{\partial T}{\partial \dot{q}}\right)\delta\dot{q}dt = \left(\frac{\partial T}{\partial \dot{q}}\right)\delta q\Bigg|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)\delta q dt \tag{9}$$

Using assumption (7), the previous equation reduces to:

$$\int_{t_1}^{t_2}\left(\frac{\partial T}{\partial \dot{q}}\right)\delta\dot{q}dt = -\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right)\delta q dt \tag{10}$$

Similarly:

$$\delta\int_{t_1}^{t_2} Udt = \int_{t_1}^{t_2}\left(\frac{\partial U}{\partial q}\right)\delta q dt \tag{11}$$

$$\int_{t_1}^{t_2} \delta W dt = \int_{t_1}^{t_2} \bar{Q}^T \delta q dt \tag{12}$$

with $\bar{Q}$ the vector of generalized forces.

Substituting equations (11), (12) and (13) into equation (6), we will have:

$$\int_{t_1}^{t_2}\left[\frac{\partial T}{\partial q} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial U}{\partial q} + \bar{Q}^T\right]\delta q dt = 0 \tag{13}$$

The coordinates q_1, q_2, …,q_n are independent, then we can write:

$$\frac{\partial T}{\partial q} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial U}{\partial q} + \bar{Q}^T = 0 \tag{14}$$

or

$$\frac{\partial T}{\partial q} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial U}{\partial q} + \bar{Q}^T = 0 \tag{15}$$

In the case of conservative systems in which all forces acting on the system can be derived from the potential function V:

$$\delta W_{nc} = 0 \tag{16}$$

Let's substitute this equation into Hamilton's principle (1):

$$\delta \int_{t_1}^{t_2} \left(T - V\right) dt = 0 \tag{17}$$

or
$$\delta \int_{t_1}^{t_2} L dt = 0 \tag{18}$$

For conservative systems, the Lagrange equation reduces to:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1,\ 2,\ ...,\ n \tag{19}$$

or
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j} \qquad j = 1,\ 2,\ ...,\ n \tag{20}$$

The total derivative of the Lagrangian is given by:

$$\frac{dL}{dt} = \sum_{j=1}^{n}\left(\frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial L}{\partial q_j}\dot{q}_j\right) \tag{21}$$

Substituting equation (20) into (21), we obtain:

$$\frac{dL}{dt} = \sum_{j=1}^{n}\left(\frac{\partial L}{\partial \dot{q}_j}\ddot{q}_j + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\dot{q}_j\right) = \sum_{j=1}^{n}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) \tag{22}$$

This equation can be written as:

$$\frac{dL}{dt} - \frac{d}{dt}\sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j = 0 \tag{23}$$

or
$$\frac{d}{dt}\left(L - \sum_{j=1}^{n}\frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) = 0 \tag{24}$$

Since the potential energy is independent of the velocity, we will have:

$$\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} \tag{25}$$

Let us substitute this equation in (23):

$$\frac{d}{dt}\left(L-\sum_{j=1}^{n}\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j\right)=0 \tag{26}$$

That is
$$L-\sum_{j=1}^{n}\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j=-H \tag{27}$$

where H is constant called Hamiltonian. Equation (26) is valid for any conservative system with one or more degrees of freedom. The Hamiltonian H can be written in a more convenient form using the following identity:

$$\sum_{j=1}^{n}\frac{\partial T}{\partial \dot{q}_j}\dot{q}_j=2T \tag{28}$$

Substituting this equation into (26) and using the Lagrangian, we obtain:

$$H=-\left(T-V-2T\right)=T+V \tag{29}$$

Which implies that the Hamiltonian is equal to the total energy of the system. Since, for conservative systems, the Hamiltonian is constant, we have:

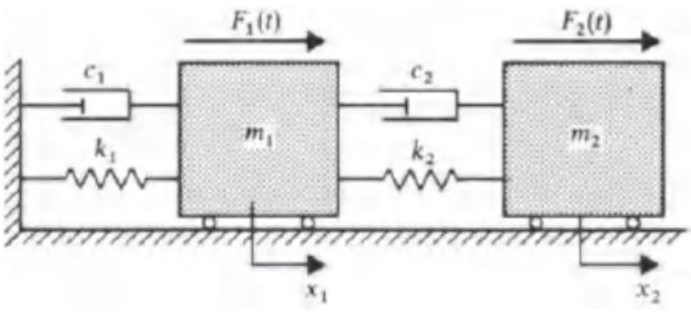
$$\frac{dH}{dt}=\frac{d}{dt}\left(T+V\right)=0 \tag{30}$$

This equation can be used to develop the equation of motion for conservative systems with one or more degrees of freedom.

Solved exercises:

Exercise 01:

Use Hamilton's principle to derive the differential equation of motion for the system in the figure below.



The kinetic and strain energies of the system can be written as follows:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2$$

The virtual works of forces and damping can be written as:

$$\delta W = F_1 \delta x_1 + F_2 \delta x_2 - c_1 \dot{x}_1 \delta x_1 - c_2 (\dot{x}_2 - \dot{x}_1) \delta (x_2 - x_1)$$

Hamilton's principle is given as:

$$\delta \int_{t_1}^{t_2} (T - U) dt + \int_{t_1}^{t_2} \delta W dt = 0$$

$$\delta \int_{t_1}^{t_2} (T - U) dt = \delta \int_{t_1}^{t_2} \left(\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 - \frac{1}{2} k_1 x_1^2 - \frac{1}{2} k_2 (x_2 - x_1)^2 \right) dt$$

$$\delta \int_{t_1}^{t_2} (T - U) dt = \int_{t_1}^{t_2} \big(m_1 \dot{x}_1 \delta \dot{x}_1 + m_2 \dot{x}_2 \delta \dot{x}_2 - k_1 x_1 \delta x_1 - k_2 (x_2 - x_1) \delta (x_2 - x_1) \big) dt$$

By integrating by parts, we will have:

$$\int_{t_1}^{t_2} m_1 \dot{x}_1 \delta \dot{x}_1 dt = m_1 \dot{x}_1 \delta x_1 \big|_{t_1}^{t_2} - \int_{t_1}^{t_2} m_1 \ddot{x}_1 \delta x_1 dt$$

By replacing in the previous expression, we will have:

$$\delta \int_{t_1}^{t_2} (T - U) dt = \int_{t_1}^{t_2} \big(-m_1 \ddot{x}_1 \delta x_1 - m_2 \ddot{x}_2 \delta x_2 - k_1 x_1 \delta x_1 - k_2 (x_2 - x_1) \delta (x_2 - x_1) \big) dt$$

Which give:

$$\delta \int_{t_1}^{t_2} (T - U) dt = \int_{t_1}^{t_2} (-m_1 \ddot{x}_1 - [k_1 + k_2] x_1 + k_2 x_2) \delta x_1 dt + (-m_2 \ddot{x}_2 - k_2 x_2 + k_1 x_1) \delta x_2 dt$$

By replacing in Hamilton's expression:

$$-\int_{t_1}^{t_2}(m_1\ddot{x}_1+[k_1+k_2]x_1-k_2x_2)\delta x_1dt+(m_2\ddot{x}_2+k_2x_2-k_1x_1)\delta x_2dt$$

$$+\int_{t_1}^{t_2}\big(F_1\delta x_1+F_2\delta x_2-c_1\dot{x}_1\delta x_1-c_2(\dot{x}_2-\dot{x}_1)\delta(x_2-x_1)\big)dt=0$$

After rearrangement:

$$\int_{t_1}^{t_2}(m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+[k_1+k_2]x_1-k_2x_2-F_1)\delta x_1dt$$

$$+(m_2\ddot{x}_2+c_2\dot{x}_2-c_2\dot{x}_1+k_2x_2-k_1x_1-F_2)\delta x_2dt=0$$

x_1 and x_2 are assumed to be independent, then we can write:

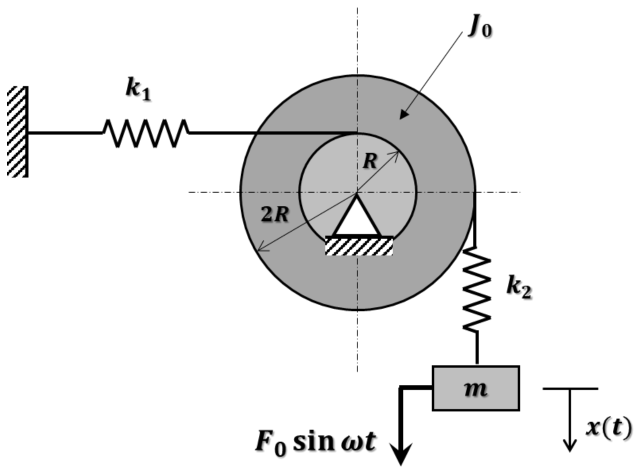
$$m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+[k_1+k_2]x_1-k_2x_2-F_1=0$$
$$m_2\ddot{x}_2+c_2\dot{x}_2-c_2\dot{x}_1+k_2x_2-k_1x_1-F_2=0$$

Or:

$$m_1\ddot{x}_1+(c_1+c_2)\dot{x}_1-c_2\dot{x}_2+[k_1+k_2]x_1-k_2x_2=F_1$$
$$m_2\ddot{x}_2+c_2\dot{x}_2-c_2\dot{x}_1+k_2x_2-k_1x_1=F_2$$

Exercise 02:

Consider the following system consisting of a mass m connected on one side to a pulley with an inertia J_0 which can be pivoted without friction by a rigid cable and on the other side by a cable with a rigidity k_2. The pulley is attached to a fixed support with a cable of stiffness k_1.



- Determine the equations of motion of the system, use the Lagrange equations.
- Calculate the natural frequencies of the system.

We give : $m = 5\ kg$, $J_0 = 0.2\ kg.m^2$, $k_1 = 5000\ N/m$, $k_2 = 4000\ N/m$, $R = 5\ cm$, $F_0 = 75\ N$ et $\omega = 5\ Hz$.

The rotation of the pulley $\theta(t)$ is independent of the displacement of the mass $x(t)$ since the cable which connects them is not rigid. Then the system is a system with 2 degrees of freedom.

The Lagrange equations are:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F(t)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$L = T - U$$

The kinetic energy:

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2$$

Potential energy:

$$U = \frac{1}{2}k_1R^2\theta^2 + \frac{1}{2}k_2(x - 2R\theta)^2 = \frac{1}{2}k_1R^2\theta^2 + \frac{1}{2}k_2(x^2 - R\theta x + 4R^2\theta^2)$$

From where:

$$L = T - U = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 - \frac{1}{2}k_1R^2\theta^2 - \frac{1}{2}k_2(x^2 - R\theta x + 4R^2\theta^2)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{\partial L}{\partial \dot{\theta}} = J_0\dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = J_0\ddot{\theta}$$

$$\frac{\partial L}{\partial x} = -k_2x + \frac{1}{2}R\theta \qquad \frac{\partial L}{\partial \theta} = -k_1R^2\theta + \frac{1}{2}Rx - 4R^2\theta$$

Hence the equations of motion are:

$$m\ddot{x} + k_2x - \frac{1}{2}R\theta k_2 = F_0 \sin \omega t$$

$$J_0\ddot{\theta} + k_1R^2\theta - \frac{1}{2}Rxk_2 + 4R^2\theta k_2 = 0$$

Assume the following harmonic solutions:

$$x(t) = A \sin \omega t$$

$$\theta(t) = \Theta \sin \omega t$$

Let's substitute these solutions into the equations of motion:

$$\begin{cases} (-m\omega^2 + k_2)A - \frac{1}{2}Rk_2\Theta = F_0 \\ \left(-J_0\omega^2 + k_1R^2 + 4R^2k_2\right)\Theta - \frac{1}{2}Rk_2A = 0 \end{cases}$$

In matrix form:

$$\begin{bmatrix} -m\omega^2 + k_2 & -\frac{1}{2}Rk_2 \\ -\frac{1}{2}Rk_2 & -J_0\omega^2 + k_1R^2 + 4R^2k_2 \end{bmatrix} \begin{Bmatrix} A \\ \Theta \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

Hence the characteristic equation is:

$$(-m\omega^2 + k_2)(-J_0\omega^2 + k_1R^2 + 4R^2k_2) - \frac{1}{4}R^2k_2^2 = 0$$

$$(-5\omega^2 + 4000)(-0.2\omega^2 + 62.5) - 10000 = 0$$

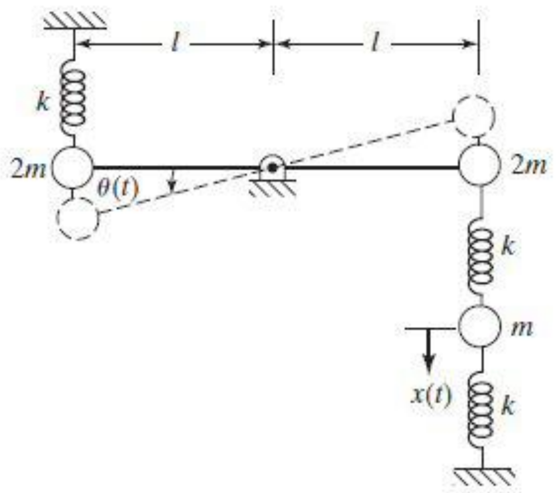
$$\omega^4 - 1112.5\omega^2 + 240000 = 0$$

$$\omega_1 = 17.11\ rd/s$$

$$\omega_2 = 28.63\ rd/s$$

Exercise 03:

A rigid rod of negligible mass and length 2l pivots at its midpoint and a mass is constrained to move in the vertical plane.



Determine the equations of motion of the system, use the Lagrange equations.

The degree of freedom of the system is 2, because we have 3 masses in motion but 2 of them are linked (masses $2m$) so the two variables are $x(t)$ and $\theta(t)$. For the variable $\theta(t)$, we must determine the equivalent inertia and the equivalent stiffness.

The kinetic energy corresponding to this variable is:

$$\frac{1}{2}J_{eq}\dot{\theta}^2 = \frac{1}{2}J_{2m}\dot{\theta}^2 + \frac{1}{2}J_{2m}\dot{\theta}^2$$

$$J_{eq} = J_{2m} + J_{2m} = 2J_{2m} = 2 \times 2m \times l^2 = 4ml^2$$

The potential energy corresponding to this variable is:

$$\frac{1}{2}k_{eq}\theta^2 = \frac{1}{2}k(\theta l)^2$$

$$k_{eq} = kl^2$$

The kinetic energy of the system:

$$T = \frac{1}{2}J_{eq}\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 = 2ml^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2$$

The potential energy of the system:

$$U = \frac{1}{2}k_{eq}\theta^2 + \frac{1}{2}k(x-\theta l)^2 + \frac{1}{2}kx^2 = \frac{1}{2}kl^2\theta^2 + \frac{1}{2}k(x-\theta l)^2 + \frac{1}{2}kx^2$$

Lagrangien:

$$L = T - U = 2ml^2\dot{\theta}^2 + \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kl^2\theta^2 - \frac{1}{2}k(x-\theta l)^2 - \frac{1}{2}kx^2$$

Let's write the Lagrange equation for each variable:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{\theta}} = 4ml^2\dot{\theta} \quad \rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = 4ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -kl^2\theta + kl(x-\theta l)$$

$$4ml^2\ddot{\theta} + kl^2\theta - kl(x-\theta l) = 0$$

$$4ml^2\ddot{\theta} + 2kl^2\theta - klx = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = -k(x-\theta l) - kx = -2kx + kl\theta$$

$$m\ddot{x} + 2kx - kl\theta = 0$$

$$\begin{bmatrix} 4ml^2 & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x} \end{Bmatrix} + \begin{bmatrix} 2kl^2 & -kl \\ -kl & 2k \end{bmatrix} \begin{Bmatrix} \theta \\ x \end{Bmatrix} = \{0\}$$

3. Energy formulation

3.1. Disk kinetic energy:

The disk is assumed to be rigid and therefore characterized by its kinetic energy. Let $R_0(X,Y,Z)$ be a fixed reference and $R(x,y,z)$ a rotating reference linked to the disk (figure 09).

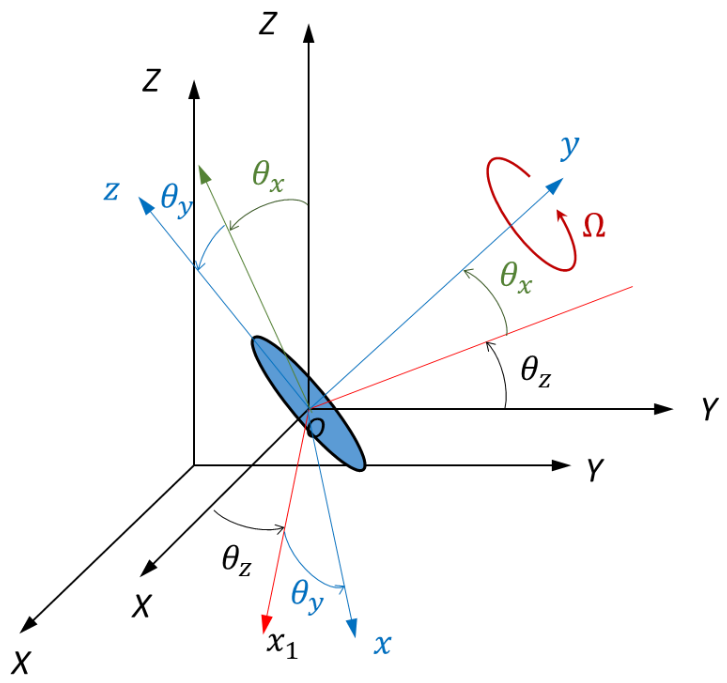


Figure 21. A rotating disk in space.

The instantaneous rotation vector reflecting the position of the xyz coordinate is:

$$\overrightarrow{\omega_{R/R_O}} = \dot{\theta}_z \vec{Z} + \dot{\theta}_x \vec{x}_1 + \dot{\theta}_y \vec{y} \tag{31}$$

where Z, x_1 and y are the unit vectors of the axes $\overrightarrow{OZ}$, $\overrightarrow{Ox_1}$, $\overrightarrow{Oy}$.

The kinetic energy of the disk corresponding to its movement around the center of mass O is calculated using the rotating reference R. In these coordinates, the instantaneous rotation vector is:

$$\overrightarrow{\omega_{R/R_O}} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -\dot{\theta}_z \cos\theta_x \sin\theta_y + \dot{\theta}_x \cos\theta_y \\ \dot{\theta}_y + \dot{\theta}_z \sin\theta_x \\ \dot{\theta}_z \cos\theta_x \cos\theta_y + \dot{\theta}_x \sin\theta_y \end{bmatrix} \tag{32}$$

Let u and w be the displacements along X and Y coordinates of O in fixed reference R_0, the coordinate along Y is constant. The mass of the disk is M_d, its inertia tensor in O has the expression:

$$J_O = \begin{bmatrix} J_{dx} & 0 & 0 \\ 0 & J_{dy} & 0 \\ 0 & 0 & J_{dz} \end{bmatrix} \tag{33}$$

The general expression for the kinetic energy of the disk is then written:

$$T_d = \frac{1}{2} M_d \left(\dot{u}^2 + \dot{w}^2 \right) + \frac{1}{2} \left(J_{dx} \omega_x^2 + J_{dy} \omega_y^2 + J_{dz} \omega_z^2 \right) \tag{34}$$

which can be simplified because the disk is symmetrical ($J_{dx} = J_{dz}$).

When the angles θ_x and θ_z are small and the angular speed constant ($\dot{\theta}_y = \Omega$), equation (33) becomes:

$$T_d = \frac{1}{2} M_d \left(\dot{u}^2 + \dot{w}^2 \right) + \frac{1}{2} J_{dx} \left(\dot{\theta}_z^2 + \dot{\theta}_x^2 \right) + \frac{1}{2} J_{dy} \left(\Omega^2 + 2\Omega \dot{\theta}_z \theta_x \right) \tag{35}$$

The term $\frac{1}{2} I_{dy} \Omega^2$, which is constant, has no influence on the equations of motion and represents the kinetic energy of the disk rotating at speed Ω, in the case where all other displacements are zero (since energy is derived to obtain the equation of motion). The last term, $I_{dy} \Omega \dot{\theta}_z \theta_x$ represents the gyroscopic effect (Coriolis).

3.2. Shaft kinetic energy:

The general formulation of the shaft kinetic energy is an extension of that of the disk (35). For an element of length L, of constant cross section, the expression of the kinetic energy is:

$$T_s = \frac{\rho A}{2} \int_0^L \left(\dot{u}^2 + \dot{w}^2 \right) dy + \frac{\rho I_s}{2} \int_0^L \left(\dot{\theta}_z^2 + \dot{\theta}_x^2 \right) dy + \rho I_s L \Omega^2 + 2\rho I_s \Omega \int_0^L \dot{\theta}_z \theta_x dy \tag{36}$$

where ρ is the mass density, A is the cross-sectional area of the beam and I_s the diametrical moment of inertia.

3.3. Shaft deformation energy:

Point C is the geometric center of the tree (figure 22), $B(x, z)$ is a point of the cross section, E is Young's modulus, ε and σ are the deformations and stresses, u^* and w^* are the displacements of the geometric center relative to the x, z axes (mobile reference).

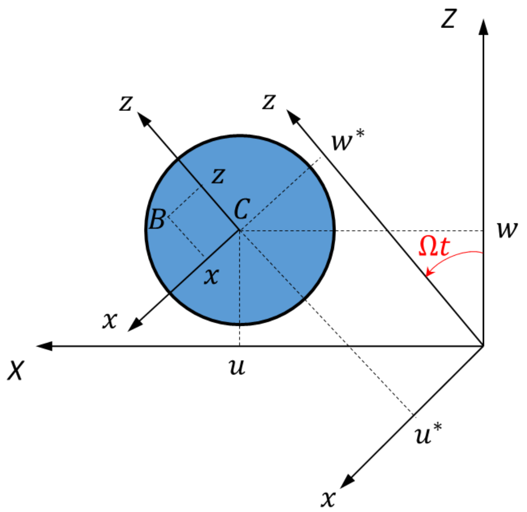


Figure 22. Rotor cross section in space.

The displacement of a point of the cross section is expressed by considering the shear in the moving reference frame in the form:

$$\{u(x,y,z)\} = \begin{cases} u_x(x,y,z) = u^* \\ u_y(x,y,z) = -z\theta_x + x\theta_z \\ u_z(x,y,z) = w^* \end{cases} \tag{37}$$

The associated strain field is then:

$$\{\varepsilon\} = \begin{cases} \varepsilon_{yy} = -z\dfrac{\partial\theta_x}{\partial y} + x\dfrac{\partial\theta_z}{\partial y} \\ \gamma_{yz} = -\theta_x + \dfrac{\partial w^*}{\partial y} \\ \gamma_{yx} = \theta_z + \dfrac{\partial u^*}{\partial y} \end{cases} \tag{38}$$

According to beam theory it is assumed that $\sigma_{xx} = \sigma_{zz} = \sigma_{xz} = 0$. According to this hypothesis the stress-strain relationship is written:

$$\{\sigma\} = \begin{cases} \sigma_{yy} = E_y\varepsilon_{yy} \\ \tau_{yz} = G_{yz}\gamma_{yz} \\ \tau_{yx} = G_{yx}\gamma_{yx} \end{cases} \tag{39}$$

Using equation (36), the equation (37) becomes:

$$
\{\sigma\} = \begin{cases} \sigma_{yy} = E_y \left(-z \dfrac{\partial \theta_x}{\partial y} + x \dfrac{\partial \theta_z}{\partial y} \right) \\ \tau_{yz} = G_{yz} \left(-\theta_x + \dfrac{\partial w^*}{\partial y} \right) \\ \tau_{yx} = G_{yx} \left(\theta_z + \dfrac{\partial u^*}{\partial y} \right) \end{cases} \tag{40}
$$

The general expression for the deformation energy of the rotor shaft in bending is then:

$$
U = \frac{1}{2} \int_V \{\varepsilon\}^t [\sigma] dV \tag{41}
$$

either:

$$
U = \frac{1}{2} \int_0^L \int_A \left(\sigma_{yy} \varepsilon_{yy} + \tau_{yz} \gamma_{yz} + \tau_{yx} \gamma_{yx} \right) dA.dy \tag{42}
$$

Taking into account relations (39) and (40) the equation (42) can be written in the form:

$$
\begin{aligned} U = \frac{1}{2} \int_0^L \int_A E_y \left[z \left(\frac{\partial \theta_x}{\partial y} \right) + x \left(\frac{\partial \theta_z}{\partial y} \right) \right]^2 dAdy + \\ \frac{1}{2} \int_0^L \int_A \left[G_{yz} \left(-\theta_x + \frac{\partial w^*}{\partial y} \right)^2 + G_{yx} \left(-\theta_z + \frac{\partial u^*}{\partial y} \right)^2 \right] dAdy \end{aligned} \tag{43}
$$

$$
\begin{aligned} U = \frac{1}{2} \int_0^L \int_A E_y \left[z^2 \left(\frac{\partial \theta_x}{\partial y} \right)^2 + x^2 \left(\frac{\partial \theta_z}{\partial y} \right)^2 + 2xz \frac{\partial \theta_x}{\partial y} \frac{\partial \theta_z}{\partial y} \right]^2 dAdy + \\ \frac{1}{2} \int_0^L \int_A \left[G_{yz} \left(-\theta_x + \frac{\partial w^*}{\partial y} \right)^2 + G_{yx} \left(-\theta_z + \frac{\partial u^*}{\partial y} \right)^2 \right] dAdy \end{aligned} \tag{44}
$$

The second moment of area of the cross section of the shaft with respect to x and z are given by:

$$
\begin{cases} I_x = \int_A z^2 dA \\ I_z = \int_A x^2 dA \\ \int_A xzdA = 0 \end{cases} \tag{45}
$$

Finally, the deformation energy has the expression:

$$U = \frac{1}{2}\int_0^L E_y \left[I_x \left(\frac{\partial \theta_x}{\partial y} \right)^2 + I_z \left(\frac{\partial \theta_z}{\partial y} \right)^2 \right] dy + \frac{1}{2}\int_0^L \kappa A \left[G_{yz} \left(-\theta_x + \frac{\partial w^*}{\partial y} \right)^2 + G_{yx} \left(-\theta_z + \frac{\partial u^*}{\partial y} \right)^2 \right] dy \tag{46}$$

where κ is the shear correction factor.

By neglecting the effect of shear, equation (46) can be written as:

$$U = \frac{1}{2}\int_0^L E_y \left[I_x \left(\frac{\partial \theta_x}{\partial y} \right)^2 + I_z \left(\frac{\partial \theta_z}{\partial y} \right)^2 \right] dy \tag{47}$$

To avoid periodic terms, explicitly time-dependent, it is necessary, taking into account the bearings properties, to express the strain energy as a function of u and w components of the displacement in R_o. The transition from u, w to u^*, w^* is written:

$$\begin{cases} u^* = u\cos\Omega t - w\sin\Omega t \\ w^* = u\sin\Omega t + w\cos\Omega t \end{cases} \tag{48}$$

Equation (47) then becomes:

$$U = \frac{E_y}{2}\int_0^L \left[I_x \left(\sin\Omega t \frac{\partial^2 u}{\partial y^2} + \cos\Omega t \frac{\partial^2 w}{\partial y^2} \right)^2 + I_z \left(\cos\Omega t \frac{\partial^2 u}{\partial y^2} - \sin\Omega t \frac{\partial^2 w}{\partial y^2} \right)^2 \right] dy \tag{49}$$

Finally, for the most common case of a symmetrical shaft (for a circular section, $I_x = I_z = I_s$), the deformation energy is written:

$$U_s = \frac{EI_s}{2}\int_0^L \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial y^2} \right)^2 dy \tag{50}$$

3.4. Bearings forces work:

The stiffness and damping characteristics are assumed to be known (figure 23). The virtual work δW of the external forces acting on the shaft is in the form:

$$\begin{aligned} \delta W = & -k_{XX}u.\delta u - k_{XZ}w.\delta w - k_{ZZ}w.\delta w - k_{ZX}u.\delta w \\ & -c_{XX}\dot{u}.\delta u - c_{XZ}\dot{w}.\delta u - c_{ZZ}\dot{w}.\delta w - c_{ZX}\dot{u}\delta w \end{aligned} \tag{51}$$

or in a more compact form:

$$\delta W = F_u \delta u + F_w \delta w \tag{52}$$

F_u and F_w are the components of the generalized force and are expressed in the following matrix form:

$$\begin{bmatrix} F_u \\ F_w \end{bmatrix} = -\begin{bmatrix} k_{XX} & k_{ZX} \\ k_{ZX} & k_{ZZ} \end{bmatrix} \begin{Bmatrix} u \\ w \end{Bmatrix} - \begin{bmatrix} c_{XX} & c_{ZX} \\ c_{ZX} & c_{ZZ} \end{bmatrix} \begin{Bmatrix} \dot{u} \\ \dot{w} \end{Bmatrix} \tag{53}$$

Frequently for hydrodynamic journal: $k_{XX} \neq k_{ZZ}$; $c_{XX} \neq c_{ZZ}$ and $k_{XZ} \neq k_{ZX}$; $c_{XZ} \neq c_{ZX}$

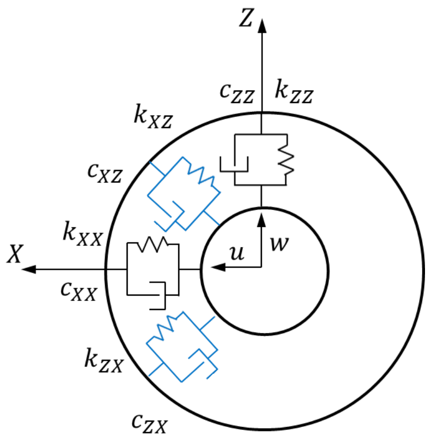


Figure 23. Bearing modeling.

3.5. Unbalance forces work:

Let calculate the unbalance kinetic energy due to a mass m_b located at a distance r from the shaft geometric center (figure 24). The mass remains in a plane perpendicular to the y axis and its coordinate along the y axis is constant.

The coordinates of the mass in the R_O reference, are:

$$\overrightarrow{OD} = \begin{Bmatrix} u + r\sin\Omega t \\ Cste \\ w + r\cos\Omega t \end{Bmatrix} \tag{54}$$

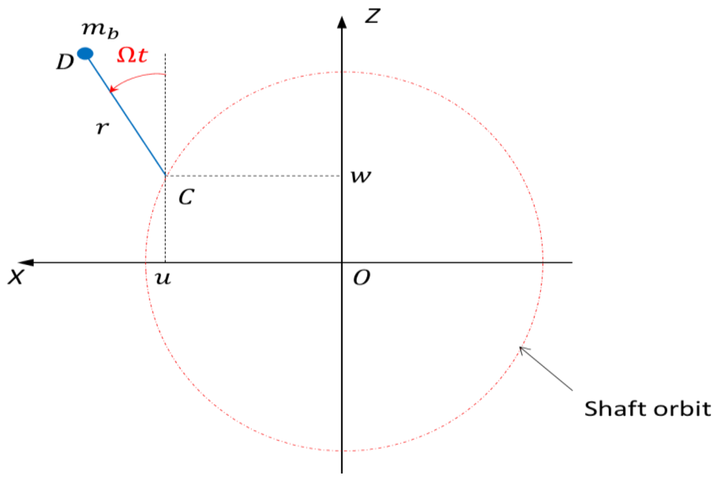


Figure 24. Unbalance modeling.

hence the velocity:

$$\vec{V} = \frac{d\overrightarrow{OD}}{dt} = \begin{Bmatrix} \dot{u} + r\Omega\cos\Omega t \\ 0 \\ \dot{w} - r\Omega\sin\Omega t \end{Bmatrix} \tag{55}$$

and the kinetic energy:

$$T_b = \frac{m_b}{2}\left(\dot{u}^2 + \dot{w}^2 + \Omega^2 r^2 + 2\Omega\dot{u}r\cos\Omega t - 2\Omega\dot{w}r\sin\Omega t\right) \tag{56}$$

The term $\frac{\Omega^2 r^2}{2}$ is constant and will not appear in the equations. The unbalance mass being negligible compared to the rotor mass, the expression of the kinetic energy can be approximated by:

$$T_b \cong m_b\Omega r\left(\dot{u}\cos\Omega t - \dot{w}\sin\Omega t\right) \tag{57}$$

4. Rotor modeling and equations of motion

4.1. Analytical model:

Consider a rotor consisting of (figure 25):

- A shaft of length L and constant circular section.
- A symmetrical disk with an unbalance at $Y = l_1$.
- A bearing located at $Y = l_2$.

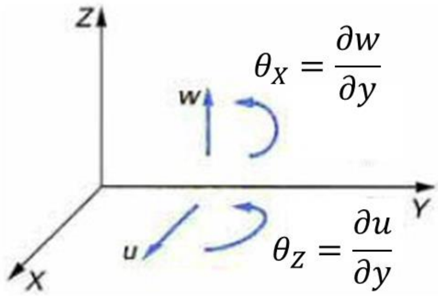
The expressions for kinetic energy, strain energy and work established are used for each element and the constant term appearing in the expression for kinetic energy is systematically neglected. The displacements expressions in the X and Z directions are:

$$\begin{cases} u(y,t) = f(y) q_1(t) = f(y) q_1 \\ w(y,t) = f(y) q_2(t) = f(y) q_2 \end{cases} \tag{58}$$

q_1 and q_2 are the independent generalized coordinates.

θ_X and θ_Z are small, they are approximated by:

$$\begin{cases} \theta_X = \dfrac{\partial w}{\partial y} = \dfrac{df(y)}{dy} q_2 = g(y) q_2 \\ \theta_Z = \dfrac{\partial u}{\partial y} = -\dfrac{df(y)}{dy} q_1 = -g(y) q_1 \end{cases} \tag{59}$$



The second order derivatives of u and w are necessary to express the strain energy; the expressions are:

$$\begin{cases} \dfrac{\partial^2 w}{\partial y^2} = \dfrac{d^2 f(y)}{dy^2} q_2 = h(y) q_2 \\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{d^2 f(y)}{dy^2} q_1 = h(y) q_1 \end{cases} \tag{60}$$

The Lagrange equations are applied in the following form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = F_{qi} \tag{61}$$

where N $(1 \le i \le N)$ is the number of degrees of freedom, q_i are the independent generalized coordinates, F_{qi} are the generalized forces.

4.1.1. Symmetrical model:

In this case, the stiffness k_{ZZ} is not taken into consideration ($k_{ZZ} = 0$) and the application of the Lagrange equations leads to the equations of motion in the general form:

$$\begin{cases} m\ddot{q}_1 - a\Omega\dot{q}_2 + kq_1 = m_b d\Omega^2 f(l_1)\sin\Omega \\ m\ddot{q}_2 - a\Omega\dot{q}_1 + kq_2 = m_b d\Omega^2 f(l_1)\cos\Omega \end{cases} \tag{62}$$

4.1.2. Natural frequencies vs rotation speed: Campbell diagram

The rotor is first considered in free movement. Only the solution of equations (62) without a second member is considered:

$$\begin{cases} m\ddot{q}_1 - a\Omega\dot{q}_2 + kq_1 = 0 \\ m\ddot{q}_2 - a\Omega\dot{q}_1 + kq_2 = 0 \end{cases} \tag{63}$$

In matrix form:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \Omega\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}\begin{Bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \{0\} \tag{64}$$

The solutions of the equations are in the form:

$$\begin{cases} q_1 = Q_1 e^{rt} \\ q_2 = Q_2 e^{rt} \end{cases} \tag{65}$$

By replacing (65) in (64), we will have:

$$\begin{bmatrix} k + mr^2 & -a\Omega r \\ a\Omega r & k + mr^2 \end{bmatrix}\begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = \{0\} \tag{66}$$

Calculating the determinant gives the characteristic equation:

$$\left(k + mr^2\right)^2 + a^2\Omega^2 r^2 = 0$$

$$m^2 r^4 + \left(2km + a^2\Omega^2\right) r^2 + k^2 = 0 \tag{67}$$

At rest ($\Omega = 0$), the roots r_{10} and r_{20} of (67) are:

$$r_{10}^2 = r_{20}^2 = j^2\omega_{10}^2 = j^2\omega_{20}^2 = -\frac{k}{m} \tag{68}$$

The frequencies are:

$$\omega_{10} = \omega_{20} = \sqrt{\frac{k}{m}} \tag{69}$$

In rotation ($\Omega \neq 0$), the roots of (65) are r_1 and r_2 and the corresponding frequencies ω_1 and ω_2.

$$\begin{cases} r_1^2 = -\left[\omega_{10}^2 + \dfrac{a^2\Omega^2}{2m^2}\left(1 - \sqrt{1 + \dfrac{4m^2\omega_{10}^2}{a^2\Omega^2}}\right)\right] = j^2\omega_1^2 \\ r_2^2 = -\left[\omega_{10}^2 + \dfrac{a^2\Omega^2}{2m^2}\left(1 + \sqrt{1 + \dfrac{4m^2\omega_{10}^2}{a^2\Omega^2}}\right)\right] = j^2\omega_2^2 \end{cases} \tag{70}$$

SO:

$$\begin{cases} \omega_1 = \sqrt{\omega_{10}^2 + \dfrac{a^2\Omega^2}{2m^2}\left(1 - \sqrt{1 + \dfrac{4m^2\omega_{10}^2}{a^2\Omega^2}}\right)} \\ \omega_2 = \sqrt{\omega_{10}^2 + \dfrac{a^2\Omega^2}{2m^2}\left(1 + \sqrt{1 + \dfrac{4m^2\omega_{10}^2}{a^2\Omega^2}}\right)} \end{cases} \tag{71}$$

Modal forms (modes) are then examined. The first equation of (66) gives:

$$Q_1 = \frac{\&\Omega r Q_2}{k + mr^2} \tag{72}$$

The modes are complex ($r = \pm j\omega$) and their interpretation delicate. Let's choose the following set of initial conditions, at $t_0 = 0$:

$$\begin{cases} q_1 = 0 \\ q_2 = q_{20} \\ \dot{q}_1 = -\omega_1 q_{20} \\ \dot{q}_2 = 0 \end{cases} \tag{73}$$

He comes:

$$\begin{cases} q_1 = -q_{20}\sin\omega_1 t \\ q_2 = q_{20}\cos\omega_1 t \end{cases} \tag{74}$$

The displacements u and w of a point located at $y=l$ from the origin of the rotor axis are given by (58), i.e.:

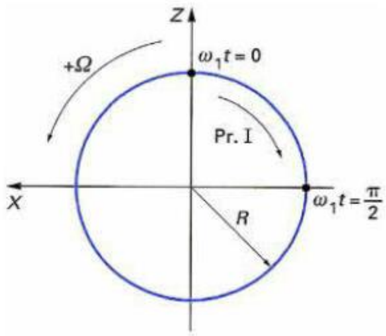
$$\begin{cases} u(l,t) = -q_{20}\sin\dfrac{\pi l}{L}\sin\omega_1 t = -R\sin\omega_1 t \\ w(l,t) = q_{20}\sin\dfrac{\pi l}{L}\cos\omega_1 t = -R\cos\omega_1 t \end{cases} \tag{75}$$

with : $f(y) = \sin\dfrac{\pi x}{L}$

These two expressions lead to:

$$R = \sqrt{u^2(l,t) + w^2(l,t)} = q_{20}\sin\frac{\pi l}{L} \tag{76}$$

The points located on the rotor axis describe circles. With all the initial conditions chosen, the orbit is described, in a direction opposite to the direction of rotations Ω; the rotor is in the reverse precession situation (Pr. I.).



Now let's choose another set of initial conditions, at $t_0 =$ 0:

$$\begin{cases} q_1 = q_{10} \\ q_2 = 0 \\ \dot{q}_1 = 0 \\ \dot{q}_2 = -\omega_2 q_{10} \end{cases} \tag{77}$$

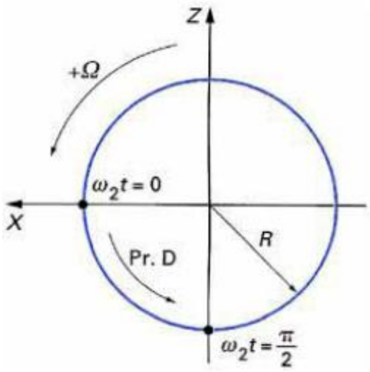
He comes:

$$\begin{cases} q_1 = q_{10}\cos\omega_2 t \\ q_2 = -q_{10}\sin\omega_2 t \end{cases} \tag{78}$$

So:

$$\begin{cases} u(l,t) = q_{10}\sin\dfrac{\pi l}{L}\cos\omega_2 t = R\cos\omega_2 t \\ w(l,t) = -q_{10}\sin\dfrac{\pi l}{L}\sin\omega_2 t = -R\sin\omega_2 t \end{cases} \tag{79}$$

The points located on the rotor axis describe circles. With all the initial conditions chosen, the orbit is described, in the same direction as that of the direction of rotations Ω; the rotor is in the direct precession situation (Pr. D.).



Campbell diagram:

The Campbell diagram is shown on Figure (25) [$f_1 = f_1(N)$, $f_2 = f_2(N)$] and the intersections of $f_1(N)$, $f_2(N)$ with two straight lines are indicated. Points A and B correspond to the intersections with the line $f = N/60$; at these two points a rotor frequency equals the rotation frequency. Points C and D correspond to the intersections with the line $f = 0.5N/60$; these two points correspond to a rotor frequency equal to half the rotation frequency. It is interesting to have a general expression of the frequencies corresponding to points A, B, C, D.

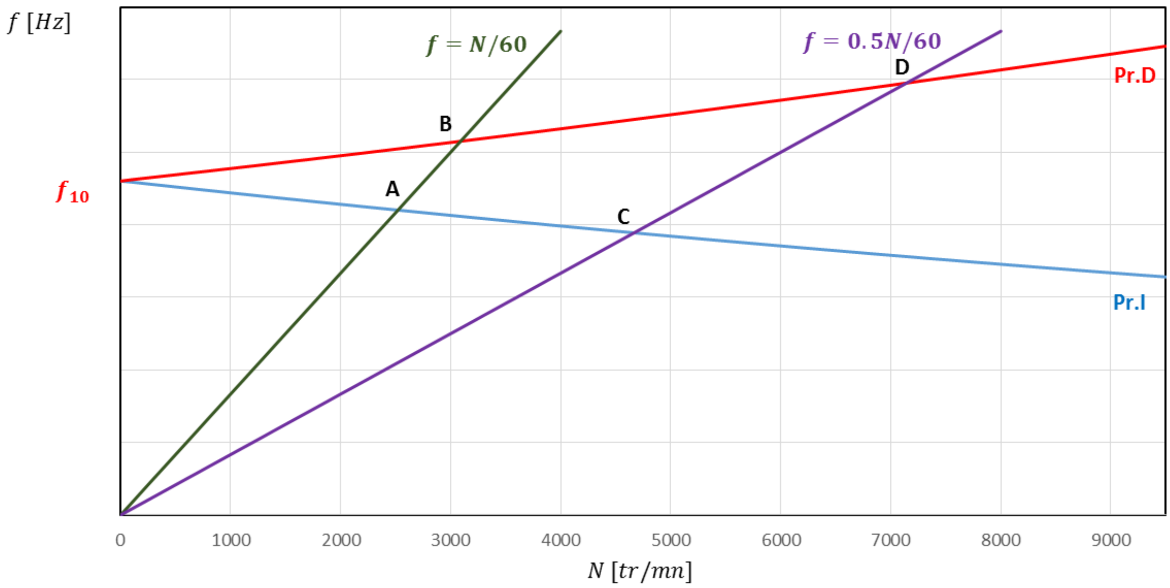


Figure 25. Campbell diagram.

The relationship between ω and Ω is:

$$\omega = s\Omega \tag{80}$$

where $s = 1$ in A and B, $s = 0.5$ in C and D. Expressions (70) allow us to write:

$$r = \pm j\omega = \pm js\Omega \tag{81}$$

In (67) gives:

$$s^2 m^2 \Omega^4 - \left(2km + a^2\Omega^2\right) s^2\Omega^2 + k^2 = 0$$

$$s^2 \left(s^2 m^2 - a^2\right)\Omega^4 - 2kms^2\Omega^2 + k^2 = 0 \tag{82}$$

The solutions of (82) are:

$$\begin{cases} \Omega_1 = \sqrt{\dfrac{k}{s\left(sm+a\right)}} \\ \Omega_2 = \sqrt{\dfrac{k}{s\left(sm-a\right)}} \end{cases} \tag{83}$$

From (80), we have:

$$\omega_1 = s\sqrt{\frac{k}{s\left(sm+a\right)}}$$

corresponding to points A and C.

$$\omega_2 = s\sqrt{\frac{k}{s\left(sm-a\right)}}$$

corresponding to points B and D.

4.2. Finite Element model:

The finite element method, widely used for the calculation of complex structures, is also effective in rotor dynamics. The gyroscopic effect must be taken into account and specific resolution methods can be advantageously used. It is necessary to define the finite elements allowing the rotors to be modeled: discs, shafts, bearings and to represent the external forces, in particular those due to unbalances.

4.2.1. Disk:

Each node has four degrees of freedom: two displacements u, w, along X and Z, and two slopes θ_X and θ_Z around X and Z. The vector δ of the nodal displacements of the center of the disk is:

$$\delta = \left\{u, w, \theta_X, \theta_Z\right\}^t \tag{84}$$

Applying the Lagrange equations to expression (35) gives:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \delta}\right)-\frac{\partial T}{\partial \delta}=\begin{bmatrix} M_d & 0 & 0 & 0 \\ 0 & M_d & 0 & 0 \\ 0 & 0 & I_{dX} & 0 \\ 0 & 0 & 0 & I_{dX} \end{bmatrix}\begin{Bmatrix} \ddot{u} \\ \ddot{w} \\ \ddot{\theta}_X \\ \ddot{\theta}_Z \end{Bmatrix}+\Omega\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{dY} \\ 0 & 0 & I_{dY} & 0 \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{w} \\ \dot{\theta}_X \\ \dot{\theta}_Z \end{Bmatrix} \quad (85)$$

The first matrix is the mass matrix and the second is the gyroscopic matrix.

4.2.2. Shaft:

The shaft is modeled by beam elements of constant circular section. The classic finite element used has 2 nodes and 4 degrees of freedom per node; the elementary matrices therefore have 8 degrees of freedom. The slope-displacement relationships are:

$$\theta_X=\frac{\partial w}{\partial Y} \quad (86)$$

$$\theta_Z=-\frac{\partial u}{\partial Y} \quad (87)$$

and the vector of nodal displacements is:

$$\delta=\left\{u_1,w_1,\theta_{X1},\theta_{Z1},u_2,w_2,\theta_{X2},\theta_{Z2}\right\}^t \quad (88)$$

In addition, we define the vectors δu, δw such that:

$$\delta u=\left\{u_1,\theta_{Z1},u_2,\theta_{Z2}\right\}^t \quad (89)$$

$$\delta w=\left\{w_1,\theta_{X1},w_2,\theta_{X2}\right\}^t \quad (90)$$

and we construct the characteristic matrices from:

$$u=N_1(Y)\delta u \quad (91)$$

$$w=N_2(Y)\delta w \quad (92)$$

where $N_1(Y)$ and $N_2(Y)$ are the classical shape functions of a beam in bending:

$$N_1(Y)=\left[1-\frac{3Y^2}{L^2}+\frac{2Y^3}{L^3};-Y+\frac{2Y^2}{L}-\frac{Y^3}{L^2};\frac{3Y^2}{L^2}-\frac{2Y^3}{L^3};\frac{Y^2}{L}-\frac{Y^3}{L^3}\right] \quad (93)$$

$$N_2(Y)=\left[1-\frac{3Y^2}{L^2}+\frac{2Y^3}{L^3};Y-\frac{2Y^2}{L}+\frac{Y^3}{L^2};\frac{3Y^2}{L^2}-\frac{2Y^3}{L^3};-\frac{Y^2}{L}+\frac{Y^3}{L^3}\right] \quad (94)$$

The kinetic energy is obtained from expression (36) which gives:

$$T_s = \frac{\rho A}{2}\int_0^L \left[\delta\dot{u}^t N_1^t N_1 \delta\dot{u} + \delta\dot{w}^t N_2^t N_2 \delta\dot{w}\right] dY + \frac{\rho I}{2}\left[\delta\dot{u}^t \frac{dN_1^t}{dY}\frac{dN_1}{dY}\delta\dot{u} + \delta\dot{w}^t \frac{dN_2^t}{dY}\frac{dN_2}{dY}\delta\dot{w}\right] dY$$
$$\ldots\ldots -2\rho I\Omega\int_0^L \delta\dot{u}^t \frac{dN_1^t}{dY}\frac{dN_2}{dY}\delta w dY + \rho I L \Omega^2 \quad (95)$$

and by substituting (93) and (94) as well as their derivatives into (95) we arrive at the compact form:

$$T_s = \frac{1}{2}\delta\dot{u}^t M_1 \delta\dot{u} + \frac{1}{2}\delta\dot{w}^t M_2 \delta\dot{w} + \frac{1}{2}\delta\dot{u}^t M_3 \delta\dot{u} + \frac{1}{2}\delta\dot{w}^t M_4 \delta\dot{w} + \Omega\delta\dot{u}^t M_5 \delta w + \rho I L \Omega^2 \quad (96)$$

where the matrices M_1 and M_2 are the classic mass matrices, M_3 and M_4 represent the secondary effect of inertia of rotation of the section in relation to a diameter and M_5 the gyroscopic effect. The last term which is a constant has a zero contribution when applying the Lagrange equations to (96); so:

$$\frac{d}{dt}\left(\frac{\partial T_s}{\partial \dot{\delta}}\right) - \frac{\partial T_s}{\partial \delta} = \left(M + M_s\right)\ddot{\delta} + C\dot{\delta} \quad (97)$$

where M and M_s are deduced respectively from M_1, M_2 and M_3, M_4 and where C comes from M_5. The matrices are:

$$C = \frac{\rho I \Omega}{15L}\begin{bmatrix} 0 & -36 & -3L & 0 & 0 & 36 & -3L & 0 \\ 36 & 0 & 0 & -3L & -36 & 0 & 0 & -3L \\ 3L & 0 & 0 & -4L^2 & -3L & 0 & 0 & L^2 \\ 0 & 3L & 4L^2 & 0 & 0 & -3L & -L^2 & 0 \\ 0 & 36 & 3L & 0 & 0 & -36 & 3L & 0 \\ -36 & 0 & 0 & 3L & 36 & 0 & 0 & 3L \\ 3L & 0 & 0 & L^2 & -3L & 0 & 0 & -4L^2 \\ 0 & 3L & -L^2 & 0 & 0 & -3L & 4L^2 & 0 \end{bmatrix} \quad (98)$$

$$M_s = \frac{\rho I}{30L}\begin{bmatrix} 36 & 0 & 0 & -3L & -36 & 0 & 0 & -3L \\ 0 & 36 & 3L & 0 & 0 & -36 & 3L & 0 \\ 0 & 3L & 4L^2 & 0 & 0 & -3L & -L^2 & 0 \\ -3L & 0 & 0 & 4L^2 & 3L & 0 & 0 & -L^2 \\ -36 & 0 & 0 & 3L & 36 & 0 & 0 & 3L \\ 0 & -36 & -3L & 0 & 0 & 36 & -3L & 0 \\ 0 & 3L & -L^2 & 0 & 0 & -3L & 4L^2 & 0 \\ -3L & 0 & 0 & -L^2 & 3L & 0 & 0 & 4L^2 \end{bmatrix} \quad (99)$$

$$M=\frac{\rho AL}{420}\begin{bmatrix} 156 & 0 & 0 & -22L & 54 & 0 & 0 & 13L \\ 0 & 156 & 22L & 0 & 0 & 54 & -13L & 0 \\ 0 & 22L & 4L^2 & 0 & 0 & 13L & -3L^2 & 0 \\ -22L & 0 & 0 & 4L^2 & -13L & 0 & 0 & -3L^2 \\ 54 & 0 & 0 & -13L & 156 & 0 & 0 & 22L \\ 0 & 54 & 13L & 0 & 0 & 156 & -22L & 0 \\ 0 & -13L & -3L^2 & 0 & 0 & -22L & 4L^2 & 0 \\ 13L & 0 & 0 & -3L^2 & 22l & 0 & 0 & 4L^2 \end{bmatrix} \tag{100}$$

The strain energy is obtained from expression (50) which gives:

$$U_s=\frac{EI}{2}\int_0^L\left[\delta u^t\,\frac{d^2N_1^t}{dY^2}\frac{d^2N_1}{dY^2}\delta u+\delta w^t\,\frac{d^2N_2^t}{dY^2}\frac{d^2N_2}{dY^2}\delta w\right]dY \tag{101}$$

After integration, we have in compact form:

$$U_s=\frac{1}{2}\delta u^t K_1\delta u+\frac{1}{2}\delta w^t K_2\delta w \tag{102}$$

where K_1 and K_2 are the classical stiffness matrices.

It is frequently necessary to take into account the shear effect which is characterized by the quantity:

$$a=\frac{12EI}{GA_rL^2} \tag{103}$$

where G the shear modulus is:

$$G=\frac{E}{2\left(1+\nu\right)} \tag{104}$$

ν the Poisson's ratio and $A_r \cong A$, the reduced section.

The shear effect obviously modifies the classical matrix which can be transformed to also include shear. The stiffness matrix then has the expression:

$$K=\frac{EI}{(1+a)L^2}\begin{bmatrix} 12 & 0 & 0 & -6L & -12 & 0 & 0 & -6L \\ 0 & 12 & 6L & 0 & 0 & -12 & 6L & 0 \\ 0 & 6L & (4+a)L^2 & 0 & 0 & -6L & (2-a)L^2 & 0 \\ -6L & 0 & 0 & (4+a)L^2 & 6L & 0 & 0 & (2-a)L^2 \\ -12 & 0 & 0 & 6L & 12 & 0 & 0 & 6L \\ 0 & -12 & -6L & 0 & 0 & 12 & -6L & 0 \\ 0 & 6L & (2-a)L^2 & 0 & 0 & -6L & (4+a)L^2 & 0 \\ -6l & 0 & 0 & (2-a)L^2 & 6L & 0 & 0 & (4+a)L^2 \end{bmatrix} \quad (105)$$

When the shear effect is not considered, $a = 0$.

4.2.3. Bearings:

Stiffness and damping characteristics relate forces to displacements and speeds. The influence of slopes and moments is usually neglected and taking into account (53) we have:

$$\begin{Bmatrix} F_u \\ F_{\theta_x} \\ F_w \\ F_{\theta_z} \end{Bmatrix} = -\begin{bmatrix} k_{XX} & 0 & k_{XZ} & 0 \\ 0 & 0 & 0 & 0 \\ k_{ZX} & 0 & k_{ZZ} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} u \\ \theta_X \\ w \\ \theta_Z \end{Bmatrix} - \begin{bmatrix} c_{XX} & 0 & c_{XZ} & 0 \\ 0 & 0 & 0 & 0 \\ c_{ZX} & 0 & c_{ZZ} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{u} \\ \dot{\theta}_X \\ \dot{w} \\ \dot{\theta}_Z \end{Bmatrix} \quad (106)$$

The first matrix is a stiffness matrix, the second a viscous damping matrix. These matrices are generally not symmetrical (hydrodynamic bearings) and the terms can vary significantly depending on the rotation speed.

4.2.4. Unbalance:

The general expression for the kinetic energy due to an unbalance is given in equation (56). Applying the Lagrange equations gives:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\delta}}\right) - \frac{\partial T}{\partial \delta} = -m_b d \Omega^2 \begin{bmatrix} \sin\Omega t \\ \cos\Omega t \end{bmatrix} \quad (107)$$

with $\delta = \{u, w\}^t$

.2.5. Single rotor equations:

Applying Lagrange's equations gives the general equation of motion:

$$M\ddot{\delta} + C(\Omega)\dot{\delta} + K\delta = F(t) \quad (108)$$

with: $F(t) = F_1 + F_2\sin\Omega t + F_3\cos\Omega t + F_4\sin s\Omega t + F_5\cos s\Omega t$ (109)

where δ is the vector now containing all the nodal displacements, M is the symmetric mass matrix, $C(\Omega)$ is a non-symmetric matrix, function of Ω, including the gyroscopic (anti-symmetric) effect, and the damping characteristics of the frequently non-symmetrical bearings. $F(t)$ is the vector including constant forces (F_1) including those due to gravity, unbalance effects (F_2, F_3) and asynchronous forces (F_4, F_5).

4.2.6. Resolution by pseudo-modal method:

The pseudo-modal method, used here, allows a significant reduction in the order of the system of equations. Generally a few hundred degrees of freedom are sufficient to model a rotor and we are interested in around ten modes: the reduction factor is therefore significant (≈ 20 to 30). This reduction leads, of course, to an appreciable gain in calculation time and occupied memory space. In addition, this method allows the taking into account of modal damping known experimentally. Numerous tests have shown that the accuracy of this method is excellent; the error, compared to an exact numerical method, is less than a percent. A modal base is defined from the solutions of:

$$M\ddot{\delta} + K^{*}\delta = 0 \tag{110}$$

where M is the mass matrix, K^{*} is a stiffness matrix obtained from K where the terms of type k_{XZ}, k_{ZX} introduced by the bearings are deleted in order to preserve the symmetry. The lowest frequencies and modes are obtained by a method of simultaneous iterations. The first n modes φ_1... φ_n form a matrix:

$$\varphi = (\varphi_1 ... \varphi_n) \tag{111}$$

used as a base change:

$$\delta = \varphi p \tag{112}$$

where p is the vector of modal variables.

By carrying out the change of base (112) and premultiplying (108) by φ^t, it comes:

$$\varphi^t M \varphi \ddot{p} + \varphi^t C(\Omega)\varphi \dot{p} + \varphi^t K \varphi p = \varphi^t F(t) \tag{113}$$

At this point, modal damping can be introduced. By analogy with a system with one degree of freedom (mass, spring, viscous damper), we define terms c_i such as:

$$c_i = 2\alpha_i \sqrt{\varphi_i^t K \varphi_i \varphi_i^t M \varphi_i} \tag{114}$$

which will be added to the diagonal of the matrix $\varphi^t C(\Omega)\varphi$. The values of α_i, modal damping factors, are essentially estimated by the machine construction specialist or known experimentally.

By assuming:

$$m = \varphi^t M \varphi \tag{115}$$

$$k = \varphi^t K \varphi \tag{116}$$

$$c = \varphi^t C\left(\Omega\right)\varphi + C_i \tag{117}$$

where C_i is a diagonal matrix containing terms of type (114),

$$\mathrm{f} = \varphi^t F(t) \tag{118}$$

equation (246) is written: $m\ddot{p} + c\dot{p} + kp = \mathrm{f}$ (119)

4.2.7. Natural frequencies. Campbell diagram:

For a fixed rotation speed Ω, the solutions of (119), without a second member, are of the form:

$$p = Pe^{rt} \tag{120}$$

which give :

$$\left[r^2 m + rc + k\right]p = 0 \tag{121}$$

and this equation can be written in the form:

$$\begin{bmatrix} 0 & I \\ k^{-1}m & -k^{-1}c \end{bmatrix}\begin{Bmatrix} rP \\ P \end{Bmatrix} = \frac{1}{r}\begin{Bmatrix} rP \\ P \end{Bmatrix} \tag{122}$$

The problem with eigenvalues and eigenvectors defined in (122) is solved, by the QR algorithm, for a few rotation speeds (generally around ten). As the matrix of (122) is a real matrix, the eigenvectors and eigenvalues are conjugate complex quantities. Consider for example the first two eigenvalues:

$$r_1 = -\frac{\alpha_1\omega_1}{\sqrt{1-\alpha_1^2}} \pm j\omega_1 \tag{123}$$

and the corresponding eigenvectors:

$$\mathrm{A}_1 = \mathrm{Re}_1 \pm j\,\mathrm{Im}_1 \tag{124}$$

The physical meaning of the modes is not obvious. The solution of free motion from expressions (123) and (124) is:

$$\delta=\left(\mathrm{Re}_1+j\,\mathrm{Im}_1\right)\exp\left[\left(-\frac{\alpha_1\omega_1}{\sqrt{1-\alpha_1^2}}+j\omega_1\right)t\right]+\left(\mathrm{Re}_1+j\,\mathrm{Im}_1\right)\exp\left[\left(-\frac{\alpha_1\omega_1}{\sqrt{1-\alpha_1^2}}-j\omega_1\right)t\right] \tag{125}$$

which can also be written:

$$\delta=\exp\left[\left(-\frac{\alpha_1\omega_1}{\sqrt{1-\alpha_1^2}}\right)t\right]\left[\left(\mathrm{Re}_1+j\,\mathrm{Im}_1\right)\left(\cos\omega_1 t+j\sin\omega_1 t\right)+\left(\mathrm{Re}_1-j\,\mathrm{Im}_1\right)\left(\cos\omega_1 t-j\sin\omega_1 t\right)\right] \tag{126}$$

either:

$$\delta=2\exp\left[\left(-\frac{\alpha_1\omega_1}{\sqrt{1-\alpha_1^2}}\right)t\right]\left[\mathrm{Re}_1\cos\omega_1 t-\mathrm{Im}_1\sin\omega_1 t\right] \tag{127}$$

The expression (127) does not include a complex quantity and the exponential (for a stable system) tends towards zero as time increases. The quantity:

$$\frac{\delta}{2\exp\left[\left(-\frac{\alpha_1\omega_1}{\sqrt{1-\alpha_1^2}}\right)t\right]}=\mathrm{Re}_1\cos\omega_1 t-\mathrm{Im}_1\sin\omega_1 t \tag{128}$$

is the mode. For a node of the structure, the components of the displacement can be put in the general form:

$$\begin{cases} u=u_c\cos\omega_1 t+u_s\sin\omega_1 t \\ w=w_c\cos\omega_1 t+w_s\sin\omega_1 t \end{cases} \tag{129}$$

and the orbits are the ellipses. The direction of precession, obtained from the positions of the node for $\omega_1 t=0$ and $\omega_1 t=\frac{\pi}{2}$, is given by the sign of the quantity:

$$S=w_c u_s-u_c w_s \tag{130}$$

4.2.8. Response to unbalance forces:

The equations to solve are:

$$m\ddot{p}+c\dot{p}+kp=\mathrm{f}_2\sin\Omega t+\mathrm{f}_3\cos\Omega t \tag{131}$$

with: $\mathrm{f}_2=\varphi^t F_2$ and $\mathrm{f}_3=\varphi^t F_3$

F_2 and F_3 are defined in (109). The solutions are search for in the form:

$$p = p_2 \sin \Omega t + p_3 \cos \Omega t \qquad (132)$$

and the identification of the terms in $\sin \Omega t$ and $\cos \Omega t$ leads to:

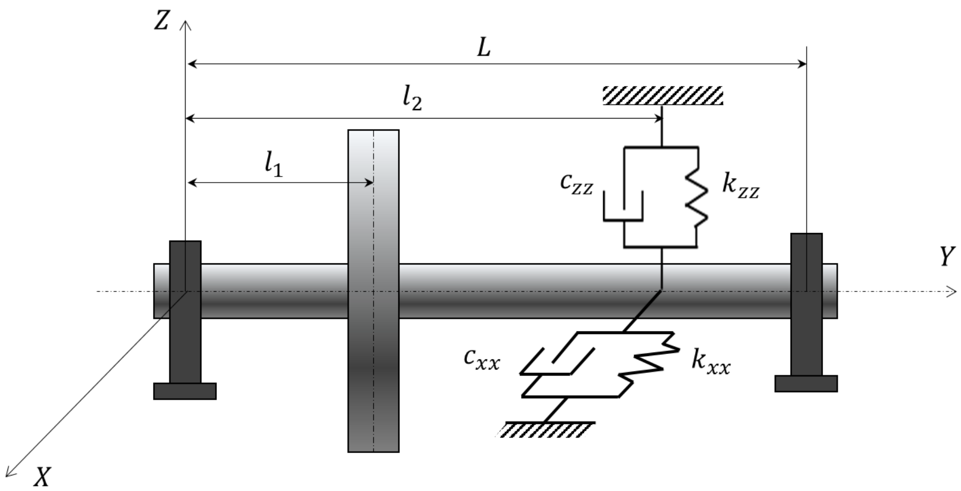
$$\begin{bmatrix} k - m\Omega^2 & -\Omega c \\ \Omega c & k - m\Omega^2 \end{bmatrix} \begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} \mathrm{f}_2 \\ \mathrm{f}_3 \end{Bmatrix} \qquad (133)$$

System (133) must be solved for a large number of rotation speeds (of the order of 1,000) and the nodal displacements are obtained from (112):

$$\delta = \varphi\left(p_2(\Omega) \sin \Omega t + p_3(\Omega) \cos \Omega t \right) \qquad (134)$$

The displacements of a node have the same form as (129); the sense of precession is given by (130).

Solved exercise:



Write the equations of free motion, extract the natural frequencies and draw the Campbell diagram of the rotor with the following data:

Disk: Inner radius $R_1 = 0.01\, m$, outer radius $R_2 = 0.15\, m$, thickness $h = 0.03\, m$, density $\rho = 7800\, kg/m^3$ and $L_1 = L/3$.

Shaft: Length $L = 0.4m$, Radius of straight section $R_1 = 0.01\, m$, density $\rho = 7800\, kg/m^3$ and $E = 200\, GPa$.

Kinetic energy of the disk:

$$T_d = \frac{1}{2} M_d(\dot{u}^2 + \dot{w}^2) + \frac{1}{2} I_{dx}\left(\dot{\theta}_z^2 + \dot{\theta}_x^2\right) + \frac{1}{2} I_{dy}\left(\Omega^2 + 2\Omega\dot{\theta}_z\theta_x\right)$$

To do this, we start by calculating the mass of the disk and the moments of inertia:

$$M_d = \pi\rho h(R_2^2 - R_1^2) = 16.47\ kg$$

$$I_{dy} = \frac{M_d}{2}(R_2^2 + R_1^2) = 0.1861\ kg.m^2$$

$$I_{dx} = \frac{M_d}{12}(3R_2^2 + 3R_1^2 + h^2) = 0.1342\ kg.m^2$$

The expressions of displacements in the X and Z directions:

$$\begin{cases} u(y,t) = f(y)q_1(t) = f(y)q_1 \\ w(y,t) = f(y)q_2(t) = f(y)q_2 \end{cases}$$

$$\begin{cases} \theta_X = \dfrac{\partial w}{\partial y} = \dfrac{df(y)}{dy}q_2 = g(y)q_2 \\ \theta_Z = \dfrac{\partial u}{\partial y} = -\dfrac{df(y)}{dy}q_1 = -g(y)q_1 \end{cases}$$

$$\begin{cases} \dfrac{\partial^2 w}{\partial y^2} = \dfrac{d^2 f(y)}{dy^2}q_2 = h(y)q_2 \\ \dfrac{\partial^2 u}{\partial y^2} = \dfrac{d^2 f(y)}{dy^2}q_1 = h(y)q_1 \end{cases}$$

If we take the expressions of the displacement those of the first mode of a beam, of constant section, in bending and supported at both ends as:

$$f(y) = \sin\frac{\pi y}{L}$$

So:

$$\begin{cases} u(y,t) = f(y)q_1 = \sin\dfrac{\pi y}{L}q_1 = 0.866q_1 \\ w(y,t) = f(y)q_2 = \sin\dfrac{\pi y}{L}q_2 = 0.866q_2 \end{cases}$$

with: $y = \frac{L}{3}$

and:

$$\begin{cases} \theta_X = \dfrac{\partial w}{\partial y} = \dfrac{df(y)}{dy}q_2 = \dfrac{\pi}{L}\cos\dfrac{\pi y}{L}q_2 = -53.42q_2 \\ \theta_Z = \dfrac{\partial u}{\partial y} = -\dfrac{df(y)}{dy}q_1 = -\dfrac{\pi}{L}\cos\dfrac{\pi y}{L}q_1 = 30.84q_1 \end{cases}$$

we will have after simplifications:

$$T_d = 7.2(\dot{q}_1^2 + \dot{q}_2^2) + 0.093\Omega^2 - 15.42\Omega q_2\dot{q}_1$$

Kinetic energy of the shaft:

$$T_s = \frac{\rho S}{2}\int_0^L (\dot{u}^2 + \dot{w}^2)dy + \frac{\rho I_a}{2}\int_0^L \left(\dot{\theta}_z^2 + \dot{\theta}_x^2\right)dy + \rho I_a L\Omega^2 + 2\rho I_a \Omega \int_0^L \dot{\theta}_z\, \theta_x dy$$

$$I_s = \frac{\pi R_1^4}{4} = 7.85 \times 10^{-9} m^4$$

$$A = \pi R_1^2 = 3.14 \times 10^{-4} m^2$$

$$\dot{u}(y,t) = \sin\frac{\pi y}{L}\dot{q}_1$$

$$\dot{w}(y,t) = \sin\frac{\pi y}{L}\dot{q}_2$$

$$\dot{\theta}_X = \frac{\pi}{L}\cos\frac{\pi y}{L}\dot{q}_2$$

$$\dot{\theta}_Z = -\frac{\pi}{L}\cos\frac{\pi y}{L}\dot{q}_1$$

$$T_s = \frac{\rho S}{2}\int_0^L \left(\left(\sin\frac{\pi y}{L}\dot{q}_1\right)^2 + \left(\sin\frac{\pi y}{L}\dot{q}_2\right)^2\right)dy + \frac{\rho I_a}{2}\int_0^L \left(\left(\frac{\pi}{L}\cos\frac{\pi y}{L}\dot{q}_1\right)^2 + \left(\frac{\pi}{L}\cos\frac{\pi y}{L}\dot{q}_2\right)^2\right)dy$$

$$+ \rho I_a L\Omega^2 - 2\rho I_a \Omega \int_0^L \frac{\pi}{L}\cos\frac{\pi y}{L}\dot{q}_1 \times \frac{\pi}{L}\cos\frac{\pi y}{L} q_2\, dy$$

$$T_s = \frac{\rho S}{2}(\dot{q}_1^2 + \dot{q}_2^2)\int_0^L \left(\sin\frac{\pi y}{L}\right)^2 dy + \frac{\rho I_a \pi^2}{2L^2}(\dot{q}_1^2 + \dot{q}_2^2)\int_0^L \left(\cos\frac{\pi y}{L}\right)^2 dy + \rho I_a L\Omega^2$$

$$- \frac{2\rho I_a \Omega \pi^2}{L^2}\dot{q}_1 q_2 \int_0^L \left(\cos\frac{\pi y}{L}\right)^2 dy$$

$$T_s = \frac{\rho S L}{4}(\dot{q}_1^2 + \dot{q}_2^2) + \frac{\rho I_a \pi^2}{4L}(\dot{q}_1^2 + \dot{q}_2^2) + \rho I_a L\Omega^2 - \frac{\rho I_a \Omega \pi^2}{L}\dot{q}_1 q_2$$

$$T_s = 0.245(\dot{q}_1^2 + \dot{q}_2^2) + 2.45 \times 10^{-5}\Omega^2 - 1.51 \times 10^{-3}\Omega \dot{q}_1 q_2$$

So the total kinetic energy:

$$T = 7.445(\dot{q}_1^2 + \dot{q}_2^2) + 0.093\Omega^2 - 15.42\Omega \dot{q}_1 q_2$$

We notice that the kinetic energy of the shaft has little influence on the total energy.

Potential energy of the shaft:

$$U_s = \frac{E I_a}{2}\int_0^L \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 dy$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\pi^2}{L^2}\sin\frac{\pi y}{L} q_1$$

$$\frac{\partial^2 w}{\partial y^2} = -\frac{\pi^2}{L^2}\sin\frac{\pi y}{L}q_2$$

$$U_s = \frac{EI_a\pi^4}{2L^4}(q_1+q_2)^2\int_0^L\left(\sin\frac{\pi y}{L}\right)^2 dy = \frac{EI_a\pi^4}{4L^3}(q_1+q_2)^2 = 5.97\times 10^5(q_1+q_2)^2$$

Lagrangien:

$$L = T - U = 7.445(\dot{q}_1^2+\dot{q}_2^2) + 0.093\Omega^2 - 15.42\Omega\dot{q}_1 q_2 - 5.97\times 10^5(q_1+q_2)^2$$

Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0$$

$$\frac{\partial L}{\partial \dot{q}_1} = 14.89\dot{q}_1 - 15.42\Omega q_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = 14.89\ddot{q}_1 - 15.42\Omega\dot{q}_2$$

$$\frac{\partial L}{\partial q_1} = -11.94\times 10^5(q_1+q_2)$$

$$14.89\ddot{q}_1 - 15.42\Omega\dot{q}_2 + 11.94\times 10^5(q_1+q_2) = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = 0$$

$$\frac{\partial L}{\partial \dot{q}_2} = 14.89\dot{q}_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = 14.89\ddot{q}_2$$

$$\frac{\partial L}{\partial q_2} = -15.42\Omega\dot{q}_1 - 11.94\times 10^5(q_1+q_2)$$

$$14.89\ddot{q}_2 + 15.42\Omega\dot{q}_1 + 11.94\times 10^5(q_1+q_2) = 0$$

Equations of motion:

$$\begin{cases}14.89\ddot{q}_1 - 15.42\Omega\dot{q}_2 + 11.94\times 10^5(q_1+q_2) = 0\\ 14.89\ddot{q}_2 + 15.42\Omega\dot{q}_1 + 11.94\times 10^5(q_1+q_2) = 0\end{cases}$$

$$\begin{bmatrix}14.89 & 0\\ 0 & 14.89\end{bmatrix}\begin{Bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{Bmatrix} + \Omega\begin{bmatrix}0 & -15.42\\ 15.42 & 0\end{bmatrix}\begin{Bmatrix}\dot{q}_1\\ \dot{q}_2\end{Bmatrix} + \begin{bmatrix}11.94\times 10^5 & 11.94\times 10^5\\ 11.94\times 10^5 & 11.94\times 10^5\end{bmatrix}\begin{Bmatrix}q_1\\ q_2\end{Bmatrix} = 0$$

The solutions of the equations are in the form:

$$\begin{cases}q_1 = Q_1 e^{rt}\\ q_2 = Q_2 e^{rt}\end{cases}$$

By replacing these solutions in the equations of motion, we will have:

$$\begin{bmatrix} 14.89r^2 + 11.94 \times 10^5 & -15.42\Omega r + 11.94 \times 10^5 \\ 15.42\Omega r + 11.94 \times 10^5 & 14.89r^2 + 11.94 \times 10^5 \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} = 0$$

The solution is in the matrix:

$$221.71r^4 + (237.77\Omega^2 + 35557320)r^2 = 0$$

From where:

$$r^2 = -\frac{237.77\Omega^2 + 35557320}{221.71}$$

At $\Omega = 0$, the roots r_{10} and r_{20} are:

$$r_{10}^2 = r_{20}^2 = j^2\omega_{10}^2 = j^2\omega_{20}^2 = -\frac{35557320}{221.71}$$

The frequencies are:

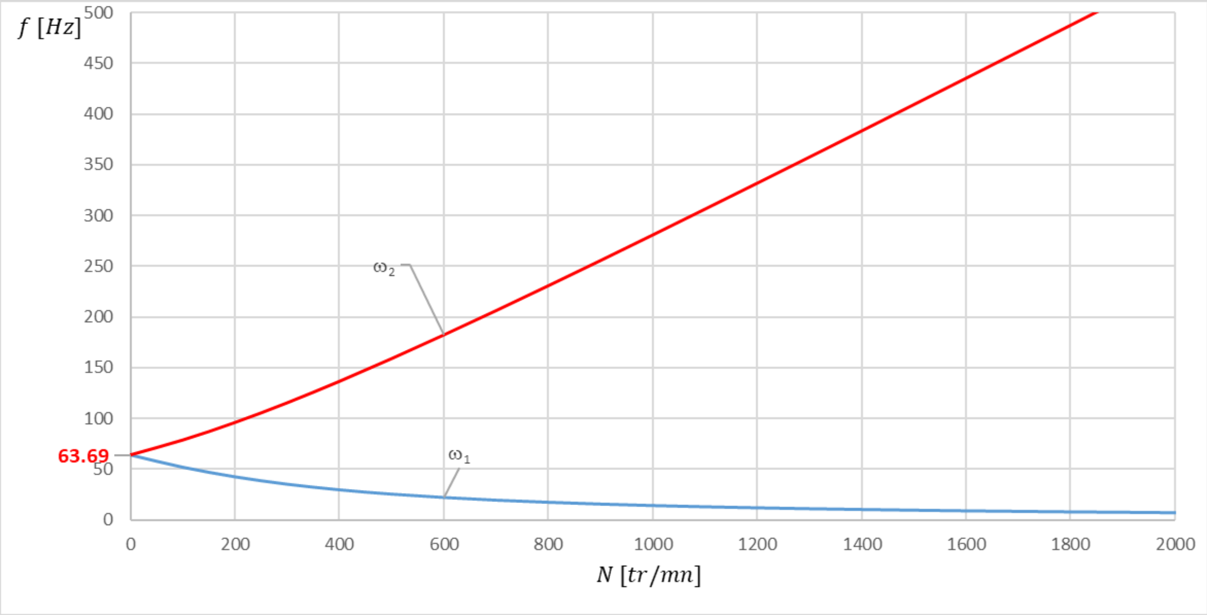
$$\omega_{10} = \omega_{20} = \sqrt{\frac{35557320}{221.71}} = 400\, rd/s$$

In rotation ($\Omega \neq 0$), the roots and the corresponding frequencies are:

$$\begin{cases} r_1^2 = -\left[\omega_{10}^2 + 127.5\Omega^2\left(1 - \sqrt{1 + \dfrac{2509.89}{\Omega^2}}\right)\right] = j^2\omega_1^2 \\ r_2^2 = -\left[\omega_{10}^2 + 127.5\Omega^2\left(1 + \sqrt{1 + \dfrac{2509.89}{\Omega^2}}\right)\right] = j^2\omega_2^2 \end{cases}$$

$$\begin{cases} \omega_1 = \sqrt{\omega_{10}^2 + 127.5\Omega^2\left(1 - \sqrt{1 + \dfrac{2509.89}{\Omega^2}}\right)} \\ \omega_2 = \sqrt{\omega_{10}^2 + 127.5\Omega^2\left(1 + \sqrt{1 + \dfrac{2509.89}{\Omega^2}}\right)} \end{cases}$$

The corresponding Campbell diagram is:



$f\ [Hz]$
500
450
400
350
300
250
200
150
100
63.69
50
0
0
200
400
600
800
1000
1200
1400
1600
1800
2000
$N\ [tr/mn]$
ω_2
ω_1

5. Jeffcott Rotor

The Jeffcott rotor is equipped with a disc mounted on the central axis of the uniform shaft. The mass of the shaft can be included in this analysis by placing ½ of the total mass of the shaft as acting at the center of the disk. This assumption is accurate to <1% error regardless of tree size.

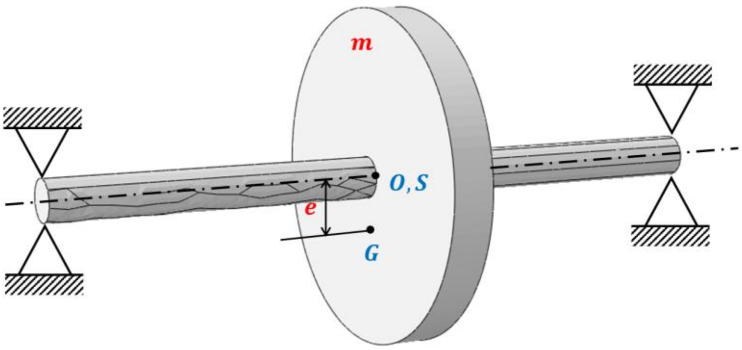


Figure 26. Jeffcott rotor model.

The single mass Jeffcott rotor mounted on a massless elastic uniform shaft. The rotor mass m is considered concentrated at the center of the rotor. The center of mass of the rotor is displaced from the elastic axis of the rotor by the distance e .

Consider the Jeffcott rotor shown (figure 26), a simple rotating elastic shaft carrying an unbalanced disk in its middle. In the Jeffcott rotor, the effects of gyroscopic or rotary inertia are neglected, since the disk is located in the middle of the shaft, allowing only its translational movement.

The mass of the shaft is neglected compared to that of the disk, and the flexibility of the (rigid) bearings is neglected compared to that of the shaft. We assume that the external damping force, which may be the air friction opposed to the rotation of the shaft, is proportional to the linear velocity of the geometric center of the disk S .

When the rotor rotates, the disk moves in the transverse direction, hence the bending deformation of the shaft. Under these conditions, the disk is subjected to the centrifugal force F_c and the restoring force F_a, where:

$$\begin{cases} F_c = m\omega^2 (e + y) \\ F_a = ky \end{cases} \tag{135}$$

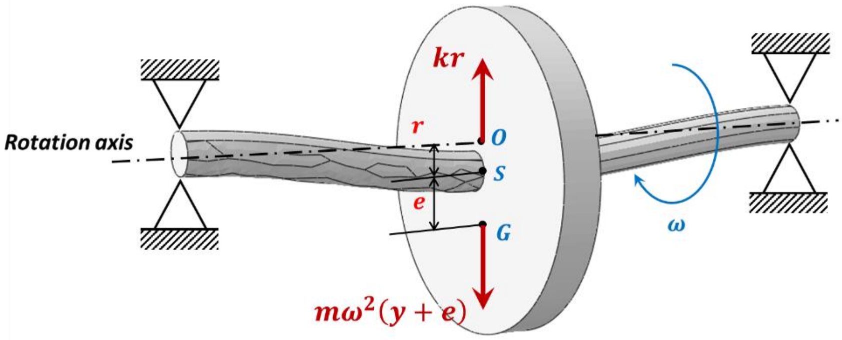


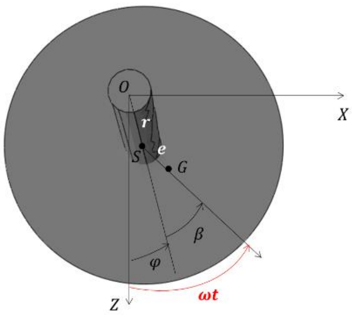
Figure 27. Jeffcott rotor in bending.

The figure (27) shows the Jeffcott rotor rotating, with coordinates describing its motion. The center of mass of the unbalanced disk is at G . Point S locates the geometric center of the disk. Thus, the amount of static unbalance is denoted by $e = SG$, and the bending deflection of the shaft due to dynamic loads is $OS = r$. The forces of gravity are neglected in this analysis. They are insignificant compared to the dynamic loads in most high-speed machines.

The shaft has bending stiffness k , the disk has mass m , and the air drag on the swirling disk and shaft is approximated by a viscous damping coefficient c . The dynamic system has three degrees of freedom; an assumption of constant speed reduces them to two.

The equations of rotor motion can be expressed in terms of Cartesian coordinates of the elastic axis (X , Z).

$$\begin{cases} m\ddot{X} + c\dot{X} + kX = me\omega^2 \cos(\omega t - \varphi) \\ m\ddot{Z} + c\dot{Z} + kZ = me\omega^2 \sin(\omega t - \varphi) \end{cases} \quad (136)$$



Let's introduce the complex rotation radius:

$$r = X + jZ \quad (137)$$

We combine the equations to:

$$m\ddot{r} + c\dot{r} + kr = me\omega^2 e^{j\omega t} \quad (138)$$

if $r = r_0 e^{\lambda t}$, the equation of motion becomes:

$$\left(m\lambda^2 + c\lambda + k\right) r_0 = 0$$

So:

$$m\lambda^2 + c\lambda + k = 0 \tag{139}$$

$$\lambda_{1,2} = \left(-\zeta \pm \sqrt{1-\zeta^2}\right)\omega_n \tag{140}$$

ω_n the natural frequency of the undamped rotor and ζ is the damping rate:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \text{and} \qquad \zeta = \frac{c}{2\sqrt{km}}$$

If we set $r = r_u e^{j\omega t}$ where ω is the swirl frequency, we look for the steady state equation:

$$r_u = \frac{me\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + \left(j\omega c\right)^2}} = \frac{ev^2}{\sqrt{\left(1-v^2\right)^2 + \left(j2\zeta v\right)^2}} \tag{145}$$

with: $v = \dfrac{\omega}{\omega_n}$

The dimensionless amplitude and phase are given by:

$$\begin{cases} \dfrac{|r_u|}{e} = \dfrac{v^2}{\sqrt{\left(1-v^2\right)^2 + \left(j2\zeta v\right)^2}} \\ \varphi = \tan^{-1}\left(\dfrac{2\zeta v}{1-v^2}\right) \end{cases} \tag{146}$$

5.1. Critical rotor speed:

The critical speed (in tr/mn) of the rotor can be determined as follows:

$$N_{cr} = 60 \times f_n = \frac{60\omega_n}{2\pi} = \frac{30\omega_n}{\pi} \tag{147}$$

Depending on the rotation speed of the rotor, the following cases can be observed:

- $N < N_{cr}$: the eccentricity e and the deformation of the shaft r_u are in the same side (figure 27) (the geometric center of the disk S is between the center of masses G and the center of rotation O).
- $N \cong N_{cr}$: the resonance of the rotor, in this case the deformation of the shaft becomes high (if there is no damping it tends towards infinity).

- $N > N_{cr}$: the eccentricity e and the deformation of the shaft r_u are not in the same side (figure 28) (the center of masses G is between the geometric center of the disk S and the center of rotation O).

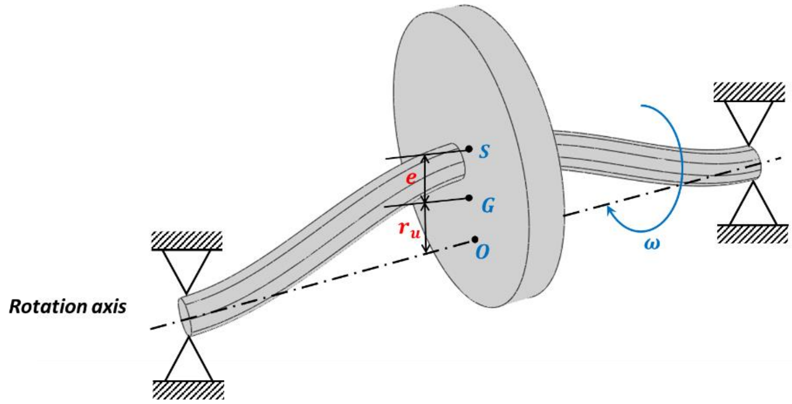
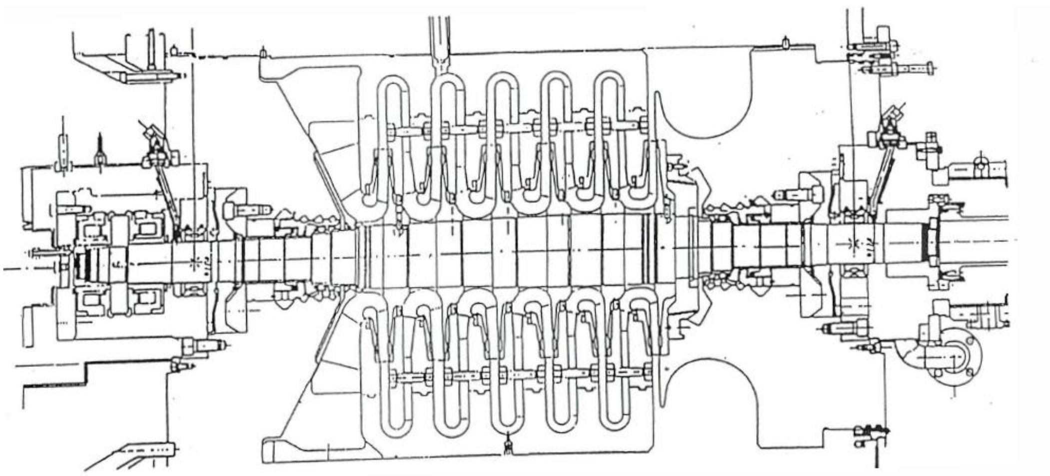


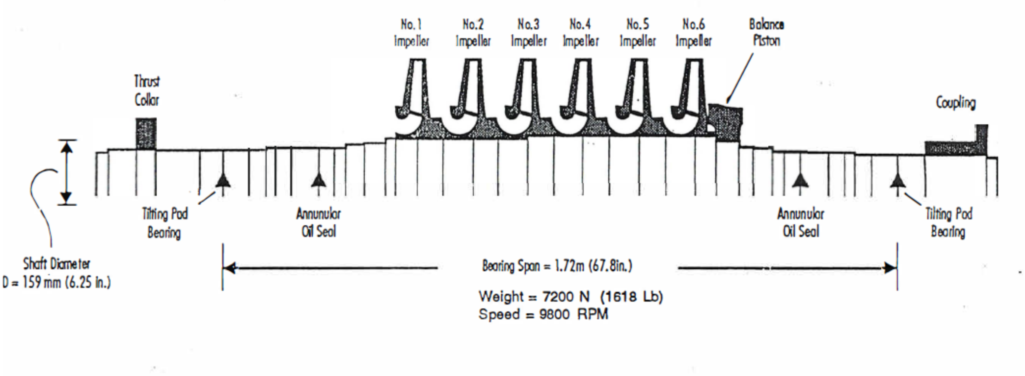
Figure 28. Shaft deformation when $N > N_{cr}$.

Solved exercise:

A six-stage compressor, shown by the figures above, has all stages located between the bearings and is fairly uniform along the shaft. The rotor has the following properties: $D = 159\ mm$, $L = 1.72\ m$, $E = 207000\ N/mm^2$; Poids $7200\ N$, speed $9800\ RPM$.

Determine the first critical speed in rpm. Compare this speed with the rotor speed. We give $k = \frac{48\ EI}{L^3}$.





The first critical speed is:

$$N_{cr} = \frac{30\omega_n}{\pi} = \frac{30}{\pi}\sqrt{\frac{2k}{m}} = \frac{30}{\pi}\sqrt{\frac{96EI}{mL^2}} = \frac{30}{\pi}\sqrt{\frac{96\pi ED^4}{64mL^2}} = \sqrt{\frac{1350ED^4}{\pi mL^2}}$$

$$= \sqrt{\frac{1350\times 207\times 10^9 \times 0.159^4}{\pi 720\times 1.72^2}} = 5166.27 \text{ tr/mn}$$

with: $g \cong 10\ m/s^2$.

Comparison of this speed with the rotor speed.

$$N > N_{cr}$$

Then the eccentricity e and the deformation of the shaft r_u are not in the same side.

6. Instability

The great majority of rotordynamic problems encountered involve synchronous whirl, i.e., response to unbalance. The remaining minority of problems involving nonsynchronous whirl or can be subdivided into three classifications:

- Supersynchronous Vs due to shaft misalignment (the dominant frequency is often twice shaft speed).
- Subsynchronous and supersynchronous Vs due to cyclic variations of parameters, mainly caused by loose bearing housings or shaft rubs, or by nonlinear force coefficients
- Nonsynchronous (usually subsynchronous) rotor whirling that becomes unstable, or has the potential to become unstable, typically when a certain speed called the threshold speed is reached.

Problems of the first and second classifications have solutions that are obvious: align the shafts, tighten the bearing housings, or eliminate the rub. Problems of the third classification, although relatively uncommon, have a history of causing some very expensive failures, with elusive causes and cures. This is the main classification of problems referred to as rotordynamic instability.

Note that rotordynamic instability occurs mostly in high-speed turbomachinery, but not in reciprocating internal combustion engines, as used in automobiles. Reciprocating machines are characterized by (1) lower speeds, (2) multiple interior bearing supports, and (3) high natural frequencies of the rotor (crankshaft). “High speed” and “lower speeds” are relative terms here. High speed could be only a few hundred revolutions per minute, provided it is significantly higher than a natural whirling frequency (eigenvalue) of the rotor–bearing system. Rotordynamic instability is manifested by shaft whirling, and the shaft will tend to whirl at its natural frequency (as modified by gyroscopic moments). Since it has already been said that instability frequencies are subsynchronous, it follows that they almost always occur when shaft speeds are higher than the natural whirling frequency.

Rotordynamic instability is a special case of the more general theory of dynamic instability, or instability of dynamic systems. Both the classical theory and experiments show that the amplitude of free V in a linear system grows exponentially with time if the damping is negative. In mathematical terms, the system is unstable if the real part of the eigenvalue is positive.

Rotordynamic instability is seldom produced by negative direct damping. Instead, it is usually produced by a follower force that is modeled by cross-coupled stiffness coefficients. Follower forces are tangential to the rotor whirl orbit, acting in the same direction as the instantaneous velocity and following the rotor around in its orbit. In the rare case where the magnitude of the follower force is proportional to the instantaneous whirl velocity, it is classified as a negative direct damping force, just as in classical V theory. More typically the force is proportional to the rotor displacement (instantaneous orbit radius), and therefore is classified as a cross-coupled stiffness force. The “cross-coupled” terminology comes from the form of the force expressions in a nonrotating $X - Z$ coordinate system. A rotor displacement in the X direction produces a force in the Z direction, and vice versa. Figure (29) shows how a tangential follower force F produced by cross-coupled stiffness coefficients $K_{XZ} = -K_{ZX}$ is resolved into F_X and F_Z components.

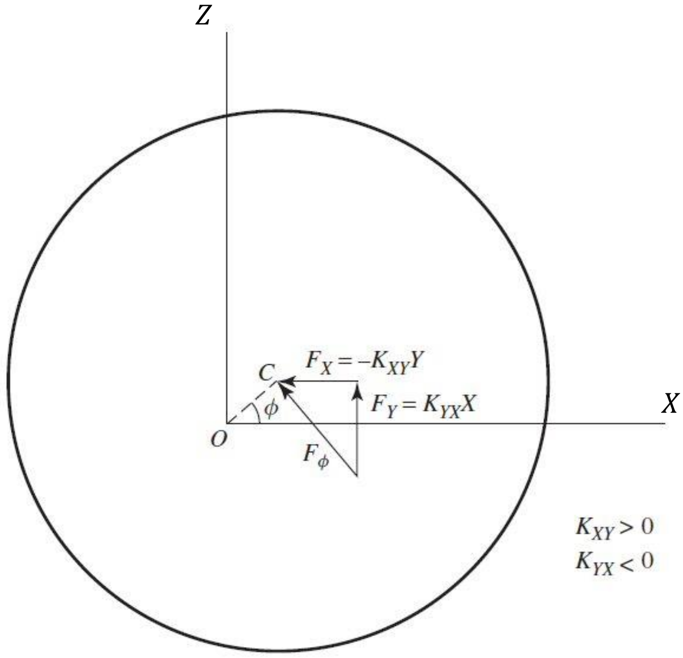


Figure 29. Cross-coupled stiffness representation of destabilizing force on a deflected rotor disk.

Forces acting on the rotor are usually fluid forces from bearings, seals, impellers, and turbine stages. These forces have radial and tangential (or normal to radial) components relative to the whirl orbit. The force component F_ϕ as shown in figure (30) is destabilizing to forward whirl, because it is acting in the direction of instantaneous velocity. Direct damping produces F_ϕ opposite to the whirl direction.

Most rotordynamic computer codes require F_r and F_ϕ to be converted into $X - Z$ (Newtonian coordinates). Thus, we have stiffness and damping force coefficients K_{XX}, K_{XZ}, C_{XX}, C_{XZ}, etc. The representation of forces on the rotor by these stiffness and damping coefficients implies that the forces are linearly proportional to the rotor displacement or velocity, which further implies that the rotor motions are small. The usefulness of this assumption for analysis is based on the fact that motions growing larger are generally unacceptable. If there is any interest in accepting or analyzing larger rotor motions they require nonlinear analysis.

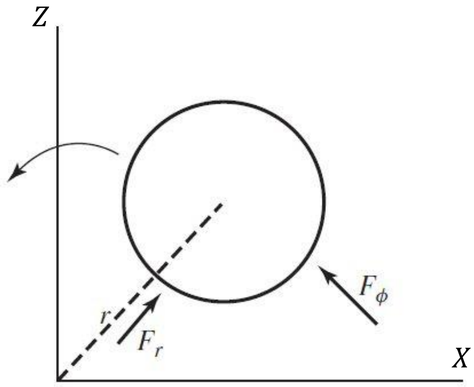


Figure 30. Components of a fluid force on a rotor.

Figure (31) illustrates Alford's force, a type of destabilizing force that can occur in axial-flow turbomachinery. It is the resultant of aerodynamic forces on the blades, produced by the variation of blade tip clearance around an unshrouded axial flow stage. A simple rotordynamics model to illustrate the destabilizing effect of Alford's force can be constructed by mounting the bladed disk (of mass m) on a flexible shaft midway between two hard-mounted bearings, i.e., a Jeffcott rotor with a bladed disk.

If the direct stiffness and damping properties are completely symmetrical (i.e., the same in X and Z directions), then the free motion of this rotor–bearing system is described by solutions to the following two coupled differential equations:

$$\begin{cases} m\ddot{X} + c\dot{X} + kX + K_{XZ}Z = 0 \\ m\ddot{Z} + c\dot{Z} + kZ + K_{ZX}X = 0 \end{cases} \tag{148}$$

where c is the external aerodynamic damping coefficient, k is the shaft stiffness, and the coefficients $K_{XZ} = -K_{ZX}$ produce Alford's force. Alford hypothesized that it would be proportional to the eccentricity (X or Z), the stage torque (T), and the efficiency factor β, and inversely proportional to the pitch diameter (D) and vane height (H) of the blades. Thus, the magnitude κ of the cross-coupled stiffness coefficients K_{XZ} and K_{XZ} in this case can be expressed as:

$$\kappa = \frac{\beta T}{DH} \tag{149}$$

Equations (148) are linear, homogeneous, and coupled, and have constant coefficients. Their general solution is:

$$\begin{cases} X = A_1 e^{st} \\ Z = A_2 e^{st} \end{cases} \tag{150}$$

where s is the eigenvalue and A_1, A_2 are determined by the initial amplitude of a perturbation.

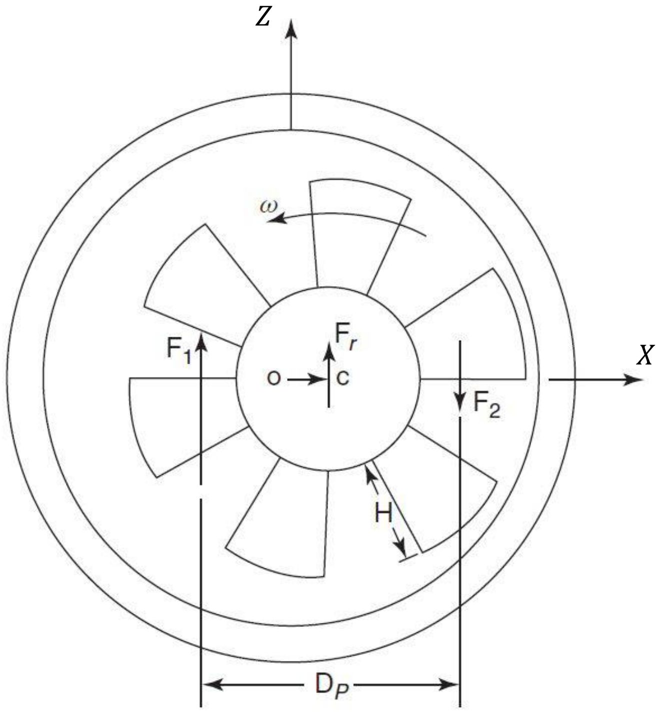


Figure 31. An axial flow stage eccentric in its housing.

Substitution of (150) into (148) transforms the differential equations into algebraic equations with unknowns A_1 and A_2. Expressed in matrix form, these equations are:

$$\begin{bmatrix} \left(ms^2 + cs + k\right) & K_{XZ} \\ K_{ZX} & \left(ms^2 + cs + k\right) \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{151}$$

Since the equations are homogeneous, nonzero solutions for the ratio A_1/A_2 can exist only if the determinant of the matrix is zero. Equating the determinant to zero gives the characteristic polynomial in the eigenvalue s:

$$\left(ms^2 + cs + k\right)^2 - K_{XZ} K_{ZX} = 0 \tag{152}$$

The eigenvalues of the rotordynamic system are the roots of the characteristic polynomial. They are generally complex numbers. That is, each root will have the form:

$$s = \lambda + j\omega_d \tag{153}$$

where λ is the damping exponent, and ω_d is the damped natural frequency (i.e., the whirling frequency) due to the form of the solutions (148). For example, Equation (148) becomes:

$$X(t) = A_1 e^{\lambda t}\left(\cos\omega_d t + j\sin\omega_d t\right) \tag{154}$$

If $\lambda > 0$, the perturbed motion grows exponentially with time and is therefore said to be unstable. The algebraic sign of λ depends on the relative magnitude of the cross-coupled stiffness (i.e., the destabilizing force) in (152). This follower force is modeled with the cross-coupled stiffness coefficients:

$$K_{XZ} = -K_{ZX} \tag{155}$$

Representing the magnitude of K_{XZ} and K_{XZ} by κ, the real and imaginary parts of the roots (153) are found to be:

$$\lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 + \left(\omega_d^2 - \frac{k}{m}\right)} \tag{156}$$

$$\omega_d^2 = \frac{k}{2m} - \frac{c^2 \pm \sqrt{\left(c^2 - 4km\right)^2 + 16\kappa^2 m^2}}{2m^2} \tag{157}$$

Figures (32 and 33) show how the whirl frequency ω_d and the damping exponent λ vary with the strength of the cross-coupled stiffness κ for three different values of damping ratio . The latter is defined by:

$$\xi = \frac{c}{2\sqrt{km}} \tag{158}$$

where c is the direct damping coefficient. Figure (32) helps to explain why the measured whirling frequency in violently unstable machines is usually higher than the associated critical speed, since the whirling frequency increases with the magnitude of the destabilizing force. Figure (33) shows that a rotor-bearing system with 5 percent damping (a typical value) can become unstable with a cross-coupled stiffness κ of only 10 percent of the effective shaft stiffness k.

Using Equations (156) and (157) or Routh's method it can be shown that a necessary condition to prevent the real part λ of the complex eigenvalues from becoming positive (i.e., a necessary condition for stability) is:

$$\kappa < c\sqrt{\frac{k}{m}} \tag{159}$$

which can also be written as:

$$\frac{\beta T}{DH} < c\omega_n \tag{160}$$

where ω_n is the undamped critical speed.

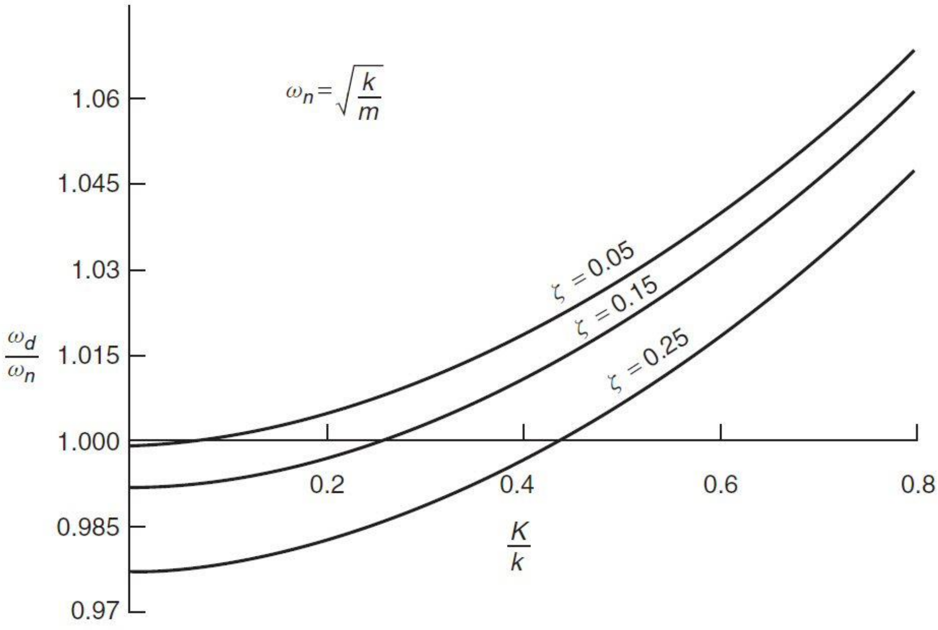


Figure 32. Effect of cross-coupled stiffness κ on whirl frequency.

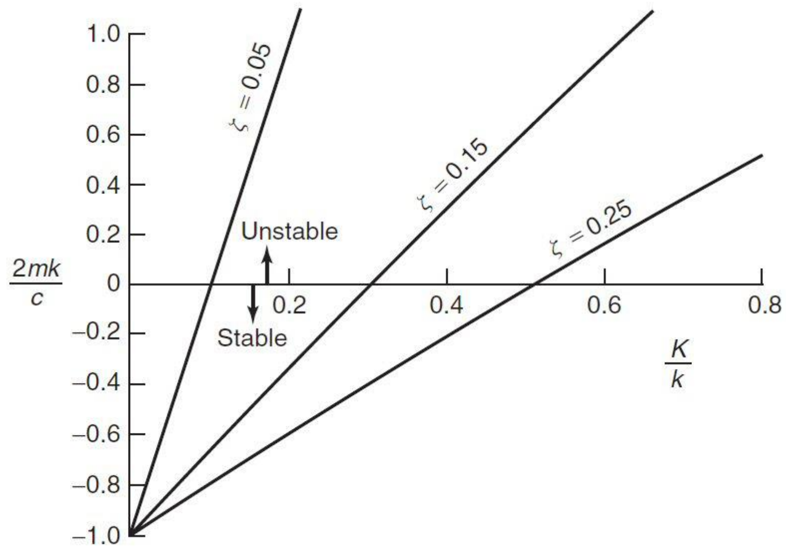


Figure 33. Effect of cross-coupled stiffness κ on stability.

If the aerodynamic load torque T of the wheel increases with speed, then (160) shows that there could be a threshold speed of instability, above which the inequality is no longer satisfied.

Note that the threshold speed can be raised by stiffening the shaft (raising ω_n). Increasing the effective damping coefficient c would also raise the threshold speed, but this would require a damper seal near midspan since this model has rigid bearings. Reducing the bearing support stiffness could increase the effective damping coefficient by allowing any bearing damping to become effective.

Most destabilizing forces in rotating machines can be represented by cross-coupled stiffness, as in the example just presented. Many engineers and the API specifications for rotating machinery now refer to all of them as Alford forces and many use Alford's equation to model them, even though Alford applied the equation only to axial-flow compressor and turbine stages.

A number of destabilizing mechanisms have been identified or hypothesized to explain incidents of rotordynamic instability. Some of the known or hypothesized sources of destabilizing forces are listed below:

- Hydrodynamic bearings (oil whip).
- Fluid ring seals (similar to oil whip).
- Internal friction in rotating parts (cross-coupled internal moment stiffness).
- Aerodynamic forces due to blade-tip clearance eccentricity (Alford's force).
- Trapped liquids inside a hollow shaft or rotor.
- Dry friction whip (backward whirl driven by rubbing friction between rotor and stator).
- Labyrinth seals.
- Torque whirl (the direct effect of very high-stage torque when misaligned by the mode shape).

7. Shaft Balancing

7.1. Introduction:

Balancing is a process for controlling the distribution of the masses of a rotor and improving it to keep the forces and Vs caused by unbalances within acceptable limits.

A rotor is considered to be all the parts which rotate in operation and those which, for functional reasons, are mounted on an axis.

Balancing is considered absolutely necessary today for all rotors, whether to extend the life of the machine, improve its operation...

The balancing problem appeared a few thousand years ago with the first water wheels. Problems arose when these wheels were not constructed sufficiently symmetrically, or when the material was not of constant thickness or identical dimensions: the wheel tended to rotate to a given position (heaviest point toward the bottom), and remained stuck there when the flow was weak. This so-called static unbalance could be counterbalanced empirically by additional masses m on the radius r (at a standstill above the axis), to finally obtain a continuous rotation (figure 29).

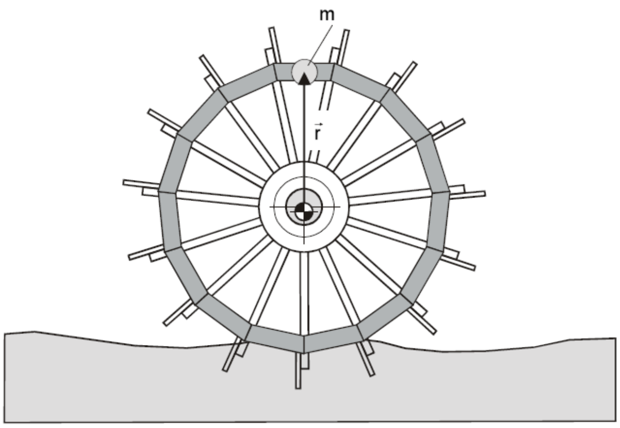


Figure 29. Static unbalance on water wheel.

With the first high-speed machines of the second half of the 19th century, a new unbalance problem appeared, hitherto unknown; the unbalance moment which can only be revealed in rotation (figure 30).

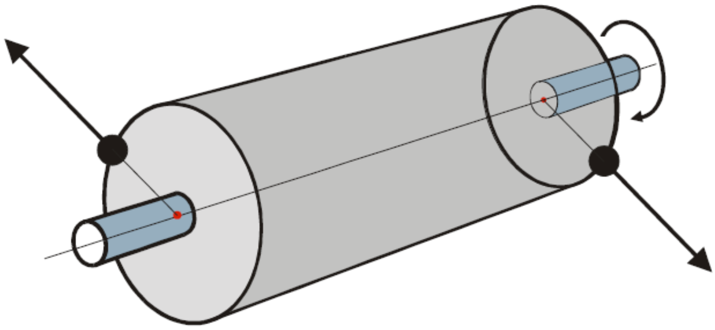


Figure 30. Moment unbalance.

In the first decades of the 20th century, new balancing problems still appeared. Rotors balanced with experience have serious vibration problems. These are always rotors whose nominal speed is a little below or even above a critical bending speed, and therefore exhibit typical resonance phenomena. For these rotors, additional, even very specific, balancing procedures are required, most often operating close to these resonances to reduce bending through selective corrections in several planes. These specific unbalances were called “modal” unbalances.

A certificate concerning balancing was filed in 1870; four years after the invention of the dynamo by W. VON SIEMENS, in Canada by H. MARTINSON. At the beginning of the 20th century, N.W. AKIMOFF in the United States and A. STODOLA in Switzerland gave new push to the balancing technique. In Germany, it was in 1907 that F. LAWACZEK filed the certificate for a two-plane balancing machine which was built by Carl SCHENCK in Darmstadt. The first model still presented some problems, but the idea was then perfected (certificate for a horizontal balancing machine in 1912) and successfully transformed by H. HEYMANN.

7.2. Unbalance (balourd):

According to the ISO 1925 (1996) definition, there is the presence of unbalance in a rotating system (rotor) when, following uncompensated centrifugal forces, oscillating forces or oscillation movements are transmitted to the bearings.

$$\vec{U} = u.\vec{r} \quad [g.mm] \tag{93}$$

The unbalance is independent of the rotation speed provided that the radius r is constant (rigid rotor).

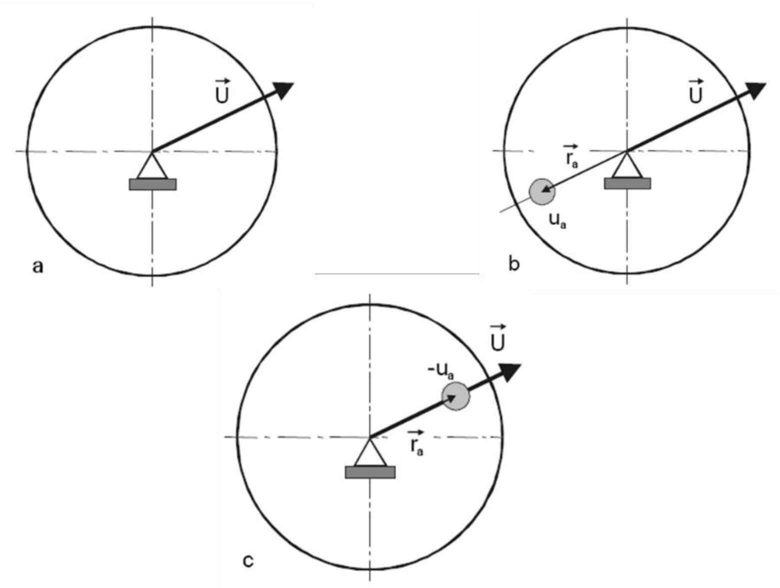
7.3. Rigid rotor:

Most rotors are constructed in such a way that their unbalance and shape do not change, or only insignificantly, up to their operating speed. These rotors are called rigid rotors. This means that the unbalance of the rotor can be given in the form of a constant value, which will not be a function of a given speed, and that it can be measured and corrected for any speed lower than the service speed.

7.4. Unbalance correction:

The correction is most often carried out by adding or removing material (figure 31), such that the sum of the centrifugal forces – and therefore the sum of the unbalances – is equal to zero.

$$\vec{U} + u_a \vec{r}_a = \vec{0} \qquad (94)$$



$$u_a = \frac{U}{r_a} \quad \text{or} \quad r_a = \frac{U}{u_a}$$

Figure 31. Polar correction.

If, depending on the characteristics of the rotor or the type of correction, we can only correct in given directions (at defined locations), we say correction in fixed points (or components) (figure 32).

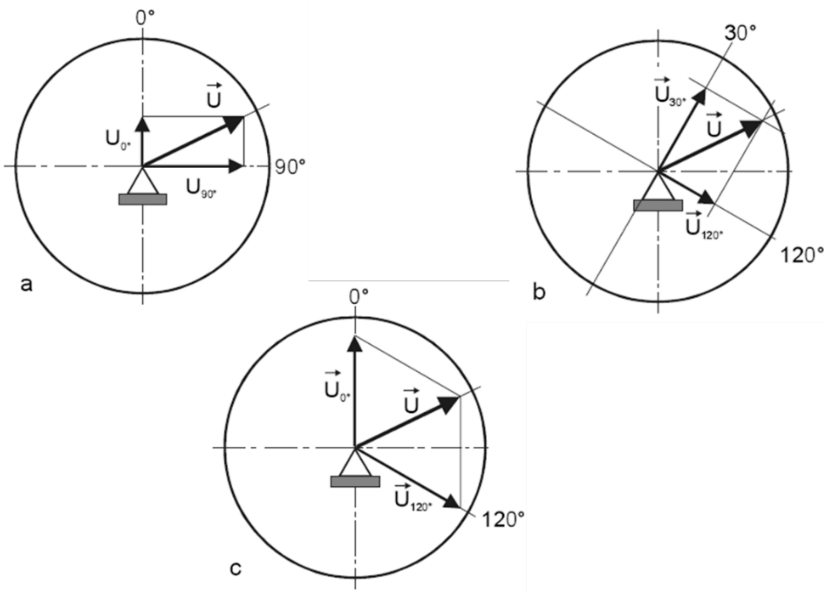


Figure 32. Correction in fixed points.

7.5. Unbalance of a disc rotor:

If the rotor rotates with an angular frequency Ω, each elementary mass u_i generates on its radius r_i a centrifugal force F_i.

$$\vec{F}_i = m_i \vec{r}_i \Omega^2 \tag{95}$$

The vector sum of the centrifugal forces of all elements is the centrifugal force acting on the bearings.

$$\vec{F} = \sum_{i=1}^{n} m_i \vec{r}_i \Omega^2 \tag{96}$$

$$\sum_{i=1}^{n} m_i \vec{r}_i \Omega^2 = u\vec{r}\Omega^2$$

$$\sum_{i=1}^{n} m_i \vec{r}_i = u\vec{r} = \vec{U} \tag{97}$$

- The unbalance of a disc rotor (and mounted perpendicular to the axis of the shaft) can be perfectly described by an unbalance vector;
- Unbalance correction only requires correction in a single plane.

7.6. Unbalance of a cylindrical rotor:

Unbalances can occur anywhere along the rotor. The individual centrifugal forces caused by unbalances can be transferred to two freely selectable planes I and II (e.g. end planes) and transformed there again into unbalances (complementary unbalances) (figure 33).

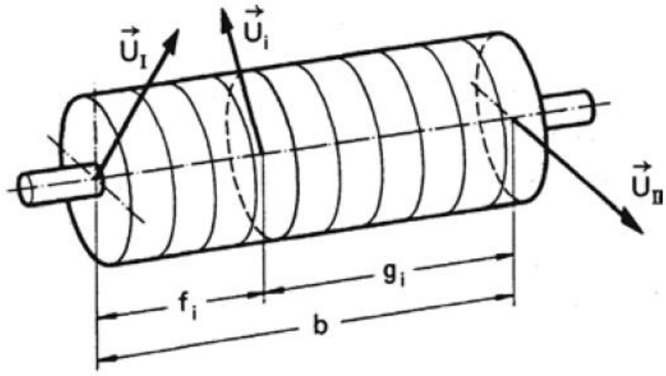


Figure 33. Cylindrical rotor unbalance.

$$\vec{U}_I = \frac{\sum_{i=1}^{n} \vec{U}_i g_i}{b} \quad \text{and} \quad \vec{U}_{II} = \frac{\sum_{i=1}^{n} \vec{U}_i f_i}{b}$$

7.7. Static unbalance:

If we add to a perfectly balanced rotor an individual unbalance in the radial plane in which its center of gravity is located, we speak of a static unbalance U_S (figure 34).

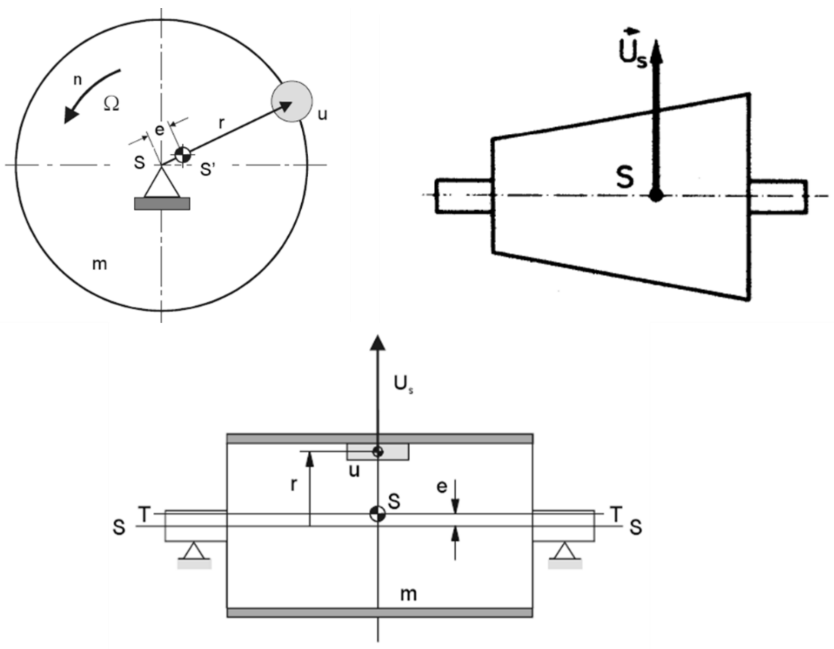


Figure 34. Static unbalance.

- For a perfectly balanced rotor the center of gravity must be on the axis of the shaft (otherwise a centrifugal force would be exerted).
- Following the addition of the unbalance mass, the center of gravity leaves the axis of the shaft.

The equilibrium condition is:

$$(m+u)\vec{e} = ur$$

$$\vec{e} = \frac{u\vec{r}}{m+u} \quad \text{eccentricity}$$

By neglecting u (often $u \ll m$):

$$\vec{e} = \frac{u\vec{r}}{m} = \frac{\vec{U}_s}{m} \quad [\mu m] \tag{98}$$

For the correction of static unbalance, only one correction plane is necessary, the plane of the center of gravity. If a correction is made in another plane, this causes, as a side effect, an unbalance moment. The correction mass can also be distributed over two planes so as to obtain the effect of a single mass in the plane of the center of graIVty (figure 35).

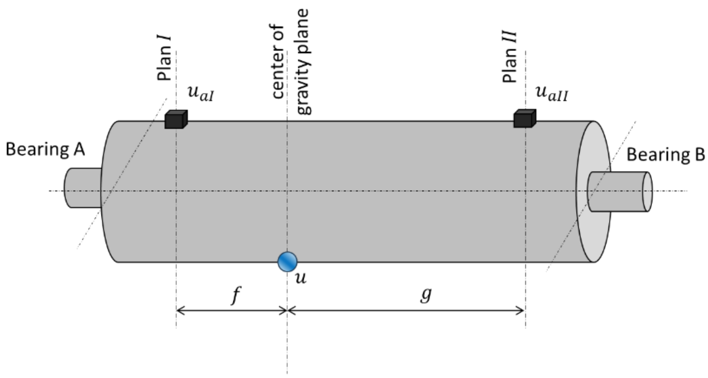


Figure 35. Distribution of static unbalance on two planes.

$$u_{aI} + u_{aII} = u \quad \text{and} \quad u_{aI} f + u_{aII} g = 0$$

$$u_{aII} = u_a \frac{f}{g+f} \quad \text{and} \quad u_{aI} = u_a \frac{g}{g+f}$$

7.8. Unbalance moment:

If, on a perfectly balanced rotor, we add two unbalances of the same value so that they are arranged face to face in two different radial planes (couple of unbalances), we then say an unbalance moment (figure 36).

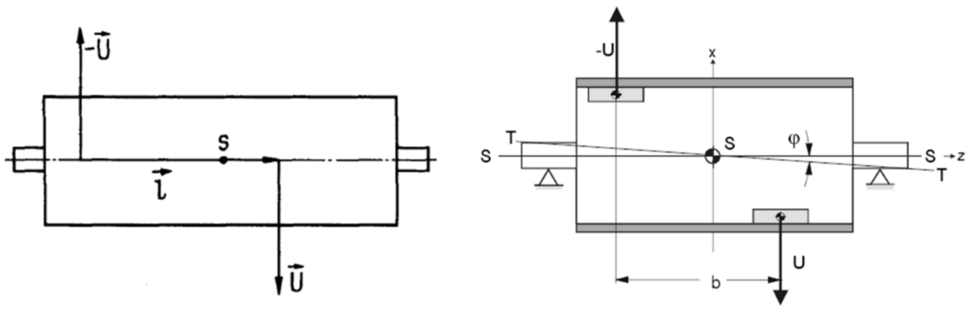


Figure 36. Unbalance moment.

$$\vec{U}_m = \vec{l} \times \vec{U} \tag{99}$$

$$\vec{\varphi} = \frac{\vec{U}_m}{J_x - J_z} \tag{100}$$

7.9. Quasi-static unbalance:

When, on a perfectly balanced rotor, we add a single unbalance in a plane different from that of the center of gravity, we call it quasi-static unbalance (figure 37).

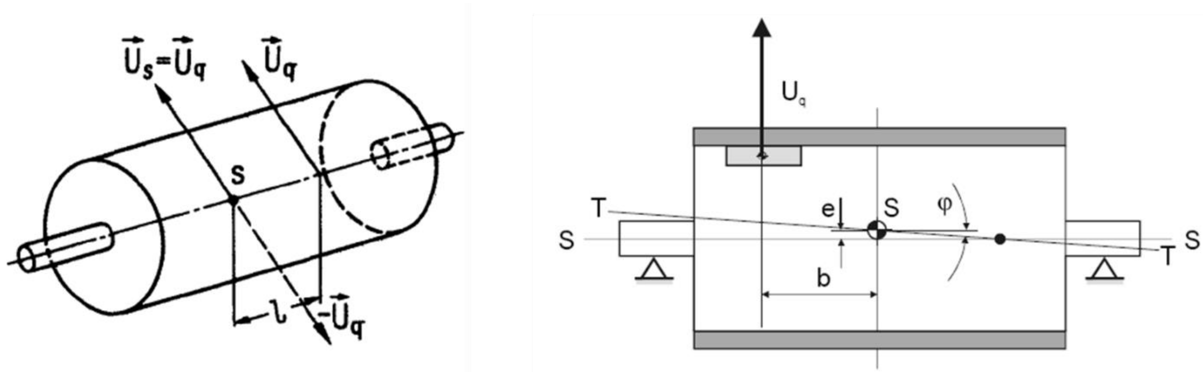


Figure 37. Quasi-static unbalance.

If we apply the same unbalance vector to the center of graIVty, and in addition the same vector with a negative sign (opposite direction), the two added unbalances compensate for each other and the initial situation has not changed.

7.10. Dynamic unbalance:

A superposition of a static unbalance with an unbalance moment.

7.11. Resulting unbalance and resulting unbalance moment:

The resulting unbalance is the vector sum of all the unbalances U_Z which are distributed over the length of the rotor.

$$\vec{U}_r = \sum_{z=1}^{k} \vec{U}_z \tag{101}$$

The resulting unbalance moment represents the vector sum of the products of each unbalance by its distance from the plane of the resulting unbalance.

$$\vec{P}_r = \sum_{z=1}^{k} \left(\vec{z}_{Ur} - \vec{z}_z \right) \vec{U}_z \tag{102}$$

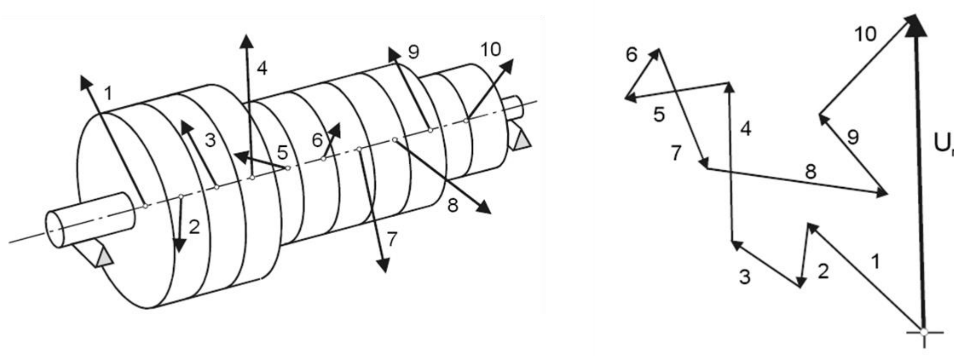


Figure 38. Distribution of unbalance vectors.

7.12. Flexible rotor:

It is a rotor whose unbalance and shape change with speed.

7.13. Tolerance and evaluation (rigid rotors):

Balancing does not consist of obtaining a “perfectly balanced” rotor: starting from the initial unbalance presented by each rotor, we simply aim to go below a certain tolerance value.

7.13.1. ISO1940-1 Evaluation criteria:

In general, the permissible unbalance is proportional to the mass of the rotor. The admissible specific unbalance corresponds to the eccentricity of the center of gravity, when we take the static unbalance as the permissible residual unbalance.

$$e_{per} = \frac{U_{per}}{m} \qquad \text{with} \qquad e_{per}\omega = const \tag{103}$$

The tangential and radial stresses of rotors of similar geometry are proportional to the square of the peripheral speed; their distribution is also similar. If the peripheral speed is kept constant, the tangential and radial stresses at the same locations will remain constant, and with them all the quantities of dimension N/m^2, in particular the surface stress on the bearings.

7.13.2. Determination of permissible residual unbalance:

Quality classes and rotor groups

The product $e_{per}\omega$ could take any value; for the sake of simplification, we agreed on a certain number of fixed values, each presenting a ratio of 2.5 with the previous one. In some cases, especially for high degrees of balancing (small unbalance tolerance), it may be necessary to use a finer graduation. Table (01) presents guidance for balance quality grades for rotors in a constant (rigid) state.

Distribution on correction plans

The quality grades indicate a maximum allowable unbalance for the complete rotor, while the approach based on the forces exerted on the bearings indicates allowable residual unbalances in the bearing planes. We must therefore carry out a distribution or an allocation.

For a given rotor, we search for the situation in which the unbalances apply the greatest stress. If permissible residual unbalance values are defined in this case, the influence of these residual unbalances is always lower in the other cases. The advantage is that we always have a margin of safety.

For the permissible unbalance U_{per} (determined from the quality degrees), the ISO 1940-1 standard gives different distribution rules. If we start from the forces applying to the bearings, the permissible residual unbalances in the bearing plans can be brought together into an overall unbalance U_{per}, so as to be able to use the same distribution rules as for the quality degrees.

For disk rotors, balancing in a single correction plane may be sufficient provided that the distance between bearings is large enough and the disc rotates with a sufficiently small axial runout. After a sufficient number of rotors of this type have been balanced in one plane, the largest remaining unbalance torque is determined and divided by the distance between bearings. When this unbalance is not greater than half of the maximum permissible residual unbalance U_{per} in the most unfavorable case, balancing on a single plane is generally sufficient.

For common rotors (two correction planes) (figure 39), the ISO 1940-1 standard describes “simplified approximation methods” for the distribution of the admissible residual unbalance on the correction planes. Four criteria will make it possible to determine the different distribution procedures (figure 40).

Table 01. Guidance for balance quality grades for rotors in a constant (rigid) state (ISO 1940-1).

Machinery types: General examples	**Balance quality grade G**	**Magnitude** $e_{per}\omega$ mm/s
Crankshaft drives for large slow marine diesel engines (piston speed below 9 m/s), inherently unbalanced	G 4000	4000
Crankshaft drives for large slow marine diesel engines (piston speed below 9 m/s), inherently balanced	G 1600	1600
Crankshaft drives, inherently unbalanced, elastically mounted	G 630	630
Crankshaft drives, inherently unbalanced, rigidly mounted	G 250	250
Complete reciprocating engines for cars, trucks and locomotives	G 100	100
Cars: wheels, wheel rims, wheel sets, drive shafts Crankshaft drives, inherently balanced, elastically mounted	G 40	40
Agricultural machinery Crankshaft drives, inherently balanced, rigidly mounted Crushing machines Drive shafts (cardan shafts, propeller shafts)	G 16	16
Aircraft gas turbines Centrifuges (separators, decanters) Electric motors and generators (of at least 80 mm shaft height), of maximum rated speeds up to 950 r/min Electric motors of shaft heights smaller than 80 mm Fans Gears Machinery, general Machine-tools Paper machines Process plant machines Pumps Turbo-chargers Water turbines	G 6.3	6.3
Compressors Computer drives Electric motors and generators (of at least 80 mm shaft height), of maximum rated speeds above 950 r/min Gas turbines and steam turbines Machine-tool drives Textile machines	G 2.5	2.5
Audio and IVdeo drives Grinding machine drives	G 1	1
Gyroscopes Spindles and drives of high-precision systems	G 0.4	0.4

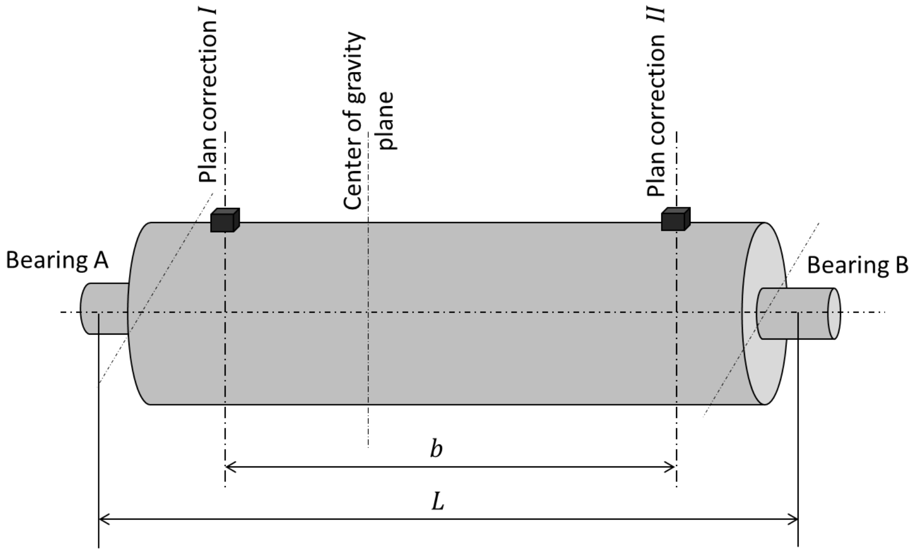


Figure 39. Standard rotor with two correction planes.

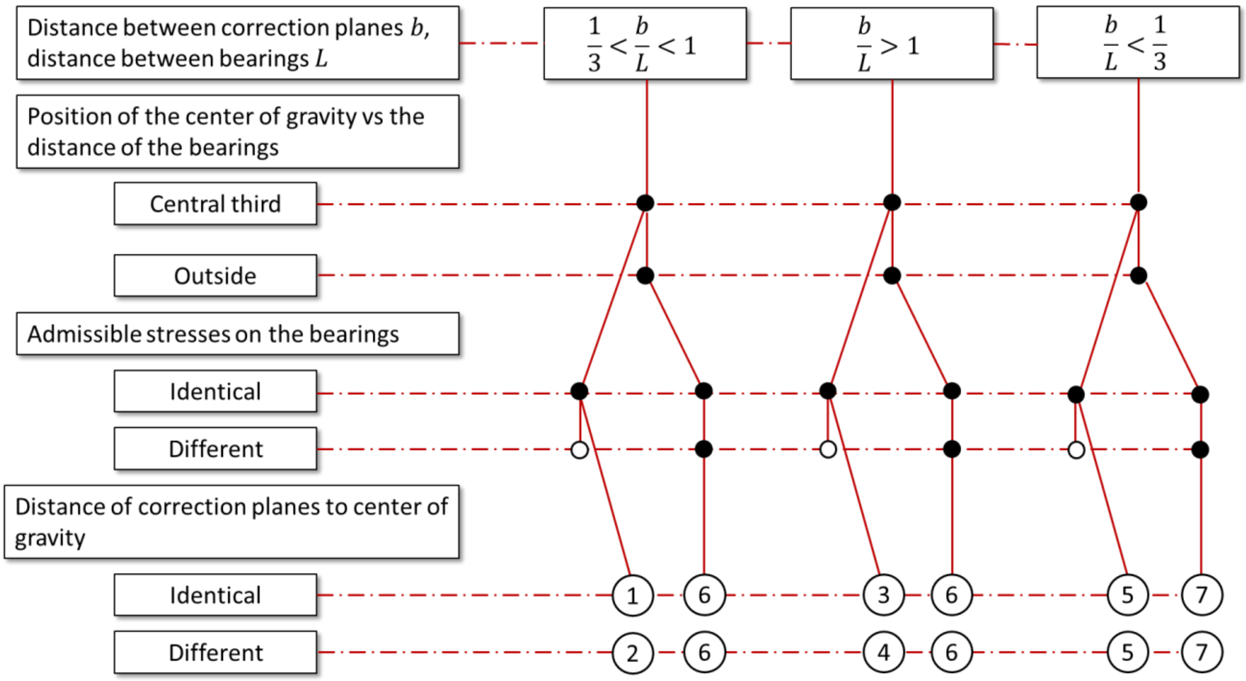


Figure 40. Diagram for choosing the procedure according to four criteria.

Procedure 01:

In the simplest and most common case, the unbalances mases are distributed uniformly on the two planes.

$$U_{perI} = U_{perII} = \frac{U_{per}}{2} \tag{104}$$

Procedure 02:

The permissible unbalance is distributed in such a way that the sum of the unbalance moments is zero. The closer a plane is to the center of gravity, the smaller the theoretical tolerance of the other plane (figure 35).

$$\begin{cases} 0.3U_{per} \le U_{perI} = U_{per}\dfrac{f}{b} \le 0.7U_{per} \\ 0.3U_{per} \le U_{perII} = U_{per}\dfrac{g}{b} \le 0.7U_{per} \end{cases} \tag{105}$$

Procedure 03:

If the distance between the correction planes is greater than the distance between the bearings. We deduce a reduced permissible residual unbalance, then it can be distributed according to procedure 01 on the correction plans.

$$U_{per}^{r} = U_{per}\frac{L}{b} \tag{106}$$

Procedure 04:

The distance between the correction planes being greater than the distance between the bearings, we again adopt a reduced admissible residual unbalance, which will then be distributed according to the asymmetrical position of the center of gravity according to procedure 02.

Procedure 05:

In this case, admissible residual unbalances are obtained according to procedure 01 or procedure 02 which are so small that they are difficult to achieve when they occur in the form of unbalance moments. The values required for the unbalance moment are also much smaller than necessary. It is therefore advisable to determine separately the admitted values for the static unbalance and for the unbalance moment: we choose a correction plane III (which can coincide with I or II) and we call d the distance of this plane with the further bearing (figure 41).

- Static unbalance : $U_{perIII} = \dfrac{U_{per}}{2}\dfrac{L}{2d}$
- Unbalance moment: $U_{perI,II} = \dfrac{U_{per}}{2}\dfrac{3L}{4d}$

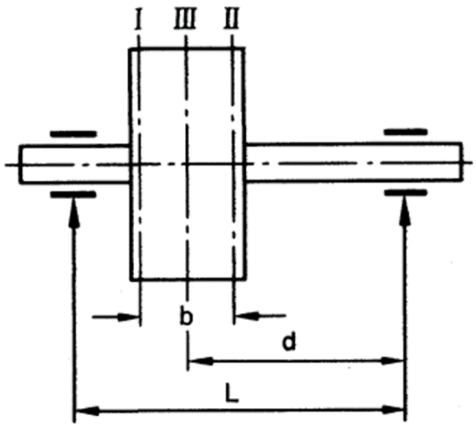


Figure 41. Rotor according to procedure 05.

Procedure 06:

We calculate here, after adoption of two quantities – the distribution of U_{per} on the two bearing planes and the ratio of admissible unbalances on the correction planes – four different values for U_{per} of which the smallest will be retained.

Procedure 07:

For a cantilever mounted rotor, the furthest bearing with lower resistance is sometimes chosen as if it were the closest. Procedure 05 can then be applied. If the bearing resistance ratio is less than 0.5, the factor $3/4$ used for calculating the admissible unbalances in planes I and II (unbalance torque) must be reduced accordingly, so that the furthest bearing not be overloaded.

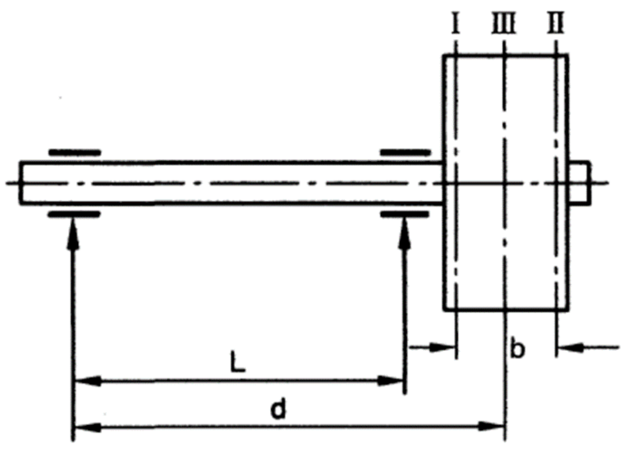
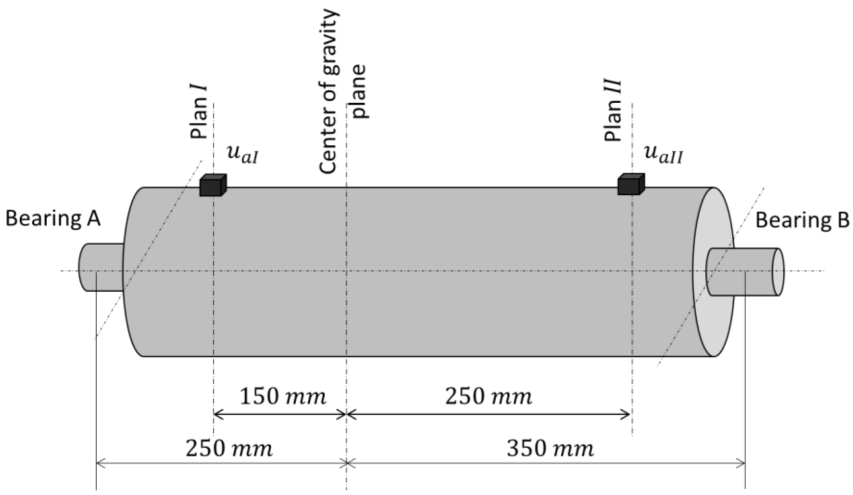


Figure 42. Rotor according to procedure 07.

Solved exercises:

Exercise 01:

A rotor must be corrected by a mass $= 40\ g$. The distances between correction planes I and II and the plane of the center of gravity are shown in the figure below. What is the value of the necessary correction masses u_{aI}, and u_{aI} in planes I and II. What is the value of the forces $F_A$and F_B on the bearings generated by these masses for a rotor speed N $= 1000\ RPM$ (Shaft diameter is $50\ cm$).



$$u_{aI} + u_{aII} = 40$$
$$u_{aI} \times 150 - u_{aII} \times 250 = 0$$

from where : $u_{aI} = 25$ g and $u_{aII} = 15$ g.

The forces generated by these masses:

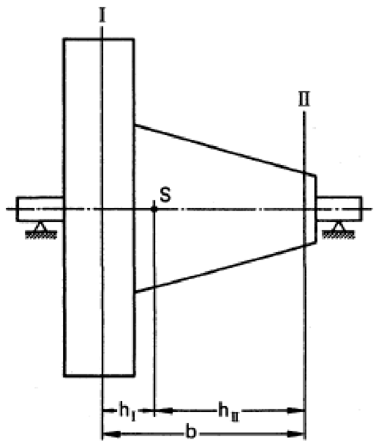
$$F_B \times 350 - F_A \times 250 = u_{aI}\frac{d}{2}\omega^2 \times 150 - u_{aII}\frac{d}{2}\omega^2 \times 250 = (u_{aI} \times 150 - u_{aII}250)\frac{d}{2}\omega^2 = 0$$
$$F_A + F_B + u_{aI}\frac{d}{2}\omega^2 + u_{aII}\frac{d}{2}\omega^2 = 0$$
$$F_A = 1.4F_B$$

$$F_A + F_B = -40 \times 10^{-3} \times 0.25 \times \left(\frac{\pi \times 1000}{30}\right)^2 = -109.66\ N$$

from where : $F_A = -63.97\ N$ and $F_B = -45.69\ N$.

Exercise 02:

In the example in the figure below, the following residual unbalances were measured in planes I and II: $U_I = 4000\ g.mm$, $U_{II} = 1200\ g.mm$. The rotor has a mass of $m = 230\ kg$, its operating speed is $N = 1500\ RPM$. What level of quality has been achieved?.



The residual unbalance at the plane of the center of gravity is:

$$U = U_I + U_{II} = 4000 + 1200 = 5200 \text{ g.mm}$$

The eccentricity is then:

$$e = \frac{U}{m} = \frac{5200}{230 \times 10^3} = 0.022\ mm$$

The circumferential speed will be:

$$e\omega = \frac{e\pi n}{30} = 3.55\ mm/s$$

Which can be considered as quality degree.

Exercise 03:

We assume an automobile wheel that weighs 10.6 kg (rim and tire 205/65/16, width in mm/height-width ratio in%/rim diameter in inches). Determine the admissible unbalance corresponding to the speed of 90 km/h knowing that the quality grade of a car wheel is G40.

For a G40 quality level, we have:

$$e_{adm}\omega = 40\ mm/s$$

$$e_{adm} = \frac{40}{\omega} = \frac{U_{adm}}{m}$$

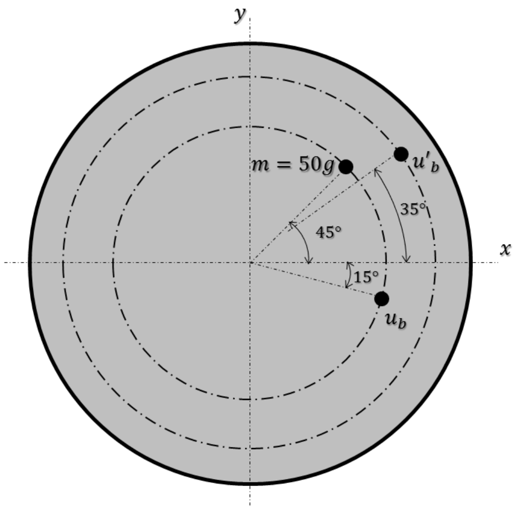
$$U_{adm} = \frac{40m}{\omega} = \frac{20mD}{V} = \frac{20 \times 10.6 \times 672.9 \times 3600}{90 \times 1000} = 3712.772\ g.mm$$

with: $V = \frac{D}{2} \times \omega \quad \rightarrow \quad \omega = \frac{2V}{D}$

$D = 0.65 \times 205 \times 2 + 16 \times 25.4 = 672.9\ mm$

Exercise 04:

A steering wheel with an unbalance u_b at the radial position $0.165\ mm$ and a phase angle of $15°$ clockwise from the phase mark x. When a test weight of magnitude $m = 50\ g$ is added to an angular position $45°$ counterclockwise of the phase mark, the radial position and phase angle become $0.225\ mm$ and $35°$ respectively counterclockwise. Find the magnitude and angular position of the required balance weight. Assume the weights are added at $0.165\ mm$.



By adding the mass $50g$ we created an unbalance u'_b which implies that:

$$\vec{U}_b + \vec{U}_m = \vec{U}'_b$$

$$\begin{cases} -u_b r_b \sin 15° + m r_b \sin 45° = u'_b r'_b \sin 35° \\ u_b r_b \cos 15° + m r_b \cos 45° = u'_b r'_b \cos 35° \end{cases}$$

$$\begin{cases} -0.165 u_b \sin 15° + 8.25 \sin 45° = 0.225 u'_b \sin 35° \\ 0.165 u_b \cos 15° + 8.25 \cos 45° = 0.225 u'_b \cos 35° \end{cases}$$

$$\begin{cases} -0.0427 u_b + 5.8336 = 0.1291 u'_b \\ 1594 u_b + 5.8336 = 0.1843 u'_b \end{cases}$$

from where: $u_b = 11.32g$

therefore the steering wheel must be balanced with a mass of $11.32\ g$ at the angular position $195°$ clockwise ($165°$ counterclockwise) and at the radial position $0.165\ mm$.

References

1. Singiresu S. Rao - Mechanical Vibrations in SI Units, Global Edition-Pearson, 2017.
2. Michel Lalanne & Guy Ferraris – Dynamique des rotors en flexion, Techniques de l'ingénieur B 5110, 1996.
3. A. A. Shabana – Theory of vibration, Volume I: Introduction, Springer Verlac, 1991.
4. Agnieszka Muszynska - Rotordynamics (Dekker Mechanical Engineering)-CRC Press, 2005.
5. John M. Vance, Fouad Y. Zeidan, Brian Murphy - Machinery Vibration and Rotordynamics, 2010.
6. J. S. Rao - History of Rotating Machinery Dynamics-Springer Netherlands, 2011.
7. Hatto Schneider – Techniques d'équilibrage, Springer, 2006.
8. Jean-Marc Pugnet – Equilibrage des rotors rigides et flexibles, Techniques de l'ingénieur BM 5130, 1997.