الجمهورية الجزائرية الديمقراطية الشعبية
PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
وزارة التعليم العالي و البحث العلمي
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

جامعة عمار ثليجي بالأغواط
AMAR TELIDJI UNIVERSITY - LAGHOUAT
كلية العلوم
FACULTY OF SCIENCE
قسم الرياضيات
DEPARTMENT OF MATHEMATICS

Item Number: .....
Serial Number: .....

A dissertation submitted in partial fulfilment of the requirements for the degree of Master in mathematic

Domain : Mathematics and informatics
Field : Mathematics
Specialty : Analysis of functions and applications

Presented by: **Ms.** Teggar Fatima Zohra

TITLE

Global existence and decay results of a viscoelastic wave equation with variable exponent and logarithmic nonlinearities

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Academic year: 2024/2025

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Acknowledgement

First of all, I thank « **Allah** » Almighty for allowing me to reach this modest scientific level and for giving me the courage and patience to successfully complete the work in this thesis.

I thank Professor **Abdelaziz Rahmone** for the honor he bestowed upon me by agreeing to judge this work and chair the defense committee.

I also extend my sincere gratitude to Professor **Saf salim**, who agreed to serve on my committee. Their presence is a great honor.

Thank you for your comments, your critiques, your advice, and simply for the interest you have shown in my work.

My sincere thanks and deep gratitude to my supervisor, Professor **Mohamed Saadaoui**, for his scientific support, his availability, and his judicious guidance, and whose intellectual skills, experience, modesty, and patience have greatly contributed to the completion of this thesis. May he find here the expression of my deepest respect.

My thanks also go to all the teachers and all the people who supported me until the end and who continued to give me very important advice as a sign of gratitude.

Dedication

I dedicate this work,

My dear father , the first educator and teacher, who gifted me his wisdom and experiences that shaped me into who I am today.

My beloved mother, my warm embrace and first teacher, who, through her prayers and patience, turned my dreams into reality.

My dear siblings [Zainab, Ibrahim, Mohammed, Lamia, and Fares], my companions in life who supported me at every step.

My beloved grandmother (may God have mercy on her), I ask God to elevate her ranks in the gardens of bliss.

To my classmate and friend Noussiba Nebeg for her support and encouragements.

To all my beloved ones.

Notations

Ω	An open set of $\mathbb{R}^n$.
$\partial\Omega$	Boundary of Ω.
$\Omega \times [0,T]$	The cylinder of $\mathbb{R}^n \times \mathbb{R}^+$.
$p'(.)$	Conjugate exponent of $p(.)$, $p'(.) = \frac{p(.)}{p(.)-1}$
$p^*(.)$	Sobolev conjugate exponent $p(.)$, $p^*(.) = \frac{Np(.)}{N-p(.)}$
$\frac{\partial}{\partial x_i}$	Partial derivative with respect to the variable x_i.
$\nabla u = \left(\frac{\partial u}{\partial x_i}\right)_{i=1,N}$	Gradient of u
$\|\Omega\|$	Lebesgue measure of the set Ω.
$\sup pu$	Support of a function u.
$C(\Omega)$	Space of continuous functions on Ω.
$C_c(\Omega)$	Space of continuous functions on Ω with compact support.
$C^k(\Omega)$	Space of class k functions in Ω.
$D(\Omega)$	The space of functions of class $C^\infty$with compact support in Ω.
$D'(\Omega)$	Space of distributions.
$L^p(\Omega)$	Lebesgue Space.
$W^{s,p}(\Omega)$	The Sobolev space,$1 \leq p \leq +\infty$ and s is a nonnegative integer.
$H^s(\Omega)$	Sobolev space of order s greater than 1
$H_0^s(\Omega)$	The closure of $D(\Omega)$ in $H^s(\Omega)$
$H^{-s}(\Omega)$	The dual of the sobolev space $H_0^s(\Omega)$.
X	Banach space
$L(X;Y)$	Space of continuous linear applications of X into Y .
$L^{\beta(\cdot)}(\Omega)$	The Lebesgue space with a variable exponent $\beta(\cdot)$
$W^{\beta(\cdot)}(\Omega)$	The variable-exponent Sobolev space
$W_0^{1,\beta(\cdot)}(\Omega)$	The closure of $C_0^\infty(\Omega)$ in $L^{1,\beta(\cdot)}(\Omega)$

Chapter 1

Introduction and mathematical tools

1.1 Introduction

This memoir divides its objective into two parts.

The first part: To prepare the student to conduct future research, as well as to refine and consolidate the scientific knowledge acquired in the previous two phases and to reinforce it with mathematical concepts, thus qualifying him to obtain this degree and allowing him to continue in the future.

This memoir follows the following structure :

Chapter 1: In the first chapter we introduce some necessary notation and set out some basic definitions and functional analysis theorems that will be needed in the body of the work.

Chapter 2: In this chapter, we use the Faedo-Galerkin method to prove the local existence of the weak solution , and the global existence of the solution is established for the problem (2.1).

Chapter 3: In the third chaptre we study general decay results for a broad class of relaxation functions and some special conditions for the variable exponent function. Our results supplement and generalize many previous results, and we support them with examples.

Finally : we present a summary and conclusion of the results obtained in this memoir.

1.2 Preliminary

In this section, we introduce some fundamental definitions, theorems and properties in functional analysis to be used throughout this work.

1.2.1 Fondamental spaces

Definition 1. *Hilbert space H is a vector space H equipped with a scalar product such that H is complete for the norm $||$.*

In what follows, H will always denote a Hilbert space.

Banach Space

Definition 2. *A sequence (a_n) in a metric space (X, d) is called a Cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer N such that for all $m, n \geq N$, the distance between the terms of the sequence is less than ε :*

$$\forall \varepsilon > 0, \ \exists N \in N, \forall m, n \geq N, \ d(a_n, a_m) < \varepsilon.$$

In other words, the terms of a Cauchy sequence become arbitrarily close to each other as the sequence progresses.

Definition 3. *(Complete Space) A metric space X (or* normed space*) is called* complete *if for every Cauchy sequence $\{x_n\}$ in X, there exists an element $x \in X$ such that:*

$$\lim_{x \to \infty} x_n = x$$

This means that the space is "closed under limits" for sequences that get arbitrarily close to each other as they progress.

Definition 4. *A Banach Space is a complete normed vector space, which means that every Cauchy sequence in this space converges to an element of the space. It is a fundamental structure in functional analysis. A normed vector space $(X. \|.\|)$ is called a Banach space if:*

$$\forall (x_n)_{n \in N} \subset X, \| x_n - x_m \| \rightarrow 0 (asn, m \rightarrow \infty) \Rightarrow \exists x \in X : \| x_n - x \| \rightarrow 0.$$

Lebesgue Spaces

Definition 5. *For $1 \leq p \leq \infty$, the space $L^p(\Omega)$ of measurable functions $f : \Omega \to \mathbb{R}$ or $\mathbb{C}$ is a Banach space with the norm:*

For $1 \leq p \leq \infty$,

$$\| f \|_p = \left(\int_\Omega |f(x)|^p \, d\mu(x) \right)^{\frac{1}{p}}, (1 \leq p \leq \infty),$$

And for $p = \infty$,

$$\| f \|_\infty = \underset{x \in \Omega}{esssup} |f(x)|.$$

Definition 6. *Let $1 \leq p \leq +\infty$, we set*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \ \ is \ \ measurable \ and \ |f|^p \in L^1(\Omega) \right\} with$$

$$\|f\|_{L^p} = \|f\|_p = \left[\int_\Omega |f(x)|^p \, d\mu \right]^{\frac{1}{p}}.$$

We shall check later on that $\|\|_p$is a norm. We set

$$L^\infty(\Omega) = \left\{ f : \Omega \to R \;\middle|\; \begin{array}{c} \text{f is measurable and there is aconstant C} \\ \text{such that } |f(x)| \leq C \text{ a.e.on } \Omega \end{array} \right\}$$

with

$$\|f\|_{L^\infty} = \|f\|_\infty = \inf \{ C; |f(x)| \leq C \ \ a.e.on \ \Omega \}.$$

Definition 7. *$L^2(\Omega)$ equipped with the scalar product $(u, v) = \int_\Omega u(x)v(x)d\mu$ is a Hilbert space. In particular, L^2 is a Hilbert space.*

Reflexive Spaces

Definition 8. *Let E be a Banach space and let $J : E \to E^{**}$ be the canonical injection from E into E^{**}. The space E is said to be reflexive if J is surjective,i.e., $J(E) = E^{**}$.*

When E is reflexive, E^{**} is usually identified with E.

1.2.2 The Sobolev space

Sobolev spaces were introduced by the Russian mathematician Sergei Lvovich Sobolev (1908-1989) during the 1930s. These spaces, denoted as $W^{m,p}(\Omega)$ consist of functions for which all $m-th$ order generalized derivatives belong to $L^p(\Omega)$ space, and the partial derivatives within these spaces adhere to specific integrability criteria. It is important to note that the term "generalized derivative" pertains to the weak derivative, as defined in the preceding chapter. In this section, we will discuss the essential properties of Sobolev spaces

Definition 9. *The Sobolev space $W^{1,p}(\Omega)$ is defined by*

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega) \middle| \begin{array}{c} \exists g_1, g_2, ..., g_N \in L^p(\Omega) \text{ such that} \\ \int_\Omega u \frac{\partial \varphi}{\partial x_i} = -\int_\Omega g_i \varphi \forall \varphi \in C_c^\infty(\Omega), \forall i = 1, 2, ..., N \end{array} \right\}$$

We set

$$H^1(\Omega) = W^{1,2}(\Omega)$$

For $u \in W^{1,2}(\Omega)$we define 2 $\frac{\partial u}{\partial x_i} = g_i$, and we write

$\nabla u = gradu = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_N}\right)$

Definition 10. *The space $W^{1,2}(\Omega)$ is equipped with the norm*

$$\|u\|_{W^{1,2}(\Omega)} = \|u\|_p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_p$$

or sometimes with the equivalent norm $\left(\|u\|_p^p + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_p^p\right)^{\frac{1}{p}}$ (if $1 \leq p \leq \infty$).

The examination of specific partial differential equations, especially those involving the bi-Laplacian, may necessitate the application of Sobolev spaces of order s exceeding 1. Therefore, we extend the definition of the Sobolev space $H^1(\Omega)$:

Definition 11. *Let $\Omega \subset \mathbb{R}^n$ be an open domain. The Sobolev space of order s on Ω the space*

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega), \forall \alpha \in \mathbb{N}^n. |\alpha| \leq s \right\}$$

$H_0^1(\Omega)$ consists of functions u that satisfy the following conditions:

Belonging to $H^1(\Omega)$: The function u must belong to the Sobolev space $H^1(\Omega)$: , which means:

$.u \in L^2(\Omega)$, i.e., u is square-integrable over Ω.

$.\nabla u \in L^2(\Omega)$ meaning the weak derivatives of u are also square-integrable.

Boundary condition : The function u must satisfy the condition $u = 0$ on the boundary $\partial\Omega$ in the weak sense. This means that u vanishes on average (in the sense of distribution) on the boundary of the domain, although it may not be strictly zero at every point on the boundary.

1.2.3 The Sobolev space with variable exponents

Let $\beta : \Omega \to [1, \infty]$ be a measurable function, where Ω is a bounded domain of $\mathbb{R}^n$,then we have the following definitions:

Definition 12. *The Lebesgue space with a variable exponent $\beta(\cdot)$ is defined by*

$$L^{\beta(\cdot)}(\Omega) := \left\{u : \Omega \to \mathbb{R};\ \text{measurable in}\ \ \Omega : \varrho_{\beta(\cdot)}(ku) < \infty, \text{for some } k > 0, \right\}$$

equipped with the norm $\varrho_{\beta(\cdot)}(ku) = \int_\Omega \frac{1}{\beta(x)} |v(x)|^{\beta(x)} dx$.

Definition 13. *The variable-exponent Sobolev space $W^{1,\beta(\cdot)}(\Omega)$ is:*

$$W^{1,\beta(\cdot)}(\Omega) := \left\{u \in L^{\beta(\cdot)}(\Omega) \text{ such that } \nabla v \text{ exists and } \ |\nabla v| \in L^{\beta(\cdot)}(\Omega)\right\}$$

Remark 14. *$W_0^{1,\beta(\cdot)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $L^{1,\beta(\cdot)}(\Omega)$.*

Remark 15. *$L^{\beta(\cdot)}(\Omega)$ is a Banach space equipped with the following Luxembourg-type norm*

$$|v|_{\beta(\cdot)} := \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{v(x)}{\lambda}\right|^{\beta(\cdot)} dx \le 1\right\}$$

Remark 16. $W^{1,\beta(\cdot)}(\Omega)$ *is a Banach space with respect to the norm*

$$\|v\|_{W^{1,\beta(\cdot)}(\Omega)} = \|v\|_{\beta(\cdot)} + \|\Delta v\|_{\beta(\cdot)}$$

Lemma 17. *If* $\beta : \Omega \to [1, \infty)$ *is a measurable function with* $\beta_2 < \infty$, *then* $C_0^\infty(\Omega)$ *is dense in* $L^{\beta(\cdot)}(\Omega)$.[[8]]

1.2.4 Log-Hölder continuity condition

The exponent $p(\cdot) : \Omega \to [1, \infty]$ is said to be satisfying the log-Hölder continuity condition; if there exists a constant $c > 0$ such that, for all δ with $0 < \delta < 1$,

$$|p(x) - p(y)| \leq -\frac{c}{\log|x-y|}, \tag{1.1}$$

for all $x, y \in \Omega$, with $|x - y| < \delta$.

1.2.5 Embedding Property

Lemma 18 (07)**.** *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *with a smooth boundary* $\partial\Omega$. *Assume that* $p,k \in C(\Omega)$ *such that,* $1 < p_1 \leq p(x) \leq p_2 < +\infty$, $1 < k_1 \leq k(x) \leq k_2 < +\infty$, $\forall x \in \overline{\Omega}$ *and* $k(x) < p^*(x)$ *in* $\overline{\Omega}$ *with*

$$p^*(x) = \begin{cases} \frac{np(x)}{n-p(x)}, \text{ if } p_2 < n; \\ +\infty, \text{ if } p_2 \geq n \end{cases}$$

then we have continuous and compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{k(\cdot)}(\Omega)$. So, there exists $c_e > 0$ such that

$$\|u\|_k \leq c_e \|u\|_{W^{1,p(.)}}, \forall u \in W^{1,p(\cdot)}(\Omega).$$

1.2.6 Some inequalities

Inequalities play a crucial role across various fields of mathematics. For example, they are instrumental in assessing whether a particular space qualifies as a metric or normed space, as well as in demonstrating the existence and uniqueness of solutions to differential equations.

Young's Inequality

Theorem 19. [[7]]*Let* $1 \leq p \leq +\infty$, *then* $a, b > 0$, *Then for any* $\varepsilon > 0$, *we have*

$$ab \leq \varepsilon a^p + C_\varepsilon b^{p'}, \tag{1.2}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$ *and* $C_\varepsilon = \frac{1}{p'(\varepsilon p)^{\frac{p'}{p}}}$.

1) For $p = \frac{1}{\varepsilon} > 1$

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}, \tag{1.3}$$

2) For $p = p' = 2$

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}. \tag{1.4}$$

Poincare's Inequality

Poincare's inequality, named after the French mathematician Henri Poincare, provides a means to constrain a function by utilizing estimates of its derivatives along with the geometric properties of its domain. These estimates play a crucial role in the field of calculus of variations.

Lemma 20. *Let* Ω *be a bounded domain of* $\mathbb{R}^n$ *and* $p(\cdot)$ *satisfies* (1.1),*the*

$$\|u\|_{p(\cdot)} \leq c_* \|\nabla u\|_{p(\cdot)}, \text{ for all } \quad u \in W_0^{1,p(\cdot)}(\Omega). \tag{1.5}$$

Remark 21. *In particular, the space* $W_0^{1,p(\cdot)}(\Omega)$ *has an equivalent norm given by*

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{p(\cdot)}$$

Hölder's Inequality

Theorem 22. *Let* $1 \leq p \leq +\infty$,*if* $v \in L^p(\Omega)$ *and* $w \in L^{p'}(\Omega)$, *then* $vw \in L^1(\Omega)$ *and*

$$\|vw\|_{L^1} \leq \|v\|_{L^p} \|w\|_{L^{p'}}, \tag{1.6}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

For $p = p' = 2$,

Theorem 23. *Let* $v, w \in L^2(\Omega)$ *and* $w \in L^2(\Omega)$, *then* $vw \in L^1(\Omega)$ *and*

$$\|vw\|_{L^1} \leq \|v\|_{L^2} \|w\|_{L^2}. \tag{1.7}$$

General inequality

Theorem 24. *Given* $p, q, r > 0$ *with* $\frac{1}{p} + \frac{1}{p'} = \frac{1}{r}$, *if* $v \in L^p(\Omega)$ *and* $w \in L^q(\Omega)$ *then* $vw \in L^r(\Omega)$ *and the inequality*

$$\|vw\|_{L^r} \leq \|v\|_{L^p} \|w\|_{L^q}, \tag{1.8}$$

holds.

Minkowski's Inequality

Theorem 25. *Let* $1 \leq p < +\infty$ *be given. If* $v, w \in L^p(\Omega)$, *then* $v + w \in L^p(\Omega)$ *and*

$$\|v + w\|_{L^p} \leq \|v\|_{L^p} + \|w\|_{L^p}. \tag{1.9}$$

Komornik's Inequality I

Theorem 26. *Let* $h : \mathbb{R}^+ \to \mathbb{R}^+$ *be a non-increasing function and assume that there exists a constant* $c > 0$ *such that*

$$\int_t^{+\infty} h(s)ds \leq ch(t) \tag{1.10}$$

holds for all $t \geq 0$. *Then, we have*

$$h(t) \leq h(0) e^{1-\frac{t}{c}}, \forall t \geq c. \tag{1.11}$$

Komornik's Inequality II

Theorem 27. *Let* $h : \mathbb{R}^+ \to \mathbb{R}^+$ *be a non-increasing function and assume that there are two positive constants* α, β *such that the inequality*

$$\int_t^{+\infty} h^{\alpha+1}(s)ds \leq \beta h^{\alpha}(0)\, h(t) \tag{1.12}$$

holds for all $t \geq 0$. Then, we have

$$h(t) \leq h(0) \left(\frac{\beta+\alpha t}{\beta+\alpha\beta}\right)^{-\frac{1}{\alpha}}. \tag{1.13}$$

Definition 28. *Let K be a convex function on $]0,r]$, then the convex conjugate of K, in the sense of Young (see [[27]]), is defined as follows:*

$$K^{*}(s) = s(K^{'})^{-1}(s) - K\left[(K^{'})^{-1}(s)\right], \qquad \text{if } s \in (0, K^{'}(r)] \tag{1.14}$$

and K^{*}satisfies the following generalized Young's inequality

$$\alpha_1\alpha_2 \leq K*(\alpha_1) + K(\alpha_2), if \alpha_1 \in (0, K^{'}(r)], \alpha 2 \in (0, r]. \tag{1.15}$$

Let

$$\beta_1 := es\sin f_{x\in\Omega}\beta(x), \beta_2 := esssup_{x\in\Omega}\beta(x)$$

Lemma 29. (*see*[[22]]) *Let $\epsilon \in]0,1[$. Then there exists $\beta_\epsilon > 0$ such that*

$$s|lns| \leq s^2 + \beta_\epsilon s^{1-\epsilon}, \forall s > 0. \tag{1.16}$$

Lemma 30. *Let u be any function in $H_0^1(\Omega)$ and d be any positive real number. Then*

$$\int_\Omega u^2 \ln|u|\, dx \leq \frac{1}{2}\|u\|_2^2 \left(\ln\|u\|_2^2 - 2(1+\ln d)\right) + \frac{d^2}{2\pi}\|\nabla u\|_2^2 \tag{1.17}$$

Logarithmic Sobolev inequality

Lemma 31 (10)**.** *Let $\varpi > 0$, there exists a unique $\eta_2 > 0$ such that*

$$e^{-\frac{1}{2}-\frac{1}{s}} < \sqrt{\frac{2\pi\varpi}{s}},, \forall s \in (0, \eta_2). \tag{1.18}$$

where $d = e^{-\frac{1}{2}-\frac{1}{s}}$.

Proof. Let $g(s) = \sqrt{\frac{2\pi\varpi}{s}} - e^{-\frac{1}{2}-\frac{1}{s}}$, then g is a continuous and decreasing function on $(0, \infty)$, with $\lim\limits_{s\to 0^+} g(s) = +\infty$ and $\lim\limits_{x\to +\infty} g(x) = -e^{-\frac{1}{2}}$.

Then, there exists a unique $\eta > 0$ such that $g(\eta) = 0$ and (1.18) hold ■

Remark 32. *Lemma* (10) *shows that the selection of η in (H3) is possible*

Chapter 2

Existence

2.1 Presentation of the problem

In this chapter we are concerned with the following problem

$$\begin{cases} u_{tt}-\mu_1\Delta u+\int_0^t h(t-s)\Delta u(s)ds+|u_t|^{\sigma(\cdot)-2}u_t=u\ln|u|^{\eta(x)} \text{ in } \Omega\times]0,+\infty[\ , \\ u=\frac{\partial u}{\partial v}=0 \qquad \text{on } \partial\Omega\times]0,+\infty[\\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x) \quad \text{in } \Omega \end{cases} \tag{2.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, 3bd is the unit outer normal to $\partial\Omega$, u_0,and u_1 are the given data, b is a relaxation function and $\sigma(.), \eta(.)$ is a variable exponent.

Problem (2.1) contains three types of problems:

1) **Viscoelasticitic problems**

The importance of viscoelastic characteristics in materials is increasingly recognised due to rapid improvements in the rubber and plastics sectors. In the recent two decades, extensive advancements in the examination of constitutive relations, failure theories, and life expectancy forecasts for viscoelastic materials and structures have been recorded and assessed (see [**?**]). A significant corpus of work pertains to the stabilisation of viscoelastic wave equations, with numerous findings documented in this domain. Multiple contributions have sought to generalise decay rates by integrating a wider array of relaxation functions, thus offering complete decay rates. Indeed, the evolution of relaxation function generalisation has undergone several phases, which we outline as follows:

A) As in (see [1]), the relaxation function h satisfies , for two positive constants a_1 and a_2,

$$-a_1 h(t) \leq h^{'}(t) \leq -a_2 h(t), \; t \geq 0$$

B) As in (see [2], [3]), the relaxation function h satisfies

$$h^{'}(t) \leq -a(t) h(t), t \geq 0,$$

where $a : R^+ \to R^+$is a nonincreasing differentiable function.

C) As in (see [4]), the relaxation function h satisfies

$$h^{'}(t) \leq -\chi(h(t)),$$

where χ is a positive function, $\chi(0) = \chi'(0) = 0$, and χ is strictly increasing and strictly convex near the origin.

D) As in (see [5]), the relaxation function h satisfies

$$h^{'}(t) \leq -a(t) b^p(t), \forall t \geq 0, 1 \leq p < \frac{3}{2}$$

E) As in (see [6]), the relaxation function h satisfies

$$h^{'}(t) \leq -a(t) H(h(t)), \tag{2.2}$$

where $H \in C^1(\mathbb{R})$, with $H(0) = 0$ and H is linear or strictly increasing and strictly convex function C^2 near the origin.

2) Variable-exponent nonlinearity problems .

The advancement of science and technology has necessitated the development of more sophisticated mathematical functional spaces to properly analyse and understand the intricate physical and engineering models. Consider fluid dynamics; the behaviour of electrorheological fluids, or smart fluids, demonstrates a marked alteration in viscosity when exposed to an electric field.

Lebesgue and Sobolev spaces with variable exponents have demonstrated efficacy in addressing a range of intricate issues, such as those related to fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration through porous media, and image processing. For additional insights on these subjects, please consult references (see [7] – [8]). For hyperbolic issues involving variable-exponent nonlinearities, we suggest consulting references (see [19] – [10]). If you need additional results concerning issues related to power-type nonlinearity, the references (see [11] – [13]) will offer pertinent information.

3) **Logarithmic source term.**

The logarithmic nonlinearity intrinsically arises in inflationary cosmology and within the frameworks of supersymmetric field theories, quantum mechanics, and nuclear physics (see [14] – [15]). Logarithmic nonlinearity issues have widespread applications in multiple domains of physics, such as nuclear physics, optics, and geophysics (see [16] – [18]).

2.2 Processing and tarming the problem

In this part, we examine problem (2.1) and demonstrate the global existence of solutions through the Faedo-Galerkin technique. We subsequently derive explicit and general decay results of the solution under appropriate conditions on the variable exponent $\sigma(.)$, $\eta(.)$and a very broad assumption regarding the relaxation function (2.2). To the best of our knowledge, this particular issue has not been previously explored in the realm of nonlinearity with variable exponents, we need to consider the following three hypotheses :

(H1) The relaxation function $h:\mathbb{R}^+\to\mathbb{R}^+$ is a C^1 nonincreasing function satisfying

$$h'(0)>0, \mu_1-\int_0^\infty h'(s)ds=\varpi>0, \tag{2.3}$$

and there exists a C^1function $H:(0,\infty)\to(0,\infty)$ which is strictly increasing and strictly convex C^2 function on $]0,r]$, $r\leq h(0)$, with $H(0)=0$, such that

$$h^{'}(t)\leq -a(t)H(h(t)), \forall t\geq 0, \tag{2.4}$$

where a is a positive nonincreasing differentiable function.

(H2) Let $\sigma : \overline{\Omega} \to [1, \infty)$ is a continuous function satisfies the log-Holder continuity condition such that $\sigma_1 := essinf_{x\in\Omega}\sigma(x)$, $\sigma 2 := esssup_{x\in\Omega}\sigma(x)$ and $1 < \sigma_1 < \sigma(x) \leq \sigma_2$, where

$$\begin{cases} \sigma_2 < \infty, n = 1, 2; \\ \sigma_2 \leq \frac{2\pi}{n-2}, n \geq 3 \end{cases}$$

(H3) Let $\eta : \overline{\Omega} \to [0, \infty)$ is a continuous function satisfies the log-Holder continuity condition. such that $\eta_1 := essinf_{x\in\Omega}\eta(x)$, $\eta_2 := esssup_{x\in\Omega}\eta(x)$ and $0 \leq \eta_1 < \eta(x) \leq \eta_2$, where

$$\sqrt{\frac{2\pi\varpi}{\eta_2}} = e^{-\frac{3}{2}-\frac{1}{\eta_2}} \tag{2.5}$$

Remark 33. *Using the facts that $H(0) = 0$ and H is strictly convex on $(0, r]$, then*

$$H(\theta s) \leq \theta H(s), 0 \leq \theta \leq 1 \text{ and } s \in (0, r]. \tag{2.6}$$

Remark 34. *If H is a strictly increasing and strictly convex C^2function on (0,r], with $H(0) = H^{'}(0) = 0$, then there is a strictly convex and strictly increasing C^2function $\overline{H}$: $[0, +\infty) - \to [0, +\infty)$ which is an extension of H. For simplicity, in the rest of this paper, we use H instead of $\overline{H}$*

2.3 Existence

This section presents the local existence theorem, the proof of which can be derived by integrating the arguments from references ([19], [26]). Furthermore, we assert and demonstrate a global existence theorem contingent upon smallness conditions of the initial data (u_0, u_1).

Definition 35. *We define the following functionals which are needed for establishing the global existence*

$$\begin{aligned} E(t) &= \frac{1}{2}\left[\|u_t\|_2^2 + \left(1 - \int_0^{+\infty} h(s)\,ds\right)\|\Delta u\|_2^2 - (h \circ \Delta u)(t) + \frac{\eta(x)+2}{2}\|u\|_2^2\right] \\ &\quad -\frac{1}{2}\int_\Omega \left(u^2 \ln |u|^{\eta(x)}\right) dx \end{aligned} \tag{2.7}$$

where $(h \circ \phi)(t) : = \int_0^t h(t-s)\|\phi(t) - \phi(t)\|_2^2\,ds$, for $v \in L^2_{loc}(R^+; L^2(\Omega))$,

$E(t)$ represents the modified energy functional associated to problem (2.1).

The energy functional associated to problem (2.1) satisfies, for any $t \geq 0$,

$$E'(t) = \frac{1}{2}(h' \circ \nabla u)(t) - \frac{1}{2}h(t)\|\nabla u\|_2^2 - \int_\Omega |u_t|^{\sigma(x)}\,dx \tag{2.8}$$

Lemma 36. *(see [[6]]) Assume (H1) holds. Then, for any $t \geq t_0$,, we have*

$$E'(t) \leq -ca(t)\int_0^{t_0} h(s)\|\nabla u(t) - \nabla u(t-s)\|_2^2\,ds < 0.$$

We can use,

Lemma 37. *Assume that h satisfies (H1). Then, for $u \in H_0^1(\Omega)$,*

$$\begin{cases} \int_\Omega \left(\int_0^t h(t-s)\left(u((t) - u(s)\,ds)^2\right) dx\right) \leq c(h \circ \Delta u)(t), \\ \int_\Omega \left(\int_0^t h'(t-s)\left(u((t) - u(s)\,ds)^2\right) dx\right) \leq c(h' \circ \Delta u)(t) \end{cases} \tag{2.9}$$

Multiplying (2.1) by u_t, integrating over Ω and using the boundary conditions, imply (2.8).

Since $E'(t) \leq 0$, imply $E(t)$ is decreasing on $[0, T[$, then

$$E(t) \leq E(0), \forall t \in [0, T[.$$

2.4 Local existence

Our main result in this section is to show the existence and the uniqueness of the solution of problem(2.1). Our proof methodology integrates the Faedo–Galerkin approximation, the compactness technique, and the fixed point theorem. The proof technique closely follows the arguments of (see[[**?**], [5]]). We consider the following theorem :

Theorem 38. *Suppose conditions (H1)–(H3) hold and* $(u_0, u_1) \in H_0^1(\Omega)\times\ L^2(\Omega)$ *Then, there exists* $T > 0$*, such that problem* (2.1) *has a weak solution*

$$\begin{aligned}
u &\in L^\infty((0,T), H_0^1(\Omega)),\\
u_t &\in L^\infty((0,T), L^2(\Omega)) \cap L^{\sigma(.)}(\Omega \times (0,T))\\
u_{tt} &\in L^\infty\left([0,T], L^2(\Omega) \cap L^\infty\left([0,T] \times L^2(\Omega)\right)\right).
\end{aligned}$$

will be established through several lemmas. Our method of the proof is based on the combination of the Faedo–Galerkin approximation, the compactness method and the fixed point theorem. We use Faedo–Galerkin's method to construct approximate solution. Let $T > 0$ be fixed and denote byV_n the space generated by the set $\{w_n, n \in IN\}$ is a basis of $H^2(\Omega) \cap H^2(\partial\Omega(\Omega))$, we define also for $1 \leq j \leq n$, the sequence $\phi_j(x,n)$ as follows $\phi_j(x,0) = w_j(x)$. Then we may extend $\phi_j(x,0)$ by $\phi_j(x,n)$ over $L^2(\partial\Omega \times [0,1])$ and denote Un to be the space generated by $\{\phi_1, ...\phi_n\}, (n = 1,2,3,...)$. We construct approximate solutions $(u_n(t)\ n = 1,2,3,...)$ in the form

$$u^n(t) = \sum_{j=1}^{n} u^{n,j}(t)\, w_j \tag{2.10}$$

where $u^n(t)$ is the solution of the following approximate problem corresponding to (2.1). By using Green's formula,

we deduce that $u^n(t)$ verifies the following system of ODEs:

$$\begin{aligned}
&\langle u_{tt}^n, w_j\rangle_\Omega + \langle \nabla u^n, \nabla w_j\rangle_\Omega + \left\langle \int_0^t g(t-s)\,\nabla v^n(s)\,ds, \nabla w_j \right\rangle_\Omega - \langle |\nabla u_t^n|^{\sigma(\cdot)-2}\nabla u_t^n, \nabla w_j\rangle_\Omega\\
&= \left\langle u^n(t) \ln |u^n|^{\eta(x)}, w_j \right\rangle_\Omega
\end{aligned} \tag{2.11}$$

for $j = 1, ..., n$. More specifically

$$u^n(0) = \sum_{j=1}^{n} u^{n,j}(0)\, w_j \tag{2.12}$$

where $u^n(0) = \langle u^0, w_j \rangle, u_t^n(0) = \langle u^1, w_j \rangle, j = 1, ..., n$. Obviously, $u^n(0) \to u^0$ strongly in $H^1_{\partial\Omega}(\Omega)$, $u_t^n(0) \to u^0$ strongly in $L^2(\Omega)$ as $n \to \infty$.

By virtue of the theory of ordinary differential equations, the system (2.11)–(2.12) has a unique local solution which is extended to a maximal interval $[0, T_n[$ (with $0 < T_n \le +\infty$). In the next step, we obtain a priori estimates for the solution of the system (2.11)–(2.12), so that it can be extended beyond $[0, T_n[$ to obtain a solution defined for all $t > 0$.

First estimate. By the same procedure as in (2.10) , for u fixed in $C([0,T], H^1_{\partial\Omega}(\Omega))$, putting $w_j = u_n$into (2.11),

we obtain

$$\begin{aligned} &\langle u_{tt}^n(t), u_t^n \rangle_\Omega + \langle \nabla u^n, \nabla u_t^n \rangle_\Omega + \left\langle \nabla |u_t^n(t)|^{\sigma(x)-2} u_t^n(t), \nabla u_t^n \right\rangle_\Omega + \left\langle \int_0^t h(t-s) \nabla u^n(s)\, ds, \nabla u_t^n \right\rangle_\Omega \\ = &\left\langle u^n(t) \ln |u^n(t)|^{\eta(t)}, u_t^n \right\rangle_\Omega \qquad (2.13) \end{aligned}$$

by using Lemma and (2.10), we get easily

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\|u_t^n(t)\|_2^2 + \frac{1}{2}\|u_t^n(t)\|_{2,\partial\Omega}^2 + \frac{1}{2}\|\nabla u^n(t)\|_2^2 + (h \circ \nabla u^n)(t)\right] \\ &+ \int_\Omega (\nabla u_t^n)^2(x,t)\, dx + \frac{1}{2}(h' \circ \nabla u^n)(t) \\ &+ \frac{1}{2}h(t)\|\nabla u^n(t)\|_2^2 \qquad (2.14) \\ = &\left\langle \ln |u^n(t)|^{\sigma(x)-2} u^n(t), u_t^n \right\rangle_\Omega \end{aligned}$$

Integrating (2.14) over $(0,t)$, we get to

$$\begin{aligned} &\frac{1}{2}\|u_t^n(t)\|_2^2 + \frac{1}{2}\|u_t^n(t)\|_{2,\partial\Omega}^2 + \frac{1}{2}\|\nabla u^n(t)\|_2^2 - \frac{1}{p}\|u_t^n\|_p^p + \mu_1 \|u_t^n(t)\|_{2,\partial\Omega}^2 \\ &(h' \circ \nabla u^n)(t) \\ &+ \int_0^t \int_\Omega (\nabla u_s^n)^2(x,t)\, dx ds - \int_0^t \frac{1}{2}h(s)\|\nabla u^n(s)\|_2^2\, ds \\ = &\frac{1}{2}\|u_t^n(0)\|_2^2 + \frac{1}{2}\|u_t^n(0)\|_{2,\partial\Omega}^2 + \frac{1}{2}\|\nabla u_t^n(0)\|_2^2 - \frac{1}{p+1}\|u^n(0)\|_p^p \qquad (2.15) \end{aligned}$$

$$E^n(t) - E^n(0)$$

$$- (h' \circ \nabla u^n)(t) - \int_0^t \int_\Omega (\nabla u_s^n)^2 (x,t)\, dxds$$

$$+ \int_0^t \frac{1}{2} h(s) \|\nabla u^n(s)\|_2^2 \, ds - \tag{2.16}$$

After deriving(2.16), inserting (2.15) in (2.14) and taking ε sufficiently small, we get

$$(E^n(t))' + \left(\mu_1 - \frac{1}{2}\right) \|u_t^n(t)\|_{2,\partial\Omega}^2 \leq L_1 \tag{2.17}$$

where L_1 is a positive constant depending on $(E^n(0))'$, T arbitrary positive .

Second estimate. First, we estimate $(u_{tt}^n(0))$ in (2.11) and taking $t = 0$, we obtain

$$\begin{aligned}
\|u_t^n(0)\|_2^2 + \|u_t^n(0)\|_{2,\partial\Omega}^2 &\leq \left\|\nabla u^{0n}(0)\right\|_2^2 + 2\mu_1 \left\|u^{1n}\right\|_{2,\partial\Omega}^2 + \left\|u^{0n} \ln\left(u^{0n}\right)\right\|_2^2 \\
&\leq \left\|\nabla u^0(0)\right\|_2^2 + 2\mu_1 \left\|u^1\right\|_{2,\partial\Omega}^2 + \left\|u^{0n} \ln\left(u^{0n}\right)\right\|_2^2 \\
&\leq C
\end{aligned} \tag{2.18}$$

where C is a positive constant. Now, differentiating (2.11) with respect to t

$$\begin{aligned}
&\langle u_{tt.t}^n, w_j \rangle_\Omega + \langle u_{tt.t}^n, w_j \rangle_{\partial\Omega} + \left\langle \Delta^N u_t^n, w_j \right\rangle_\Omega + \langle \nabla u_{tt}^n, w_j \rangle_\Omega \qquad (2.19) \\
&+ \left\langle \frac{d}{dt} \int_0^t h(t-s) \nabla u^n(s)\, ds, \nabla w_j \right\rangle_\Omega \\
=\ & \left\langle \frac{d}{dt} \left(u^n(t) \ln |u^n(t)|^{\eta(x)} \right), w_j \right\rangle_\Omega
\end{aligned}$$

Also as in(see[[17], [18]]) , we have

$$\langle u_{tt}^n(t), u_{tt}^n(t) \rangle_{\partial\Omega} = \int_{\partial\Omega} \left(\frac{\partial}{\partial t} (u_t^n(t)) \right)^2 d\gamma \tag{2.20}$$

Multiplying (2.19) by $u_{tt}^{n.j}(t)$,

$$\left\langle u_{tt}^{n}, u_{tt}^{n.j}\right\rangle_{\Omega} + \left\langle u_{tt}^{n}, u_{tt}^{n.j}\right\rangle_{\partial\Omega} + \left\langle \Delta u_{t}^{n}, u_{tt}^{n.j}\right\rangle_{\Omega} + \left\langle \nabla u_{tt}^{n}, u_{tt}^{n.j}\right\rangle_{\Omega} + \left\langle \frac{d}{dt}\int_{0}^{t} h\left(t-s\right)\nabla u^{n}\left(s\right)ds, \nabla u_{tt}^{n.j}\right\rangle_{\Omega}$$

$$+\mu_{1}\left\langle u_{tt}^{n}, u_{tt}^{n.j}\right\rangle_{\partial\Omega} = \left\langle \frac{d}{dt}\left(u^{n}\left(t\right)\ln\left|u^{n}\left(t\right)\right|^{\eta(x)}\right), u_{tt}^{n.j}\right\rangle_{\Omega}$$

summing over j from 1 to n, and inserting (2.15)–(2.18) in (2.16), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|u_{t}^{n}\left(t\right)\right\|_{2}^{2} + \left\|u_{t}^{n}\left(t\right)\right\|_{2,\partial\Omega}^{2} + \left\|\nabla u_{t}^{n}\left(t\right)\right\|_{2}^{2} + \left\|\nabla u_{t}^{n}\left(t\right)\right\|_{2}^{2}\right] + h\left(0\right)\frac{d}{dt}\left\langle \nabla u^{n}\left(t\right), \nabla u_{t}^{n}\left(t\right)\right\rangle_{\Omega} \\
&+h\left(0\right)\left\|\nabla u_{t}^{n}\left(t\right)\right\|_{2}^{2} - \frac{d}{dt}\int_{0}^{t} h'\left(t-s\right)\left\langle \nabla u^{n}\left(t\right), \nabla u_{t}^{n}\left(t\right)\right\rangle_{\Omega} ds + h'\left(0\right)\left\langle \nabla u^{n}\left(t\right), \nabla u_{t}^{n}\left(t\right)\right\rangle_{\Omega} \\
&+\int_{0}^{t} h''\left(t-s\right)\left\langle \nabla u^{n}\left(t\right), \nabla u_{t}^{n}\left(t\right)\right\rangle_{\Omega} ds + \mu\left\|u_{tt}^{n}\left(t\right)\right\|_{2,\partial\Omega}^{2} + \left\|u_{tt}^{n}\left(t\right)\right\|_{2,\partial\Omega}^{2} \\
= &\left\langle \frac{d}{dt}\left(u^{n}\left(t\right)\ln\left|u^{n}\left(t\right)\right|^{\eta(x)}\right), u_{t}^{n}\right\rangle_{\Omega} \qquad (2.21)
\end{aligned}$$

Taking the sum of (2.21) , we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\left\|u_{t}^{n}\left(t\right)\right\|_{2}^{2} + \left\|\nabla u_{t}^{n}\left(t\right)\right\|_{2}^{2} + \left\|\nabla u_{t}^{n}\left(t\right)\right\|_{2}^{2}\right] \\
&+h\left(0\right)\left\|\nabla u_{t}^{n}\left(t\right)\right\|_{2}^{2} + \frac{1}{2}\left\|\nabla u_{tt}^{n}\left(t\right)\right\|_{2}^{2} \\
= \;& h\left(0\right)\frac{d}{dt}\left\langle \nabla u^{n}\left(t\right), \nabla u_{t}^{n}\left(t\right)\right\rangle_{\Omega} + \frac{d}{dt}\int_{0}^{t} h'\left(t-s\right)\left\langle \nabla u^{n}\left(t\right), \nabla u_{t}^{n}\left(t\right)\right\rangle_{\Omega} ds \\
&-h'\left(0\right)\left\langle \nabla u^{n}\left(t\right), \nabla u_{t}^{n}\left(t\right)\right\rangle_{\Omega} - \int_{0}^{t} h''\left(t-s\right)\left\langle \nabla u^{n}\left(t\right), \nabla u_{t}^{n}\left(t\right)\right\rangle_{\Omega} ds + \frac{1}{2}\left\|u_{tt}^{n}\left(t\right)\right\|_{2}^{2} \\
&+\left\langle \frac{d}{dt}\left(u^{n}\left(t\right)\ln\left|u^{n}\left(t\right)\right|^{\eta(x)}\right), u_{t}^{n}\right\rangle_{\Omega} \qquad (2.22)
\end{aligned}$$

Using Cauchy–Schwarz and Young's inequalities, we conclude the following estimates:

$$\left|\, h'\left(0\right)\left\langle \nabla u^{n}\left(t\right), \nabla u_{t}^{n}\left(t\right)\right\rangle_{\Omega}\right| \leq c'\epsilon\left\|\nabla u^{n}\left(t\right)\right\|_{2}^{2} + c'\frac{\left(h'\left(0\right)\right)^{2}}{4\epsilon}\left\|\nabla u_{t}^{n}\left(t\right)\right\|_{2}^{2} \qquad (2.23)$$

and

$$\left|\int_0^t h''(t-s)\langle \nabla u^n(t), \nabla u_t^n(t)\rangle_\Omega ds\right| \leq c'\|\nabla u_t^n(t)\|_2 \int_0^t h''(t-s)\|\nabla u_t^n(s)\|_2^2 ds$$

$$\leq c'\epsilon \|h''\|_{L^1} \int_0^t h''(t-s)\|\nabla u^n(s)\|_2^2 ds + \frac{c'}{4\epsilon}\|\nabla u_t^n(t)\|_2^2 \tag{2.24}$$

Replacing (2.23)–(2.24) in (2.22), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left[\|u_{tt}^n(t)\|_2^2 + \|\nabla u_t^n(t)\|_2^2 - \|\nabla u_t^n(t)\|_2^2\right] \\
&+h(0)\|\nabla u_t^n(t)\|_2^2 + \frac{1}{2}\|\nabla u_{tt}^n(t)\|_2^2 \\
\leq\ & c\|u_{tt}^n(t)\|_2^2 + c'\|\nabla u^n(t)\|_2^2 + \frac{c'(h(0))^2}{4\epsilon}\|\nabla u_t^n(t)\|_2^2 \\
&+\frac{c'}{4\epsilon}\|\nabla u_{tt}^n(t)\|_2^2 + c'\epsilon\int_0^t h''(t-s)\|\nabla u^n(t)\|_2^2 ds - h(0)\frac{d}{dt}\langle \nabla u^n(t), \nabla u_t^n(t)\rangle_\Omega \\
&+\frac{d}{dt}\int_0^t h'(t-s)\langle \nabla u^n(t), \nabla u_t^n(t)\rangle_\Omega ds + \frac{c'}{4\epsilon}\|\nabla u^n(t)\|_2^2 + c'\epsilon\|h''\|_{L^1}\int_0^t h''(t-s)\|\nabla u^n(s)\|_2^2 ds \\
&+\frac{1}{4\lambda}\left\|\frac{d}{dt}f(u^n(t))\right\|_2^2 + \lambda\|u_{tt}^n(t)\|_2^2 + \frac{c}{\mu}\|z_t^n(\gamma,1,t)\|_2^2 + \mu\|u_{tt}^n(t)\|_{2,\Gamma_1}^2
\end{aligned} \tag{2.25}$$

integrating the last inequality over $(0,t)$ and using Gronwall's lemma, we obtain

$$
\begin{aligned}
& \frac{1}{2}\left[\left\|u_{tt}^{n}(t)\right\|_2^2+\left\|\nabla u_t^n(t)\right\|_2^2\right] \\
& +\left(\frac{1}{2}-\frac{c'}{4\epsilon}\right)\int_0^t\left\|\nabla u_{ss}^n(s)\right\|_2^2 ds-\frac{1}{2}\int_0^t\left\|\nabla u_{ss}^n(s)\right\|_2^2 \\
\leq\ & \frac{1}{2}\left[\left\|u_{tt}^{n}(0)\right\|_2^2+\left\|\nabla u_t^n(0)\right\|_2^2\right] \\
& +\int_0^t h(0)\frac{d}{ds}\left\langle\nabla u^n(s),\nabla u_t^n(s)\right\rangle_\Omega ds+\int_0^t h'(t-s)\left\langle\nabla u^n(s),\nabla u_t^n(s)\right\rangle_\Omega ds \\
& +\left(\frac{c'(h(0))^2}{4\epsilon}+\frac{c'}{4\epsilon}-c'h(0)\right)\int_0^t\left\|\nabla u_s^n(s)\right\|_2^2 ds+\left(c'\epsilon\left\|h''\right\|_{L^1}+\frac{c'}{4\epsilon}+c'\epsilon\right)\int_0^t\left\|\nabla u^n(s)\right\|_2^2 ds \\
& +\lambda\int_0^t\left\|u_{ss}^n(s)\right\|_{2,\Omega}^2 ds+c\int_0^t\left\|u_{ss}^n(s)\right\|_2^2 ds \qquad (2.26)
\end{aligned}
$$

As in [[22]], using again Cauchy–Schwarz and Young's inequalities, we conclude the following estimates:

$$
\left|\int_0^t h'(t-s)\left\langle\nabla u^n(s),\nabla u_t^n(s)\right\rangle_\Omega ds\right|\leq\epsilon\left\|\nabla u_t^n(t)\right\|_2^2+\frac{r\left\|h''\right\|_{L^1}\left\|h''\right\|_{L^\infty}}{4\epsilon}\int_0^t\left\|\nabla u^n(s)\right\|_2^2 ds \qquad (2.27)
$$

$$
\int_0^t\frac{d}{ds}\left\langle\nabla u^n(s),\nabla u_t^n(s)\right\rangle_\Omega ds=\left\langle\nabla u^n(t),\nabla u_t^n(t)\right\rangle_\Omega-\left\langle\nabla u^n(0),\nabla u_t^n(0)\right\rangle_\Omega \qquad (2.28)
$$

$$
\begin{aligned}
\left|\left\langle\nabla u^n(t),\nabla u_t^n(t)\right\rangle_\Omega\right| & \leq\left\|\nabla u^n(t)\right\|_2\left\|\nabla u_t^n(t)\right\|_2 \\
& \leq\frac{c'}{4\epsilon}\left\|\nabla u^n(t)\right\|_2+c'\epsilon\left\|\nabla u_t^n(t)\right\|_2 \qquad (2.29)
\end{aligned}
$$

$$
\left|\left\langle\nabla u^n(t),\nabla u_t^n(t)\right\rangle_\Omega\right|\leq\frac{1}{4\epsilon}\left\|\nabla u^n(0)\right\|_2+\epsilon\left\|\nabla u_t^n(0)\right\|_2 \qquad (2.30)
$$

Then from (2.27)–(2.30), after choosing ϵ,μ,λ small enough and using Gronwall's lemma, we obtain

$$
\begin{aligned}
& \left[\|u_{tt}^{n}(t)\|_{2}^{2}+\|\nabla u_{t}^{n}(t)\|_{2}^{2}-\|\nabla u_{t}^{n}(t)\|_{2}^{2}\right] \\
& +\left(\frac{1}{2}-\frac{c'}{4\epsilon}\right)\int_{0}^{t}\|\nabla u_{ss}^{n}(s)\|_{2}^{2}\,ds-\frac{1}{2}\int_{0}^{t}\|\nabla u_{ss}^{n}(s)\|\,ds \\
\leq\ & \left[\|u_{tt}^{n}(0)\|_{2}^{2}+\|\nabla u_{t}^{n}(0)\|_{2}^{2}\right]+c\int_{0}^{t}\|u_{ss}^{n}(s)\|_{2}^{2}\,ds \\
& +\left(\frac{r\|h''\|_{L^{1}}\|h''\|_{L^{\infty}}}{4\epsilon}+\frac{c'(h(0))^{2}}{4\epsilon}+c'\left(\epsilon\|h''\|_{L^{1}}+\frac{1}{4\epsilon}+\epsilon\right)\int_{0}^{t}\|\nabla u^{n}(s)\|_{2}^{2}\,ds\right) \\
& +c'\left(\frac{(h(0))^{2}}{4\epsilon}+\frac{1}{4\epsilon}+\epsilon-h(0)\right)\int_{0}^{t}\|\nabla u_{s}^{n}(s)\|_{2}^{2}\,ds+(c+\mu)\int_{0}^{t}\|u_{ss}^{n}(s)\|_{2,\Omega}^{2}\,ds \\
& +\frac{c'(h(0))^{2}}{4\epsilon}\|\nabla u^{n}(t)\|_{2}^{2}+(c'\epsilon+\epsilon)\|\nabla u_{t}^{n}(t)\|_{2}^{2} \qquad (2.31)
\end{aligned}
$$

For ϵ sufficiently small and by taking all constants above positive. Finally, using Gronwall's lemma, we deduce that

$$
\|u_{tt}^{n}(t)\|_{2}^{2}+\|\nabla u_{t}^{n}(t)\|_{2}^{2}-\|\nabla u_{t}^{n}(t)\|_{2}^{2}-\frac{1}{2}\int_{0}^{t}\|\nabla u_{ss}^{n}(s)\|\,ds\leq C \qquad (2.32)
$$

where C is a positive constant.

Third estimate. As in(see[[20], [21]]). Replacing w_j by $-\Delta w_j$ in (2.11), multiplying the result by $u_t^{n,j}(t)$, summing over j from 1 to n,implies

$$
\begin{aligned}
& \frac{1}{2}\frac{d}{dt}\left[\|\nabla u_{t}^{n}(t)\|_{2}^{2}+(h\circ\Delta u^{n})(t)+\|\Delta u_{t}^{n}(t)\|_{2}^{2}\left(1-\int_{0}^{t}h(s)\,ds\right)\right] \\
& +\left\|\Delta^{N}u_{t}^{n}(t)\right\|_{2}^{2}+ \\
=\ & \left\langle u^{n}(t)\ln|u^{n}(t)|^{\eta(x)},\nabla u_{t}^{n}\right\rangle_{\Omega} \qquad (2.33)
\end{aligned}
$$

Replacing φ_j by $-\Delta\varphi_j$ in (2.11), Let us define

$$
L^{n}(t)=\|\Delta u_{t}^{n}(t)\|_{2}^{2}\left(1-\int_{0}^{t}h(s)\,ds\right)\left[\|\nabla u_{t}^{n}(t)\|_{2}^{2}+(h\circ\Delta u^{n})(t)\right] \qquad (2.34)
$$

Combining (2.33) together, yields

$$\frac{1}{2}\frac{d}{dt}L^n(t) + \|\Delta u_t^n(t)\|_2^2 = \left\langle u^n(t)\ln|u^n(t)|^{\eta(x)}, \nabla u_t^n\right\rangle_\Omega \tag{2.35}$$

$$+\frac{1}{2}\|\nabla u_t^n(t)\|_2^2 - \left\|\Delta^N u_t^n(t)\right\|_2^2 + \frac{1}{2}(h'\circ\Delta u^n)(t) \tag{2.36}$$

Using (H0), (H1) and integrating (2.35) over $(0,t)$, we obtain

$$\begin{aligned} & L^n(t) + \int_0^t \|\Delta u_s^n(s)\|_2^2\, ds \\ \leq\ & \left[\|\nabla u^n(0)\|_2^2 + \|\Delta u_t^n(0)\|_2^2\right] \\ & + \int_0^t \left\langle u^n(s)\ln|u^n(s)|^{\eta(x)}, \nabla u_s^n\right\rangle_\Omega ds \end{aligned} \tag{2.37}$$

Let us consider $u^n(t)\ln|u^n(t)|^{\eta(x)}$. Obviously, by using Young's inequality, the last terms in (2.37) can be estimated as follows

$$\int_0^t \left\langle u^n(s)\ln|u^n(s)|^{\eta(x)}, \nabla u_s^n\right\rangle_\Omega ds \leq \frac{1}{4\beta}\int_0^t \left\|\nabla u^n(s)\ln|u^n(s)|^{\eta(x)}\right\|_2^2 ds + \beta\int_0^t \|\Delta u_s^n(s)\|_2^2\, ds \tag{2.38}$$

Inserting (2.38) into (2.37), for ϵ and β small enough and every $n > 1$, we get

$$L^n(t) + \int_0^t \|\Delta u_s^n(s)\|_2^2\, ds \leq C \tag{2.39}$$

where $C > 0$ is positive constant independent of n and T.

We observe that estimates (2.17), (2.32) and (2.39) hold that for all $t \geq 0$. Therefore, we conclude that

$$(u^n) \quad \text{is bounded in} \quad L^\infty\left((0,T); H^1_{\partial\Omega}(\Omega)\right) \tag{2.40}$$

$$(u_t^n) \quad \text{is bounded in} \quad L^\infty\left((0,T); H^1_{\partial\Omega}(\Omega)\right) \tag{2.41}$$

$$(u_{tt}^n) \quad \text{is bounded in} \quad L^\infty\left((0,T); L^2(\Omega)\right) \tag{2.42}$$

We can pass to the limit when $n \to \infty$, we deduce from (2.40)-(2.42),

$$u^n \to u \text{ weakly star in } L^\infty\left((0,T);H^2(\Omega)\cap H^1_{\partial\Omega}(\Omega)\right) \tag{2.43}$$

$$u^n_{tt} \to u_{tt} \text{ weakly star in } L^\infty\left((0,T);L^2(\Omega)\right) \tag{2.44}$$

For suitable functions $u \in L^\infty((0,T);H^1_{\partial\Omega}(\Omega))$. We have to

show that (u is a solution of (2.1). Using the embedding

$$L^\infty\left((0,T);H^1_{\partial\Omega}(\Omega)\right) \hookrightarrow L^2\left((0,T);H^1_{\partial\Omega}(\Omega)\right)$$

$$H^1((0,T)\times\Omega) \hookrightarrow L^2((0,T)\times\Omega)$$

From (2.41) we have that (u^n_t) is bounded in

$$L^\infty\left((0,T);H^1_{\partial\Omega}(\Omega)\right) \hookrightarrow L^2\left((0,T);H^1_{\partial\Omega}(\Omega)\right)$$

since (u^n_{tt}) is bounded in

$$L^\infty\left((0,T);H^1_{\partial\Omega}(\Omega)\right) \hookrightarrow L^2\left((0,T);H^1_{\partial\Omega}(\Omega)\right)$$

Consequently, (u^n) is bounded in $H^1(\Omega\times(0,T))$

using Aubin–Lions theorem , we can extract a subsequence $\left(u^\xi\right)$ of (u^n) such that

$$u^n_t \to u_t \text{ strongly in } L^2(\Omega\times(0,T)) \tag{2.45}$$

therefore

$$u^n_t \to u_t \text{ strongly a.e in } \Omega\times(0,T) \tag{2.46}$$

Now, we will pass to the limit in (2.11) . By the same techniques as in(see[[17], [18]]) , taking $n=\xi$, $\forall w_j \in u_n$, $\forall\phi_j \in u$ and fixed $j<\xi$, to get

$$\begin{aligned}
&\int_\Omega u^\xi_{tt}(t).w_j dx + \int_\Omega \nabla u^\xi(t).\nabla w_j dx + \int_\Omega \nabla u^\xi_t(t).\nabla w_j dx \\
&+ \int_\Omega \int_0^t h(t-s)\nabla u^\xi(s).\nabla w_j ds dx \\
= &\int_\Omega u^\xi(s)\ln\left|u^\xi(s)\right|^{\eta(x)}.w_j dx
\end{aligned} \tag{2.47}$$

by using the property of continuous of the operator in the distributions space

$$\int_{\Omega} u_{tt}^{\xi}.w_j dx \rightharpoonup^* \int_{\Omega} u_{tt}.w_j dx \text{ in } D'(0,T) \tag{2.48}$$

$$\int_{\Omega} \nabla u^{\xi}.w_j dx \rightharpoonup^* \int_{\Omega} \nabla u_{tt}.w_j dx \text{ in } L^{\infty}(0,T) \tag{2.49}$$

$$\int_{\Omega}\int_0^t h(t-s)\nabla u^{\xi}(s).\nabla w_j ds dx \rightharpoonup^* \int_{\Omega}\int_0^t h(t-s)\nabla u(s).\nabla w_j ds dx \text{ in } L^{\infty}(0,T) \tag{2.50}$$

$\forall \psi \in H^1_{\partial\Omega}(\Omega) \cap H^2(\Omega)$, the convergence (2.48)–(2.50) permits us to deduce that

$$\begin{aligned}
& \int_{\Omega} u_{tt}^{\xi}(t).\psi dx + \int_{\Omega} \nabla u^{\xi}(t).\nabla\psi dx + \int_{\Omega} \nabla u_t^{\xi}(t).\nabla\psi dx \\
& + \int_{\Omega}\int_0^t h(t-s)\nabla u^{\xi}(s).\nabla\psi ds dx \\
= & \int_{\Omega} u^{\xi}(s)\ln\left|u^{\xi}(s)\right|^{\eta(x)}.\psi dx \\
\to & \int_{\Omega} u_{tt}(t).\psi dx + \int_{\Omega} \nabla u(t).\nabla\psi dx + \int_{\Omega} \nabla u_t(t).\nabla\psi dx \\
& + \int_{\Omega}\int_0^t h(t-s)\nabla u(s).\nabla\psi ds dx \\
= & \int_{\Omega} u\ln|u|^{\eta(x)}.\psi dx
\end{aligned} \tag{2.51}$$

Hence, this completes our proof of existence result of problem (2.11)

2.4.1 Global existence

Now we shall prove that the solution obtained above is global and bounded in time, for this purpose we define

$$I(u) = I(u(t)) = \left(1 - \int_0^t h(s)\,ds\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (h \circ \Delta u)(t) - \int_{\Omega} u^2 \ln|u|^{\eta}\,dx \tag{2.52}$$

$$J(u) = J(u(t)) = \frac{1}{2}I(u(t)) + \frac{\eta(x)}{4}\|u\|_2^2, \tag{2.53}$$

then

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)), \tag{2.54}$$

Lemma 39. *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$,*suppose that* $(H1)$ *holds. Let* u *be the solution of the problem* (2.1). *Assume further that* $I(0) > 0$ *and* $0 < E(t) \leq E(0) < C$ *and* $\|u_0\| < \rho_0$

.

Then

$$I(t) > 0 \quad \text{for all} \quad t \in [0, T[\,. \tag{2.55}$$

Proof. we show by use contradiction principe .Suppose that

$$\|u(x,t)\|_2 < \rho_0\ ,\ \ 0 < E(t) \leq E(0) < C \quad \text{is not truein} \quad [0.T[\tag{2.56}$$

Since $I(0) > 0$, then there exists (by continuity of $u(t)$) $T^* < T$ such that

$$I(t) \geq 0$$

for all $t \in [0, T^*]$. From (2.52) and (2.53) we have

$$E(t) = \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)),$$

■

By (1.17), (1.18)and (2.54), we obtain

$$\begin{aligned} E(t) &= \frac{1}{2}\|u_t(t)\|_2^2 + J(u(t)) \geq J(u(t)) \\ &\geq \frac{\varpi}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}(h \circ \nabla u)(t) - \frac{1}{2}\int_\Omega u^2 ln|u|^\eta dx + \frac{\eta}{4}\|u\|_2^2 \\ &\geq \frac{\varpi}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|u\|_2^2 + \frac{1}{2}(h \circ \nabla u)(t) - \frac{1}{4}\|u\|_2^2 \ln\|u\|_2^2 - \frac{d^2}{4\pi}\|\Delta u\|_2^2 \\ &\quad + \frac{1}{2}(1 + \ln d)\|\Delta u\|_2^2 + \frac{\eta}{4}\|u\|_2^2 \\ &\geq \frac{1}{2}\left(\varpi - \frac{\eta d^2}{2\pi}\right)\|\Delta u\|_2^2 + \frac{1}{2}\left(\frac{\eta+2}{2} + \eta(1 + \ln d) - \frac{\eta}{2}\ln\|u\|_2^2\right)\|u\|_2^2 \end{aligned} \tag{2.57}$$

So, for $\varpi - \frac{\eta d^2}{2\pi} > 0$, this means that $d < \sqrt{\frac{2\pi\varpi}{\eta}}$

$$\begin{aligned} E(t) &\geq \frac{1}{2}\left(\varpi - \frac{\eta d^2}{2\pi}\right)\|\Delta u\|_2^2 + \frac{1}{2}\left(\frac{\eta+2}{2} + \eta\,(1+\ln d) - \frac{\eta}{2}\ln\|u\|_2^2\right)\|u\|_2^2 \\ &\geq \frac{1}{2}\left(\frac{\eta+2}{2} + \eta\,(1+\ln d) - \frac{\eta}{2}\ln\|u\|_2^2\right)\|u\|_2^2 \\ &\geq \frac{1}{2}\left(D_0 - \frac{\eta}{2}\ln\|u\|_2^2\right)\|u\|_2^2 \end{aligned}$$

where $D_0 = \frac{\eta+2}{2} + \eta\,(1+\ln d) =$.

Should we choose $d < \sqrt{\frac{2\pi\varpi}{\eta}}$, then (2.57) becomes

$$E(t) \geq \xi\,(\rho) = \frac{1}{2}D_0\rho^2 - \frac{\eta}{4}\rho^2\ln\rho^2 \tag{2.58}$$

Let the function $\xi\,(\rho) = \frac{1}{2}D_0\rho^2 - \frac{\eta}{4}\rho^2\ln\rho^2$, where $\rho = \|u\|_2$.

Using (2.58), we have

$$\begin{aligned} \xi'\,(\rho) &= D_0 p - \frac{\eta}{2}\left(2p\ln p^2 + p\right) \\ &= \frac{\eta+2}{2}\rho + \eta\rho + \eta\rho\ln d - \eta\rho\ln\rho^2 - \frac{\eta}{2}p \\ &= \rho\left(1 + \eta\left(1 + \ln d - \ln\rho^2\right)\right) \end{aligned}$$

so, there existe $\rho_0 > 0$ such that

$$\xi'\,(\rho_0) = 0$$

we can deduce that : ξ is increasing on $]0.\rho_0[$and decreasing on $]\rho_0, +\infty[$ and $\lim_{\rho\to+\infty}\xi\,(\rho) = -\infty$.

Moreover,

$$\max_{0<\rho<+\infty}\xi\,(\rho) = \xi\,(\rho_0) = \frac{1}{2}\rho_0^2\,(D_0 - \eta\ln\rho_0) = C$$

Therefore, using the continuity of $u(t)$ and (2.56), it follows that there exists $0 < t_0 < T$ such that $\|u(x,t_0)\|_2 = \rho_0$. From Equation (2.58), we can see that.

$$E(t_0) \geq Z(\|u(x,t_0)\|_2) = Z(\rho_0) = C$$

, which is a contradiction with $E(t) \leq E(0) < C$,$\forall\ t \geq 0$.

By recalling the concept of $I(u(t))$ and employing (2.58) with $d < \sqrt{\frac{2\pi\varpi}{\eta}}$ for any $t \in [0, T)$, it leads to

$$\begin{aligned} I(u(t)) &\geq \varpi \|\nabla u\|_2^2 - \int_\Omega u^2 ln|u|^\eta dx \\ &\geq \left(\varpi - \frac{\eta d^2}{2\pi}\right) \|\Delta u\|_2^2 + \left(1 + \eta(1 + \ln d) - \frac{\eta}{2} \ln \|u\|_2^2\right) \|u\|_2^2 \\ &\geq \left(\varpi - \frac{\eta d^2}{2\pi}\right) \|\Delta u\|_2^2 + \|u\|_2^2 \geq 0 \end{aligned} \tag{2.59}$$

This completes the proof.

Theorem 40. *Let $2 \leq p \leq \bar{q}$ and $\max\left(1, \frac{\bar{q}}{\bar{q}-1+p}\right) \leq m \leq \bar{q}$.Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.*

satisfying (2.59). Suppose that (2.52), (2.54) and $I(0) > 0$ hold. Then the solution of (2.1) is global and boundedin time.

Proof. To prove Theorem , using the definition of T^*, we have to verify that

$$\begin{aligned} \|u_t\|_2^2 &\leq cE(0), \\ \|\nabla u\|_2^2 &\leq cE(0) \end{aligned}$$

is uniformly bounded in time. To do this, we use (2.53), (2.54) and

$$\|\nabla u_t(t)\|_2^2 \leq c \|\nabla u\|_2^2$$

to get

$$\begin{aligned} E(0) &\geq E(t) = \frac{1}{2} \|u_t(t)\|_2^2 + J(u(t)) \\ &= \frac{1}{2} \|u_t(t)\|_2^2 + \frac{1}{2} I(u(t)) \\ &\quad + \frac{\eta(x)}{4} \|u\|_2^2 \\ &\geq \frac{1}{2} \|u_t(t)\|_2^2 + \frac{\eta(x)}{4} \|u\|_2^2 \end{aligned} \tag{2.60}$$

So,

$$\begin{aligned} \|u_t\|_2^2 &\leq 2E(t) \leq 2E(0), \qquad (2.61) \\ \|\nabla u\|_2^2 &\leq \frac{2\pi}{2\pi\varpi - \eta d^2} I(t) \leq \frac{4\pi}{2\pi\varpi - \eta d^2} E(t) \leq \frac{4\pi}{2\pi\varpi - \eta d^2} E(0) \end{aligned}$$

Therefore

$$\|\nabla u\|_2^2 + \|u_t\|_2^2 \leq CE(0)$$

then where C is positive constant, which depends only on p. Thus, we obtain the global existence result. ■

Chapter 3

Decay results

In this section, we establish our main decay results. To achieve exponential decay, it is necessary to construct a suitable Lyapunov functional, ensuring that all energy terms are negative. However, due to the quadratic nature of the energy, its direct application is challenging; thus, we partition the energy to derive the Lyapunov functional $L(t)$. Subsequently, we establish the equivalence between them L and E, we mean

$$\eta_1 E(t) \leq L(t) \leq \alpha_2 E(t), \forall t \geq 0.$$

Where, α_1 and α_2 are two positive constants.

Lemma 41. *Given $t_0 > 0$. Assume that (H1)–(H3) and (2.9) hold. Then,*

$$L(t) := N_1 E(t) + N_2 I_1(t) + I_2(t)$$

satisfies, for a suitable choice of $N_1, N_2 > 0$ and certain positive constants λ_0 and c, the estimates for any $t \geq t_0$, for $\sigma_1 \geq 2$, are applicable.

$$L'(t) \leq -\lambda_0 E(t) + c(h \circ \nabla u)(t) . \tag{3.1}$$

And, for $1 < \sigma_1 < 2$.

$$L'(t) \leq -cE(t) + c(h \circ \nabla u)(t) + c\left(-E'(t)\right)^{\sigma_1 - 1}, \tag{3.2}$$

Since h is positive and $h(0) > 0$ then, for any $t_0 > 0$, we have

$$\int_0^t h(s)ds \geq \int_0^{t_0} h(s)ds = b_0 > 0, \forall t \geq t_0.$$

where

$$\begin{aligned} I_1(t) &:= \int_\Omega u u_t dx \text{ , and} \\ I_2(t) &:= -\int_\Omega u_t \int_0^t h(t-s)(u(t)-u(s))\, ds dx. \end{aligned}$$

Lemma 42. *Assume that (H1)–(H3) and (2.9) hold, then the functional*

$$I_1(t) := \int_\Omega u u_t dx,$$

satisfies, along with the solution of (2.1), the estimates:

For $\sigma_1 \geq 2$,

$$\begin{aligned} I_1'(t) &\leq \|u_t\|_2^2 - \|u\|_2^2 - \frac{\varpi}{4}\|\nabla u(t)\|_2^2 + c(h \circ \Delta u)(t) \\ &\quad + c\int_\Omega |u_t|^{\sigma(x)} dx + \int_\Omega u^2 \ln |u|^\eta dx, \text{ for } \sigma_1 \geq 2 \end{aligned} \tag{3.3}$$

and , for $1 < \sigma_1 < 2$.

$$\begin{aligned} I_1'(t) &\leq \|u_t\|_2^2 - \|u\|_2^2 - \frac{\varpi}{4}\|\nabla u(t)\|_2^2 + c(h \circ \Delta u)(t) + c\int_\Omega |u_t|^{\sigma(x)} dx \\ &\quad + \left(\int_\Omega |u_t|^{\sigma(x)}\right)^{\sigma_1 - 1} + \int_\Omega u^2 ln|u|^\eta dx, \end{aligned} \tag{3.4}$$

Proof. Differentiate I_1 and use the differential equation in (2.1), to get ■

$$\begin{aligned} I_1'(t) &= \int_\Omega \frac{d}{dt} u u_t dx = \int_\Omega u_t^2 dx + \int_\Omega u u_{tt} dx \\ &= \int_\Omega u_t^2 dx + \int_\Omega \left(\mu_1 \Delta u - \int_0^t h(t-s)\Delta u(s) ds - |u_t|^{\sigma(\cdot)-2} u_t + u \ln |u|^{\eta(x)}\right) u dx \\ &\leq \|u_t\|_2^2 - \|u\|_2^2 - \left(1 - \int_0^t h(s)\, ds\right)\|\nabla u(t)\|_2^2 \\ &\quad + \int_\Omega \Delta u(t) \int_\Omega h(t-s)(\nabla u(s) - \nabla u(t)) ds dx \\ &\quad - \int_\Omega u |u_t|^{\sigma(x)-2} u_t dx + \int_\Omega u^2 \ln |u|^\eta dx \end{aligned} \tag{3.5}$$

$$\begin{aligned}
I_1'(t) &= \int_\Omega \frac{d}{dt} u u_t dx = \int_\Omega u_t^2 dx + \int_\Omega u u_{tt} dx \\
&\leq \|u_t\|_2^2 - \|u\|_2^2 - \frac{\varpi}{4} \|\nabla u(t)\|_2^2 + c(h \circ \Delta u)(t) \\
&\quad + c\int_\Omega |u_t|^{\sigma(x)} dx + \int_\Omega u^2 \ln |u|^\eta dx, \text{ for } \sigma_1 \geq 2 \qquad (3.6)
\end{aligned}$$

Young's inequality and (3.3) give

$$\int_\Omega \Delta u. \int_0^t h(t-s)(\nabla u(s) - \nabla u(t)) ds dx \leq \delta_0 \int_\Omega |\nabla u|^2 dx + \frac{c}{4\delta_0} (h \circ \Delta u)(t) \qquad (3.7)$$

Estimation of the term $-\int_\Omega u|u_t|^{\sigma(x)-2} u_t dx$:

We use Young's inequality with $p(x) = \frac{\sigma(x)}{\sigma(x)-1}$ and $p'(x) = \sigma(x)$ so, for all $x \in \Omega$, we have

$$|u_t|^{\sigma(x)-2} u_t u \leq \delta |u|^{\sigma(x)} + c_\delta(x) |u_t|^{\sigma(x)}$$

where

$$c_\delta(x) = \delta^{1-\sigma(x)} (\sigma(x))^{-\sigma(x)} (\sigma(x)-1)^{\sigma(x)-1}$$

Hence,

$$-\int_\Omega u|u_t|^{\sigma(x)-2} u_t dx \leq \delta \int_\Omega |u|^{\sigma(x)} dx + \int_\Omega c_\delta(x) |u_t|^{\sigma(x)} dx. \qquad (3.8)$$

Now, using (2.7), (2.8), (2.61) and Lemma (Embedding Property), we obtain

$$\begin{aligned}
\int_\Omega |u|^{\sigma(x)} dx &\leq \int_{\Omega+} |u|^{\sigma(x)} dx + \int_{\Omega-} |u|^{\sigma(x)} dx \\
&\leq \int_{\Omega+} |u|^{\sigma_2} dx + \int_{\Omega-} |u|^{\sigma_1} dx \\
&\leq \int_{\Omega+} |u|^{\sigma_2} dx + \int_{\Omega-} |u|^{\sigma_1} dx \\
&\leq \int_\Omega |u|^{\sigma_2} dx + \int_\Omega |u|^{\sigma_1} dx \\
&\leq \left(c_e^{\sigma_1} \|\Delta u\|^{\sigma_1-2} + c_e^{\sigma_2} \|\Delta u\|^{\sigma_2-2}\right) \|\nabla u\|_2^2 \\
&\leq \begin{pmatrix} c_e^{\sigma_1} \left(\frac{4\pi}{2\pi\varpi - \eta d^2} E(0)\right)^{\sigma_1-2} \\ + c_e^{\sigma_2} \left(\frac{4\pi}{2\pi\varpi - \eta d^2} E(0)\right)^{\sigma_2-2} \end{pmatrix} \|\nabla u\|_2^2 \\
&\leq c \|\nabla u\|_2^2
\end{aligned}$$

where $\Omega_+ = \{x \in \Omega\colon\ |u(x,t)| \geq 1\}$ and $\Omega_- = \{x \in \Omega\colon\ |u(x,t)| < 1\}$

and $c = \left(c_e^{\sigma_1} \left(\frac{4\pi}{2\pi\varpi - \eta d^2} E(0) \right)^{\sigma_1 - 2} + c_e^{\sigma_2} \left(\frac{4\pi}{2\pi\varpi - \eta d^2} E(0) \right)^{\sigma_2 - 2} \right)$

Then, (3.8) and (**??**) yield

$$-\int_\Omega u|u_t|^{\sigma(x)} u_t dx \leq \delta c \|\nabla u\|_2^2 + \int_\Omega c_\delta(x) |u_t|^{\sigma(x)} dx \tag{3.9}$$

Combining the above results with fixing δ and $\delta = \frac{\varpi}{4c}$ completes the proof of (3.3).

For the proof of (3.4), we re-estimate the fifth term in (3.5) as follows:

First, we define

$$\Omega_1 = \{x \in \Omega\colon \sigma(x) < 2\}$$

and

$$\Omega_2 = \{x \in \Omega\colon \sigma(x) \geq 2\}$$

Then, we get

$$-\int_\Omega u|u_t|^{\sigma(x)-2} u_t dx \leq -\int_{\Omega_1} u|u_t|^{\sigma(x)-2} u_t dx - \int_{\Omega_2} u|u_t|^{\sigma(x)-2} u_t dx. \tag{3.10}$$

Using the definition of Ω_1, we have

$$2\sigma(x) - 2 < \sigma(x), \text{ and } 2\sigma(x) - 2 \geq 2\sigma_1 - 2. \tag{3.11}$$

Therefore, using Young's and Poincare's inequalities and (3.11), we obtain

$$\begin{aligned}
-\int_{\Omega_1} u|u_t|^{\sigma(x)-2} u_t dx &\leq \theta \int_{\Omega_1} |u|^2 dx + \frac{1}{4\theta} \int_{\Omega_1} |u_t|^{2\sigma(x)-2} dx \\
&\leq \theta c_*^2 \|\nabla u\|_2^2 + c \left[\int_{\Omega_1^+} |u_t|^{2\sigma(x)-2} dx + \int_{\Omega_1^-} |u_t|^{2\sigma(x)-2} dx \right] \\
&\leq \theta c_*^2 \|\nabla u\|_2^2 + c \left[\int_{\Omega_1^+} |u_t|^{\sigma(x)} dx + \int_{\Omega_1^-} |u_t|^{2\sigma_1-2} dx \right] \\
&\leq \theta c_*^2 \|\nabla u\|_2^2 + c \left[\int_\Omega |u_t|^{\sigma(x)} dx + \left(\int_{\Omega_1^-} |u_t|^2 dx \right)^{\sigma_1 - 1} \right]
\end{aligned}$$

$$\leq \theta c_*^2 \|\nabla u\|_2^2 + c\left[\int_\Omega |u_t|^{\sigma(x)}\,dx + \left(\int_{\Omega_1^-} |u_t|^{\sigma(x)}\,dx\right)^{\sigma_1-1}\right]$$
$$\leq \theta c_*^2 \|\nabla u\|_2^2 + c\left[\int_\Omega |u_t|^{\sigma(x)}\,dx + \left(\int_{\Omega} |u_t|^{\sigma(x)}\,dx\right)^{\sigma_1-1}\right] \tag{3.12}$$

where

$$\Omega_1^+ = \{x \in \Omega_1 \colon\ |u_t(x,t)| \geq 1\} \text{ and } \Omega_1^- = \{x \in \Omega_1 \colon\ |u_t(x,t)| < 1\} \tag{3.13}$$

After setting $3b8 = \frac{\varpi}{8c_*^2}$, (3.12) becomes

$$-\int_{\Omega_1} u|u_t|^{\sigma(x)-2}u_t dx \leq \frac{\varpi}{8}\|\nabla u\|_2^2 + c\left[\int_\Omega |u_t|^{\sigma(x)}\,dx + \left(\int_\Omega |u_t|^{\sigma(x)}\,dx\right)^{\sigma_1-1}\right] \tag{3.14}$$

Next, for any δ we have, by the case $\sigma(x) \geq 2$,

$$-\int_{\Omega_2} u|u_t|^{\sigma(x)-2}u_t dx \leq \delta c\|\nabla u\|_2^2 + \int_\Omega c_\delta(x)|u_t|^{\sigma(x)}\,dx. \tag{3.15}$$

Therefore, by combining (3.10)–(3.15), we get to

$$I_1'(t) \leq ||u_t||_2^2 - \left(\frac{3\varpi}{8} - c\delta\right)\|\nabla u(t)\|_2^2 + c(h\circ\nabla u)(t) + c\left[\begin{array}{c}\int_\Omega (1+c_\delta(x))|u_t|^{\sigma(x)}\,dx \\ +\left(\int_\Omega |u_t|^{\sigma(x)}\right)^{\sigma_1-1}\end{array}\right]$$
$$+\int_\Omega u^2 \ln|u|^\eta\,dx.$$

By fixing $\delta = \frac{\varpi}{8c}$, $c_\delta(x)$ remains bounded and, consequently, we obtain (3.4).

Lemma 43. *Assume that (H1)–(H3) and (2.9) hold, then for any $\delta > 0$, the functional*

$$I_2(t) := -\int_\Omega u_t \int_0^t h(t-s)(u(t)-u(s))\,dsdx. \tag{3.16}$$

satisfies, along the solution of (2.1), the estimates:

$$
\begin{aligned}
I_2'(t) &\le \delta \|\nabla u\|_2^2 - \left(\int_0^t b(s)\,ds - \delta \right) ||u_t||_2^2 + \int_\Omega c_\delta(x) |u_t|^{\sigma(x)}\,dx \\
&\quad + \frac{c}{\delta} (-h' \circ \Delta u)(t) + \frac{c}{\delta} (h \circ \Delta u)(t) + c_{\epsilon,\delta} (h \circ \Delta u)^{\frac{1}{1+\epsilon}}(t) \qquad (3.17)
\end{aligned}
$$

for $\sigma_1 \ge 2$ and for $1 < \sigma_1 < 2$, we have the following estimate

$$
\begin{aligned}
I_2'(t) &\le \delta \|\nabla u\|_2^2 - \left(\int_0^t b(s)\,ds - \delta \right) ||u_t||_2^2 + c (h \circ \Delta u)(t) + c_{\epsilon,\delta} (h \circ \Delta u)^{\frac{1}{1+\epsilon}}(t) \\
&\quad + \frac{c}{\delta} (-h' \circ \Delta u)(t) + \frac{c}{\delta} \left[\int_\Omega |u_t|^{\sigma(x)}\,dx + \left(\int_\Omega |u_t|^{\sigma(x)}\,dx \right)^{\sigma_1 - 1} \right] \qquad (3.18)
\end{aligned}
$$

Proof. When I_2 is directly differentiated using (3.16), the result is

$$
\begin{aligned}
I_2'(t) &= \int_\Omega \Delta u \int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\,ds dx \\
&\quad - \int_\Omega u \int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\,ds dx \\
&\quad - \int_\Omega \left(\int_0^t h(t-s) \Delta u(s)\,ds \right) \left(\int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\,ds \right) dx \\
&\quad - \int_\Omega u_t \int_0^t h'(t-s)(u(t) - u(s))\,ds dx - \left(\int_0^t b(s)\,ds \right) \|u_t\|_2^2 \\
&\quad + \int_\Omega |u_t|^{\sigma(x)-2} u_t \int_0^t h(t-s)(u(t) - u(s))\,ds dx \\
&\quad - \eta \int_\Omega u \ln|u| \int_0^t h(t-s)(u(t) - u(s))\,ds dx \\
&= \left(1 - \int_0^t h(s)\,ds \right) \int_\Omega \Delta u \int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\,ds dx \qquad (3.19) \\
&\quad - \int_\Omega u \int_0^t h(t-s)(u(t) - u(s))\,ds dx \\
&\quad + \int_\Omega \left(\int_0^t h(t-s)(\Delta u(t) - \Delta u(s))\,ds \right)^2 dx \\
&\quad - \int_\Omega u_t \int_0^t h'(t-s)(u(t) - u(s))\,ds dx - \left(\int_0^t h(s)\,ds \right) \|u_t\|_2^2 \\
&\quad + \int_\Omega |u_t|^{\sigma(x)-2} u_t \int_0^t h(t-s)(u(t) - u(s))\,ds dx \\
&\quad - \eta \int_\Omega u \ln|u| \int_0^t h(t-s)(u(t) - u(s))\,ds dx
\end{aligned}
$$

■

Using Young's inequality and (2.9) , we get

$$\left(1-\int_0^t h(s)\,ds\right)\int_\Omega \Delta u.\int_0^t h(t-s)\left(\Delta u(t)-\Delta u(s)\right)dsdx \le c\delta\left\|\nabla u\right\|_2^2+\frac{c}{\delta}\left(h\circ\Delta u\right)(t). \tag{3.20}$$

Using Young's, Poincare's, and (2.9) inequalities results in

$$\int_\Omega u\int_0^t h(t-s)\left(u(t)-u(s)\right)dsdx \le c\delta\left\|\nabla u\right\|_2^2+\frac{c}{\delta}\left(h\circ\Delta u\right)(t). \tag{3.21}$$

By using Young's inequality and (2.9), we get

$$-\int_\Omega u_t\int_0^t h'(t-s)\left(u(t)-u(s)\right)dsdx \le \delta\left\|u_t\right\|_2^2+\frac{c}{\delta}\left(-h'\circ\Delta u\right)(t). \tag{3.22}$$

Next, for almost every $x\in\Omega$ fixed, we have

$$\begin{aligned}\int_0^t h(t-s)\left|u(t)-u(s)\right|ds &\le \left(\int_0^t h(s)\,ds\right)^{\frac{\sigma(x)-1}{\sigma(x)}}\left(\int_0^t h(t-s)\left|u(t)-u(s)\right|^{\sigma(x)}ds\right)^{\frac{1}{\sigma(x)}} \\ &\le (1-\varpi)^{\frac{\sigma(x)-1}{\sigma(x)}}\left(\int_0^t h(t-s)\left|u(t)-u(s)\right|^{\sigma(x)}ds\right)^{\frac{1}{\sigma(x)}}\end{aligned} \tag{3.23}$$

Consequently, for nearly all $x\in\Omega$, we have

$$\left|\int_0^t h(t-s)\left|u(t)-u(s)\right|ds\right|^{\sigma(x)} \le (1-\varpi)^{\sigma_1-1}\int_0^t h(t-s)\left|u(t)-u(s)\right|^{\sigma(x)}ds. \tag{3.24}$$

Using (2.9) and the Young, Holder, and Poincare inequalities, we obtain

$$\begin{aligned}&\int_\Omega \left|u_t\right|^{\sigma(x)-2}u_t\int_0^t h(t-s)\left(u(t)-u(s)\right)dsdx. \\ \le\ &\delta\int_\Omega\left|\int_0^t h(t-s)\left(u(t)-u(s)\right)ds.\right|^{\sigma(x)}dx+\int_\Omega c_\delta(x)\left|u_t\right|^{\sigma(x)}dx \\ \le\ &\delta(1-\varpi)^{\sigma_1-1}\int_\Omega\int_0^t h(t-s)\left|u(t)-u(s)\right|^{\sigma(x)}dsdx+\int_\Omega c_\delta(x)\left|u_t\right|^{\sigma(x)}dx,\end{aligned} \tag{3.25}$$

where $\quad c_{\delta}(x)=\delta^{1-\sigma(x)}(\sigma(x))^{-\sigma(x)}(\sigma(x)-1)^{\sigma(x)-1}$

In a similar vein, we have

$$\begin{aligned}
&\int_{\Omega}\int_{0}^{t} h(t-s)\,|u(t)-u(s)|^{\sigma(x)}\,dsdx \\
\leq\ & \int_{\Omega_+}\int_{0}^{t} h(t-s)\,|u(t)-u(s)|^{\sigma_2}\,dsdx+\int_{\Omega_-}\int_{0}^{t} h(t-s)\,|u(t)-u(s)|^{\sigma_1}\,dsdx \\
\leq\ & \int_{0}^{t} h(t-s)\,\|u(t)-u(s)\|_{\sigma_2}^{\sigma_2}\,ds+\int_{0}^{t} h(t-s)\,\|u(t)-u(s)\|_{\sigma_1}^{\sigma_1}\,ds \\
\leq\ & \left[\begin{array}{c} c_e^{\sigma_2}\left(\frac{4\pi}{2\pi\varpi-\eta d^2}E(0)\right)^{\frac{\sigma_2-2}{2}} \\ +c_e^{\sigma_1}\left(\frac{4\pi}{2\pi\varpi-\eta d^2}E(0)\right)^{\frac{\sigma_1-2}{2}}\end{array}\right]\int_{0}^{t} h(t-s)\,\|u(t)-u(s)\|_{2}^{2}\,ds \\
\leq\ & c\,(h\circ\Delta u)(t)
\end{aligned} \tag{3.26}$$

where $c=\left[c_e^{\sigma_2}\left(\frac{4\pi}{2\pi\varpi-\eta d^2}E(0)\right)^{\frac{\sigma_2-2}{2}}+c_e^{\sigma_1}\left(\frac{4\pi}{2\pi\varpi-\eta d^2}E(0)\right)^{\frac{\sigma_1-2}{2}}\right]$

Therefore,

$$\begin{aligned}
&\int_{\Omega}|u_t|^{\sigma(x)-2}u_t\int_{0}^{t} h(t-s)\,(u(t)-u(s))dsdx \\
\leq\ & c\delta\,(1-\varpi)^{\sigma_1-1}(h\circ\Delta u)(t)+\int_{\Omega} c_{\delta}(x)\,|u_t|^{\sigma(x)}\,dx,
\end{aligned} \tag{3.27}$$

For the last term in (3.19), the use of (1.16), Young's, Cauchy-Schwarz's and Poincare's inequalities, the embedding theorem and (2.9) leads to, for any $\delta>0$,

$$\begin{aligned}
&\int_{\Omega} u ln|u|^{\eta(x)}\int_{0}^{t} h(t-s)\,(u(t)-u(s))dsdx \\
\leq\ & \eta(x)\int_{\Omega}\left(u^2+\beta_{\epsilon}|u|^{1-\epsilon}\right)\left|\int_{0}^{t} h(t-s)\,(u(t)-u(s))dsdx\right| \\
\leq\ & \eta\int_{\Omega}\left(u^2+\beta_{\epsilon}|u|^{1-\epsilon}\right)\left|\int_{0}^{t} h(t-s)\,(u(t)-u(s))dsdx\right| \\
\leq\ & c\int_{\Omega}|u|^2\left|\int_{0}^{t} h(t-s)\,(u(t)-u(s))ds\right|dx \\
&+\delta\int_{\Omega}u^2dx+c_{\varepsilon,\delta}\int_{\Omega}\left|\int_{0}^{t} h(t-s)\,(u(t)-u(s))ds\right|^{\frac{2}{1+\varepsilon}}dx \\
\leq\ & c\delta\,\|\nabla u\|_2^2+\frac{c}{\delta}(h\circ\Delta u)(t)+c_{\epsilon,\delta}(h\circ\nabla u)^{\frac{1}{1+\epsilon}}(t).
\end{aligned}$$

Combining the above estimates with (3.19), we obtain (3.17).

For the proof of (3.18), we re-estimate the fifth term in (3.19) as follows:

$$
\begin{aligned}
& \int_{\Omega} |u_t|^{\sigma(x)-2} u_t \int_0^t h(t-s)(u(t)-u(s))dsdx \\
\leq\ & \delta \int_{\Omega} \left| \int_0^t h(t-s)(u(t)-u(s))ds \right|^2 dx + \frac{c}{\delta} \int_{\Omega} |u_t|^{2\sigma(x)-2} dx \\
\leq\ & \delta (1-\varpi)(h \circ \Delta u)(t) + \frac{c}{\delta} \int_{\Omega} |u_t|^{2\sigma(x)-2} dx \\
\leq\ & c\delta (h \circ \Delta u)(t) + \frac{c}{\delta} \int_{\Omega_1} |u_t|^{2\sigma(x)-2} dx + \frac{c}{\delta} \int_{\Omega_2} |u_t|^{2\sigma(x)-2} dx \qquad (3.28) \\
\leq\ & c\delta (h \circ \Delta u)(t) + \frac{c}{\delta}(\int_{\Omega} |u_t|^{\sigma(x)} dx + (\int_{\Omega} |u_t|^{\sigma(x)} dx)^{\sigma_1 - 1}).
\end{aligned}
$$

Then (3.18) is established

Lemma 44. *By using (2.8), (3.3) and (3.17), then, for $t \geq t_0$ and any $\lambda_0 > 0$, we have*

$$
\begin{aligned}
L'(t) \leq\ & -\lambda_0 E(t) - (N_2\delta - \frac{\varpi}{2} + \frac{\lambda_0(1-b_0)}{2})||\nabla u||_2^2 \\
& -(N_2(b_0-\delta) - 1 - \frac{\lambda_0}{2})||u_t||_2^2 \\
& +c(h \circ \nabla u)(t) + (\frac{1}{2}N_1 - \frac{4c}{\ell} N_2^2 (H' o \nabla u)(t) \\
& + \left(1 - \frac{\lambda_0}{2}\right) \int_{\Omega} u^2 ln|u|^{\eta} dx + (1 - \frac{\lambda_0(\eta+2)}{4})||u||_2^2
\end{aligned}
$$

Using the Logarithmic Sobolev inequality, for $0 < \lambda_0 < \frac{1}{2}$, we get

$$
\begin{aligned}
L'(t) \leq\ & -\lambda_0 E(t) - \left[N_2\delta - \frac{\varpi}{2} + \frac{\lambda_0(1-b_0)}{2} - (1 - \frac{\lambda_0}{2})\frac{\eta d^2}{2\pi} \right] ||\nabla u||_2^2 \\
& + \left[\frac{1}{2}N_1 - \frac{4c}{\varpi} N_2^2 \right])(H' o \nabla u)(t) \\
& - \left[N_2(b_0-\delta) - 1 - \frac{\lambda_0}{2} \right] ||u_t||_2^2 + c(h \circ \nabla u)(t) \\
& -(1 - \frac{\eta}{2}(1 - \frac{\lambda_0}{2}) ln||u||_2^2 + \eta(1 + lnd)(1 - \frac{\lambda_0}{2}) - \frac{\lambda_0(\eta+2)}{4})||u||_2^2
\end{aligned}
$$

At this point, we choose :

1) N_2 large enough

$$N_2 > \max\left\{\left(1-\frac{\lambda_0}{2}\right)\frac{\eta d^2}{\delta 23c0}, \left(1+\frac{\lambda_0}{2}\right)\frac{1}{(b_0-\delta)}\right\}$$

Then,

$$N_2\delta - \frac{\varpi}{2} + \frac{\lambda_0\,(1-b_0)}{2} - \left(1-\frac{\lambda_0}{2}\right)\frac{\eta d^2}{23c0} > 0$$

and,

$$N_2(b_0-\delta) - 1 - \frac{\lambda_0}{2} > 0,$$

2) λ_0 and η so small that

$$1 - \frac{\eta}{2}(1-\frac{\lambda_0}{2})ln||u||_2^2 + \eta(1+lnd)(1-\frac{\lambda_0}{2}) - \frac{\lambda_0\,(\eta+2)}{4} > 0.$$

and then N_1 large enough that

$$N_1 > \frac{4c}{\varpi}N_2^2$$

Therefore, we get to the desired result (3.1). On the other hand, we can choose N_1 even larger so that

$$L \sim E. \tag{3.29}$$

Using (2.57) and (2.8) , we get

$$\begin{aligned}(h\circ\nabla u)(t) &= (h\circ\nabla u)^{\frac{1+\varepsilon}{1+\varepsilon}}(t) = (h\circ\nabla u)^{\frac{\varepsilon}{1+\varepsilon}}(t)(h\circ\nabla u)^{\frac{1}{1+\varepsilon}}(t) &(3.30)\\ &\le c(h\circ\nabla u)^{\frac{1}{1+\varepsilon}}(t) &(3.31)\end{aligned}$$

Remark 45. *In the case of H is linear and since a is nonincreasing, we have*

$$(h\circ\nabla u)^{\frac{1}{1+\varepsilon}}(t) \le c(-E'(t))^{\frac{1}{1+\varepsilon}} \tag{3.32}$$

Lemma 46. *If (H1)–(H2) are satisfied, then we have the following estimate*

$$(h \circ \nabla u)(t) \leq \frac{t}{\varepsilon_0} H^{-1}\left(\frac{\varepsilon_0((-h' \circ \nabla u)(t))}{ta(t)}\right), \forall t > 0, \tag{3.33}$$

where ε_0 is small enough and ,

$$(-h' \circ \nabla u)(t) \leq -cE'(t), \tag{3.34}$$

Proof. To establish (3.33), let us define the following functional

$$\Gamma(t) := \frac{\varepsilon_0}{t} \int_0^t ||\nabla u(t) - \nabla u(t-s)||_2^2 ds, \qquad \forall t > 0. \tag{3.35}$$

■

Then, using (2.7), (2.8) and the dentition of $\Gamma(t)$, we have

$$\begin{aligned}
\Gamma(t) &\leq \frac{2\varepsilon_0}{t}\left(\int_0^t ||\nabla u(t)||_2^2 + ||\nabla u(t-s)||_2^2 ds\right) \\
&\leq \frac{4\varepsilon_0}{\varpi t}\left(\int_0^t (E(t) + E(t-s)) ds\right) \\
&\leq \frac{8\varepsilon_0}{\varpi t} \int_0^t E(s) ds \\
&\leq \frac{8\varepsilon_0}{\varpi t} \int_0^t E(0) ds = \frac{8\varepsilon_0}{\varpi} E(0) < +\infty
\end{aligned} \tag{3.36}$$

Thus, ε_0 can be chosen so small so that, for all $t > 0$,

$$\Gamma(t) < 1. \tag{3.37}$$

Without loss of the generality, for all $t > 0$, we assume that $\Gamma(t) > 0$, otherwise we get a exponential decay from (3.1). The use of Jensen's inequality and using (3.34), (2.6) and (3.37) gives

$$
\begin{aligned}
(-h' \circ \nabla u)(t) &= \frac{1}{\varepsilon_0 \Gamma(t)} \int_0^t \Gamma(t)\,(-H'(s)) \int_\Omega \varepsilon_0 |\nabla u(t) - \nabla u(t-s)|^2 dx ds \\
&\geq \frac{1}{\varepsilon_0 \Gamma(t)} \int_0^t \Gamma(t)\, a(s) H(h(s)) \int_\Omega \varepsilon_0 |\nabla u(t) - \nabla u(t-s)|^2 dx ds \\
&\geq \frac{a(t)}{\varepsilon_0 \Gamma(t)} \int_0^t H(\Gamma(t)\, h(s)) \int_\Omega \varepsilon_0 |\nabla u(t) - \nabla u(t-s)|^2 dx ds \\
&\geq \frac{ta(t)}{\varepsilon_0} H(\frac{\varepsilon_0}{t} \int_0^t h(s) \int_\Omega |\nabla u(t) - \nabla u(t-s)|^2 dx ds), \qquad (3.38)
\end{aligned}
$$

hence (3.33) is established.

3.0.2 The case $\sigma \geq 2$.

Theorem 47. *Assume that (H1)–(H3) and* (2.9) *hold. Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Then, there exist positive constants c, t_0 and t_1 such that the solution of (2.1) satisfies:

1) if H is linear,

$$
E(t) \leq c(1 + \int_{t_0}^t a^{1+\varepsilon}(s) ds)^{\frac{-1}{\varepsilon}}, \forall t \geq t_0, \qquad (3.39)
$$

2) If H is nonlinear,

$$
E(t) \leq c t^{\frac{1}{1+\varepsilon}} \beta_2^{-1} \left(\frac{c}{t^{\frac{1}{1+\varepsilon}} \int_{t_1}^t a(s)\, ds} \right), \forall t \geq t_1 \qquad (3.40)
$$

where $H_2(s) = sH'(\varepsilon_1 s)$ and $H(t) = ([H^{-1}]^{\frac{1}{1+\varepsilon}})^{-1}(t)$

Case H is linear

We multiply (3.1) by $a(t)$ and use (3.30) and (3.32) to get

$$
a(t)L'(t) \leq -\lambda_0 a(t) E(t) + c(-E'(t))^{\frac{1}{1+\varepsilon}} \forall t \geq t_0. \qquad (3.41)
$$

Multiply (3.41) by $a^\varepsilon(t) E^\varepsilon(t)$, and recall that $a\prime \leq 0$, to obtain

$$
a^{\varepsilon+1}(t) E^\varepsilon(t) L'(t) \leq -\lambda_0 a^{\varepsilon+1}(t) E^{\varepsilon+1}(t) + c(aE)^\varepsilon(t)(-E'(t))^{\frac{1}{1+\varepsilon}} \forall t \geq t_0
$$

Use of Young's inequality, with $\varepsilon+1$ and $\frac{\varepsilon+1}{\varepsilon}$, gives, for any $\varepsilon'>0$,

$$\begin{aligned} a^{\varepsilon+1}(t)E^{\varepsilon}(t)L'(t) &\leq -\lambda_0 a^{\varepsilon+1}(t)E^{\varepsilon+1}(t) + c(\varepsilon' a^{\varepsilon+1}(t)E^{\varepsilon+1} - c_{\varepsilon'}E'(t)) \\ &= -(\lambda_0 - \varepsilon' c)a^{\varepsilon+1}(t)E^{\varepsilon+1} - cE'(t), \forall t \geq t_0. \end{aligned}$$

so,

We then choose $0 < c_1 = \lambda_0 - \varepsilon' c$ and use that $a' \leq 0$ and $E' \leq 0$, to get,

$$(a^{\varepsilon+1}E^{\varepsilon}L)(t) \leq a^{\varepsilon+1}(t)E^{\varepsilon}(t)L_1'(t) \leq -c_1 a^{\varepsilon+1}(t)E^{\varepsilon+1}(t) - cE'(t), \forall t \geq t_0,$$

which implies

$$a^{\varepsilon+1}(t)E^{\varepsilon}(t)L_1'(t) + cE'(t) = (a^{\varepsilon+1}E^{\varepsilon}L + cE)'(t) \leq -c_1 a^{\varepsilon+1}(t)E^{\varepsilon+1}(t), \forall t \geq t_0,$$

where

$$L_1 = a^{\varepsilon+1}E^{\varepsilon}L + cE.$$

by (3.29) we have $L \sim E$, Then

$$L_1 \sim E$$

So,

$$L_1'(t) \leq -ca^{\varepsilon+1}(t)L_1^{\varepsilon+1}(t), \forall t \geq t_0.$$

so,

$$L_1'(t)L_1^{-\varepsilon-1}(t) \leq -ca^{\varepsilon+1}(t), \forall t \geq t_0.$$

Integrating over (t_0, t) and using the fact that $L_1 \sim E$, we get

$$\begin{aligned} \int_{t_0}^{t} L_1'(t)L_1^{-\varepsilon-1}(t)dt &= \frac{1}{\varepsilon}\left[L_1^{-\varepsilon}(t_0) - L_1^{-\varepsilon}(t)\right]_{t_0} \\ &= \\ &\leq -c\int_{t_0}^{t} a^{\varepsilon+1}(t)dt, \forall t \geq t_0. \end{aligned}$$

$$E^{\varepsilon}(t) \leq c(1 + \int_{t_0}^{t} a^{1+\varepsilon}(s)ds), \forall t \geq t_0, \tag{3.42}$$

we obtain (3.39).

Case H is non-linear.

Using (3.1), (3.30) and (3.33), we obtain, $\forall t \geq t_0$,

$$L'(t) \leq -\lambda_0 E(t) + ct^{\frac{1}{1+\varepsilon}} \left[H^{-1}\left(\frac{\varepsilon_0(-h' \circ \nabla u)(t)}{ta\,(t)} \right) \right]^{\frac{1}{1+\varepsilon}} \tag{3.43}$$

Combining the strictly increasing property of $\overline{H}$ and the fact that $\frac{1}{t} < 1$ whenever $t > 1$, we obtain

$$H^{-1}\left(\frac{\varepsilon_0(-h' \circ \nabla u)(t)}{ta\,(t)} \right) \leq H^{-1}\left(\frac{\varepsilon_0(-h' \circ \nabla u)(t)}{t^{\frac{1}{1+\varepsilon}}a\,(t)} \right) \tag{3.44}$$

then, (3.43) becomes, for $\forall t \geq t_1 = max\{t_0, 1\}$,

$$L(t) \leq -\lambda_0 E(t) + ct^{\frac{1}{1+\varepsilon}} \left[H^{-1}\left(\frac{\varepsilon_0(-h' \circ \nabla u)(t)}{t^{\frac{1}{1+\varepsilon}}a\,(t)} \right) \right]^{\frac{1}{1+\varepsilon}} \tag{3.45}$$

Set

$$H(t) = \left(\left[H^{-1}\right]^{\frac{1}{1+\varepsilon}}\right)^{-1}(t)\,, \chi\,(t) = \frac{\varepsilon_0(-h' \circ \nabla u)(t)}{t^{\frac{1}{1+\varepsilon}}a\,(t)} \tag{3.46}$$

Using the facts that $H' > 0$ and $H'' > 0$ on $]0, r]$, (3.45) reduces to

$$L'(t) \leq -\lambda_0 E(t) + ct^{\frac{1}{1+\varepsilon}} H^{-1}(3c7(t)), \forall t \geq t_1 \tag{3.47}$$

Now, for $\varepsilon_1 < r$ and using (3.47) and the fact that $E' \leq 0, H' > 0, H^{''} > 0$, on $]0, r]$, we find that the functional L_2, defined by

$$L_2(t) := H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)} \right) L(t),$$

satisfies, for some $c_1, c_2 > 0$.

$$c_1 L_2(t) \leq E(t) \leq c_2 L_2(t) \tag{3.48}$$

and, for all $t \geq t_1$,

$$L_2'(t) \leq -\lambda_0 E(t) H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right) + ct^{\frac{1}{1+\varepsilon}} H'(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}) H^{-1}(3c7(t)) \tag{3.49}$$

So, using (1.14) and (1.15) with $\eta_1 = H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right)$ and $3b1_2 = H^{-1}(3c7(t))$, we get to

$$\begin{aligned} L_2'(t) &\leq -\lambda_0 E(t) H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right) + ct^{\frac{1}{1+\varepsilon_0}} H^*(h'(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)})) + ct^{\frac{1}{1+\varepsilon}} 3c7(t) \\ &\leq -\lambda_0 E(t) H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_1 H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) + ct^{\frac{1}{1+\varepsilon_0}} 3c7(t) \end{aligned} \tag{3.50}$$

Then, multiplying (3.50) by $a(t)$ and using (3.34), (3.46), we get

$$a(t)L_2'(t) \leq -\lambda_0 a(t) E(t) H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_1 a(t) \frac{E(t)}{E(0)} H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) + cE'(t), \forall t \geq t_1.$$

Using the non-increasing property of a, we obtain, for all $t \geq t_1$,

$$\begin{aligned} (aL_2 + cE)\prime(t) \leq\ & -\lambda_0 a(t) E(t) H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right) \\ & + c\varepsilon_1 a(t) \frac{E(t)}{E(0)} H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) \end{aligned}$$

Therefore, by setting

$$L_3 := aL_2 + cE \sim E,$$

we conclude that

$$L_3'(t) \leq -\lambda_0 a(t) E(t) H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right) + c\varepsilon_1 a(t) . \frac{E(t)}{E(0)} H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right).$$

This gives, for a suitable choice of ε_1,

$$L_3'(t) \leq -ca(t)\left(\frac{E(t)}{E(0)}\right) H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right), \forall t \geq t_1$$

or

$$c\left(\frac{E(t)}{E(0)}\right) H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right) a(t) \leq -L_3'(t), \forall t \geq t_1 \tag{3.51}$$

An integration of (3.51) yields

$$\int_{t1}^{t} c\left(\frac{E(s)}{E(0)}\right) H'\left(\frac{\varepsilon_1}{s^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(s)}{E(0)}\right) a(s)ds \leq -\int_{t1}^{t} L_3'(s)ds \leq L_3(t_1). \tag{3.52}$$

Utilising the conditions $H', H'' > 0$ and the non-increasing nature of E, we infer that the function $t \to E(t)H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right)$ is non-increasing; hence, we conclude that

$$\begin{aligned} & c\left(\frac{E(t)}{E(0)}\right) H'\left(\frac{\varepsilon_1}{s^{\frac{1}{1+\varepsilon_0}}} \cdot \frac{E(t)}{E(0)}\right) \int_{t1}^{t} a(s)ds \\ \leq & \int_{t1}^{t} c\left(\frac{E(s)}{E(0)}\right) H'\left(\frac{\varepsilon_1}{s^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(s)}{E(0)}\right) a(s)ds \leq L_3(t_1). \qquad \forall t \geq t_1 \end{aligned} \tag{3.53}$$

Multiplying each side of (3.53) by $\frac{1}{t^{\frac{1}{1+\varepsilon}}}$, we have

$$\left(\frac{1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right) H'\left(\frac{\varepsilon_1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right) \int_{t1}^{t} a(s)ds \leq \frac{c}{t^{\frac{1}{1+\varepsilon}}}, \forall t \geq t_1 \tag{3.54}$$

Next, we set $H_2(s) = sH'(\varepsilon_1 s)$which is strictly increasing, and consequently we obtain,

$$H_2\left(\frac{1}{t^{\frac{1}{1+\varepsilon}}} \cdot \frac{E(t)}{E(0)}\right) \int_{t_1}^{t} a(s)ds \leq \frac{c}{t^{\frac{1}{1+\varepsilon}}}, \forall t \geq t_1 \tag{3.55}$$

Finally, we infer

$$E(t) \leq ct^{\frac{1}{1+\varepsilon}} H_2^{-1}\left(\frac{c}{t^{\frac{1}{1+\varepsilon}} \int_{t_1}^{t} a(s)ds}\right) \tag{3.56}$$

This finishes the proof.

3.0.3 Example

The following examples illustrate the results of Theorem 5.4,

Example 48. *Let $h(t) = c_1 e^{-c_2(1+t)}$, where $c_2 > 0$ and $c_1 > 0$is small enough so that (H1) holds.Then $H'(t) = -a(t)H(h(t))$ where $H(t) = t$ and $a(t) = c$. Therefore, we can use (3.39) to deduce*

$$E(t) \leq \frac{c}{(1+t)^{\frac{1}{\varepsilon}}} \tag{3.57}$$

Example 49. *Let* $h(t) = \frac{c}{(1+t)^q}$, *where* $q > 1+\varepsilon$ *and* c_1 *is chosen so that hypothesis (H1) is satisfied. Then* $H'(t) = -aH(h(t))$, *with* $H(s) = s^{\frac{q+1}{q}}$ *,where a is a fixed constant. Then,* (3.40) *gives*

$$E(t) \leq \frac{c}{t^{\frac{q-1-\varepsilon}{(1+\varepsilon)^2(q+1)}}} \tag{3.58}$$

3.0.4 The case : $1 < \sigma_1 < 2$

To establish the stability result in the case $1 < \sigma_1 < 2$, we need the following lemma:

Lemma 50. *The energy functional E(t) satisfies the following estimate:*

$$[-E'(t)]^{\frac{1}{1+\varepsilon}} + [-E'(t)]^{\sigma_1 - 1} \leq c\,[-E'(t)]^{\sigma_3} \tag{3.59}$$

where $\sigma_\varepsilon = min\{\sigma_1 - 1, \frac{1}{1+\varepsilon}\}$.

Proof. Using (2.3), (2.7), (2.53), (2.57) and Lemma 3.3, we have

$$E(t) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2 \geq J(t) \geq \frac{\varpi}{2}\|\Delta u(t)\|_2^2$$

then, using (2.8),

$$\|\Delta u(t)\|_2^2 \leq \frac{2}{\varpi}E(t) \leq \frac{2}{\varpi}E(0)$$

■

So, from (2.8), (**??**) and using Young's inequality, we get

$$\begin{aligned}
|E'(t)| &= \frac{1}{2}h(t)\|\Delta u(t)\|_2^2 - \frac{1}{2}(h' \circ \Delta u)(t) - \int_\Omega |u_t|^{\sigma(x)}\,dx \\
&\leq \frac{1}{2}h(t)\|\Delta u(t)\|_2^2 - \int_0^t h'(t-s)(\|\Delta u(t)\|_2^2 + \|\Delta u(s)\|_2^2)ds + c\|\Delta u\|_2^2 \\
&\leq \frac{2}{\iota}\left(\frac{1}{2}h(t) + 2h(0) + 2h(t) + c\right)E(0) \quad \leq cE(0)
\end{aligned} \tag{3.60}$$

Setting $\sigma_\varepsilon = min\{\sigma_1 - 1, \frac{1}{1+\varepsilon}\}$ and using (5.30), we obtain

$$\begin{aligned}
[-E'(t)]^{\frac{1}{1+\varepsilon}} + [-E'(t)]^{\sigma_1-1} &\leq c\,[-E'(t)]^{\sigma_3}[-E'(t)]^{\frac{1}{1+\varepsilon}-\sigma_3} + [-E'(t)]^{\sigma_3}[-E'(t)]^{\sigma_1-1-\sigma_3} \\
&\leq ((cE(0))^{\frac{1}{1+\varepsilon}-\sigma_3} + (cE(0))^{\sigma_1-1-\sigma_3})\,[-E'(t)]^{\sigma_3}
\end{aligned} \tag{3.61}$$

which completes the proof of Lemma 5.5.

Theorem 51. *Assume that (H1)–(H3) and* (2.9) *hold. Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Then, there exist positive constants C, k_2, k_3 such that the energy functional associated to problem (2.1) satisfies

$$E(t) \leq C\left(\int_{t_0}^{t} a^{\frac{1}{\sigma_3}}(s)ds\right)^{\frac{\sigma_\varepsilon - 1}{\sigma_\varepsilon}} \quad \forall t \geq t_0, \tag{3.62}$$

if H is linear.

If H is nonlinear, we have

$$E(t) \leq k_3 t^{\frac{1}{1+\varepsilon}} H_3^{-1}\left(\frac{k_2}{t^{\frac{1}{1+\varepsilon}} \int_{t_1}^{t} a(s)ds}\right), \forall t \geq t_1 \tag{3.63}$$

where $\sigma_\varepsilon = min\{\sigma_1 - 1, \frac{1}{1+\varepsilon}\}$,$H_3(s) = sH'(\varepsilon_3 s)$ and $H(s) = \left([H^{-1}]^{\frac{1}{1+\varepsilon}}\right)^{-1}(s)$.

Case H is linear:

Multiplying (3.2) by $a(t)$ and combining (2.4), (2.7), (3.32) and (3.59), we obtain, for some $m_1 > 0$,

$$\begin{aligned} a(t)L'(t) &\leq -m_1 a(t)E(t) + c\left[-E'(t)\right]^{\frac{1}{1+\varepsilon}} + ca(t)\left[-E'(t)\right]^{\sigma_1 - 1} \\ &\leq -m_1 a(t)E(t)c + c\left[-E'(t)\right]^{\sigma_3} \forall t \geq t_0 \end{aligned} \tag{3.64}$$

Let $L := aL + cE \sim E$, multiply both sides of the above estimate by $a^q E^q$, with $q = \frac{1}{\sigma_\varepsilon} - 1$ and applyYoung's inequality, to get,

$$a^q E^q(t)\check{L}(t) \leq -(m_1 - \varepsilon_2)a^{q+1}(t)E^{q+1}(t) - cE^{'}(t), \forall t \geq t_0$$

Set

$$L_1 := a^q E^q L + cE \sim E$$

Select ε_2 sufficiently small and, utilising the non-increasing feature of E, we derive, for some $m_2, m_3 > 0$,

$$L_1'(t) \le -m_2 a^{q+1}(t) E^{q+1}(t) \le -m_3 a^{q+1}(t) L_2^{q+1}(t), \forall t \ge t_0$$

Using the equivalence $L \sim E$ and a straightforward integration over (t_0, t), we get,

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$$E(t) \le C \left(\int_{t_0}^{t} a^{\frac{1}{\sigma_3}}(s) ds \right)^{\frac{\sigma_\varepsilon - 1}{\sigma_\varepsilon}}, \forall t \ge t_0$$

Case H is nonlinear :

Using (3.1), (3.30) and (3.33), we obtain, $\forall t \ge t_0$,

$$L'(t) \le -\lambda_0 E(t) + ct^{\frac{1}{1+\varepsilon}} \left[H^{-1} \left(\frac{\varepsilon_0 I(t)}{ta(t)} \right) \right]^{\frac{1}{1+\varepsilon}} + c \left[-E'(t) \right]^{\sigma_1 - 1} \tag{3.65}$$

Using $(3.44 - 3.46)$, (3.65) reduces to

$$L'(t) \le -\lambda_0 E(t) + ct^{\frac{1}{1+\varepsilon}} H^{-1}(3c7(t)) + c \left[-E'(t) \right]^{\sigma_1 - 1}, \forall t \ge t_1 \tag{3.66}$$

Now, for $\varepsilon_3 < r$ and using (3.66) and the fact that $E' \le 0, H' > 0, H'' > 0$ on $]0, r]$, we find that the functional F , defined by

$$F(t) := H' \left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}} . \frac{E(t)}{E(0)} \right) L(t) \tag{3.67}$$

satisfies $\qquad F \sim E$

and, for all $\qquad \forall\, t \ge t_1$,

$$\begin{aligned} F'(t) \quad &\le \quad -\lambda_0 E(t) H' \left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}} . \frac{E(t)}{E(0)} \right) + ct^{\frac{1}{1+\varepsilon}} H' \left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}} . \frac{E(t)}{E(0)} \right) H'(3c7(t)) \\ &\quad + cH' \left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}} . \frac{E(t)}{E(0)} \right) \left[-E'(t) \right]^{\sigma_1 - 1} \end{aligned} \tag{3.68}$$

Upon employing the generalised Young inequality, we obtain

$$
\begin{aligned}
F'(t) &\leq -\lambda_0 E(t) H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) + ct^{\frac{1}{1+\varepsilon_0}} H^*\left(H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)\right) \\
&\quad + cH'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)[-E'(t)]^{\sigma_1-1} + ct^{\frac{1}{1+\varepsilon}} 3c7(t) \qquad (3.69) \\
&\leq -\lambda_0 E(t) H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) + c\varepsilon_1 \frac{E(t)}{E(0)} H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) \\
&\quad + ct^{\frac{1}{1+\varepsilon}} 3c7(t) - c_\varepsilon E' + \varepsilon\, [H']^{\frac{1}{2-\sigma_1}}\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)
\end{aligned}
$$

.

Using the facts that $\frac{1}{2-\sigma_1} > 1$ and $H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)$ is bounded, we have

$$
[H']^{\frac{1}{2-\sigma_1}}\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) \leq cH'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) \qquad (3.70)
$$

Then, multiplying (3.69) by $a(t)$, using (3.46), (3.70) and the fact that $E(t) > 0$, we get

$$
\begin{aligned}
a(t)F_1'(t) \leq -\lambda_0 a(t) E(t) H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) + c\varepsilon_5 a(t).\frac{E(t)}{E(0)} H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) \\
+ c\varepsilon a(t) E(t) H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) - cE'(t)\ ,\ \forall t \geq t_1,
\end{aligned}
$$

where $F_1 = F + c_\varepsilon E$'. Using the non-increasing property of a, we obtain, for all $t \geq t_1$,

$$
\begin{aligned}
(aF_1 + cE)'(t) &\leq -\lambda_0 a(t) E(t) H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) + c\varepsilon_5 a(t)\frac{E(t)}{E(0)} H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) \\
&\quad + c\varepsilon a(t) E(t) H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)
\end{aligned}
$$

Therefore, by setting $F_2 := aF_1 + cE \sim E$, we conclude that

$$
\begin{aligned}
F_2'(t) &\leq -\lambda_0 a(t) E(t) H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) + c\varepsilon_3 a(t).\frac{E(t)}{E(0)} H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right) \\
&\quad + c\varepsilon a(t) E(t) H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)
\end{aligned}
$$

This gives, for a suitable choice of ε_3 and ε

$$F_2'(t) \leq -Ka(t)\left(\frac{E(t)}{E(0)}\right)H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right), \forall t \geq t_1$$

or

$$K\left(\frac{E(t)}{E(0)}\right)H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon_0}}}.\frac{E(t)}{E(0)}\right)a(t) \leq -F_2'(t), \forall t \geq t_1 \tag{3.71}$$

An integration of (3.71) yields

$$\int_{t_1}^{t} K\left(\frac{E(s)}{E(0)}\right)H'\left(\frac{\varepsilon_3}{s^{\frac{1}{1+\varepsilon_0}}}.\frac{E(t)}{E(0)}\right)a(s)ds \leq -\int_{t_1}^{t} F_2'(t)ds \leq F_2(t_1) \tag{3.72}$$

Using the facts that $H', H'' > 0$ and the non-increasing property of E, we deduce that the map $t \to E(t)H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon_0}}}.\frac{E(t)}{E(0)}\right)$ is non-increasing and consequently, we have

$$\begin{aligned} & K\left(\frac{E(t)}{E(0)}\right)H'\left(\frac{\varepsilon_3}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)\int_{t_1}^{t} a(s)ds \\ \leq & \int_{t_1}^{t} K\left(\frac{E(s)}{E(0)}\right)H'\left(\frac{\varepsilon_3}{s^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)a(s)ds \leq F_2(t_1), \forall t \geq t_1 \end{aligned} \tag{3.73}$$

Multiplying each side of (3.73) by $\frac{1}{t^{\frac{1}{1+\varepsilon}}}$, we have

$$\left(\frac{K}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)H'\left(\frac{\varepsilon_3}{s^{\frac{1}{1+\varepsilon_0}}}.\frac{E(t)}{E(0)}\right)\int_{t_1}^{t} a(s)ds \leq \frac{K_2}{t^{\frac{1}{1+\varepsilon}}}, \forall t \geq t_1 \tag{3.74}$$

Using the fact that $B_3(s) = H'(\varepsilon_3)$ is strictly increasing, we obtain

$$kB_3\left(\frac{1}{t^{\frac{1}{1+\varepsilon}}}.\frac{E(t)}{E(0)}\right)\int_{t_1}^{t} a(s)ds \leq \frac{K_2}{t^{\frac{1}{1+\varepsilon}}}, \forall t \geq t_1 \tag{3.75}$$

Finally, we infer

$$E(t) \leq K_3 t^{\frac{1}{1+\varepsilon}} B_3^{-1}\left(\frac{K_2}{t^{\frac{1}{1+\varepsilon}} \int_{t_1}^{t} a(s)ds}\right) \tag{3.76}$$

This finishes the proof.

3.0.5 Example

The following examples illustrate the results of Theorem 5.6

Example 52. *Let* $h(t) = c_1 e^{-c_2(t^2+1)}$*, where* $c_2 > 0$ *and* $c_1 > 0$ *is small enough so that (H1) holds. Then* $h'(t) = -a(t)H(h(t))$ *where* $H(t) = t$ *and* $a(t) = t$*. Therefore,* (3.62) *gives for* $t > t_0$ *and* $\epsilon \in (0, 1)$*,*

$$E(t) \leq C\left(\int_{t_0}^{t} a^{\frac{1}{\sigma_3}}(s)ds\right)^{\frac{\sigma_\varepsilon - 1}{\sigma_\varepsilon}} \leq C\left(\int_{t_0}^{t} s^{\frac{1}{\sigma_3}}ds\right)^{\frac{\sigma_\varepsilon - 1}{\sigma_\varepsilon}}, \forall t \geq t_0 \tag{3.77}$$

so,

$$E(t) \leq c(t^{\frac{1+\sigma_3}{\sigma_3}} - t_0^{\frac{1+\sigma_3}{\sigma_3}})^{\frac{\sigma_\epsilon - 1}{\sigma_\epsilon}} \tag{3.78}$$

Example 53. *Let* $h(t) = \frac{c_1}{(1+t)^q}$ *where* $q > 1 + \varepsilon$ *and* c_1 *is chosen so that hypothesis (H1) is satisfied. Then* $h'(t) = -aH(h(t))$*, with* $H(s) = s^{\frac{q+1}{q}}$

where a *is a fixed constant. Then,* (3.63) *gives, for* $t > t_1$ *and* $\varepsilon \in]0, 1[$*,*

$$E(t) \leq K_3 t^{\frac{1}{1+\varepsilon}} (H')^{-1}\left(\frac{K_2}{t^{\frac{1}{1+\varepsilon}} \int_{t_1}^{t} a(s)ds}\right) \tag{3.79}$$

$$E(t) \leq \frac{c}{t^{\frac{q-1-\epsilon}{(1+\epsilon)^2(q+1)}}} \tag{3.80}$$

Example 54. *Let* $h(t) = \frac{c_1}{(1+t)^{q+1}}$ *where* $q > 1 + \varepsilon$ *and* c_1 *is chosen so that hypothesis (H1) is satisfied. Then* $h'(t) = -aH(h(t))$*, with* $H(s) = s^{\frac{q+1}{q}}$

where a *is a fixed constant. Then,* (3.63) *gives, for* $t > t_1$ *and* $\varepsilon \in]0, 1[$*,*

$$E(t) \leq K_3 t^{\frac{1}{1+\varepsilon}} (H')^{-1}\left(\frac{K_2}{t^{\frac{1}{1+\varepsilon}} \int_{t_1}^{t} a(s)ds}\right) \tag{3.81}$$

so,

$$E(t) \leq \frac{c}{t^{\frac{q-1-\epsilon}{(1+\epsilon)^2(q+1)}}} \tag{3.82}$$

3.1 Conclusion

we use the Faedo-Galerkin method to prove the local existence of the weak solution in $[0,T] \times \Omega$, and the global existence of the solution is established for the problem (2.1.1) in $[0,+\infty[\times\Omega$. And we study general decay results for a broad class of relaxation functions and some special conditions for the variable exponent function in two cas $\sigma(x)$, $\sigma(x)$. Our results supplement and generalise many previous findings, and we support them with examples.

Bibliography

[1] M. M. Cavalcanti, V. N. Domingos Cavalcanti, J. A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, Electron. J. Differ. Eq., 44 (2002),1–14.

[2] S. A. Messaoudi, General decay of solution energy in a viscoelastic equation with a nonlinear source, Nonlinear Anal. Theor., 69 (2008), 2589–2598.

[3] S. A. Messaoudi, General decay of solutions of a viscoelastic equation, J. Math. Anal. Appl., 341 (2008), 1457–1467.

[4] F. Alabau-Boussouira, P. Cannarsa, A general method for proving sharp energy decay rates for memory-dissipative evolution equations, C. R. Acad Sci. Paris, Ser I, 347 (2009), 867–872.

[5] S. A. Messaoudi, W. Al-Khulaifi, General and optimal decay for a quasilinear viscoelastic equation, Appl. Math. Lett., 66 (2017), 16–22.

[6] M. Mustafa, Optimal decay rates for the viscoelastic wave equation, Math. Method. Appl. Sci., 41 (2017), 192–204.

[7] S. Antontsev, S. Shmarev, Evolution PDEs with nonstandard growth conditions: Existence, uniqueness, localization, blow-up, Springer, 2015.

[8] L. Diening, P. Harjulehto, P. Hasto, M. Ruzicka, Lebesgue and Sobolev spaces with variable exponents, Springer, 2011.

[9] S. A. Messaoudi, J. Al-Smail, A. Talahmeh, Decay for solutions of a nonlinear damped wave equation with variable-exponent nonlinearities, Comput. Math. Appl., 76 (2018), 1863–1875.

[10] S. A. Messaoudi, A. Talahmeh, On wave equation: Review and recent results, Arab. J. Math., 7 (2018), 113–145.

[11] A. Palmieri, H. Takamura, Semi-linear wave models with power non-linearity and scale-invariant time-dependent mass and dissipation II, Math. Nachr., 291 (2018), 1859–1892.

[12] A. Palmieri, M. Reissig, Blow-up for a weakly coupled system of semilinear damped wave equations in the scattering case with power nonlinearities, Nonlinear Anal., 187 (2019), 467–492.

[13] W. Chen, R. Ikehata, The Cauchy problem for the Moore-Gibson-Thompson equation in the dissipative case, J. Differ. Equations, 292 (2021), 176–219.

[14] J. Barrow, P. Parsons, Inflationary models with logarithmic potentials, Phys. Rev. D, 52 (1995), 5576–5587.

[15] K. Enqvist, J. McDonald, Q-balls and baryogenesis in the MSSM, Phys. Lett. B, 425 (1998), 309–321 .

[16] K. Bartkowski, P. Gorka, One-dimensional Klein-Gordon equation with logarithmic nonlinearities, J. Phys. A Math. Theor., 41 (2008), 355201.

[17] I. Bialynicki-Birula, J. Mycielski, Wave equations with logarithmic nonlinearities, Bull. Acad. Pol. Sc., 23 (1975), 461–466.

[18] P. Gorka, Logarithmic Klein-Gordon equation, Acta Phys. Pol. B, 40 (2009), 59–66.

[19] D. Edmunds, J. Rakosnik, Sobolev embeddings with variable exponent, Stud. Math., 143 (2000), 267–293.

[20] D. Edmunds, J. Rakosnik, Sobolev embeddings with variable exponent II, Math. Nachr., 246 (2002), 53–67.

[21] M. Al-Gharabli1, Adel M. Al-Mahdi, M. Kafin, Global existence and new decay results of a viscoelastic wave equation with variable exponent and logarithmic nonlinearities, AIMS Mathematics, 6(9): 2021,10105–10129.

[22] M. Al-Gharabli, A. Guesmia, S. A. Messaoudi, Well-posedness and asymptotic stability results for a viscoelastic plate equation with a logarithmic nonlinearity, Appl. Anal., 99 (2020), 50–74.

[23] L. Gross, Logarithmic Sobolev inequalities, Am. J. Math., 97 (1975), 1061–1083.

[24] H. Chen, P. Luo, G. W. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity, J. Math. Anal. Appl., 422 (2015), 84–98.

[25] F. Belhannache, M. Algharabli, S. A. Messaoudi, Asymptotic stability for a viscoelastic equation with nonlinear damping and very general type of relaxation functions, J. Dyn. Control Syst., 26 (2020), 45–67.

[26] M. Al-Gharabli, A. Guesmia, Messaoudi S. A. Messaoudi, Existence and a general decay results for a viscoelastic plate equation with a logarithmic nonlinearity, Commun. Pure Appl. Anal., 18 (2019), 159–180.

[27] V. Arnold, Mathematical methods of classical mechanics, Springer Science & Business Media, 1989.

Summary

In this work, we consider the following viscoelastic problem with variable exponents and logarithmic nonlinearities:

$$u_{tt} - \mu_1 \Delta u + \int_0^t h(t-s)\Delta u(s)ds + |u_t|^{\sigma(\cdot)-2}u_t = u \ln |u|^{\eta(x)}$$

We use the Faedo-Galerkin method to show that a weak solution exists locally, and we also demonstrate that a solution exists globally for the problem . Additionally, we explore general decay results for a wide range of relaxation functions and specific conditions for the variable exponent function. We conclude our results by presenting examples illustrating these results.

Résumé

Dans ce travail, nous considérons le problème viscoélastique suivant avec des exposants variables et des non-linéarités logarithmiques :

$$u_{tt} - \mu_1 \Delta u + \int_0^t h(t-s)\Delta u(s)ds + |u_t|^{\sigma(\cdot)-2}u_t = u \ln |u|^{\eta(x)}$$

Nous utilisons la méthode de Faedo-Galerkin pour montrer qu'une solution faible existe localement, et nous démontrons également qu'une solution existe globalement pour le problème . De plus, nous explorons des résultats de décroissance générale pour une large gamme de fonctions de relaxation et des conditions spécifiques pour la fonction à exposant variable. Nous concluons nos résultats en présentant des exemples illustrant ces résultats.

الملخص

في هذا العمل، نعتبر المشكلة اللزجة المرنة التالية ذات الأسس المتغيرة واللاخطية اللوغاريتمية

$$u_{tt} - \mu_1 \Delta u + \int_0^t h(t-s)\Delta u(s)ds + |u_t|^{\sigma(\cdot)-2}u_t = u \ln |u|^{\eta(x)}$$

نستخدم طريقة فايدوغاليركين لإظهار أن هناك حلاً ضعيفاً موجوداً محلياً، ونوضح أيضاً أن هناك حلاً موجوداً كلياً للمشكلة .بالإضافة إلى ذلك، نبرهن نتائج الانحلال العامة لمجموعة واسعة من دوال الاسترخاء والشروط المحددة لدالة الأس المتغيرة ,ونختتم نتائجنا بتقديم أمثلة توضح هذه النتائج.