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Course Handout Lectures

Intended for students of: First year LMD ST

PHYSICS 1

Course and Exercises with Solutions

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بسم الله الرحمن الرحيم

"يأبى اللهُ العصمة لكتابٍ غير كتابه، والمنصف من اغتفر قليل خطأ المرء في كثير صوابه"

ابن رجب الحنبلي

Preface

The course presented in this text is designed for first-year university students pursuing studies in technical and material sciences, in accordance with the Algerian LMD educational framework.

This course offers a clear and simplified approach to object point mechanics, accompanied by illustrations and exercises with detailed solutions.

The primary aim is to support students in addressing difficulties encountered in their studies while facilitating a thorough comprehension of the object point mechanics subject matter.

The exercises presented at the end of each chapter are systematically arranged in increasing order of complexity, progressing from basic applications of the concepts to more intricate problems.

It is my sincere hope that this course will be beneficial and will significantly contribute to the success of first-year university students.

Mourad ROUGAB

Contents

DIMENSIONAL ANALYSIS 1

1.1 Introduction

Physics is the scientific discipline dedicated to the examination and study of natural phenomena. The term Physics is derived from the Greek word "phúsis", meaning "nature". Its primary objective is to describe the interactions of matter, energy, space, and time, thereby revealing the fundamental principles and laws that govern the universe across various temporal and spatial scales. Physics seeks to provide rational explanations for the diverse phenomena observed in the natural world. To achieve this comprehension, the discipline is based on a limited set of quantitative laws (fundamental laws). These laws pertain to various branches of physics, beginning with classical physics, which includes mechanics, electromagnetism, optics, thermodynamics, and concluding with Albert Einstein's theory of special relativity.

Prior to engaging in the study of physics in depth, it is essential to become acquainted with certain frequently used terminologies and the methodologies employed in the investigation of physical phenomena and their implications.

In physics, quantities that are measured or calculated are generally expressed as numerical values accompanied by their corresponding units. These units must be accurate and consistent, which underscores the importance of a coherent system of units. The coherence and relationships between units can be verified through dimensional analysis, a method that helps establish the general form of an equation and validates the correctness of laws in terms of dimensions.

In this chapter, we will define the units required to express the results of measurements and calculations, and we will also discuss dimensional analysis.

1.2 Units and dimensions

System of Units

A **system of units** is defined as a complete set of units used to measure all kinds of fundamental and derived quantities. Below are some of the frequently used systems of units:

(a) **FPS system**[1]**:** In this system, the units of length, mass and time are **foot (ft)**, **pound (lb)** and **second (s)**, respectively. The unit of force in this system is **poundal**.
(b) **CGS System**[2]**:** In this system, the units of length, mass, and time are **centimetre (cm)**, **gram (g)**, and **second (s)** respectively. The unit of force in this system is **dyne**, and that of work or energy is **erg**.
(c) **MKS system** In this system, the units of length, mass and time are **metre (m)**, **kilogram (kg)** and **second (s)**, respectively. The unit of force in this system is **newton (N)** and that of work or energy is **joule (J)**.
(d) **International system (SI)** This unit system facilitates significant advancements beyond the MKS system and is known as **rationalised MKS system**. It is useful for deriving all physical quantities in the field of physics.

1: The FPS system is not a metric system. This system is not in much use these days.

2: The drawback of CGS system is that many of the derived units on this system are inconveniently small.

International system of units

The General Conference on Weights and Measures (**CGPM** from the french term "La Conférence générale des poids et mesures") convened and established a system of units known as **the International System of Units**[3], commonly referred to as **SI**, which comes from the French term "**Le Système International d'unités**". This system is widely used around the world.

3: The name *Système International d'Unités*, and the abbreviation SI, were established by the 11th CGPM in 1960.

Figure 1.1: The SI Base Units

The CGPM officially adopts formal definitions for all SI base units. The first two definitions were established in 1889, while the latest one was adopted in 1983. These definitions are periodically updated to reflect advancements in science.

The International System of Units (SI) consists of seven independent base units, also known as fundamental units: second (s), meter (m), kilogram (kg), ampere (A), kelvin (K), mole (mol), and candela (cd). Table 1.1 lists the seven fundamental quantities, along with their corresponding symbols and SI units.

Table 1.1: The SI Base Units

Quantity	Typical symbols	Unit name
Length	l, x, r, etc.	metre (m)
Mass	m	kilogram (kg)
Time	t	second (s)
Electric current intensity	I, i	ampere (A)
Thermodynamic temperature	T	kelvin (K)
Amount of substance	n	mole (mol)
Luminous intensity	I_v	candela (cd)

Additionally, the International System of Units (SI) includes supplementary units: the radian (rad), which measures plane angles, and the steradian (sr), which measures solid angles (Table 1.2).

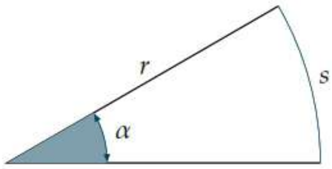


Figure 1.2: Plane angle $\alpha = s/r$.

Table 1.2: The SI supplementary units

Quantity	Unit name	Symbol of unit
Plane angle	radian	rad
Solid angle	steradian	sr

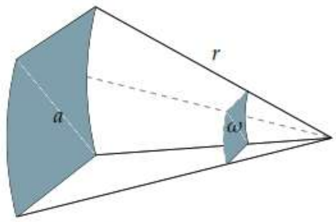


Figure 1.3: Solid angle $\omega = a/r$.

The second

The SI unit for time, the second, has evolved from being defined as 1/86,400 of a mean solar day to a more precise definition based on atomic vibrations. In 1967, the second was redefined as the duration required for 9,192,631,770 vibrations of the unperturbed ground state of the caesium 133 atom (caesium frequency $\Delta\nu_{Cs}$). This definition provides a constant and reliable measure of time, unlike the solar day, which is gradually lengthening due to the Earth's slowing rotation. This high level of precision is crucial for technologies such as GPS, which rely on accurate timekeeping from atomic clocks to provide precise location information.

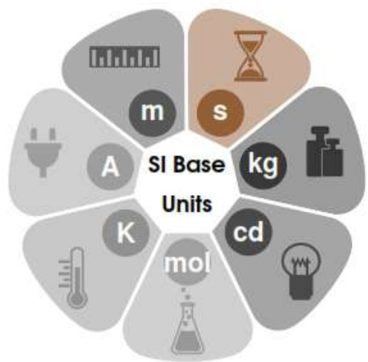


Figure 1.4: The second

The new definition of the second implies the exact relation $\Delta\nu_{Cs}$= 9,192,631,770 Hz. By rearranging this relationship, one can derive an expression for the unit of time, the second, in terms of the defining constant $\Delta\nu_{Cs}$:

$$1\mathrm{s} = \frac{9,192,631,770}{\Delta\nu_{Cs}}$$

The meter

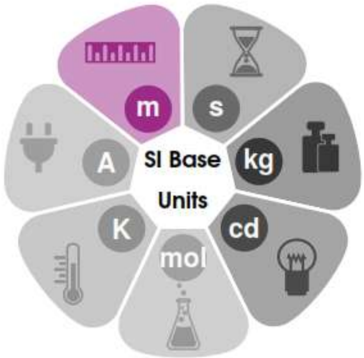


Figure 1.5: The meter

The SI unit for length, the meter, has undergone several definitions to enhance precision since its inception in 1791, when it was originally defined as $1/10{,}000{,}000$ of the distance from the equator to the North Pole. In 1889, the meter was redefined based on the distance between two engraved lines on a platinum-iridium bar. By 1960, it was further refined to correspond to 1,650,763.73 wavelengths of orange light emitted by krypton atoms. The current definition, established in 1983, defines the meter as the distance light travels in a vacuum in $1/299{,}792{,}458$ of a second, based on the exact speed of light being 299,792,458 m/s.

The kilogram

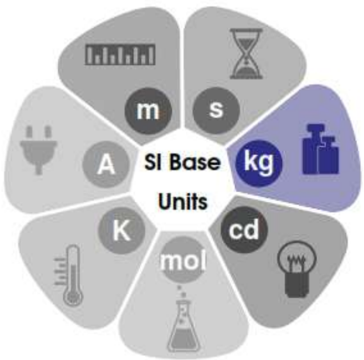


Figure 1.6: The kilogram

The kilogram is the SI unit of mass, defined by the mass of a platinum-iridium cylinder stored at the International Bureau of Weights and Measures in Paris. Replicas of this standard kilogram are maintained at the U.S. National Institute of Standards and Technology (NIST) in Maryland and at various locations worldwide. Currently, NIST scientists are exploring two methods to redefine the kilogram, emphasizing that all other mass measurements are ultimately based on comparisons with this standard mass.

The ampere

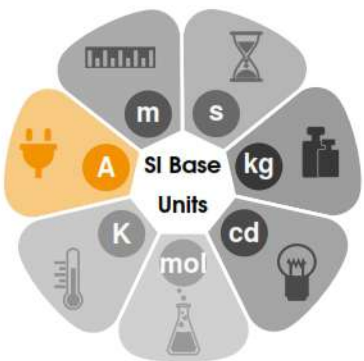


Figure 1.7: The ampere

Since 1948, the ampere has been defined as the intensity of a constant current that, when maintained in two infinitely long, parallel conductors with negligible cross-sectional areas and separated by one meter in a vacuum, produces a force per unit length of 2×10^{-7} newtons. In 2019, at the 26^{th} General Conference on Weights and Measures, the definition of the ampere was revised to be based on the elementary electric charge, $e = 1,602,176,634 \times 10^{-19}$ C (coulombs), and the second:

$$1\,C = 1\,A.s$$

The kelvin

Since 1968, the kelvin, the unit of thermodynamic temperature, has been defined as the fraction $1/273.16$ of the thermodynamic temperature of the triple point of water. Consequently, the thermodynamic temperature of the triple point of water is exactly 273.16 kelvins (T_{tpw} = 273.16 K). In 2019, the kelvin was redefined by taking the fixed numerical value of the Boltzmann constant (k) as 1.380649×10^{-23} when expressed in the unit $J.K^{-1}$. This definition implies the exact relationship $k = 1.380649 \times 10^{-23}$ $\mathrm{J.K^{-1}}$. As a result, one kelvin corresponds to a change in thermodynamic temperature (T) that leads to a change in thermal energy (kT) of 1.380649×10^{-23} J.

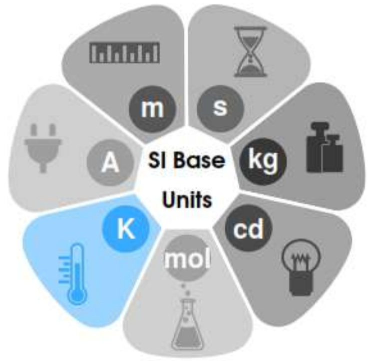


Figure 1.8: The kelvin

The mole

The mole, as defined in 1971, is characterized as "the amount of substance of a system containing as many elementary entities as there are atoms in 0.012 kilogram of carbon 12". At the 26^{th} General Conference on Weights and Measures held in 2019, a revised definition of the mole was established, stipulating that one mole contains exactly $6.02214076 \times 10^{23}$ entities. This number is the fixed numerical value of the Avogadro constant (NA) when expressed in the unit mol^{-1} and is commonly referred to as Avogadro's number. The quantity of substance, denoted as n, within a system serves as an indicator of the number of specified elementary entities present. An elementary entity may encompass an atom, a molecule, an ion, an electron, or any other particle or defined collection of particles.

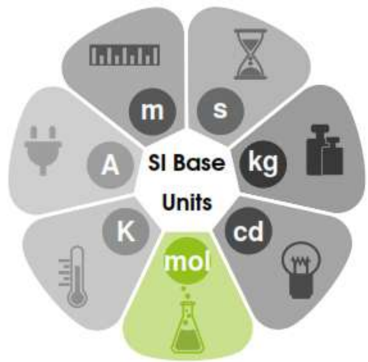


Figure 1.9: The mole

The candela

Since 1979, the candela has been defined as the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and whose energy intensity in this direction is 1/683 watt per steradian. Consequently, the spectral luminous efficacy of monochromatic radiation of frequency 540×10^{12} hertz is precisely 683 lumens per watt (the wavelength λ of radiation of this frequency is approximately 555 nm), $\mathrm{K}(\lambda_{555})$ = 683 lm/W = 683 cd sr/W.

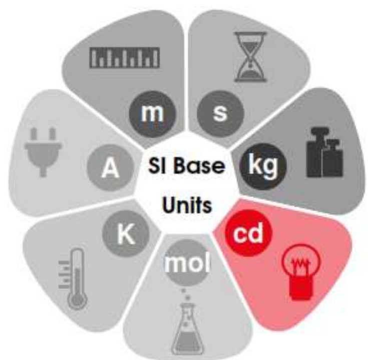


Figure 1.10: The candela

SI derived units

In the International System of Units (SI), there exists an additional category of units known as **derived units**. These units are formulated through algebraic expressions that involve multiplication and/or division.

Example of derived units

- Frequency is expressed in hertz (Hz) in the International System of Units. Since frequency represents the inverse of a period, it can also be expressed in s^{-1}.
- Speed is expressed in $m.s^{-1}$.
- SI unit of force is Newton (N).
- SI unit of energy is Joule (J).
- SI unit of density is kg.m^{-3}.

It is important to note that there are several unit systems, including the CGS system, the MKSA system, and the FPS (or British) system. For instance, the CGS system comprises three base quantities: length measured in centimeters, mass measured in grams, and time measured in seconds. The MKSA system, on the other hand, is based on four fundamental quantities and units: length in meters, mass in kilograms, time in seconds, and electric current intensity in amperes.

Some practical units

Units of length

1. **Parallactic Second (Parsec):** It is defined as the distance at which an arc measuring one astronomical unit (1.496×10^{11} m) subtends an angle of one arcsecond[4] (arcsec). The parsec is considered the largest practical unit of distance.

4: Also known as one second of an arc.

$$r = \frac{l}{\theta} = \frac{1.496 \times 10^{11}\ \text{m}}{1\ \text{second}}$$

or,

$$\begin{aligned} 1\ \text{parsec} &= \frac{1.496 \times 10^{11}\ \text{m}}{\pi/(60 \times 60 \times 180)\ \text{radian}} \\ &= \frac{1.469 \times 10^{11} \times 60 \times 60 \times 180}{\pi} \end{aligned}$$

$$\therefore\ \mathbf{1\ parsec = 3.08 \times 10^{16} m.}$$

2. **Light Year:** It is also a practical unit of distance used in astronomy to measure the distances to nearby stars. A light year is defined as the distance that light travels in one year through a vacuum, free space, or air.

$$1 \text{ Light year} = 3 \times 10^8 \text{ m/s} \times 365 \times 24 \times 60 \times 60 \text{ s}$$

$$\therefore\ \mathbf{1\ Light\ year = 9.5 \times 10^{15}\ m = 9.5 \times 10^{12}\ km.}$$

3. **Astronomical Unit (A.U.):** It is defined as the average distance from the Sun to the Earth, which is the radius of the Earth's orbit.

$$\mathbf{1\ A.U. = 1.496 \times 10^{11}\ m \approx 1.5 \times 10^{8}\ km.}$$

   This unit is also used in astronomy to measure distances of planets.
4. **Microns (μm):** It is an unit of distance defined as **micrometre**.

$$\mathbf{1\ micron = 1\mu m = 10^{-6}\ m.}$$

5. **Angstrom (Å):** It is also a practical unit of length used in atomic physics.

$$\mathbf{1\ \text{Å} = 10^{-10}\ m}$$

6. **Fermi:** It is the smallest practical unit of distance used in nuclear physics.

$$\mathbf{1\ Fermi = 10^{-15}\ m.}$$

Units of mass

1. **Chandra Shekhar Limit (C.S.L.):** It is considered the largest practical unit of mass.

$$\mathbf{1\ \ C.S.L. = 1.4 \times mass\ of\ the\ sun.}$$

2. **Metric Ton:**

$$\mathbf{1\ \ Metric\ Ton = 1000\ kg.}$$

3. **Atomic Mass Unit (a.m.u.):** It is considered the smallest unit of mass. This unit is not the atomic standard for mass, but rather a practical mass unit in atomic and nuclear physics. It is defined as one-twelfth (1/12th) of the mass

of a single carbon-12 (C-12) atom.

$$1 \text{ a.m.u.} = \frac{1}{12} \times \text{ mass of one C-12 atom}$$
$$= \frac{1}{12} \times \frac{12}{N} = \frac{1}{N}$$

$$\mathbf{1\ a.m.u.} = \frac{\mathbf{1}}{\mathbf{6.023 \times 10^{23}}}\ \mathbf{g = 1.67 \times 10^{-27}\ kg}.$$

1 a.m.u is just equal to the reciprocal of Avogadro's Number.

1.3 Common prefixes of SI units

Table 1.3 presents the prefixes and symbols used for converting units into multiples and sub-multiples. For instance, one thousand ohms is referred to as a kilo-ohm, denoted as $k\Omega$. Similarly, $M\Omega$ represents the mega-ohm (one million Ohms), while $m\Omega$ denotes the milli-ohm (one thousandth of an ohm).

Table 1.3: Prefixes for multiples and sub-multiples.

Prefix	Symbol	Factor	Prefix	Symbol	Factor
yotta	Y	10^{24}	deci	d	10^{-1}
zetta	Z	10^{21}	centi	c	10^{-2}
exa	E	10^{18}	milli	m	10^{-3}
peta	P	10^{15}	micro	μ	10^{-6}
tera	T	10^{12}	nano	n	10^{-9}
giga	G	10^{9}	pico	p	10^{-12}
mega	M	10^{6}	femto	f	10^{-15}
kilo	k	10^{3}	atto	a	10^{-18}
hecto	h	10^{2}	zepto	z	10^{-21}
deca	da	10	yocto	y	10^{-24}

1.4 Physical quantities

A physical quantity is defined as any measurable physical property. The measurement process involves comparing the quantity in question with a standard quantity of the same nature, known as a **standard** or **etalon**, which is conventionally accepted as a unit of measurement. The ratio between the measured quantity and this standard provides the numerical value of the physical quantity in terms of the chosen unit.

We distinguish between **base quantities** (fundamental quantities), which cannot be calculated using equations based on other already determined values, and **derived quantities**, which can be expressed as algebraic combinations of the base quantities. Each physical quantity consists of a numerical value accompanied by a corresponding unit of measurement. For example, the distance "d" traveled by a moving object is equal to ten meters, denoted mathematically as $d = 10\,m$. Similarly, the mass m of an object is quantified as $m = 66\,kg$, and so on.

1.5 Dimension of physical quantity

By convention, physical quantities are organized according to a system of dimensions. Each of the seven base quantities of the international system of units is assigned a unique dimension, symbolically represented by a single uppercase letter in Roman type without serifs. The symbols used to indicate the dimension of each base quantity are displayed in Table 1.4.

Quantity	Dimension symbol	SI unit
Length	L	m
Mass	M	kg
Time	T	s
electric current	I	A
Thermodynamic temperature	Θ	K
Amount of substance	N	mol
Luminous intensity	J	cd

Table 1.4: Dimensions of the base quantities

There exist also physical quantities classified as **dimensionless** (also known as "non-dimensional" or "of dimension 1"), indicating that they do not possess any associated dimensions.

1.6 Dimensional equation

The dimensional equation of a physical quantity X is an expression derived from a physical formula in which X is included. In this equation, the dimension of X, denoted as $[X]$, is expressed in terms of the dimensions of the seven base SI units. The dimension of a quantity X is commonly expressed as a dimensional product, namely:

$$\boxed{[X] = L^{\alpha}\, M^{\beta}\, T^{\gamma}\, I^{\delta}\, \Theta^{\varepsilon}\, N^{\zeta}\, J^{\eta}} \tag{1.1}$$

the exponents $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, and η are referred to as **dimensional exponents**. These exponents are typically small integers that may take on positive, negative, or zero values. It is important to note that for dimensionless physical quantities, the dimensional representation is expressed as $[X] = 1$.

1.7 Dimensional analysis

Dimensional analysis is a qualitative investigative method that involves identifying all relevant parameters of a physical phenomenon to deduce how a quantity depends on these parameters. This analysis, however, does not allow for the determination of numerical factors, which can only be obtained through quantitative or experimental studies.

Dimensional analysis allows:

- To determine the composite unit of the specified quantity based on the base units.
- The eventual resolution of unit conversion.
- Error detection in an equation (Any non-homogeneous formula is necessarily false).
- The prediction of the form of a law, provided that the pertinent parameters of the problem are identified.

Example 1.1

1. Find the dimension of a force F.
2. Determine the composite unit of force in terms of the base units.

Solution

1. $F = m\,a \Rightarrow [F] = [M]\,[a]$,
   We have:
   $a = \dfrac{dv}{dt} \Rightarrow [a] = [v]\,[T]^{-1}$ and $v = \dfrac{dx}{dt} \Rightarrow [v] = [L]\,[T]^{-1}$.
   Then:
   $[F] = [M]\,[L]\,[T]^{-1}\,[T]^{-1}$, therefore: $[F] = M\,L\,T^{-2}$.
2. We have:
   $[F] = M\,L\,T^{-2}$,
   we know that the unit of force is the Newton, using the

Table 1.4 we find:
$1\ Newton = 1\ kg.m.s^{-2}$.

Example 1.2

Unit conversion:
In the CGS system (centimeters, grams, seconds), force is expressed in dynes. How many Newtons are equivalent to 1 dyne?

Solution

The dimensional equation for force, expressed as $[F] = M\,L\,T^{-2}$, must be validated across all systems of units.
So we have: $1\ Newton = 1\ kg.m.s^{-2}$ and $1\ dyne = 1\ g.cm.s^{-2}$
Thus, we can deduce the conversion: $1\ Newton = 10^5\ dynes$.

Example 1.3

Which of these formulas are homogeneous (dimensionally correct)?

(a) $T = 2\pi\sqrt{\frac{\ell}{g}}$;

(b) $T = 2\pi\sqrt{\frac{g}{\ell}}$;

(c) $T = \frac{1}{2\pi}\sqrt{\frac{\ell}{g}}$;

(d) $T = 2\pi\sqrt{\frac{(\ell + g)}{\ell}}$.

T represents the period of oscillation of a simple pendulum (time) of length ℓ. g is the acceleration due to gravity.

Solution

(a) $[2\pi] = [1]$, $\left[\sqrt{\frac{\ell}{g}}\right] = \sqrt{\frac{[L]}{[L]\ [T]^{-2}}} = \frac{1}{\sqrt{[T]^{-2}}} = \sqrt{[T]^2} = [T]$.
(a) is a homogeneous formula.

(b) $\left[\sqrt{\frac{g}{\ell}}\right] = \sqrt{\frac{[L]\ [T]^{-2}}{[L]}} = \sqrt{[T]^{-2}} = [T]^{-1}$.

(b) is a non-homogeneous formula.

(c) $\left[\frac{1}{2\pi}\right] = [1]$, $\left[\sqrt{\frac{\ell}{g}}\right] = [T]$.

(c) is a homogeneous formula.

(d) Non-homogeneous : The terms in the addition (ℓ and g) must have the same dimensions.

Refer to Example 1.2

The prediction of the form of a law:

We assume that the period T is a function of the mass m, the gravitational field g, and the length ℓ of the pendulum, expressed as $T = f(m, g, \ell)$. We can then write:

$$T = Cte\ \left(m^{\alpha}\, g^{\beta}\, \ell^{\gamma}\right)$$

where Cte is a dimensionless factor. This leads us to the following dimensional equation:

$$T = M^{\alpha}\ \left(L\,T^{-2}\right)^{\beta}\ L^{\gamma} = M^{\alpha}\, L^{(\beta+\gamma)}\, T^{-2\beta}$$

For the law to be homogeneous, the following conditions must be met:

$$\begin{aligned} &\alpha = 0, \\ &-2\beta = 1 \Rightarrow \beta = \frac{-1}{2}, \\ &\beta + \gamma = 0 \Leftrightarrow \beta = -\gamma, \\ &\gamma = \frac{1}{2}. \end{aligned}$$

The general formula for the oscillation period is, therefore:

$$T = Cte\sqrt{\frac{\ell}{g}}$$

It is important to note that the discovery of a law does not inherently validate its accuracy. Dimensional analysis serves to verify that the law is dimensionally consistent; however, empirical experimentation is necessary to substantiate or challenge the findings of the analysis.

CHAPTER 1 SUMMARY

System of Units

Below are some of the frequently used systems of units:

(a) **FPS system** In this system, the units of length, mass and time are **foot (ft)**, **pound (lb)** and **second (s)**, respectively.
(b) **CGS System** In this system, the units of length, mass, and time are **centimetre (cm)**, **gram (g)**, and **second (s)** respectively.
(c) **MKS system** In this system, the units of length, mass and time are **metre (m)**, **kilogram (kg)** and **second (s)**, respectively.
(d) **International system (SI)** This unit system facilitates significant advancements beyond the MKS system and is known as **rationalised MKS system**. It is useful for deriving all physical quantities in the field of physics.

International system of units

Quantity	Typical symbols	Unit name
Length	l, x, r, etc.	metre (m)
Mass	m	kilogram (kg)
Time	t	second (s)
Electric current intensity	I, i	ampere (A)
Thermodynamic temperature	T	kelvin (K)
Amount of substance	n	mole (mol)
Luminous intensity	I_v	candela (cd)

Quantity	Unit name	Symbol of unit
Plane angle	radian	rad
Solid angle	steradian	sr

SI derived units

- Frequency is expressed in hertz (Hz) in the International System of Units. Since frequency represents the inverse of a period, it can also be expressed in s^{-1}.
- Speed is expressed in $m.s^{-1}$.
- SI unit of force is Newton (N).
- SI unit of energy is Joule (J).
- SI unit of density is kg.m^{-3}.

Some practical units

Parallactic Second (Parsec): 1parsec = 3.08×10^{16}m.
Light Year: 1 Light year = 9.5×10^{15} m.
Astronomical Unit (A.U.): 1 A.U. = 1.496×10^{11} m.
Microns (μm): 1 micron = 1μm = 10^{-6} m.
Angstrom (Å): 1 Å = 10^{-10} m.
Fermi: 1 Fermi = 10^{-15} m.
Chandra Shekhar Limit (C.S.L.): 1 C.S.L. = 1.4 × mass of the sun.
Metric Ton: 1 Metric Ton = 1000 kg.
Atomic Mass Unit (a.m.u.): 1 a.m.u. = 1.67×10^{-27} kg.

Dimension of physical quantity

Quantity	Dimension symbol	SI unit
Length	L	m
Mass	M	kg
Time	T	s
electric current	I	A
Thermodynamic temperature	Θ	K
Amount of substance	N	mol
Luminous intensity	J	cd

Dimensional equation

$[X] = L^{\alpha}\, M^{\beta}\, T^{\gamma}\, I^{\delta}\, \Theta^{\varepsilon}\, N^{\zeta}\, J^{\eta}$.

1.8 Exercises

Exercise 1.1

Find the dimensions of the following physical quantities:

(a) Volume (V),
(b) Density (D),
(c) Work (W),
(d) Kinetic energy (K),
(e) Potential energy (U).

Sol : page 16

Exercise 1.2

Using dimensional analysis, determine in terms of base units the composite unit for the following SI derived units:

(a) joule (J),
(b) watt (W),
(c) pascal (Pa),
(d) coulomb (C),
(e) volt (V),
(f) ohm (Ω),
(g) katal (kat).

Sol : page 16

Exercise 1.3

The gravitational constant G is a key physical constant found in Newton's law of universal gravitation.

1. Find the dimensional formula for G.
2. Derive its SI units.
3. Convert it into the CGS units system.

Sol : page 17

Exercise 1.4

In uniformly variable motion, there is an important relationship that is applicable exclusively to this type of motion, defined as follows:

$$v^2 - v_0^2 = 2a(x - x_0).$$

where v and v_0 are velocities, a is acceleration, and x and x_0 represent positions. Using the dimensional analysis, check the accuracy of this equation.

Sol : page 18

Exercise 1.5

In a spring-mass system, the angular frequency ω is found to be: $\omega = \sqrt{\frac{k}{m}}$. What is the dimension of the spring stiffness k?

Sol : page 18

Exercise 1.6

The equation for velocity as a function of time for a moving object is given by $v = a + bt + \dfrac{c}{d+t}$.
Write the dimensions of a, b, c, and d.

Sol : page 18

1.9 Solutions

Solution of exercise 1.1 (Page 15)

(a) Volume

$$\text{Volume} = \text{length} \times \text{breadth} \times \text{height}$$
$$= \text{length} \times \text{length} \times \text{length}$$
$$\therefore [\text{V}] = [L] \times [L] \times [L] = [L^3]$$

(b) Density

$$\text{Density} = \frac{\text{mass}}{\text{volume}}$$
$$\therefore [\text{D}] = [M]/[L^3] = [ML^{-3}]$$

(c) Work

$$\text{Work} = \text{force} \times \text{displacement}$$
$$= \text{force} \times \text{length}$$

$[\text{Force}] = [MLT^{-2}]$ (See Example 1.2)

$$\therefore [\text{W}] = [MLT^{-2}] \times [L] = [ML^2T^{-2}]$$

(d) Kinetic energy

Kinetic energy $= \frac{1}{2}$mass × velocity^2

$$\text{Velocity} = \frac{\text{displacement}}{\text{time}} = \frac{\text{length}}{\text{time}}$$
$$[\text{Velocity}] = \frac{[L]}{[T]} = [LT^{-1}]$$
$$[\text{Velocity}^2] = [L^2T^{-2}]$$
$$\therefore [\text{K}] = [M] \times [L^2T^{-2}] = [ML^2T^{-2}]$$

(e) Potential energy

Potential energy = mass × height × gravitational acceleration (g)

$$g = \frac{\text{velocity}}{\text{time}} = \frac{[LT^{-1}]}{[T]} = [LT^{-2}]$$

$$\therefore [\text{U}] = [M]\times[L]\times[LT^{-2}] = [ML^2T^{-2}]$$

Solution of exercise 1.2 (Page 15)

(a) joule (J):

The joule is the unit of **energy** or **work**. From Exercise 1.1:

$$[\text{W}] = [ML^2T^{-2}]$$

Thus, the joule in base units is:

$$1\,\text{J} = 1\,\text{N}\cdot\text{m} = 1\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-2}.$$

(b) watt (W):

The watt is the unit of **power**, defined as **energy per unit time**:

$$\text{Power} = \frac{\text{Energy}}{\text{Time}} = \frac{\text{J}}{\text{s}}$$

Using the joule from part (a):

$$1\,\text{W} = 1\,\text{J}\cdot\text{s}^{-1} = 1\,\text{kg}\cdot\text{m}^2\cdot\text{s}^{-3}.$$

(c) pascal (Pa):

The pascal is the unit of **pressure**, defined as **force per unit area**:

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} = \frac{\text{N}}{\text{m}^2}$$

$\because [\text{Force}] = [MLT^{-2}]$ (See Example 1.2), thus:

$$[\text{Pressure}] = \frac{[MLT^{-2}]}{[L^2]} = [ML^{-1}T^{-2}]$$

$$\therefore\ 1\,\text{Pa} = 1\,\text{N}\cdot\text{m}^{-2} = 1\,\text{kg}\cdot\text{m}^{-1}\cdot\text{s}^{-2}.$$

(d) coulomb (C):

The coulomb is the unit of **electric charge** (Q), defined as **current × time**:

$$Q = I\cdot T$$

Thus, its dimensional formula is:

$$[Q] = [IT]$$

$$\therefore\ 1\,\text{C} = 1\,\text{A}\cdot\text{s}.$$

(e) volt (V):
The volt is the unit of **electric potential**, defined as the work done per unit charge (or energy per unit charge):

$$\text{Voltage} = \frac{\text{Work}}{\text{Charge}}$$

Using the work and charge dimensions from parts (a) and (d), respectively, we can deduce the dimensional formula for voltage:

$$[V] = \frac{[\mathrm{M\,L^2T^{-2}}]}{[\mathrm{I\,T}]} = [\mathrm{M\,L^2T^{-3}I^{-1}}].$$

$$\therefore\ 1\,\mathrm{V} = \frac{1\,\mathrm{J}}{1\,\mathrm{C}} = 1\,\mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-3}\cdot\mathrm{A}^{-1}.$$

(f) ohm (Ω):
The ohm is the unit of **electrical resistance (R)**. From Ohm's Law, R is defined as **voltage per current**:

$$\text{Resistance} = \frac{\text{Voltage}}{\text{Current}} = \frac{\mathrm{V}}{\mathrm{A}}$$

Using the voltage dimensional formula from part (d):

$$[R] = \frac{[\mathrm{M\,L^2T^{-3}I^{-1}}]}{[\mathrm{I}]} = [\mathrm{M\,L^2T^{-3}I^{-2}}]$$

$$\therefore\ 1\,\Omega = \frac{1\,\mathrm{V}}{1\,\mathrm{A}} = \mathrm{kg}\cdot\mathrm{m}^2\cdot\mathrm{s}^{-3}\cdot\mathrm{A}^{-2}.$$

(g) katal (kat):
The katal is the unit of **catalytic activity**, which is defined as :

$$\frac{[\text{Amount of substance}]}{[\text{Time}]}$$

$$[\text{Catalytic Activity}] = \frac{[N]}{[T]} = [NT^{-1}]$$

$$\therefore\ 1\,\mathrm{kat} = \frac{1\,\mathrm{mol}}{1\,\mathrm{s}} = \mathrm{mol}\cdot\mathrm{s}^{-1}.$$

Solution of exercise 1.3 (Page 15)

1. The Newton's law of universal gravitation states that the force of attraction F between two masses m_1 and m_2 separated by a distance r can be expressed as:

$$F = G\frac{m_1 m_2}{r^2}$$

Rearranging for G:

$$G = \frac{F\cdot r^2}{m_1 m_2}$$

$$[G] = \frac{[MLT^{-2}]\cdot[L^2]}{[M]\cdot[M]} = [M^{-1}L^3T^{-2}].$$

2. Using the dimensional formula above, we find:

$$G = \mathrm{kg}^{-1}\cdot\mathrm{m}^3\cdot\mathrm{s}^{-2}.$$

The unit of G can also be written as:

$$\mathrm{N}\cdot\mathrm{m}^2\cdot\mathrm{kg}^{-2}.$$

3. The value of G in the SI system is:

$$G = 6.67\times10^{-11}\,\mathrm{kg}^{-1}\cdot\mathrm{m}^3\cdot\mathrm{s}^{-2}$$

Let $[M_1^{-1}L_1^3T_1^{-2}]$ be the dimensional formula of G in the SI unit system, and $[M_2^{-1}L_2^3T_2^{-2}]$ be the dimensional formula of G in the CGS unit system. We will refer to the gravitational constant in the SI unit system as G_1 and in the CGS unit system as G_2.
Then, we can write:

$$G_1[M_1^{-1}L_1^3T_1^{-2}] = G_2[M_2^{-1}L_2^3T_2^{-2}]$$

$$G_2 = G_1\left[\frac{M_1}{M_2}\right]^{-1}\left[\frac{L_1}{L_2}\right]^{3}\left[\frac{T_1}{T_2}\right]^{-2}$$

Here,

$$G_1 = 6.67\times10^{-11}.$$

The conversion between SI and CGS

units give:

- **Length**: 1 cm = 10^2 cm
- **Mass**: 1 kg = 10^3 g

Time is the same in both systems (s). Thus,

- $M_1 = 1\,\mathrm{kg}, \quad M_2 = 1\,\mathrm{g} = 10^{-3}\,\mathrm{kg},$
- $L_1 = 1\,\mathrm{m}, L_2 = 1\,\mathrm{cm} = 10^{-2}\,\mathrm{m},$
- $T_1 = T_2 = 1\,\mathrm{s}.$

Substituting these values in the above equation, we get:

$$G_2 = G_1 \left[\frac{1\,\mathrm{kg}}{10^{-3}\,\mathrm{kg}}\right]^{-1} \left[\frac{1\,\mathrm{m}}{10^{-2}\,\mathrm{m}}\right]^{3} \left[\frac{1\,\mathrm{s}}{1\,\mathrm{s}}\right]^{-2}$$
$$= 6.67 \times 10^{-11} \times 10^3$$
$$= 6.67 \times 10^{-8}$$

Thus, the value of G in the CGS system of units is

$$G = 6.67430 \times 10^{-8}\ \mathrm{cm}^3 \cdot \mathrm{g}^{-1} \cdot \mathrm{s}^{-2}.$$

Generally, G is expressed in $\mathrm{N} \cdot \mathrm{m}^2 \cdot \mathrm{kg}^{-2}$. In this case, we find:

$$G = 6.67 \times 10^{-8}\ \mathrm{dyne} \cdot \mathrm{cm}^2 \cdot \mathrm{g}^{-2}.$$

Solution of exercise 1.4 (Page 15)

To check the accuracy of an equation, we need to ensure that both sides of the equation have the same dimensions.

Determining the dimensions of the first side of the equation:

v^2 and v_0^2 has the dimension $[L/T]^2 = [L^2T^{-2}]$.

$$\therefore [v^2 - v_0^2] = [L^2T^{-2}]$$

Determining the dimensions of the second side of the equation:

$$[2a(x - x_0)] = [a][(x - x_0)]$$
$$= [L \times T^{-2}][L] = [L^2T^{-2}]$$

The dimensions of both sides of the equation are identical. Therefore, the equation is dimensionally correct.

Solution of exercise 1.5 (Page 15)

We know that ω has units that are the inverse of time.

$$[\omega] = [T^{-1}]$$

$$\therefore [\omega^2] = [T^{-2}]$$

On the other hand,

$$\omega^2 = \frac{k}{m} \Rightarrow k = \omega^2 \times m$$

Therefore,

$$[k] = [\omega^2][m] = [T^{-2}][M] = [MT^{-2}].$$

Solution of exercise 1.6 (Page 15)

According to the principle of dimensional homogeneity,

$$[a] = [v] \quad \therefore\ [a] = [\mathrm{LT}^{-1}]$$

$$[bt] = [v] \quad \therefore\ [b] = \frac{[v]}{[t]} = \frac{[\mathrm{LT}^{-1}]}{[\mathrm{T}]}$$

$$\therefore\ [b] = [\mathrm{LT}^{-2}]$$

Similarly,

$$[d] = [t] = [\mathrm{T}]$$

Further,

$$\frac{[c]}{[d+t]} = [v] \quad \text{or} \quad [c] = [v][d+t]$$

or

$$[c] = [\mathrm{LT}^{-1}][\mathrm{T}] \quad \text{or} \quad [c] = [\mathrm{L}]$$

VECTOR CALCULUS 2

2.1 Introduction

Several physical quantities can be fully described by providing a single numerical value along with the appropriate unit of measurement. For example, we can specify a mass of 32 kg, a duration of 40 minutes, and a surface area of 100 m^2. These values—32 kg, 40 min, and 100 m^2—sufficiently describe their respective quantities. Physical quantities that can be defined in this manner are referred to as **scalar quantities**. The term "scalar" is synonymous with "number." However, certain physical quantities necessitate both a magnitude and a direction for their complete determination; these quantities are known as **vectors**. The most obvious example is displacement, which is defined by the effective distance a body has traveled along with the specific direction of that movement. Other examples of vector quantities include force, acceleration, velocity, and momentum. These quantities are represented by vectors, which are Euclidean entities that can be geometrically represented as arrows. Vectors can undergo operations such as addition, subtraction, and multiplication. This chapter explores the fundamental aspects of vector algebra, particularly in relation to its applications in mechanics.

2.2 Definition of a vector

A vector is a specific quantity that has a magnitude and direction. It is represented by an arrow and is labeled with lowercase or uppercase letters, with a small arrowhead on top. For example: $\overrightarrow{w}$, $\overrightarrow{V}$, $\overrightarrow{AB}$, etc.
The magnitude of a vector represents its length and the direction of a vector is from its tail (initial point) to its head (terminal point) (see Figure 2.1).

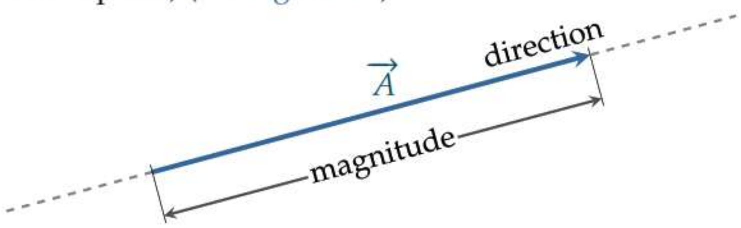


Figure 2.1: The magnitude and direction of a vector

2.3 Types of vectors

Zero vector

The zero vector is a vector that has a magnitude equal to 0. It has no direction and is denoted as $\vec{0}$.

Equal vectors

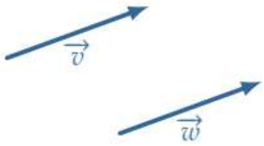


Figure 2.2: Equal vectors.

Two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ are considered equal if they have the same magnitude (length) and the same direction. Given that a vector is represented by an arrow, any translation of this arrow represents the same vector.

Negative of a vector (opposite vector)

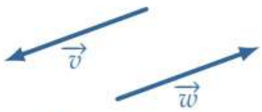


Figure 2.3: Opposite vector.

The vector $\overrightarrow{v}$ is said to be the negative of vector $\overrightarrow{w}$ if they have the same magnitude but are opposite in direction. This is denoted as: $\overrightarrow{v} = -\overrightarrow{w}$.

Unit vector

A vector $\overrightarrow{u}$ for which the magnitude is equal to 1 unit length is called a **unit vector** and denoted by u cap or hat: $\hat{u}$.

Theorem

For any non-zero vector $\overrightarrow{v}$, the vector $\hat{v} = \dfrac{\overrightarrow{v}}{\|\overrightarrow{v}\|}$ is a unit vector that has the same direction as $\overrightarrow{v}$.

2.4 Components of a vector

Theorem

Let $\overrightarrow{v}$ be a vector defined in a three-dimensional space. The space is referred to the coordinate system $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$. There exists a unique ordered triplet of real numbers $(x\,; y\,; z)$ so that: $\overrightarrow{v} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$.
The real numbers x, y and z are the **coordinates** or **components** of the vector $\overrightarrow{v}$ in terms of the coordinate system $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$. x represents the abscissa, y denotes the ordinate, and z represents the altitude of the vector $\overrightarrow{v}$.

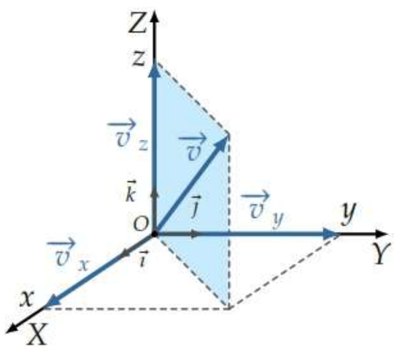


Figure 2.4: Components of a vector

The vector $\overrightarrow{v}$ shown in Figure 2.4 can be written in terms of its components: $\overrightarrow{v} = \overrightarrow{v}_x + \overrightarrow{v}_y + \overrightarrow{v}_z = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$.

2.5 Magnitude of a vector

The magnitude (length) of a vector $\overrightarrow{v} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$ is a non-negative scalar quantity represented by either double vertical lines $\|\overrightarrow{v}\|$, an absolute value $|v|$, or simply v. It is calculated based on its components by the Pythagorean theorem as: $\|\overrightarrow{v}\| = \sqrt{x^2 + y^2 + z^2}$.

Example 2.1

Consider the given vectors $\overrightarrow{v}$ and $\overrightarrow{w}$, which are defined as follows: $\overrightarrow{v} = 3\,\vec{\imath} + 4\,\vec{k}$ and $\overrightarrow{w} = -3\,\vec{\imath} - 4\,\vec{\jmath}$.
Calculate $\|\overrightarrow{v}\|$ and $\|\overrightarrow{w}\|$.

Solution

$\|\overrightarrow{v}\| = \sqrt{3^2 + 0^2 + 4^2} = 5$, $\|\overrightarrow{w}\| = \sqrt{(-3)^2 + (-4)^2 + 0^2} = 5$.

Properties of magnitude

Since the magnitude is defined as the length of the vector, it must possess the following properties:

Properties of magnitude

Let $\overrightarrow{v}$ and $\overrightarrow{w}$ be two vectors and λ be a real number, then:

- $\|\overrightarrow{v}\| \geqslant 0$;
- $\|\overrightarrow{v}\| = 0$, if and only if $\overrightarrow{v} = \overrightarrow{0}$;
- $\|-\overrightarrow{v}\| = \|\overrightarrow{v}\|$;
- $\|\lambda\,\overrightarrow{v}\| = |\lambda\,|\|\overrightarrow{v}\|$;
- $\|\overrightarrow{v} + \overrightarrow{w}\| \leqslant \|\overrightarrow{v}\| + \|\overrightarrow{w}\|$.

2.6 Operations on vectors

Vector addition and subtraction

Algebraic (analytical) method

Given two vectors $\overrightarrow{A} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$ and $\overrightarrow{B} = x'\,\vec{\imath} + y'\,\vec{\jmath} + z'\,\vec{k}$. The sum of these two vectors is the vector $\overrightarrow{C}$ noted:

$$\overrightarrow{C} = \overrightarrow{A} + \overrightarrow{B} = (x + x')\,\vec{\imath} + (y + y')\,\vec{\jmath} + (z + z')\,\vec{k}.$$

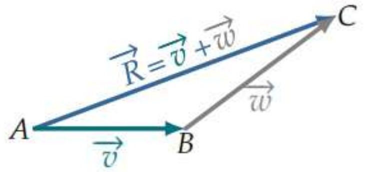
Example 2.2

Using the vectors provided in Example 2.5, find: $\|\overrightarrow{v}\|+\|\overrightarrow{w}\|$ and $\|\overrightarrow{v}+\overrightarrow{w}\|$.

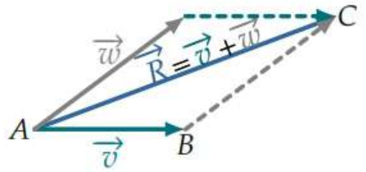
Solution

$\|\overrightarrow{v}\| + \|\overrightarrow{w}\| = 5 + 5 = 10.$
$\overrightarrow{v} + \overrightarrow{w} = -4\,\vec{j} + 4\,\vec{k}$, so: $\|\overrightarrow{v} + \overrightarrow{w}\| = \sqrt{32}$.

Graphical (geometric) methods



(a) Triangle law



(b) Parallelogram law

Figure 2.5: Graphical method of vectors addition

We can graphically obtain the sum $\overrightarrow{R}$ (the **resultant vector**) using what is called the **Chasles' relation**. It consists of placing the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ in such a way that the tail of $\overrightarrow{w}$ coincides with the head of $\overrightarrow{v}$. The resultant $\overrightarrow{R}$ is the vector whose tail is the same as the first vector (point A) and its head is the same as the last vector (point C). This method is called the **Triangle law**.
We can also use another geometric method known as the **Parallelogram law**. This method consists of placing the tails of the two vectors to be added ($\overrightarrow{v}$ and $\overrightarrow{w}$ in our example) at the same point. The resultant vector $\overrightarrow{R}$ is obtained from the diagonal of the parallelogram formed by $\overrightarrow{v}$ and $\overrightarrow{w}$ (see Figure 2.5).

Properties of vector addition

Suppose the vectors $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ are defined in a three-dimensional space, we have:

Properties of vector addition

- **Associativity:** $\left(\overrightarrow{u}+\overrightarrow{v}\right)+\overrightarrow{w}=\overrightarrow{u}+\left(\overrightarrow{v}+\overrightarrow{w}\right)$.
- **Commutativity:** $\overrightarrow{v}+\overrightarrow{w}=\overrightarrow{w}+\overrightarrow{v}$.
- **Additive identity:** This is the **zero vector**. We have: $\overrightarrow{v}+\overrightarrow{0}=\overrightarrow{0}+\overrightarrow{v}=\overrightarrow{v}$.
- **Additive inverse:** $-\overrightarrow{v}$ is the **opposite vector** of vector $\overrightarrow{v}$. So: $\overrightarrow{v}+\left(-\overrightarrow{v}\right)=\overrightarrow{0}$.

The **subtraction** of the two vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ can be defined as the addition of the two vectors $\overrightarrow{v}$ and $-\overrightarrow{w}$, where $-\overrightarrow{w}$ represents the negative vector of $\overrightarrow{w}$. Then, we write: $\overrightarrow{v}-\overrightarrow{w}=\overrightarrow{v}+\left(-\overrightarrow{w}\right)$.

Multiplying vectors by a scalar (scalar multiplication)

Given a vector $\overrightarrow{w}$ with (x, y, z) components, and λ a scalar (real number). The multiplication of λ by $\overrightarrow{w}$ is a vector $\overrightarrow{B}$ defined as follows:

$$\boxed{\overrightarrow{B} = \lambda\,\overrightarrow{w} = (\lambda x)\,\vec{\imath} + (\lambda y)\,\vec{\jmath} + (\lambda z)\,\vec{k}} \tag{2.1}$$

- If $\|\overrightarrow{w}\| = a$, then $\|\overrightarrow{B}\| = |\lambda|\, a$,
- If λ is positive, $\overrightarrow{B}$ keeps the same direction as $\overrightarrow{w}$,
- If λ is negative, the direction of $\overrightarrow{B}$ is reversed (opposite direction of $\overrightarrow{w}$),
- $\overrightarrow{B}$ is a zero vector, if $\lambda = 0$ or if $\overrightarrow{w}$ is a zero vector.

Properties of scalar multiplication

Let $\overrightarrow{v}$ and $\overrightarrow{w}$ be two vectors, and let λ and μ be two real numbers. Then:

Properties of scalar multiplication

- $\lambda\left(\overrightarrow{v} + \overrightarrow{w}\right) = \lambda\,\overrightarrow{v} + \lambda\,\overrightarrow{w}$;
- $(\lambda + \mu)\,\overrightarrow{v} = \lambda\,\overrightarrow{v} + \mu\,\overrightarrow{v}$;
- $\lambda\left(\mu\,\overrightarrow{v}\right) = (\lambda\,\mu)\,\overrightarrow{v}$.

Collinear Vectors

Theorem

Let $\overrightarrow{v}$ and $\overrightarrow{w}$ be two non-zero vectors, we say that $\overrightarrow{v}$ and $\overrightarrow{w}$ are **collinear** if there exists a scalar $\lambda \in \mathbb{R}^*$ such that: $\overrightarrow{v} = \lambda\,\overrightarrow{w}$. In other words, two vectors are **collinear** if one is a scalar multiple of the other.

2.7 Scalar product

The scalar product (also known as the **dot product** or the **inner product**) of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is an algebraic operation denoted as $\overrightarrow{A} \cdot \overrightarrow{B}$[1] which takes two vectors as arguments and returns a scalar as a result (hence its name).

1: Read as $\overrightarrow{A}$ dot $\overrightarrow{B}$

Geometric definition

The dot product of two non-zero vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is given by the geometric definition as follows:

$$\boxed{\overrightarrow{A} \cdot \overrightarrow{B} = \|\overrightarrow{A}\| \, \|\overrightarrow{B}\| \ \cos\theta} \tag{2.2}$$

where θ ($0 \leqslant \theta \leqslant \pi$) represents the angle between $\overrightarrow{A}$ and $\overrightarrow{B}$. If one of the two vectors is null (zero), then the dot product is null.

Example 2.3

Suppose 2 and 3 are the magnitudes of the two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ respectively, which form an angle of 180° between them. Find $\overrightarrow{u} \cdot \overrightarrow{v}$

Solution

$\overrightarrow{u} \cdot \overrightarrow{v} = 2 \times 3 \ \cos 180 = -6.$

Algebraic definition (cartesian expression)

In the context of an orthonormal coordinate system $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$, the dot product of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ can be expressed in terms of their respective components (x, y, z) and (x', y', z'). This is defined by the following relation:

$$\boxed{\overrightarrow{A} \cdot \overrightarrow{B} = xx' + yy' + zz'} \tag{2.3}$$

Example 2.4

Consider the vectors, $\overrightarrow{u}$ and $\overrightarrow{v}$ such that: $\overrightarrow{u} = 3 \ \vec{\imath} + 2 \ \vec{\jmath}$ and $\overrightarrow{v} = \vec{\imath} - \vec{\jmath} + \vec{k}$. Compute $\overrightarrow{u} \cdot \overrightarrow{v}$.

Solution

$\overrightarrow{u}.\overrightarrow{v} = (3 \times 1) + (2 \times -1) + (0 \times 1) = 1.$

Properties of the scalar product

For any vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ and any scalar $\lambda \in \mathbb{R}$, we have:

Properties of the scalar product

- **Commutativity:** $\overrightarrow{v} \cdot \overrightarrow{w} = \overrightarrow{w} \cdot \overrightarrow{v}$;
- **Distributivity:** $\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} . \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}$;
- **Orthogonality:** If $\overrightarrow{v} \perp \overrightarrow{w}$, then $\overrightarrow{v} \cdot \overrightarrow{w} = 0$;
- $(\lambda \overrightarrow{v}).\overrightarrow{w} = \lambda(\overrightarrow{v}.\overrightarrow{w})$;
- $\overrightarrow{v} \cdot \overrightarrow{v} = \|\overrightarrow{v}\|^2$;
- $\overrightarrow{v} \cdot \overrightarrow{0} = 0.$

The angle between two vectors

The dot product allows, among other things, to measure the angle between two vectors.

Theorem

Let $\overrightarrow{A}$ and $\overrightarrow{B}$ be two non-zero vectors, the angle θ, $0 \leqslant \theta \leqslant \pi$, between the vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is given by the formula:

$$\theta = \arccos\left(\frac{\overrightarrow{A} \cdot \overrightarrow{B}}{\|\overrightarrow{A}\| \, \|\overrightarrow{B}\|}\right).$$

$\theta = \arccos\left(\dfrac{|\overrightarrow{A} \cdot \overrightarrow{B}|}{\|\overrightarrow{A}\| \, \|\overrightarrow{B}\|}\right)$ give the acute angle.

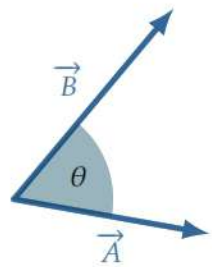


Figure 2.6: Angle between two vectors.

Example 2.5

Find the angle α between the two vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ from Example 1.

Solution

$$\cos\alpha = \frac{\overrightarrow{u} \cdot \overrightarrow{v}}{\|\overrightarrow{u}\|\|\overrightarrow{v}\|} = \frac{1}{\sqrt{13}\sqrt{3}} = 0.18 \Rightarrow \alpha = 79.18°$$

2.8 Vector product

The vector product (also known as the **cross product** or the **outer product**) of two vectors $\overrightarrow{A}$ and $\overrightarrow{B}$ is a vector operation, denoted as $\overrightarrow{A} \times \overrightarrow{B}$ (read as $\overrightarrow{A}$ cross $\overrightarrow{B}$), performed in space[2] that takes two vectors as arguments and returns a vector as a result (hence its name).

2: The cross product does not exist in two-dimensions

Geometric definition

Let $\overrightarrow{v} = \overrightarrow{OA}$ and $\overrightarrow{w} = \overrightarrow{OB}$ be two vectors in an orthonormal coordinate system $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$, forming an angle α between them. By definition, the cross product $\overrightarrow{v} \times \overrightarrow{w}$ is the vector $\overrightarrow{a}$ whose magnitude is equal to:

$$\boxed{\|\overrightarrow{a}\| = \|\overrightarrow{v}\|\|\overrightarrow{w}\||\sin\alpha|} \tag{2.4}$$

the direction of $\vec{a}$ is perpendicular to the plane formed by $\overrightarrow{v}$ and $\overrightarrow{w}$ (see Figure 2.7). This is given by the right-hand rule (the right-handed screw rotated from $\overrightarrow{v}$ to $\overrightarrow{w}$).
The area of parallelogram $OACB$ is:

$$S_{OACB} = OB\ .\ hA = \|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{a}\| = \|\overrightarrow{v}\|\|\overrightarrow{w}\||\sin\alpha|.$$

The area of triangl OAB (pounded by vectors $\overrightarrow{v}$ and $\overrightarrow{w}$) is:

$$S_{OAB} = \frac{1}{2}(OB.hA) = \frac{1}{2}\|\overrightarrow{v} \times \overrightarrow{w}\| = \frac{1}{2}\|\overrightarrow{a}\|.$$

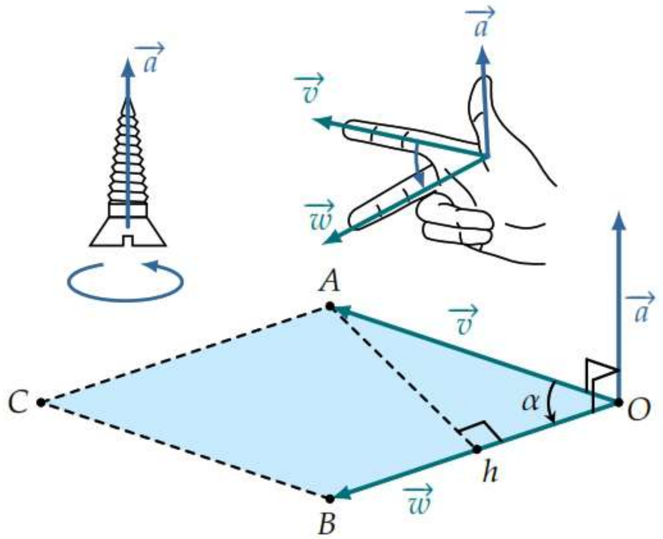


Figure 2.7: Geometric interpretation of the cross product

Algebraic definition (cartesian expression)

Let us consider two vectors $\overrightarrow{v} = (x, y, z)$ and $\overrightarrow{w} = (x', y', z')$, whose components are expressed in the orthonormal basis $(\vec{\imath}, \vec{\jmath}, \vec{k})$. The cross product $\overrightarrow{v} \times \overrightarrow{w}$ can also be defined as:

$$\overrightarrow{v} \times \overrightarrow{w} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix}$$

$$\boxed{\overrightarrow{v} \times \overrightarrow{w} = \left(yz' - zy'\right)\vec{\imath} + \left(zx' - xz'\right)\vec{\jmath} + \left(xy' - yx'\right)\vec{k}} \tag{2.5}$$

Example 2.6

Find $\overrightarrow{v_1} \times \overrightarrow{v_2}$, knowing that: $\overrightarrow{v_1} = \left(\sqrt{3}, 3\right)$ and $\overrightarrow{v_2} = (2, 0)$.

Solution

$$\overrightarrow{v_1} \times \overrightarrow{v_2} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \sqrt{3} & 3 & 0 \\ 2 & 0 & 0 \end{vmatrix} = 0\,\vec{\imath} - 0\,\vec{\jmath} - 6\,\vec{k} = -6\,\vec{k}.$$

Example 2.7

Calculate the components of the vector $\overrightarrow{C}$, knowing that: $\overrightarrow{C} = \overrightarrow{A} \times \overrightarrow{B}$, $\overrightarrow{A} = (1, -1, 2)$, and $\overrightarrow{B} = (2, 0, -2)$.

Solution

$$\overrightarrow{C} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -1 & 2 \\ 2 & 0 & -2 \end{vmatrix} = 2\vec{\imath} - 6\vec{\jmath} + 2\,\overrightarrow{k}$$

Thus, the components of the vector $\overrightarrow{C}$ are $(2, -6, 2)$.

Properties of the cross product

For vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ and scalars λ and μ, we have:

Properties of the cross product

- The cross product is **not commutative**: $\overrightarrow{v} \times \overrightarrow{w} = -\overrightarrow{w} \times \overrightarrow{v}$;
- The cross product is **distributive** over vector sum: $\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{w}$;
- $\left(\lambda\,\overrightarrow{v}\right) \times \left(\mu\,\overrightarrow{w}\right) = \lambda\mu\left(\overrightarrow{v} \times \overrightarrow{w}\right)$;
- $(\lambda\,\overrightarrow{v}) \times \overrightarrow{w} = \lambda(\overrightarrow{v} \times \overrightarrow{w}) = \overrightarrow{v} \times (\lambda\,\overrightarrow{w})$;
- $\overrightarrow{v}$ and $\overrightarrow{w}$ are **colinear** if and only if $\overrightarrow{v} \times \overrightarrow{w} = \overrightarrow{0}$.

2.9 Scalar triple product (mixed product)

In a coordinate system $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$, the scalar triple product of three vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$, taken in that order, is the **real number** (scalar quantity) denoted as $\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right]$ and defined by:

$$\boxed{\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right] = \overrightarrow{a} \,.\left(\overrightarrow{b} \times \overrightarrow{c}\right)} \tag{2.6}$$

In three-dimensional space, the components of vectors $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$ are denoted as (a_1, a_2, a_3), (b_1, b_2, b_3), and (c_1, c_2, c_3), respectively. The scalar triple product of these vectors can be expressed as a determinant involving their components,

$$\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right] = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} . \begin{pmatrix} b_1c_3 - b_3c_2 \\ b_3c1 - b_1c3 \\ b_1c_2 - b_2c_1 \end{pmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_1 & c_1 \\ a_3 & b_1 & c_1 \end{vmatrix}$$
$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

Example 2.8

Calculate the scalar triple product: $\left[\overrightarrow{v}, \overrightarrow{w}, \overrightarrow{u}\right]$, knowing that: $\overrightarrow{v} = (1, 1, 0)$, $\overrightarrow{w} = (2, 0, -1)$, and $\overrightarrow{u} = (-1, 2, -1)$.

Solution

$$\left[\overrightarrow{v}, \overrightarrow{w}, \overrightarrow{u}\right] = \overrightarrow{v} \cdot (\overrightarrow{w} \times \overrightarrow{u})$$

First, we calculate the cross product $\overrightarrow{w} \times \overrightarrow{u}$:

$$\overrightarrow{w} \times \overrightarrow{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & -1 \\ -1 & 2 & -1 \end{vmatrix} = 2\vec{i} + 3\vec{j} + 4\vec{k}$$

Next, we calculate the dot product $\overrightarrow{v} \cdot (\overrightarrow{w} \times \overrightarrow{u})$:

$$\overrightarrow{v} \cdot (\overrightarrow{w} \times \overrightarrow{u}) = 1 \cdot 2 + 1 \cdot 3 + 0 \cdot 4 = 2 + 3 + 0 = 5$$

$$\therefore \ \left[\overrightarrow{v}, \overrightarrow{w}, \overrightarrow{u}\right] = 5$$

Properties of the scalar triple product

Properties of the scalar triple product

- The scalar triple product changes sign when two vectors are swapped:
$\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right] = -\left[\overrightarrow{b}, \overrightarrow{a}, \overrightarrow{c}\right] = -\left[\overrightarrow{c}, \overrightarrow{b}, \overrightarrow{a}\right] = -\left[\overrightarrow{a}, \overrightarrow{c}, \overrightarrow{b}\right]$;
- The scalar triple product is invariant under a circular permutations of its vectors $\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right] = \left[\overrightarrow{b}, \overrightarrow{c}, \overrightarrow{a}\right] = \left[\overrightarrow{c}, \overrightarrow{a}, \overrightarrow{b}\right]$;
- $\left(\overrightarrow{a} \wedge \overrightarrow{b}\right) . \overrightarrow{c} = \overrightarrow{a} . \left(\overrightarrow{b} \wedge \overrightarrow{c}\right)$;

- $\left[\vec{a}, \vec{b}, \lambda\vec{c}\right] = \lambda\left[\vec{a}, \vec{b}, \vec{c}\right], \forall\, \lambda \in \mathbb{R};$
- $\left[\vec{a}, \vec{b}, \vec{c} + \vec{d}\right] = \left[\vec{a}, \vec{b}, \vec{c}\right] + \left[\vec{a}, \vec{b}, \vec{d}\right];$
- $\left[\vec{a}, \vec{b}, \vec{c}\right]$ represents the volume of the parallelepiped with coterminous edges $\vec{a}$, $\vec{b}$, and $\vec{c}$.

Example 2.9

Let there be a parallelepiped generated by three vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$. Demonstrate that its volume is equal to the scalar triple product of the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$.

Solution

The volume of parallelepiped with sides $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, is equal to the area of the base parallelogram $\times$ height, so: $V = S \times h$ (see Figure 2.8).

$S = \|\overrightarrow{OA}\times\overrightarrow{OB}\|$ and $h = \|\overrightarrow{OC}\| \cos\theta$. So we can write: $V = \|\overrightarrow{OA} \times \overrightarrow{OB}\|\|\overrightarrow{OC}\| \cos\theta$, this formula is the dot product of $\left(\overrightarrow{OA} \times \overrightarrow{OB}\right)$ and $\overrightarrow{OC}$ which form an angle θ between them, hence:

$$V = |(\overrightarrow{OA} \times \overrightarrow{OB}) \,.\, \overrightarrow{OC}|$$

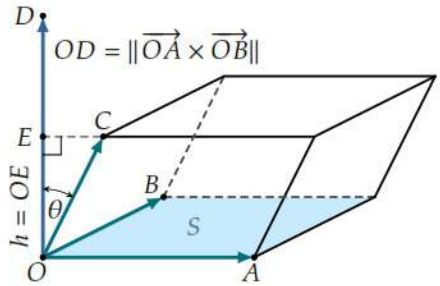


Figure 2.8: Graphical interpretation of the scalar triple product (Volume of a parallelepiped)

It is deduced that the volume of the parallelepiped constructed on $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ is equal to the absolute value of the scalar triple product of the three vectors that form it.

2.10 Derivatives of vectors

If x, y and z are the components of a vector $\vec{v}$ expressed as a function to time. The derivative of the vector $\vec{v}$ with respect to time is a vector, denoted $\frac{d\vec{v}}{dt}$, $\vec{v}'$, or $\dot{\vec{v}}$, which has the following components: $\frac{dx}{dt}, \frac{dy}{dt}$ and $\frac{dz}{dt}$. Then:

$$\frac{d\vec{v}}{dt} = \frac{dx}{dt}\,\vec{\imath} + \frac{dy}{dt}\,\vec{\jmath} + \frac{dz}{dt}\,\vec{k}.$$

The second derivative of $\vec{v}$ with respect to time can be written as:

$$\frac{d^2\vec{v}}{dt^2} = \vec{v}'' = \ddot{\vec{v}} = \frac{d^2x}{dt^2}\,\vec{\imath} + \frac{d^2y}{dt^2}\,\vec{\jmath} + \frac{d^2z}{dt^2}\,\vec{k}.$$

Time derivatives of rotating unit vectors

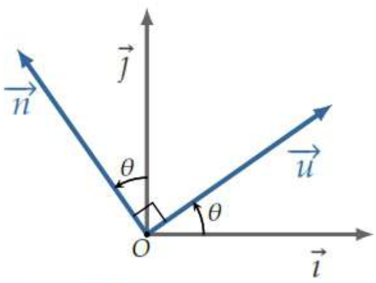


Figure 2.9: Time-derivatives of unit vectors

Given $\overrightarrow{u}$ and $\overrightarrow{n}$ two orthogonal unit vectors represented in the cartesian base system $(O, \vec{i}, \vec{j}, \vec{k})$. $\overrightarrow{u}$ and $\overrightarrow{n}$ can rotate around the Oz axis with the same angle θ (see Figure 2.9). We have:
$\overrightarrow{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$ and $\overrightarrow{n} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j}$.

$$\frac{d\overrightarrow{u}}{dt} = \frac{d\overrightarrow{u}}{d\theta}\frac{d\theta}{dt} = \underbrace{\left(\frac{d\theta}{dt}\right)}_{\dot{\theta}} \underbrace{\left(-\sin\theta\,\vec{i} + \cos\theta\,\vec{j}\right)}_{\overrightarrow{n}}$$

$$\boxed{\frac{d\overrightarrow{u}}{dt} = \dot{\theta}\,\overrightarrow{n}} \tag{2.7}$$

Similarly, we find:

$$\boxed{\frac{d\overrightarrow{n}}{dt} = -\dot{\theta}\,\overrightarrow{u}} \tag{2.8}$$

We can therefore generalize this result for any rotating unit vector, where $\dot{\theta} = \omega$ represents the angular velocity.

2.11 Function of several variables and vector operators

In simple terms, a function of several variables is a mathematical function that depends on two or more independent variables. For instance, the function $f(x, y, z) = x^2 + y^2 + 2z^2$ is a function of three variables. It assigns a real number to each triplet $(x, y, z) \in \mathbb{R}^3$.

The gradient operator

Let $f(M)$ be a scalar function of several real variables. The **gradient** of the function $f(M)$ is defined as the vector function $\overrightarrow{\text{grad}} f = \overrightarrow{\nabla} f$ ($\overrightarrow{\nabla}$ is the **del** operator or **nabla** operator). For example, the vector differential operator del in the cartesian coordinate systems is defined as:

$$\overrightarrow{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}.$$

Thus, the gradient operator of the function f can be expressed in the different coordinate systems as follows:

The Gradient Expressions

- In the cartesian coordinate system:
$\overrightarrow{\nabla} f = \dfrac{\partial f}{\partial x}\vec{\imath} + \dfrac{\partial f}{\partial y}\vec{\jmath} + \dfrac{\partial f}{\partial z}\vec{k}.$
- In the cylindrical coordinate system:
$\overrightarrow{\nabla} f = \dfrac{\partial f}{\partial \rho}\overrightarrow{e_\rho} + \dfrac{1}{\rho}\dfrac{\partial f}{\partial \theta}\overrightarrow{e_\theta} + \dfrac{\partial f}{\partial z}\overrightarrow{e_z}.$
- In the spherical coordinate system:
$\overrightarrow{\nabla} f = \dfrac{\partial f}{\partial r}\overrightarrow{e_r} + \dfrac{1}{r}\dfrac{\partial f}{\partial \theta}\overrightarrow{e_\theta} + \dfrac{1}{r\, sin\theta}\dfrac{\partial f}{\partial \varphi}\overrightarrow{e_\varphi}.$

Example 2.10

Let's consider the two functions $f(x, y, z) = x^3 + 2y + 3z^2$ and $g(\rho, \theta, z) = \rho - cos\theta$.
Calculate the gradient of each function.

Solution

$\overrightarrow{\nabla} f = 3x^2\vec{\imath} + 2\vec{\jmath} + 6z\,\vec{k}.$

$\overrightarrow{\nabla} g = \overrightarrow{e_\rho} + \dfrac{1}{\rho}\, sin\theta\,\overrightarrow{e_\theta}.$

The divergence operator

The divergence operator is a differential operator that acts on a **vector field**, yielding a **scalar field** as a result. It is read as divergence and is denoted as follows:

$$\mathrm{div}\overrightarrow{A}(\mathrm{M}, t) \quad \text{or} \quad \overrightarrow{\nabla} \cdot \overrightarrow{A}(\mathrm{M}, t)$$

The divergence of a vector field, when expressed in different coordinate systems, is given by:

The Divergence Expressions

- In the cartesian coordinate system:
$\mathrm{div}\overrightarrow{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}.$
- In the cylindrical coordinate system:
$\mathrm{div}\overrightarrow{A} = \dfrac{\partial(rA_r)}{r\partial r} + \dfrac{\partial(A_\theta)}{r\partial\theta} + \dfrac{\partial A_z}{\partial z}.$
- In the spherical coordinate system:
$\mathrm{div}\overrightarrow{A} = \dfrac{1}{r^2}\dfrac{\partial(r^2 A_r)}{\partial r} + \dfrac{1}{r\sin\theta}\dfrac{\partial(\sin\theta\, A_\theta)}{\partial\theta} + \dfrac{1}{r\sin\theta}\dfrac{\partial A_\varphi}{\partial\varphi}.$

The curl operator

The curl operator (often called **rotational operator**) is a differential operator that transforms a **vector field** into another **vector field**. It is read as rotational or curl and is denoted as follows:

$$\overrightarrow{\text{curl}}\,\overrightarrow{A}(\mathrm{M},t) \quad \text{or} \quad \overrightarrow{\text{rot}}\,\overrightarrow{A}(\mathrm{M},t) \quad \text{or} \quad \overrightarrow{\nabla}\times\overrightarrow{A}(\mathrm{M},t)$$

The curl of a vector field can be expressed in different coordinate systems as follows:

The Curl Expressions

- In the cartesian coordinate system:

$$\overrightarrow{\text{curl}}\,\overrightarrow{A} = \begin{pmatrix} \dfrac{\partial A_z}{\partial y} - \dfrac{\partial A_y}{\partial z} \\ \dfrac{\partial A_x}{\partial z} - \dfrac{\partial A_z}{\partial x} \\ \dfrac{\partial A_y}{\partial x} - \dfrac{\partial A_x}{\partial y} \end{pmatrix}.$$

- In the cylindrical coordinate system:

$$\overrightarrow{\text{curl}}\,\overrightarrow{A} = \begin{pmatrix} \dfrac{1}{r}\dfrac{\partial A_z}{\partial \theta} - \dfrac{\partial A_\theta}{\partial z} \\ \dfrac{\partial A_r}{\partial z} - \dfrac{\partial A_z}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial (rA_\theta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial A_r}{\partial \theta} \end{pmatrix}.$$

- In the spherical coordinate system:

$$\overrightarrow{\text{curl}}\,\overrightarrow{A} = \begin{pmatrix} \dfrac{1}{r\sin\theta}\dfrac{\partial(\sin\theta A_\varphi)}{\partial\theta} - \dfrac{1}{r\sin\theta}\dfrac{\partial A_\theta}{\partial\varphi} \\ \dfrac{1}{r\sin\theta}\dfrac{\partial A_r}{\partial\varphi} - \dfrac{1}{r}\dfrac{\partial(rA_\varphi)}{\partial r} \\ \dfrac{1}{r}\dfrac{\partial(rA_\theta)}{\partial r} - \dfrac{1}{r}\dfrac{\partial A_r}{\partial\theta} \end{pmatrix}.$$

CHAPTER 2 SUMMARY

Unit vector

For any non-zero vector $\overrightarrow{v}$, the vector $\hat{v} = \dfrac{\overrightarrow{v}}{\|\overrightarrow{v}\|}$ is a unit vector that has the same direction as $\overrightarrow{v}$.

Magnitude of a vector

$\|\overrightarrow{v}\| = |v| = v = \|\overrightarrow{v}\| = \sqrt{x^2 + y^2 + z^2}$.

Scalar product

$\overrightarrow{A} \cdot \overrightarrow{B} = \|\overrightarrow{A}\| \; \|\overrightarrow{B}\| \; \cos\theta$.
$\overrightarrow{A} \cdot \overrightarrow{B} = x_A \times x_B + y_A \times y_B + z_A \times z_B$.

Properties of the scalar product

- **Commutativity:** $\overrightarrow{v} \cdot \overrightarrow{w} = \overrightarrow{w} \cdot \overrightarrow{v}$;
- **Distributivity:** $\overrightarrow{u} \cdot (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} . \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{w}$;
- **Orthogonality:** If $\overrightarrow{v} \perp \overrightarrow{w}$, then $\overrightarrow{v} \cdot \overrightarrow{w} = 0$;
- $(\lambda \overrightarrow{v}).\overrightarrow{w} = \lambda(\overrightarrow{v}.\overrightarrow{w})$;
- $\overrightarrow{v} \cdot \overrightarrow{v} = \|\overrightarrow{v}\|^2$;
- $\overrightarrow{v} \cdot \overrightarrow{0} = 0$.

The angle between two vectors

$\theta = \arccos\left(\dfrac{\overrightarrow{A} \cdot \overrightarrow{B}}{\|\overrightarrow{A}\| \; \|\overrightarrow{B}\|}\right)$.

Vector product

$\|\overrightarrow{v} \times \overrightarrow{w}\| = \|\overrightarrow{a}\| = \|\overrightarrow{v}\| \|\overrightarrow{w}\| |\sin\alpha|$.

$$\overrightarrow{v} \times \overrightarrow{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix} = \left(yz' - zy'\right)\vec{i} + (zx' - xz')\vec{j} + \left(xy' - yx'\right)\vec{k}$$

Properties of the cross product

For vectors $\overrightarrow{u}$, $\overrightarrow{v}$ and $\overrightarrow{w}$ and scalars λ and μ, we have:

- The cross product is **not commutative**: $\overrightarrow{v} \times \overrightarrow{w} = -\overrightarrow{w} \times \overrightarrow{v}$;

- The cross product is **distributive** over vector sum: $\overrightarrow{u} \times (\overrightarrow{v} + \overrightarrow{w}) = \overrightarrow{u} \times \overrightarrow{v} + \overrightarrow{u} \times \overrightarrow{w}$;
- $\left(\lambda \overrightarrow{v}\right) \times \left(\mu \overrightarrow{w}\right) = \lambda\mu \left(\overrightarrow{v} \times \overrightarrow{w}\right)$;
- $(\lambda \overrightarrow{v}) \times \overrightarrow{w} = \lambda(\overrightarrow{v} \times \overrightarrow{w}) = \overrightarrow{v} \times (\lambda \overrightarrow{w})$;
- $\overrightarrow{v}$ and $\overrightarrow{w}$ are **colinear** if and only if $\overrightarrow{v} \times \overrightarrow{w} = \overrightarrow{0}$.

Scalar triple product (mixed product)

$$\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right] = \overrightarrow{a} . \left(\overrightarrow{b} \times \overrightarrow{c}\right)$$

Properties of the scalar triple product

- $\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right] = -\left[\overrightarrow{b}, \overrightarrow{a}, \overrightarrow{c}\right] = -\left[\overrightarrow{c}, \overrightarrow{b}, \overrightarrow{a}\right] = -\left[\overrightarrow{a}, \overrightarrow{c}, \overrightarrow{b}\right]$;
- $\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right] = \left[\overrightarrow{b}, \overrightarrow{c}, \overrightarrow{a}\right] = \left[\overrightarrow{c}, \overrightarrow{a}, \overrightarrow{b}\right]$;
- $\left(\overrightarrow{a} \wedge \overrightarrow{b}\right) . \overrightarrow{c} = \overrightarrow{a} . \left(\overrightarrow{b} \wedge \overrightarrow{c}\right)$;
- $\left[\overrightarrow{a}, \overrightarrow{b}, \lambda\overrightarrow{c}\right] = \lambda \left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right], \forall \lambda \in \mathbb{R}$;
- $\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c} + \overrightarrow{d}\right] = \left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right] + \left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{d}\right]$;
- $\left[\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}\right]$ represents the volume of the parallelepiped with coterminous edges $\overrightarrow{a}$, $\overrightarrow{b}$, and $\overrightarrow{c}$.

Time derivatives of rotating unit vectors

Given $\overrightarrow{u}$ and $\overrightarrow{n}$ two orthogonal unit vectors represented in the cartesian base system $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$. $\overrightarrow{u}$ and $\overrightarrow{n}$ can rotate around the Oz axis with the same angle θ.
$\dfrac{d\overrightarrow{u}}{dt} = \dot{\theta}\, \overrightarrow{n}$ and $\dfrac{d\overrightarrow{n}}{dt} = -\dot{\theta}\, \overrightarrow{u}$.

2.12 Exercises

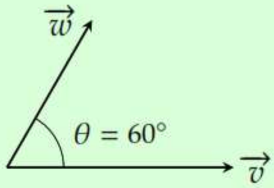
Exercise 2.1

Imagine you walk 18 meters directly west and then 25 meters directly north. Use a graphical approach to determine your distance from the starting point and the compass direction of the line connecting your starting point to your final position.

Sol : page 37

Exercise 2.2

Using the diagram shown bellow, find $\overrightarrow{v}+\overrightarrow{w}$ and $\overrightarrow{v}-\overrightarrow{w}$. Given $\|\overrightarrow{v}\| = 4$ units and $\|\overrightarrow{w}\| = 3$ units.



Sol : page 37

Exercise 2.3

Let $\overrightarrow{v}$ and $\overrightarrow{v}+\overrightarrow{w}$ be two vectors defined by their components, where $\overrightarrow{v} = (4, 6)$ and $\overrightarrow{v}+\overrightarrow{w} = (10, 9)$.
Calculate for the vector $\overrightarrow{w}$ the following: (a) its x and y components, (b) its length, and (c) the angle it makes with x-axis.

Sol : page 38

Exercise 2.4

Let the following three vectors be:
$\overrightarrow{u} = -\vec{\imath} - 2\vec{\jmath} + 3\vec{k}$,
$\overrightarrow{v} = 2\vec{\imath} - 4\vec{k}$,
$\overrightarrow{w} = -5\vec{\imath} - 2\vec{\jmath}$.
1. Calculate: $\|\overrightarrow{v}\|$, $\|\overrightarrow{w}\|$, $\|\overrightarrow{v}\| + \|\overrightarrow{w}\|$, $\|\overrightarrow{v}+\overrightarrow{w}\|$, $\overrightarrow{u}+\overrightarrow{v}$, $\overrightarrow{v}+\overrightarrow{u}$, $\overrightarrow{u}-\overrightarrow{v}$, $\left(\overrightarrow{u}+\overrightarrow{v}\right)+\overrightarrow{w}$, $\overrightarrow{u}+\left(\overrightarrow{v}+\overrightarrow{w}\right)$.
2. What can we conclude?

Sol : page 38

Exercise 2.5

Which of the following is a unit vector?

(a) $\overrightarrow{v_1} = 2\vec{\imath} - \vec{\jmath}$
(b) $\overrightarrow{v_2} = \cos\theta\vec{\imath} - \sin\theta\vec{\jmath}$
(c) $\overrightarrow{v_3} = \sin\theta\vec{\imath} + \cos\theta\vec{\jmath}$
(d) $\overrightarrow{v_4} = \dfrac{1}{\sqrt{3}}\vec{\imath} + 2\vec{\jmath}$

Sol : page 38

Exercise 2.6

We consider the vectors $\overrightarrow{u}$, $\overrightarrow{v}$, and $\overrightarrow{w}$ such that:

$\overrightarrow{u} = 3\vec{\imath} + 2\vec{\jmath} - \vec{k}$,
$\overrightarrow{v} = 2\vec{\imath} - 2\vec{\jmath} + \vec{k}$,
$\overrightarrow{w} = -\vec{\imath} - \vec{\jmath} + \vec{k}$.
1. Calculate : $\overrightarrow{v}\cdot\overrightarrow{w}$, $\overrightarrow{w}\cdot\overrightarrow{v}$, $\overrightarrow{u}\cdot\left(\overrightarrow{v}+\overrightarrow{w}\right)$, $\left(\overrightarrow{u}\cdot\overrightarrow{v}\right)+\left(\overrightarrow{u}\cdot\overrightarrow{w}\right)$, $\overrightarrow{v}\cdot\overrightarrow{v}$, and $\|\overrightarrow{v}\|^2$.
2. What can we conclude from this?

Sol : page 38

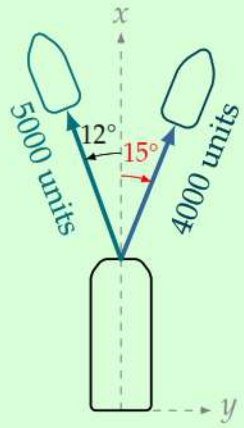
Exercise 2.7

If $\overrightarrow{a} = 6\vec{\imath} - 8\vec{\jmath}$, $\overrightarrow{b} = -8\vec{\imath} + 3\vec{\jmath}$, and $\overrightarrow{c} = 26\vec{\imath} + 19\vec{\jmath}$.
Find the unknown constants α and λ such that $\alpha\overrightarrow{a} + \lambda\overrightarrow{b} + \overrightarrow{c} = \overrightarrow{0}$.

Sol : page 39

Exercise 2.8

A barge is being towed by two tugboats. One tugboat exerts a force of magnitude 4000 units at an angle of 15°, while the other tugboat applies a force of magnitude 5000 units at an angle of 12° (see Figure bellow).



1. Determine the components of the resultant force acting on the barge.
2. What is the total magnitude of the pull?
3. What is the direction of this resultant force relative to the x-axis?

Sol : page 39

Exercise 2.9

Consider two nonzero vectors $\overrightarrow{v}$ and $\overrightarrow{w}$ satisfying the equation:
$\|\overrightarrow{v}+\overrightarrow{w}\| = \|\overrightarrow{v}-\overrightarrow{w}\|$.
Find the angle θ between the vectors $\overrightarrow{v}$ and $\overrightarrow{w}$.

Sol : page 39

Exercise 2.10

Given the vectors $\overrightarrow{A} = (1,-1,0)$, $\overrightarrow{B} = (2,-1,1)$, $\overrightarrow{C} = (-1,2,-3)$, and $\overrightarrow{D} = (0,1,-2)$:
1. Calculate the following: (a) $(\overrightarrow{A}\times\overrightarrow{B})\cdot\overrightarrow{C}$, (b) $(\overrightarrow{A}\times\overrightarrow{B})\cdot(\overrightarrow{C}\times\overrightarrow{D})$, (c) $(\overrightarrow{A}\cdot\overrightarrow{B})(\overrightarrow{C}\times\overrightarrow{D})$.
2. Determine the angle α between $\overrightarrow{B}$ and $\overrightarrow{C}$, and the angle β between $\overrightarrow{A}$ and $\overrightarrow{D}$.
3. Find the angle θ that $\overrightarrow{A}$ makes with the x-axis.

Sol : page 40

Exercise 2.11

If $\overrightarrow{u} = a\vec{\imath} + b\vec{\jmath}$ is a unit vector and it is perpendicular to $\overrightarrow{w} = \vec{\imath}+\vec{\jmath}$, then find the value of a and b.

Sol : page 40

Exercise 2.12

Given the vectors $\overrightarrow{A} = 3\vec{\imath} + 4\vec{\jmath}$ and $\overrightarrow{B} = 7\vec{\imath} + 24\vec{\jmath}$, use two different methods to find the vector $\overrightarrow{C}$ that has the same magnitude as $\overrightarrow{B}$ and parallel to $\overrightarrow{A}$.

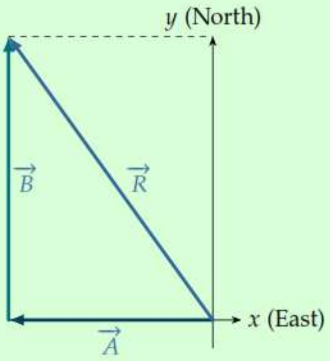
Sol : page 40

2.13 Solutions

Solution of exercise 2.1 (Page 35)

To solve the problem using graphical approach, we first need to represent the displacements as vectors and calculate the resultant vector. This will allow us to find both the distance from the starting point and the compass direction.

For the 1st movement (18 meters West) we have $\overrightarrow{A} = -18\vec{\imath}$ and for the 2nd movement (25 meters North): $\overrightarrow{B} = 25\vec{\jmath}$

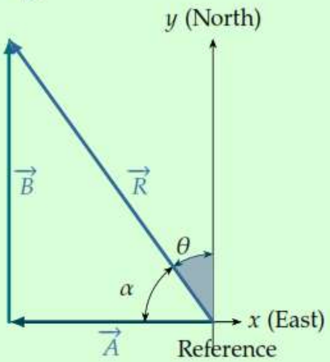


$$\overrightarrow{R} = \overrightarrow{A} + \overrightarrow{B} = -18\vec{\imath} + 25\vec{\jmath}$$

$$|\overrightarrow{R}| = R = \sqrt{(-18)^2 + (25)^2} = 30.81 \text{ m}$$

Thus, the distance from the starting point is 30.81 meters.

Generally, the north is taken as the reference (0°). Therefore, the compass direction is the angle θ that the resultant vector $\overrightarrow{R}$ makes with the positive y-axis. This angle can be calculated as follows: $\theta = 90 - \alpha$



$$\alpha = \arctan\left(\frac{B}{A}\right) = \arctan\left(\frac{25}{18}\right) \approx 54.25^\circ$$

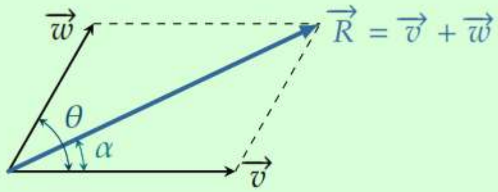
$$\therefore\ \theta = 90^\circ - 54.25^\circ = 35.75^\circ.$$

This direction can be noted in relative notation as follows: **N37.75°W** (approximately **Northwest**).

Solution of exercise 2.2 (Page 35)

Addition

$$\begin{aligned} R &= \sqrt{v^2 + w^2 + 2vw\ cos\theta} \\ &= \sqrt{16 + 9 + 2 \times 4 \times\ \cos 60^\circ} \\ &= \sqrt{37} \text{ units} \end{aligned}$$



$$\tan\alpha = \frac{|\overrightarrow{w}|\sin\theta}{|\overrightarrow{v}| + |\overrightarrow{w}|\cos\theta} = 0.472$$

$$\therefore \quad \alpha = \tan^{-1}(0.472) = 25.3^\circ$$

Thus, resultant $\overrightarrow{R}$ is $\sqrt{37}$ units at angle 25.3° from $\overrightarrow{v}$ (see the figure above).

Subtraction

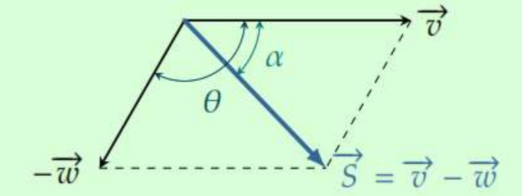
$$\begin{aligned} S &= \sqrt{|\overrightarrow{v}|^2 + |\overrightarrow{w}|^2 - 2|\overrightarrow{v}||\overrightarrow{w}|\cos\theta} \\ &= \sqrt{13} \text{ units} \end{aligned}$$

and

$$\tan\alpha = \frac{|\overrightarrow{w}|\sin\theta}{|\overrightarrow{v}| - |\overrightarrow{w}|\cos\theta} = 1.04$$

$$\therefore \quad \alpha = \tan^{-1}(1.04) = 46.1^\circ$$

Thus, $\overrightarrow{v}-\overrightarrow{w}$ is $\sqrt{13}$ units at $46.1°$ from $\overrightarrow{v}$ in the direction shown in figure below.



Solution of exercise 2.3 (Page 35)

$$\overrightarrow{v} + \overrightarrow{w} = (v_x + w_x)\vec{\imath} + (v_y + w_y)\vec{\jmath}$$

Given that: $v_x + w_x = 10$ and $v_y + w_y = 9$ and $v_x = 4$ and $v_y = 6$

Hence

(a) $w_x = 10 - 4 = 6$ and $w_y = 9 - 6 = 3$ $\overrightarrow{w} = (6, 3)$.

(b) $\|\overrightarrow{w}\| = \sqrt{(6^2 + 3^2)} = \sqrt{(45)} = 3\sqrt{5}$.

(c) $\theta = \tan^{-1}\left(\frac{w_y}{w_x}\right) = \tan^{-1}\left(\frac{3}{6}\right) = 26.6°$.

Solution of exercise 2.4 (Page 35)

1.

$\|\overrightarrow{v}\| = 2\sqrt{5}$,
$\|\overrightarrow{w}\| = \sqrt{29}$,
$\|\overrightarrow{v}\| + \|\overrightarrow{w}\| = 2\sqrt{5} + \sqrt{29}$,
$\|\overrightarrow{v} + \overrightarrow{w}\| = \sqrt{9 + 4 + 16} = \sqrt{29}$,
$\overrightarrow{u} + \overrightarrow{v} = \vec{\imath} - 2\vec{\jmath} - \vec{k}$, $\overrightarrow{v} + \overrightarrow{u} = \vec{\imath} - 2\vec{\jmath} - \vec{k}$,
$\overrightarrow{u} - \overrightarrow{v} = -3\vec{\imath} - 2\vec{\jmath} + 7\vec{k}$,
$(\overrightarrow{u} + \overrightarrow{v}) + \overrightarrow{w} = -4\vec{\imath} - 4\vec{\jmath} - \vec{k}$,
$\overrightarrow{u} + (\overrightarrow{v} + \overrightarrow{w}) = -4\vec{\imath} - 4\vec{\jmath} - \vec{k}$.

The calculations highlight important characteristics of vector operations:

- Vector addition is both commutative and associative.
- According to the triangle inequality, the norm of the sum of two vectors does not equal the sum of their individual norms.

Solution of exercise 2.5 (Page 35)

To determine which vectors are unit vectors, we need to check if their magnitudes are equal to 1.

(a) $\|\overrightarrow{v_1}\| = \sqrt{5}$, $\overrightarrow{v_1}$ is not a unit vector.
(b) $\|\overrightarrow{v_2}\| = 1$, $\overrightarrow{v_2}$ is a unit vector.
(c) $\|\overrightarrow{v_3}\| = \sqrt{(\sin\theta)^2 + (\cos\theta)^2} = 1$
$\overrightarrow{v_3}$ is a unit vector.
(d) $\|\overrightarrow{v_4}\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + (2)^2} = \sqrt{\frac{13}{3}}$
$\overrightarrow{v_4}$ is not a unit vector.

Solution of exercise 2.6 (Page 35)

1.

$* \ \overrightarrow{v} \cdot \overrightarrow{w} = (2)(-1) + (-2)(-1) + (1)(1) = 1.$

$* \ \overrightarrow{w} \cdot \overrightarrow{v} = (-1)(2) + (-1)(-2) + (1)(1) = 1.$

$* \ \overrightarrow{u} \cdot \left(\overrightarrow{v} + \overrightarrow{w}\right)$:

$\overrightarrow{v} + \overrightarrow{w} = \vec{\imath} - 3\vec{\jmath} + 2\vec{k}$.

$\overrightarrow{u} \cdot \left(\overrightarrow{v} + \overrightarrow{w}\right) = \overrightarrow{u} \cdot (\vec{\imath} - 3\vec{\jmath} + 2\vec{k}) = -5.$

$* \ \left(\overrightarrow{u} \cdot \overrightarrow{v}\right) + \left(\overrightarrow{u} \cdot \overrightarrow{w}\right)$:

$\overrightarrow{u} \cdot \overrightarrow{v} = (3)(2) + (2)(-2) + (-1)(1) = 1.$

$\overrightarrow{u} \cdot \overrightarrow{w} = (3)(-1) + (2)(-1) + (-1)(1) = -6.$

$\therefore \ \left(\overrightarrow{u} \cdot \overrightarrow{v}\right) + \left(\overrightarrow{u} \cdot \overrightarrow{w}\right) = 1 + (-6) = -5.$

$* \ \overrightarrow{v} \cdot \overrightarrow{v} = (2)^2 + (-2)^2 + (1)^2 = 9.$

$* \ \|\overrightarrow{v}\| = 3.$

$\therefore \ \|\overrightarrow{v}\|^2 = 3^2 = 9.$

2. From the calculations presented above, it is evident that:

- $\overrightarrow{v} \cdot \overrightarrow{w} = \overrightarrow{w} \cdot \overrightarrow{v}$, confirming that the dot product is commutative.

- $\overrightarrow{u} \cdot \left(\overrightarrow{v} + \overrightarrow{w}\right) = \left(\overrightarrow{u} \cdot \overrightarrow{v}\right) + \left(\overrightarrow{u} \cdot \overrightarrow{w}\right)$, confirming the distributive property of the dot product.
- $\overrightarrow{v} \cdot \overrightarrow{v} = \|\overrightarrow{v}\|^2$, this verifies that the square of a vector's norm is equal to its dot product with itself. This is a fundamental property of the dot product.

Solution of exercise 2.7 (Page 35)

$\alpha\overrightarrow{a} + \lambda\overrightarrow{b} + \overrightarrow{c} = (6\alpha - 8\lambda + 26)\vec{\imath} + (-8\alpha + 3\lambda + 19)\vec{\jmath} = \overrightarrow{0}$

This gives us two equations:

$$\begin{cases} 6\alpha - 8\lambda + 26 = 0 & \text{(i)} \\ -8\alpha + 3\lambda + 19 = 0 & \text{(ii)} \end{cases}$$

The resolution of this system of equations results in:

$$\alpha = 5 \quad \text{and} \quad \lambda = 7.$$

Solution of exercise 2.8 (Page 35)

1.

Tugboat 1 (force $\overrightarrow{F_1}$):

$$\begin{aligned}\overrightarrow{F_1} &= [4000\cos(15^\circ)]\vec{\imath} + [4000\sin(15^\circ)]\vec{\jmath} \\ &= 3863.7\vec{\imath} + 1035.3\vec{\jmath}.\end{aligned}$$

Tugboat 2 (force $\overrightarrow{F_2}$):

$$\begin{aligned}\overrightarrow{F_2} &= [5000\cos(12^\circ)]\vec{\imath} - [5000\sin(12^\circ)]\vec{\jmath} \\ &= 4890.7\vec{\imath} - 1039.6\vec{\jmath}.\end{aligned}$$

2. $\overrightarrow{R} = \overrightarrow{F_1} + \overrightarrow{F_2} = 8754.4\vec{\imath} - 4.3\vec{\jmath}$.

3. The magnitude F_R of the resultant force is given by:

$$R = 8754.4 \text{ units.}$$

4. The direction θ relative to the x-axis is given by:

$$\theta = \tan^{-1}\left(\frac{-4.3}{8754.1}\right) \approx -0.028^\circ.$$

The resultant force is -0.028° relative to the x-axis (0.028° relative to the x-axis in the counterclockwise direction).

Solution of exercise 2.9 (Page 36)

We start with the given condition:

$$\|\overrightarrow{v} + \overrightarrow{w}\| = \|\overrightarrow{v} - \overrightarrow{w}\|$$

We know that the dot product of a vector with itself is equal to the square of its magnitude. Applying this property to our vectors, we obtain:

$$\begin{aligned}\|\overrightarrow{v} + \overrightarrow{w}\|^2 &= (\overrightarrow{v} + \overrightarrow{w}) \cdot (\overrightarrow{v} + \overrightarrow{w}) \\ &= \overrightarrow{v} \cdot \overrightarrow{v} + 2\overrightarrow{v} \cdot \overrightarrow{w} + \overrightarrow{w} \cdot \overrightarrow{w} \\ &= v^2 + 2\overrightarrow{v} \cdot \overrightarrow{w} + w^2 \\ &= v^2 + 2\, v\, w\, \cos\theta + w^2\end{aligned}$$

Similarly,

$$\begin{aligned}\|\overrightarrow{v} - \overrightarrow{w}\|^2 &= (\overrightarrow{v} - \overrightarrow{w}) \cdot (\overrightarrow{v} - \overrightarrow{w}) \\ &= \overrightarrow{v} \cdot \overrightarrow{v} - 2\overrightarrow{v} \cdot \overrightarrow{w} + \overrightarrow{w} \cdot \overrightarrow{w} \\ &= v^2 - 2\overrightarrow{v} \cdot \overrightarrow{w} + w^2 \\ &= v^2 - 2\, v\, w\, \cos\theta + w^2\end{aligned}$$

Given the condition:
$\|\overrightarrow{v} + \overrightarrow{w}\| = \|\overrightarrow{v} - \overrightarrow{w}\|$
hence,
$\|\overrightarrow{v} + \overrightarrow{w}\|^2 = \|\overrightarrow{v} - \overrightarrow{w}\|^2$
or
$v^2 + 2vw\cos\theta + w^2 = v^2 - 2vw\cos\theta + w^2$
Simplify:
$4vw\cos\theta = 0$
Given $\overrightarrow{v}$ and $\overrightarrow{w}$ are nonzero vectors, therefore:
$\cos\theta = 0$, this implies: $\theta = 90^\circ$.

Solution of exercise 2.10 (Page 36)

1. (a) $(\overrightarrow{A} \times \overrightarrow{B}) \cdot \overrightarrow{C}$

$$\overrightarrow{A} \times \overrightarrow{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{vmatrix} = -\vec{\imath} - \vec{\jmath} + \vec{k}$$
$$= (-1, -1, 1)$$

$(\overrightarrow{A} \times \overrightarrow{B}) \cdot \overrightarrow{C} = -4.$

(b) $(\overrightarrow{A} \times \overrightarrow{B}) \cdot (\overrightarrow{C} \times \overrightarrow{D})$

$$\overrightarrow{C} \times \overrightarrow{D} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -1 & 2 & -3 \\ 0 & 1 & -2 \end{vmatrix} = (-1, -2, -1)$$

$$(\overrightarrow{A} \times \overrightarrow{B}) \cdot (\overrightarrow{C} \times \overrightarrow{D}) = 2.$$

(c) $(\overrightarrow{A} \cdot \overrightarrow{B})(\overrightarrow{C} \times \overrightarrow{D})$

$$\overrightarrow{A} \cdot \overrightarrow{B} = (1, -1, 0) \cdot (2, -1, 1) = 3$$

$$\begin{aligned} (\overrightarrow{A} \cdot \overrightarrow{B})(\overrightarrow{C} \times \overrightarrow{D}) &= 3(-1, -2, -1) \\ &= (-3, -6, -3) \\ &= -3\vec{\imath} - 6\vec{\jmath} - 3\vec{k}. \end{aligned}$$

2. Using the geometric definition of the dot product, the angle between two vectors $\overrightarrow{u}$ and $\overrightarrow{w}$ is expressed as follows:

$$\phi = \cos^{-1}\left(\frac{\overrightarrow{u} \cdot \overrightarrow{w}}{\|\overrightarrow{u}\|\|\overrightarrow{w}\|}\right)$$

For α:

$$\overrightarrow{B} \cdot \overrightarrow{C} = (2, -1, 1) \cdot (-1, 2, -3) = -7$$

$$\|\overrightarrow{B}\| = \sqrt{2^2 + (-1)^2 + 1^2} = \sqrt{6}$$

$$\|\overrightarrow{C}\| = \sqrt{(-1)^2 + 2^2 + (-3)^2} = \sqrt{14}$$

$$\alpha = \cos^{-1}\left(\frac{-7}{\sqrt{6}\sqrt{14}}\right) = \cos^{-1}\left(\frac{-7}{\sqrt{84}}\right)$$

$$\alpha = 139.79°.$$

Similarly, we find:

$$\beta = 108.43°.$$

3. The x-axis is represented by the unit vector $\vec{\imath} = (1, 0, 0)$.

$$\therefore\ \theta = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\theta = \frac{\pi}{4}.$$

Solution of exercise 2.11 (Page 36)

Given that $\overrightarrow{u}$ is a unit vector, we have:

$$\sqrt{a^2 + b^2} = 1 \Rightarrow a^2 + b^2 = 1 \quad \text{(i)}$$

Also, $\overrightarrow{u} \perp \overrightarrow{w} \implies \overrightarrow{u} \cdot \overrightarrow{w} = 0$
Thus,

$$\overrightarrow{u} \cdot \overrightarrow{w} = a + b = 0 \Rightarrow b = -a.$$

Substituting $b = -a$ and solving the equation (i) we get:

$$a = \pm\frac{1}{\sqrt{2}}.$$

Thus, if $a = \frac{1}{\sqrt{2}}$, then $b = -\frac{1}{\sqrt{2}}$,
and if $a = -\frac{1}{\sqrt{2}}$, then $b = \frac{1}{\sqrt{2}}$.

Solution of exercise 2.12 (Page 36)

The vector $\overrightarrow{C}$ can be written as:
$\overrightarrow{C} = x\vec{\imath} + y\vec{\jmath}$.

First method (using cross product):

As:

$$\begin{cases} \|\overrightarrow{C}\| = \sqrt{x^2 + y^2} = \|\overrightarrow{B}\| & \text{(i)} \\ \overrightarrow{C} \mathbin{/\!/} \overrightarrow{A} \Leftrightarrow \overrightarrow{C} \times \overrightarrow{A} = \overrightarrow{0} & \text{(ii)} \end{cases}$$

$\|\overrightarrow{B}\| = 25$ and $\overrightarrow{C} \times \overrightarrow{A} = (-4x + 3y)\vec{k}$.
As: $\overrightarrow{C} \times \overrightarrow{B} = \overrightarrow{0} \Leftrightarrow \|\overrightarrow{C} \times \overrightarrow{B}\| = 0$
Therefore,

$$\begin{cases} x^2 + y^2 = 625 & \text{(iii)} \\ -4x + 3y = 0 & \text{(iv)} \end{cases}$$

Thus, x can be either 15 or -15.
Therefore, the vector $\overrightarrow{C}$ can be either:

$$\overrightarrow{C} = 15\vec{\imath} + 20\vec{\jmath} \text{ or } \overrightarrow{C} = -15\vec{\imath} - 20\vec{\jmath}.$$

Second method (using collinearity of vectors):
$\overrightarrow{C} /\!/ \overrightarrow{A} \Leftrightarrow \overrightarrow{C} = \lambda\overrightarrow{A}$
$\|\overrightarrow{C}\| = |\lambda|\|\overrightarrow{A}\| = \|\overrightarrow{B}\| = 25$.
As: $\|\overrightarrow{A}\| = 5 \implies |\lambda| = 5$.
Thus, λ can be either 5 or -5. Therefore, the vector $\overrightarrow{C}$ can be either:

$$\overrightarrow{C} = 15\vec{\imath} + 20\vec{\jmath} \text{or } \overrightarrow{C} = -15\vec{\imath} - 20\vec{\jmath}.$$

KINEMATICS OF A PARTICLE 3

3.1 Introduction

Kinematics is a branch of mechanics that focuses solely on analyzing the motion of a particle (object point). It involves describing the trajectory of the point, expressing its velocity and acceleration in different coordinate systems, without considering the causes of its motion. The connection between the causes of motion (forces) and the resulting trajectories will be the subject of the next chapter (dynamics of a material point).

3.2 Point object

In the realm of physics, an object is classified as **point object** (also called **a particle** or **material point**) when its dimensions are negligible compared to the distances it travels within a reasonable time frame. Earth's radius is significantly smaller than its orbital radius around the sun, so Earth can be considered a point object when revolving around the sun.

3.3 Notion of reference frame

In a moving train, an object that is stationary for a seated passenger appears to be moving for an observer on the side of the road. This is why it is necessary to specify with respect to what this object is in motion, hence the need for a **reference frame** or **frame of reference** in which an observer analyzing the motion of this object is fixed. Thus, we define a frame of reference as a solid body assumed to be undeformable, from which we define a system of coordinate axes linked to an observer equipped with a clock defining time.

In simple terms, a frame of reference comprises a coordinate system and a clock that are utilized to analyze the motion of all material points. Terrestrial, geocentric, and heliocentric reference frames are examples.

3.4 Locating a point; usual coordinate systems

Choice of coordinate system

The space contains three dimensions; this means that it takes three coordinates to define the position of a point M in space. In order to do this, it is necessary to firstly select an origin point denoted as O. The point M is then located relative to O. We note $\overrightarrow{r} = \overrightarrow{OM}$ the **position vector** of M.
To locate the position vector $\overrightarrow{OM}$, it is imperative to establish a **vector basis** denoted as $(\overrightarrow{e_1}; \overrightarrow{e_2}; \overrightarrow{e_3})$. The vector $\overrightarrow{OM}$ can then be decomposed in this basis as:

$$\boxed{\overrightarrow{OM} = d_1\,\overrightarrow{e_1} + d_2\,\overrightarrow{e_2} + d_3\,\overrightarrow{e_3}} \tag{3.1}$$

where d_1, d_2 and d_3 are the components of the vector $\overrightarrow{OM}$ in the base $(\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3})$.

However, for practical reasons (especially when calculating dot products and cross products), it is important that the base used is a **direct orthonormal basis**.

A direct orthonormal basis means:

- Ortho (orthogonal):

$$\overrightarrow{e_1} \cdot \overrightarrow{e_2} = 0;$$
$$\overrightarrow{e_1} \cdot \overrightarrow{e_3} = 0;$$
$$\overrightarrow{e_2} \cdot \overrightarrow{e_3} = 0.$$

- Normal:

$$\|\overrightarrow{e_1}\| = \|\overrightarrow{e_2}\| = \|\overrightarrow{e_3}\|.$$

- direct:

$$\overrightarrow{e_1} \times \overrightarrow{e_2} = \overrightarrow{e_3};$$
$$\overrightarrow{e_2} \times \overrightarrow{e_3} = \overrightarrow{e_1};$$
$$\overrightarrow{e_3} \times \overrightarrow{e_1} = \overrightarrow{e_2}.$$

It is crucial to carefully select the coordinate system in which the problem description will be formulated in order to streamline the calculations.

In the plane, for example, we can use two axes x and y to locate the point M being studied. However, if the movement of M is circular, the use of x and y will be complicated, it will be better to locate M by its distance from the center O (radius r), and the angle covered (this is what we call **polar coordinates**).

For three-dimensional movement on Earth, it is customary and simpler to locate a point by its latitude, longitude, and altitude (this is what we will call the system of **spherical coordinates**).

Cartesian coordinates

We consider a coordinate system consisting of three axes Ox, Oy, and Oz passing through a point O (Origin). To this coordinate system, we associate a **direct orthonormal basis** $(\vec{\imath}, \vec{\jmath}, \vec{k})$, where $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ are the unit vectors of the Ox, Oy, and Oz axes respectively (see Figure 3.1). This coordinate system is known as the **Cartesian coordinate system** or **rectangular coordinate system**

At a given moment t, we denote the position of a point M by the position vector $\overrightarrow{OM}$, which is written as:

$$\boxed{\overrightarrow{OM} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}} \tag{3.2}$$

where x, y, and z are the **Cartesian coordinates** of the point M representing the orthogonal projections on the three axes of the Cartesian coordinate system.

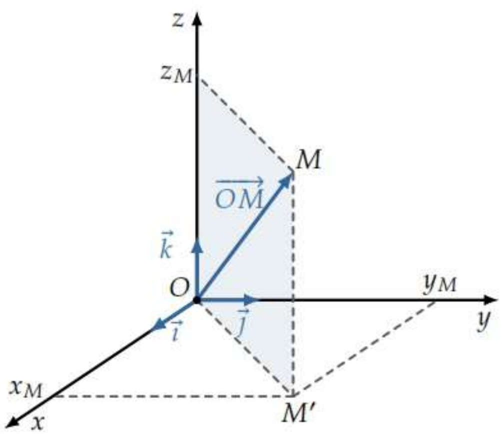


Figure 3.1: Cartesian coordinate system

One must not confuse the coordinates of point M and the components of the position vector. These two mathematical concepts, which coincide in Cartesian coordinates, are no longer the same at all in cylindrical coordinates, for example.

Cylindrical polar coordinates

Polar coordinates

In the polar coordinate system, the position of a particle M is located by the distance r ($r = OM$) called **radial coordinate** and a rotation angle θ called the **angular coordinate** or polar angle measured with respect to a fixed axis (in general, we take the Ox axis of Cartesian coordinates) as shown in Figure 3.2. The polar coordinates of M are therefore: (r, θ).

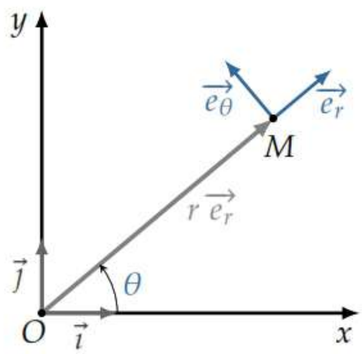


Figure 3.2: Representation of polar coordinates

- r : Is a distance so it is positive. $r \in [0, +\infty[$.
- θ : The polar angle $\left(\vec{\imath}, \overrightarrow{OM}\right)$, $\theta \in [0, 2\pi]$.

The transformation formulas between Cartesian coordinates and polar coordinates are:

$$\begin{cases} x = r\,\cos\theta \\ y = r\,\sin\theta \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\dfrac{y}{x}\right) \end{cases}$$

We define the direct orthonormal mobile projection base $(\overrightarrow{e_r}, \overrightarrow{e_\theta})$ adapted to polar coordinates as follows:

- $\overrightarrow{e_r}$ is a unit vector with the same direction as $\overrightarrow{OM}$.
- $\overrightarrow{e_\theta}$ is obtained by rotating $\overrightarrow{e_r}$ by an angle of $\dfrac{\pi}{2}$ **rad**.

The position vector is written:

$$\boxed{\overrightarrow{OM} = r\,\overrightarrow{e_r}} \tag{3.3}$$

Cylindrical coordinates

In cylindrical coordinates, a point M in space is identified as a point on a cylinder whose axis Oz is generally coincident with the Oz axis of the Cartesian coordinate system. We introduce here the direct orthonormal basis $(\overrightarrow{e_\rho}, \overrightarrow{e_\theta}, \overrightarrow{e_z})$, associated with the cylindrical coordinates (ρ, θ, z) (see Figure 3.3).

The point M (or $\overrightarrow{OM}$) is then located by:

- ρ : the radius of the cylinder, $\rho = OM'$ where M' is the orthogonal projection of M onto the plane $xOy.\rho \in [0, +\infty[$.
- θ : the angle $\left(\vec{\imath}, \overrightarrow{OM'}\right).\theta \in [0, 2\pi]$.
- z : the same as in Cartesian coordinates $z \in]-\infty, +\infty[$.

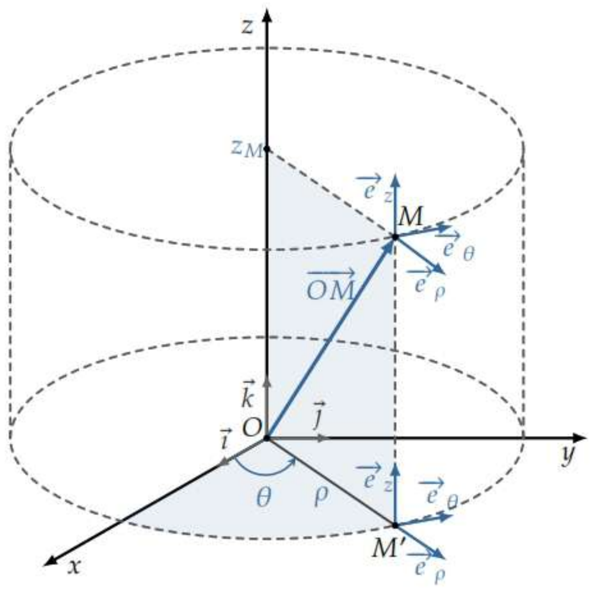


Figure 3.3: Representation of the cylindrical coordinates

In this coordinate system, the position vector $\overrightarrow{OM}$ is written as:

$$\boxed{\overrightarrow{OM} = \overrightarrow{OM'} + \overrightarrow{M'M} = \rho\,\overrightarrow{e_\rho} + z\,\overrightarrow{e_z}} \tag{3.4}$$

The cylindrical and Cartesian coordinates are related by the following formulas:

$$\begin{cases} x = \rho\,\cos\theta \\ y = \rho\,\sin\theta \\ z = z \end{cases} \quad \text{and} \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\dfrac{y}{x}\right) \\ z = z \end{cases}$$

Spherical coordinates

In spherical coordinates, a point M is considered as a point on a sphere centered at O. Here, we introduce the direct orthonormal mobile basis $\left(\overrightarrow{e_r}, \overrightarrow{e_\theta}, \overrightarrow{e_\varphi}\right)$, associated with the spherical coordinates (r, θ, φ), which consist of a distance and two angles of rotation around O (see Figure 3.4).
The point M is located in space by:

- r: the radius of the sphere, $r = \|\overrightarrow{OM}\|$. $r \in [0, +\infty[$.
- θ: the angle $\left(\vec{k}, \overrightarrow{e_r}\right)$. θ called **colatitude** is oriented by $\overrightarrow{e_\varphi}$. $\theta \in [0, \pi]$.
- φ: the angle $\left(\vec{\imath}, \overrightarrow{OM'}\right)$ where M' is the orthogonal projection of M onto the plane xOy. φ called **longitude** is oriented by $\vec{k}$. $\varphi \in [0, 2\pi]$.

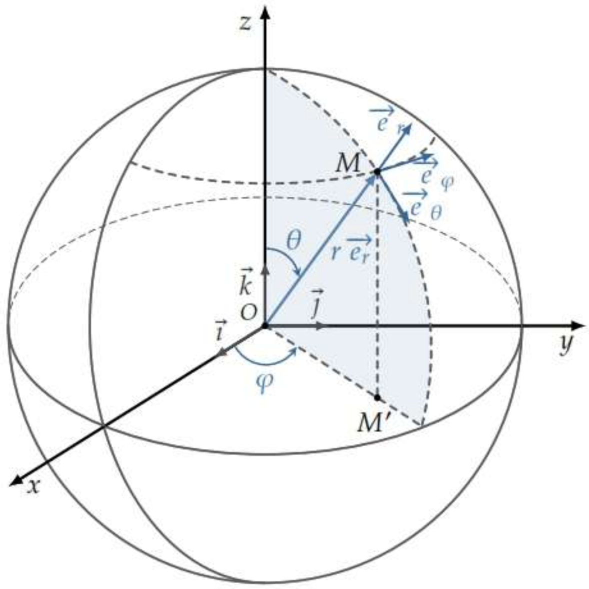


Figure 3.4: Representation of the spherical coordinates

In the spherical coordinate system, the position vector $\overrightarrow{OM}$ is defined as:

$$\boxed{\overrightarrow{OM} = r\,\overrightarrow{e_r}} \tag{3.5}$$

The spherical and Cartesian coordinates are related by the following formulas:

$$\begin{cases} x = r\,\sin\theta\,\cos\varphi \\ y = r\,\sin\theta\,\sin\varphi \\ z = r\,\cos\theta \end{cases} \quad \text{and} \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\dfrac{\sqrt{x^2+y^2}}{z}\right) \\ \varphi = \arctan\left(\dfrac{y}{x}\right) \end{cases}$$

Frenet basis (frame)

Let M be a moving object traveling a distance s (curvilinear abscissa) on a curve $(\mathscr{C})$. We define the Frenet basis (intrinsic or local basis) as the unit vectors $\overrightarrow{e_t}$ and $\overrightarrow{e_n}$, which are related to the moving object M at each time (see Figure 3.5).

- **The unit vector $\overrightarrow{e_t}$:** Called **unit tangent vector**, tangent to the curve and oriented, generally, in the direction of movement.
- **The unit vector $\overrightarrow{e_n}$:** Called **unit normal vector**, orthogonal to $\overrightarrow{e_t}$ and oriented towards the inside of the curve (concavity of the curve).

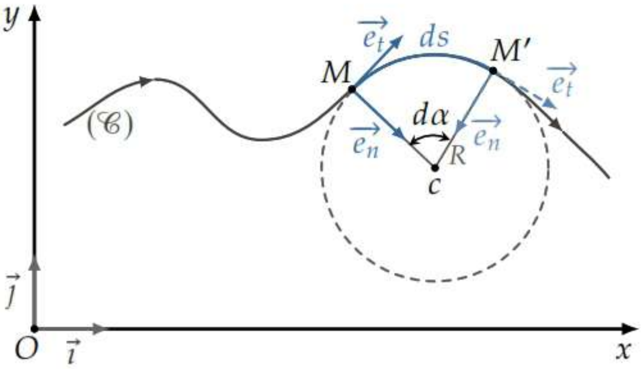


Figure 3.5: Frenet basis

In three-dimensional motion, we introduce a third unit vector $\overrightarrow{e_b}$, known as the **unit binormal vector**. This vector is perpendicular to both $\overrightarrow{e_t}$ and $\overrightarrow{e_n}$, and it can be defined by the equation $\overrightarrow{e_b} = \overrightarrow{e_t} \times \overrightarrow{e_n}$ (see Figure 3.6).

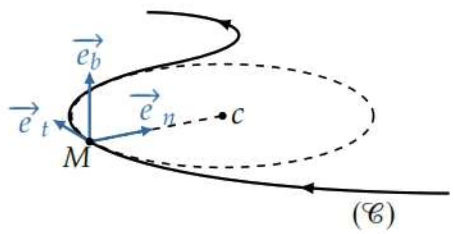


Figure 3.6: Three-dimensional Frenet basis

3.5 Terms related to motion

Trajectory (Path)

A trajectory is a curve whose points correspond to the successive positions occupied by a system during its motion. A simplification would be to define it as the path followed by an object during its displacement.

The trajectory is one of the elements that allows characterizing the movement of an object, depending on its geometric shape, we can say that the motion is rectilinear or curvilinear.

The trajectory of an object point depends on the chosen reference frame.

Example 3.1

The parametric coordinates of a particle moving in the (xy) plane are given as follows: $x = 2\sqrt{t} - 1$ and $y = 4t + 3$. What type of trajectory does this particle follow?

Solution

1. Express t in terms of x

$x = 2\sqrt{t} - 1 \Rightarrow t = \dfrac{x^2 + 2x + 1}{4}$

2. Substitute the expression for t into the equation for y:

$$y = 4\left(\frac{x^2 + 2x + 1}{4}\right) + 3 = x^2 + 2x + 4.$$

The equation $y = x^2 + 2x + 4$ is in the form of a quadratic function, which represents a **parabola**.

Displacement and Distance

In physics, displacement is a term used to describe the straight line between two positions (initial and final positions) of a moving object during a specific time period, regardless of the nature of the path (straight or curved) taken by the object. In contrast, distance refers to the actual path length covered by a moving object between its initial and final positions during a specific time interval (curve $(\mathscr{C})$ in Figure 3.7).

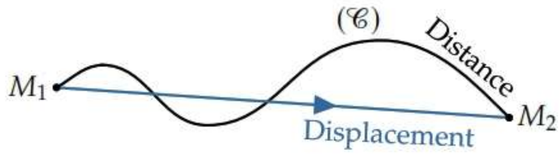


Figure 3.7: Displacement and distance representation

Displacement and elementary displacement vectors

We consider the simplest case where the object M moves along the Ox axis. Let M_1 be the position of M at time t_1 and

M_2 be its position at time t_2 ($t_2 > t_1$) (see Figure 3.8). We call the **displacement vector** the vector $\overrightarrow{M_1M_2}$.

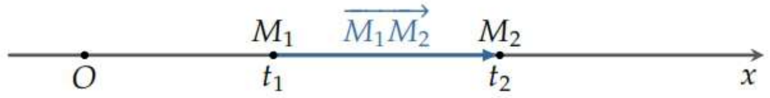


Figure 3.8: Displacement vector

We have: $\overrightarrow{M_1M_2} = \overrightarrow{M_1O} + \overrightarrow{OM_2} = \overrightarrow{OM_2} - \overrightarrow{OM_1} = \Delta\overrightarrow{OM}$.

If x_1 and x_2 are respectively the components of $\overrightarrow{OM_1}$ and $\overrightarrow{OM_2}$, then $x_2 - x_1 = \Delta x$ is the component of the displacement vector $\overrightarrow{M_1M_2}$, and we write:

$$\overrightarrow{M_1M_2} = \Delta\overrightarrow{OM} = \Delta x\,\vec{\imath}$$

Assuming now that the displacement from position M_1 to position M_2 is infinitely small (compared to the displacement being studied), this displacement, called **elementary displacement**, is represented by the vector:

$$\overrightarrow{M_1M_2} = d\,\overrightarrow{OM}$$

In the Cartesian coordinate system

In the case where M moves in three-dimensional Cartesian coordinate system $(O, \vec{\imath}, \vec{\jmath}, \vec{k})$, the displacement vector can be written as:

$$\boxed{\overrightarrow{M_1M_2} = \Delta\overrightarrow{OM} = \Delta x\,\vec{\imath} + \Delta y\,\vec{\jmath} + \Delta z\,\vec{k}} \tag{3.6}$$

The elementary displacement vector is expressed as follows:

$$\boxed{d\overrightarrow{OM} = \lim_{M_1 \to M_2} \overrightarrow{M_1M_2} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}} \tag{3.7}$$

In the cylindrical coordinate system

The position vector can be expressed in the cylindrical coordinate systems as follows:

$$\boxed{\overrightarrow{OM} = \rho\,\overrightarrow{e_\rho} + z\,\overrightarrow{e_z}} \tag{3.8}$$

In polar coordinate system:

$$\boxed{\overrightarrow{OM} = r\,\overrightarrow{e_r}} \tag{3.9}$$

To find the elementary displacement vector, it is necessary to calculate the differential of the position vector, taking into account that the unit vector $\overrightarrow{e_\rho}$ depends on the coordinate θ. The differential position vector $d\overrightarrow{OM}$ is expressed as:

$$d\overrightarrow{OM} = \frac{\partial\overrightarrow{OM}}{\partial\rho}d\rho + \frac{\partial\overrightarrow{OM}}{\partial\theta}d\theta + \frac{\partial\overrightarrow{OM}}{\partial z}dz$$

The elementary displacement vector can be written in the cylindrical coordinate system as:

$$\boxed{d\overrightarrow{OM} = d\rho\,\overrightarrow{e_\rho} + \rho\,d\theta\,\overrightarrow{e_\theta} + dz\,\overrightarrow{e_z}} \tag{3.10}$$

and in the polar coordinate system as:

$$\boxed{d\overrightarrow{OM} = dr\,\overrightarrow{e_r} + r\,d\theta\,\overrightarrow{e_\theta}} \tag{3.11}$$

In the spherical coordinate system

We have already seen that the position vector in spherical coordinates is written as: $\overrightarrow{OM} = r\,\overrightarrow{e_r}$.

The differential position vector in spherical coordinates is then written as:

$$\boxed{d\overrightarrow{OM} = \frac{\partial\overrightarrow{OM}}{\partial r}dr + \frac{\partial\overrightarrow{OM}}{\partial\theta}d\theta + \frac{\partial\overrightarrow{OM}}{\partial\varphi}d\varphi} \tag{3.12}$$

However, the unit vector $\overrightarrow{e_r}$ depends on the coordinates θ and φ, and it is written as:

$$\overrightarrow{e_r} = sin\theta\ cos\varphi\,\vec{\imath} + sin\theta\ sin\varphi\,\vec{\jmath} + cos\theta\,\vec{k}$$

So, we get:

$$\boxed{d\overrightarrow{OM} = dr\,\overrightarrow{e_r} + r\,d\theta\,\overrightarrow{e_\theta} + r\sin\theta\,d\varphi\,\overrightarrow{e_\varphi}} \tag{3.13}$$

Velocity and Speed

Average velocity

Let $\overrightarrow{M_1M_2}$ be the displacement vector between two positions M_1 and M_2 coinciding, respectively, with the instants t_1 and t_2 ($t_2 > t_1$). The **average velocity** is defined as the ratio between the displacement vector and the time interval, that is:

$$\vec{v}_{av} = \frac{\overrightarrow{M_1M_2}}{t_2 - t_1}$$

We know that : $\overrightarrow{M_1M_2} = \Delta\overrightarrow{OM}$, so:

$$\boxed{\vec{v}_{avg} = \frac{\Delta\overrightarrow{OM}}{\Delta t}} \quad (3.14)$$

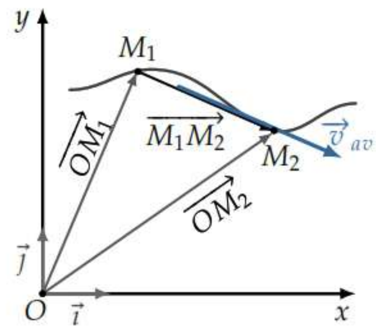


Figure 3.9: Average velocity

> Velocity is a vector quantity, meaning it has both magnitude and direction, similar to displacement.

Average speed

Let M be an object that covers a distance D during a time period Δt. The average speed v_{avg} of this object is defined as the ratio of distance travelled D to the time taken Δt, so:

$$v_{avg} = \frac{D}{\Delta t} > 0.$$

> Speed is a scalar quantity, meaning it only has magnitude and is always positive.

Example 3.2

A cyclist covers the first half of the distance between two places at a speed of 4 m/s and the second half at 6 m/s. Determine its average speed.

Solution

Let the total distance be $2D\ m$. The time taken to travel a distance is given by: $t = \frac{\text{Distance}}{\text{speed}}$. Then, the time taken for the first half $t_1 = D/4$ and the time taken for the second

half $t_2 = D/6$.
Therefore, the time taken to cover the total distance $2D$ is

$$\frac{D}{4} + \frac{D}{6} = \frac{5D}{12}\ s.$$

Hence, $v_{avg} = \frac{\text{Total distance}}{\text{Total time}} = \frac{2D}{\frac{5D}{12}} = 4.8\ m/s.$

Instantaneous velocity

In a reference frame $\mathcal{R}$, the **instantaneous velocity** of a moving object M is always defined at a particular instant t. It is given by the variation of displacement during an infinitesimal interval of time (Δt tends to zero) divided by Δt. It actually corresponds to the following limit:

$$\overrightarrow{v}(t)_{M/\mathcal{R}} = \lim_{\Delta t \to 0} \frac{\Delta \overrightarrow{OM}}{\Delta t}$$

The instantaneous velocity can also be defined simply as the time derivative of the position vector. Thus, we write the instantaneous velocity of M with respect to the reference frame $\mathcal{R}$:

$$\boxed{\overrightarrow{v}(t)_{M/\mathcal{R}} = \lim_{\Delta t \to 0} \frac{\Delta \overrightarrow{OM}}{\Delta t} = \frac{d\,\overrightarrow{OM}}{dt}} \tag{3.15}$$

The instantaneous velocity vector is always **tangent** to the trajectory at the position of M and depends on the chosen reference frame.
The expression for the instantaneous velocity vector can also be obtained simply by dividing the elemental displacement vector by dt.

Instantaneous velocity in the Cartesian coordinates

Taking into account the fact that the unit vectors of the Cartesian base are fixed, we obtain:

$$\boxed{\overrightarrow{v}_{M/\mathcal{R}} = \frac{d\,\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}} \tag{3.16}$$

To facilitate writing, we use **Newton's notation**:

$$\boxed{\overrightarrow{v}_{M/\mathcal{R}} = \dot{x}\,\vec{\imath} + \dot{y}\,\vec{\jmath} + \dot{z}\,\vec{k}} \tag{3.17}$$

Instantaneous velocity in the cylindrical polar coordinates

In the cylindrical coordinates:

$$\boxed{\overrightarrow{v}_{M/\mathcal{R}} = \dot{\rho}\,\overrightarrow{e_\rho} + \rho\,\dot{\theta}\,\overrightarrow{e_\theta} + \dot{z}\,\overrightarrow{e_z}} \tag{3.18}$$

In the polar coordinates:

$$\boxed{\overrightarrow{v}_{M/\mathcal{R}} = \dot{r}\,\overrightarrow{e_r} + r\,\dot{\theta}\,\overrightarrow{e_\theta}} \tag{3.19}$$

Instantaneous velocity in the spherical coordinates

The expression of the instantaneous velocity vector in spherical coordinates is given by:

$$\boxed{\overrightarrow{v}_{M/\mathcal{R}} = \dot{r}\,\overrightarrow{e_r} + r\,\dot{\theta}\,\overrightarrow{e_\theta} + r\,\sin\theta\,\dot{\varphi}\,\overrightarrow{e_\varphi}} \tag{3.20}$$

Velocity in Frenet basis

The velocity vector in this basis is written as: $\overrightarrow{v} = v\,\overrightarrow{e_t}$ where $v = \dfrac{ds}{dt} = \dot{s}$ (ds: it is an elementary displacement).

Instantaneous speed

In simple terms, instantaneous speed refers to the magnitude of instantaneous velocity.

Example 3.3

A car is moving along the x-axis, and its motion is described by the equation $x = 4t^2 - 10$. Determine the instantaneous velocity at time $t = 2\,s$ and deduce its instantaneous speed at that moment.

Solution

As the car travels along the x-axis, its position vector can be written as:

$$\overrightarrow{OM} = x\vec{\imath} = \left(4t^2 - 10\right)\vec{\imath}$$

Then, the instantaneous velocity is:

$$\overrightarrow{v} = \frac{d\,\overrightarrow{OM}}{dt} = 8t - 10\vec{\imath}$$

At $t = 2\,s$: $\overrightarrow{v}(2) = 6\vec{\imath}$.
Therefore,

$$v_{avg} = \|6\vec{\imath}\| = 6.$$

Acceleration

In most movements, the velocity varies over time, either in magnitude, direction, or both. This variation is characterized by the acceleration $\overrightarrow{a}$ of the moving object.

Average acceleration

Let M_1 and M_2 be the two positions of a moving particle that correspond, respectively, to the moments t_1 and t_2 and to the velocities v_1 and v_2. The average acceleration of this particle between the two moments t_1 and t_2 is equal to the change in velocity $\Delta\overrightarrow{v} = \overrightarrow{v_2} - \overrightarrow{v_1}$ divided by $\Delta t = t_2 - t_1$. Therefore:

$$\boxed{\overrightarrow{a}_{avg} = \frac{\overrightarrow{v_2} - \overrightarrow{v_1}}{t_2 - t_1} = \frac{\Delta\overrightarrow{v}}{\Delta t}} \tag{3.21}$$

Instantaneous acceleration

The instantaneous acceleration of a moving particle M at time t corresponds to the limit of the previous ratio as Δt approaches zero. By definition, we can say that the instantaneous acceleration corresponds to the first derivative of the velocity vector with respect to time, and therefore to the second derivative of the position vector with respect to time. The instantaneous acceleration vector $\overrightarrow{a}(t)$ of M in the reference frame $\mathcal{R}$ is written as:

$$\boxed{\overrightarrow{a}(t)_{M/\mathcal{R}} = \lim_{\Delta t \to 0} \frac{\Delta\overrightarrow{v}}{\Delta t} = \frac{d\,\overrightarrow{v}}{dt} = \frac{d^2\,\overrightarrow{OM}}{dt^2}} \tag{3.22}$$

Acceleration in the Cartesian coordinates

$$\boxed{\overrightarrow{a}_{M/\mathcal{R}} = \ddot{x}\,\vec{\imath} + \ddot{y}\,\vec{\jmath} + \ddot{z}\,\vec{k}} \tag{3.23}$$

Acceleration in the cylindrical polar coordinates

In the cylindrical coordinates:

$$\boxed{\overrightarrow{a}_{M/\mathcal{R}} = \left(\ddot{\rho} - \rho\,\dot{\theta}^2\right)\overrightarrow{e_\rho} + \left(2\dot{\rho}\,\dot{\theta} + \rho\,\ddot{\theta}\right)\overrightarrow{e_\theta} + \ddot{z}\,\overrightarrow{e_z}} \tag{3.24}$$

In the polar coordinates:

$$\boxed{\overrightarrow{a}_{M/\mathcal{R}} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\overrightarrow{e_r} + \left(2\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\overrightarrow{e_\theta}} \tag{3.25}$$

Acceleration in the spherical coordinates

$$\boxed{\begin{aligned}\overrightarrow{a}_{M/\mathcal{R}} = {} & \left(\ddot{r} - r\,\dot{\theta}^2 - r\,\sin^2\theta\,\dot{\varphi}^2\right)\overrightarrow{e_r} \\ & + \left(2\dot{r}\,\dot{\theta} + r\,\ddot{\theta} - r\,\sin\theta\cos\theta\,\dot{\varphi}^2\right)\overrightarrow{e_\theta} \\ & + \left(2\dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta\right)\overrightarrow{e_\varphi}\end{aligned}} \tag{3.26}$$

! It should be noted that there is no simple expression for acceleration in spherical coordinates, and this expression is rarely used.

Acceleration in Frenet basis

$$\begin{aligned}\overrightarrow{a} &= \frac{d\overrightarrow{v}}{dt} = \frac{d\left(v\,\overrightarrow{e_t}\right)}{dt} = \frac{dv}{dt}\,\overrightarrow{e_t} + v\,\frac{d\,\overrightarrow{e_t}}{dt} \\ &= \frac{dv}{dt}\,\overrightarrow{e_t} + v\,\frac{d\,\overrightarrow{e_t}}{ds}\,\frac{ds}{dt} = \frac{dv}{dt}\,\overrightarrow{e_t} + v^2\,\frac{d\,\overrightarrow{e_t}}{ds}\end{aligned}$$

We know that: $\overrightarrow{e_n} = \dfrac{d\,\overrightarrow{e_t}}{d\alpha}$[1] and $d\alpha = \dfrac{ds}{R}$, therefore:

1: Derivative of the unit vector.

$$\overrightarrow{e_n} = R\,\frac{d\,\overrightarrow{e_t}}{ds} \Rightarrow \frac{d\,\overrightarrow{e_t}}{ds} = \frac{1}{R}\,\overrightarrow{e_n}$$

Thus:

$$\boxed{\overrightarrow{a} = \left(\frac{dv}{dt}\right)\overrightarrow{e_t} + \left(\frac{v^2}{R}\right)\overrightarrow{e_n}} \tag{3.27}$$

We simply write:

$$\boxed{\overrightarrow{a} = \overrightarrow{a_t} + \overrightarrow{a_n}} \tag{3.28}$$

- $\overrightarrow{a_t}$: The tangential acceleration.
- $\overrightarrow{a_n}$: The normal acceleration.

Example 3.4

A particle M is in motion, described in polar coordinates by the following equations:

$$\begin{cases} r(t) = \lambda\, t^2 \\ \theta(t) = \omega\, t \end{cases} \qquad \lambda \text{ and } \omega \text{ are positive constants.}$$

Find the components of the velocity and acceleration vectors in polar coordinates.

Solution

$$\overrightarrow{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dr}{dt}\overrightarrow{e_r} + r\frac{d\theta}{dt}\overrightarrow{e_\theta} = \dot{r}\,\overrightarrow{e_r} + r\,\dot{\theta}\,\overrightarrow{e_\theta}$$
$$= (2\lambda t)\overrightarrow{e_r} + (\omega\lambda t^2)\overrightarrow{e_\theta}$$

$$\overrightarrow{a} = (\ddot{r} - r\dot{\theta}^2)\overrightarrow{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\overrightarrow{e_\theta}$$

$$\overrightarrow{a} = \frac{d\overrightarrow{v}}{dt} = \frac{d}{dt}\left[(2\lambda t)\overrightarrow{e_r} + (\omega\lambda t^2)\overrightarrow{e_\theta}\right]$$
$$= (2\lambda - \lambda\omega^2 t^2)\overrightarrow{e_r} + (4\lambda\omega t)\overrightarrow{e_\theta}$$

Example 3.5

An object point M is moving according to the following Cartesian coordinate equations:

$$\begin{cases} x(t) = b\ \cos\theta \\ y(t) = b\ \sin\theta \\ z(t) = h\ \theta \end{cases}$$

where, b and h are positive constants and θ is a function of time. In the cylindrical system, determine the velocity and acceleration vectors as a function of $\dot{\theta}$ and $\ddot{\theta}$.

Solution

We know that: $\overrightarrow{v} = \dot{\rho}\,\overrightarrow{e_\rho} + \rho\,\dot{\theta}\,\overrightarrow{e_\theta} + \dot{z}\,\overrightarrow{e_z}$

$$\rho = \sqrt{x^2 + y^2} = b \Rightarrow \dot{\rho} = 0, \text{so} : \overrightarrow{v} = b\,\dot{\theta}\,\overrightarrow{e_\theta} + h\dot{\theta}\,\overrightarrow{e_z}$$

$$\begin{aligned} \overrightarrow{a} &= \frac{d\overrightarrow{v}}{dt} = \frac{d}{dt}\left[b\,\dot{\theta}\,\overrightarrow{e_\theta} + h\dot{\theta}\,\overrightarrow{e_z}\right] \\ &= (-b\dot{\theta}^2)\overrightarrow{e_\rho} + (b\ddot{\theta})\overrightarrow{e_\theta} + (h\ddot{\theta})\overrightarrow{e_z} \end{aligned}$$

Equations of motion

We can consider that the movement of an object point is known as soon as we are able to locate its position at any given instance. Therefore, we need to know how to express its three coordinates as a function of time. In other words, the movement of an object is known when we have its **equations of motion** (also called **kinematic equations**). These equations are given in the cartesian, cylindrical, and spherical coordinate systems, respectively, as follows:

$$\begin{cases} x(t) \\ y(t) \\ z(t) \end{cases}, \quad \begin{cases} \rho(t) \\ \theta(t) \\ z(t) \end{cases}, \text{and} \quad \begin{cases} r(t) \\ \varphi(t) \\ \theta(t) \end{cases}$$

3.6 Types of Motion

The motion of an object can be classified based on the path it follows (straight or curved) or according to its speed (constant or variable). Based on these classifications, various types of motion can be identified.

According to the path of the moving object:

- Rectilinear motion;
- Curvilinear motion.

According to the speed of the moving object:

- Uniform motion;
- Non-uniform motion.

Rectilinear motion

The motion of an object is said to be **rectilinear** if it corresponds to a displacement along a fixed straight line in the reference frame. In this case, the velocity and acceleration vectors are aligned with the line along which the movement takes place.
The most suitable coordinate system is the one that favors an axis aligned with a straight line. It is therefore natural to choose Cartesian coordinates.

Uniform rectilinear motion

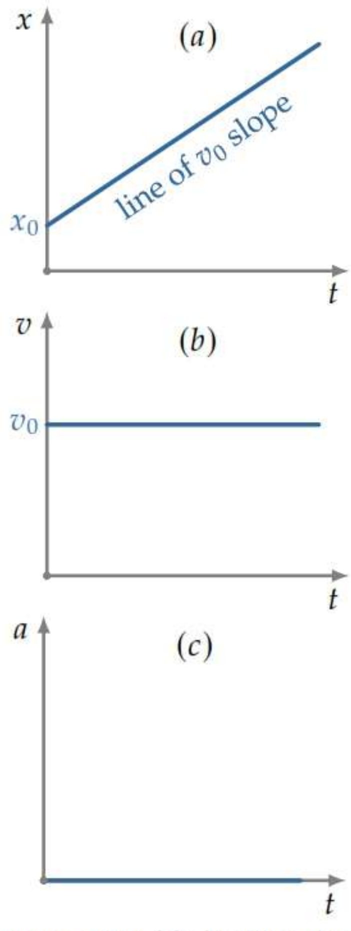


Figure 3.10: (a): Position-time graph, (b): Velocity-time graph, (c): Acceleration-time graph for the uniform motion

It is said that the motion of a material particle M is **uniform rectilinear** (URM), if M moves along a straight line with a **constant velocity**, and therefore its **acceleration is zero**.

If we choose a Cartesian coordinate system with the Ox axis as the axis along which the motion takes place, the equation of motion is written as:

$$\boxed{x(t) = v_0\,(t - t_0) + x_0} \tag{3.29}$$

where $x_0 = x(t_0)$ and $v(t_0) = v_0$ are, respectively, the **initial position** and the **initial velocity** (position and velocity at the initial time t_0).

$$\boxed{v(t) = v(t_0) = v_0 = \text{constant}} \tag{3.30}$$

This equation can be obtained by integrating the acceleration with respect to time, knowing the initial conditions $x(t_0)$ and $v(t_0)$.

$$\frac{dv}{dt} = a \Rightarrow v(t) - v(t_0) = \int_{t_0}^{t} a\,dt = 0$$
$$\Rightarrow v(t) = v(t_0) = v_0 = \text{constant}.$$

$$\frac{dx}{dt} = v(t) \Rightarrow x(t) - x(t_0) = \int_{t_0}^{t} v_0\,dt = v_0(t - t_0)$$
$$\Rightarrow x(t) = v_0(t - t_0) + x_0.$$

The use of the equations mentioned earlier enables us to plot the position-time, velocity-time, and acceleration-time graphs.

Uniformly variable rectilinear motion

We consider a particle M in rectilinear motion in the reference frame $\mathcal{R}$. The motion of M is classified as **Uniformly Variable Rectilinear Motion (UVRM)** if its acceleration remains constant.

By integrating with respect to time, we obtain the equations of motion:

$$\boxed{a(t) = \text{constant}} \tag{3.31}$$

$$\frac{dv}{dt} = a \Rightarrow v(t) - v(t_0) = \int_{t_0}^{t} a\, dt = a(t - t_0).$$

$$\boxed{v(t) = a(t - t_0) + v_0} \tag{3.32}$$

This equation, which relates velocity to time, is called **the first equation** (equation for velocity-time relation).

$$\frac{dx}{dt} = v(t) \Rightarrow x(t) - x(t_0) = \int_{t_0}^{t} v\, dt.$$

$$\boxed{x(t) = \frac{1}{2}a(t - t_0)^2 + v_0(t - t_0) + x_0} \tag{3.33}$$

This equation, which relates position to time, is called **the second equation** (equation for position-time relation).

By eliminating time between $x(t)$ and $v(t)$, it is possible to derive an important relationship that is valid only for this type of motion (Uniformly variable motion).

$$\boxed{(v^2 - v_0^2) = 2a(x - x_0)} \tag{3.34}$$

This equation, relating position to velocity is called **the third equation** (equation for position-velocity relation).

The position-time, velocity-time, and acceleration-time graphs for the uniformly variable rectilinear motion are depicted in Figure 3.11.

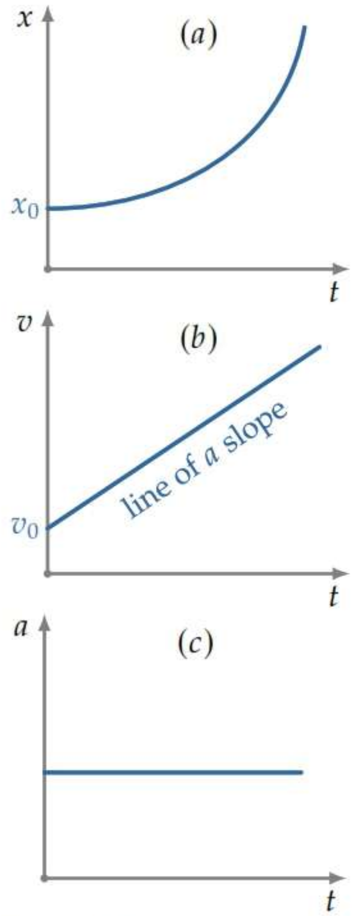


Figure 3.11: (a): Position-time graph, (b): Velocity-time graph, (c): Acceleration-time graph for the uniformly variable rectilinear motion

According to the signs of v and a, and therefore according to the directions of $\overrightarrow{v}$ and $\overrightarrow{a}$, we have two possible types of the uniformly variable rectilinear motion:

- **Uniformly accelerated rectilinear motion (UARM):** The motion of an object point is said to be uniformly accelerated rectilinear motion (slowed up) if it meets the following criteria:

1. Straight line trajectory,
2. Constant acceleration vector,
3. Speed increases with time $\|\overrightarrow{v}\| \nearrow$ or $\overrightarrow{v} \cdot \overrightarrow{a} > 0$.

- **Uniformly decelerated rectilinear motion (UDRM):** The motion of an object point is said to be uniformly accelerated rectilinear motion (slowed down, retarded) if it satisfies the following conditions:
  1. Straight line trajectory,
  2. Constant acceleration vector,
  3. Speed decreases with time $\|\overrightarrow{v}\| \searrow$ or $\overrightarrow{v} \cdot \overrightarrow{a} < 0$.

Distance covered during the n^{th} second

In the uniformly variable motion, the distance traveled by an object point in t second is given by:

$$x(t) = \frac{1}{2}at^2 + v_0 t$$

Using this equation, the distances traveled in n second and in $n - 1$ second are:

$$x(n) = \frac{1}{2}an^2 + v_0 n \tag{i}$$

$$x(n-1) = \frac{1}{2}a(n-1)^2 + v_0(n-1) \tag{ii}$$

Thus, the distance covered by the particle during the n^{th} second is given by:

$$x_{n^{th}} = x(n) - x(n-1).$$

$$\therefore \quad x_{n^{th}} = \left[\frac{1}{2}an^2 + v_0 n\right] - \left[\frac{1}{2}a(n-1)^2 + v_0(n-1)\right]$$

Rearranging, the distance covered by the particle during the n^{th} second is:

$$\boxed{x_{n^{th}} = \frac{1}{2}a(2n-1) + v_0} \tag{3.35}$$

Example 3.6

An object moving with constant acceleration covers a distance of 65 m in the 5th second and 105 m in 9th second. How far will it travel in 20 seconds, and what is the distance covered in 20th second?

Solution

We know that the distance covered in the nth second can be written as:

$$x_{n^{th}} = \frac{1}{2}a(2n-1) + v_0.$$

Therefore, in the $5th$ second, we have:

$$\frac{9a}{2} + v_0 = 65 \quad (1)$$

Similarly, in the $9th$ second, we have:

$$\frac{17a}{2} + v_0 = 105 \quad (2)$$

Solving equations (1) and (2), we get:

$$a = 10\, m/s^2.$$

Substituting this value in (1), we get:

$$v_0 = 20\, m/s.$$

Therefore, the distance covered in 20 seconds

$$x(20) = \frac{1}{2}at^2 + v_0 t = \frac{1}{2} \times 10 \times 20^2 + 20 \times 20 = 2400\, m.$$

Displacement during $20th$ second alone

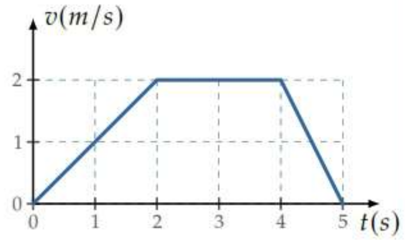
$$x_{20} = \frac{a}{2}\left[(2 \times 20) - 1\right] + v_0 = \frac{10}{2} \times 39 + 20 = 215\, m.$$

Study of motion based on the velocity-time graph

The velocity-time diagram allows us to analyze the motion of a moving object and identify its characteristics using **the area method**.

The area method

The position of the moving object at a specific time t can be determined by calculating the algebraic sum of the areas formed by the velocity curve and the abscissa axis from the origin to time t. The same approach applies when calculating the distance traveled between two times; however, in this case, the distance is equal to the absolute sum of the areas delimited by the velocity curve and the abscissa axis from times t_1 to t_2. Additionally, we can calculate the object's acceleration, establish the equation of motion, represent the position, velocity, and acceleration vectors, draw an acceleration diagram, and define the phases of motion and their natures, among other analyses.



Application exercise

The accompanying figure illustrates the speed-time graph of a mobile object moving in a straight line, starting from an initial position of $0m$. The equations of motion for this mobile object are as follows:

$$\begin{cases} x(t) = 0.5\,t^2 & \text{in the time interval } [0, 2s]\,, \\ x(t) = 2\,t - 2 & \text{in the time interval } [2, 4s]\,, \\ x(t) = -t^2 + 10\,t - 18 & \text{in the time interval } [4, 5s]\,. \end{cases}$$

I- Using the speed-time graph:

1. Calculate the object's acceleration.
2. Determine the distance traveled by the object between $2s$ and $4s$.
3. What is the position of the mobile object at $4s$.
4. Provide the different phases of the movement and their natures. Justify.
5. Represent the position and velocity vectors at $t = 4s$.

II- Answer the questions 1, 2, and 3 using the provided equations of motion.

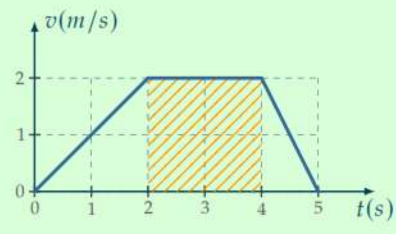
Solution

I.1. we know that: $a = \dfrac{\Delta v}{\Delta t}$.

* $0\,s \leqslant t \leqslant 2\,s : a_1 = 1\ m.s^{-2}$.
* $2\,s \leqslant t \leqslant 4\,s : a_2 = 0\ m.s^{-2}$.
* $4\,s \leqslant t \leqslant 5\,s : a_3 = -2\ m.s^{-2}$.

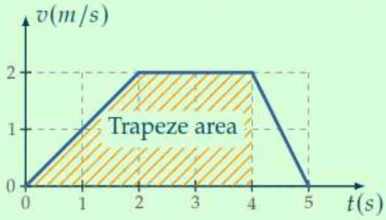
I.2. To determine the distance traveled $\mathscr{D}$, we use the area method:

$$\begin{aligned} \mathscr{D}_{2\to 4} &= \left|(\text{Area of rectangle})_{2\to 4}\right| \\ &= |2 \times 2| = 4\,\frac{m}{s} \times s = 4\ m. \end{aligned}$$



I.3. To determine the position $x(4)$, we use the area method:

$$x(4) = \text{Trapeze area} = \frac{(4+2)\times 2}{2} = 6\,\frac{m}{s} \times s = 6\,m.$$



I.4. * **$0\,s \leqslant t \leqslant 2\,s$:**

In this interval, the nature of motion is Uniformly Accelerated Rectilinear Motion (UARM) in the positive direction.

$v \cdot a > 0 \Rightarrow$ accelerated motion

$a = cst \;\Rightarrow$ uniformly

$v > 0 \Rightarrow$ positive direction

* **$2\,s \leqslant t \leqslant 4\,s$:**

In this interval, the nature of motion is Uniform Rectilinear Motion (URM) in the positive direction.

$v = cte \; ou \; a = 0 \Rightarrow$ uniform

$v > 0 \Rightarrow$ positive direction

* **$4\,s \leqslant t \leqslant 5\,s$:**

In this interval, the nature of motion is Uniformly Decelerated Rectilinear Motion (UDRM) in the positive direction.

$v \cdot a < 0 \Rightarrow$ accelerated motion

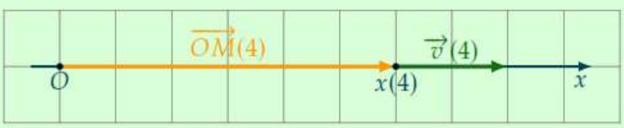
$a = cst \;\Rightarrow$ uniformly

$v > 0 \Rightarrow$ positive direction

I.5. To represent the position and velocity vectors, we first need to find the position of the mobile object at the specified time ($t = 4s$). This position represents the initial point of the velocity vector and serves as the final point of the position vector. Next, we need to calculate the magnitude of the velocity at the indicated time ($v(4)$), which represents the length of the velocity vector.

$x(4) = 6\,m \xrightarrow{1cm \,\longrightarrow\, 1\,m} x(4) = 6\,cm,$

$v(4) = 2\,m.s^{-1} \xrightarrow{1cm \longrightarrow 1\,m/s} \|\overrightarrow{v}(4)\| = 2\,cm.$



II.1. we know that: $a = \dfrac{d^2x}{dt^2}$.

* $0\,s \leqslant t \leqslant 2\,s$: in this interval we have: $x(t) = 0.5\,t^2$,
  $\therefore\ a_1 = 1\ m.s^{-2}$.
* $2\,s \leqslant t \leqslant 4\,s$: in this interval we have: $x(t) = 2\,t - 2$,
  $\therefore\ a_2 = 0\ m.s^{-2}$.
* $4\ s \leqslant t \leqslant 5\ s$: in this interval we have: $x(t) = -t^2 + 10\,t - 18$,
  $\therefore\ a_3 = -2\ m.s^{-2}$.

II.2. To determine the distance traveled $\mathcal{D}$, we first need to calculate $x(2)$ and $x(4)$ using the equation of the second phase of motion: $x(t) = 2\,t - 2$
$\mathcal{D}_{2\to 4} = x(4) - x(2) = 6 - 2 = 4\ m$.

II.3. $x(4) = (2 \times 4) - 2 = 6\ m$.

Sinusoidal rectilinear motion

Let's consider a moving particle, denoted as M, along the Ox axis. We say that its movement is sinusoidal, also called **Simple Harmonic Motion** (SHM), if the coordinate x of M can be expressed as:

$$\boxed{x(t) = A\cos(\omega t + \varphi)} \quad (3.36)$$

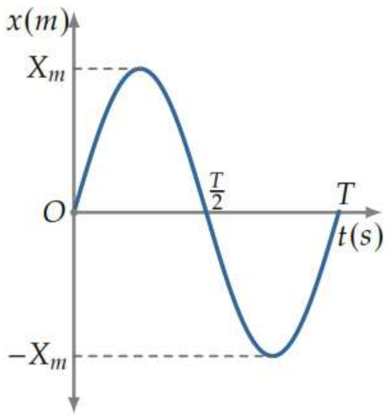


Figure 3.12: Representation of the harmonic motion

where:

- A: is the amplitude of motion,
- $(\omega t + \varphi)$: is the phase at instant t,
- φ: is the initial phase (at $t = 0$),
- ω: is the angular frequency.

The most typical examples of simple harmonic motion are simple pendulum concepts and spring-mass system. In SHM, speed and acceleration are also sinusoidal functions of time.

$$\boxed{v(t) = -\omega A\sin(\omega t + \varphi) = \omega A\cos\left(\omega t + \varphi + \frac{\pi}{2}\right)} \quad (3.37)$$

$$\boxed{\begin{aligned} a(t) &= -\omega^2 A\cos(\omega t + \varphi) = \omega^2 A\cos(\omega t + \varphi + \pi) \\ &= -\omega^2 x(t) \end{aligned}} \quad (3.38)$$

Period and frequency

In simple harmonic motion (SHM), the **period (T)** is defined as the time required to complete one oscillation. It is typically expressed in seconds.

The frequency of an event is a concept closely related to its period. **Frequency (f)** is defined as the number of events occurring per unit of time. In the context of periodic motion, frequency specifically refers to the number of oscillations that occur within a given time frame. Consequently, frequency is the inverse of the period, and its SI unit is hertz (Hz).

In the absence of friction, the period is constant, so the angular frequency is defined as: $\omega = \frac{2\pi}{T}$.

Curvilinear motion

Curvilinear motion of an object refers to movement in which the trajectory is any curve different from a straight line.

Curvilinear abscissa, velocity, and acceleration

We consider a moving point object M on a directed curve $(\mathscr{C})$. We denote P and Q the initial and final positions of M, respectively. The **curvilinear abscissa** of point M on curve $(\mathscr{C})$ is defined as the algebraic number S, which represents the length of the curvilinear arc $\overset{\frown}{PQ}$.

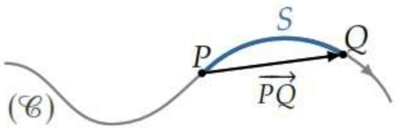


Figure 3.13: Curvilinear abscissa

Curvilinear motion can be modeled as a circle or arc of radius R, which depends on time (variable), with center A and an angle α expressed in radians (see Figure 3.14). The curvilinear abscissa S intercepted by the angle α is given by: $S = R\,\alpha$.

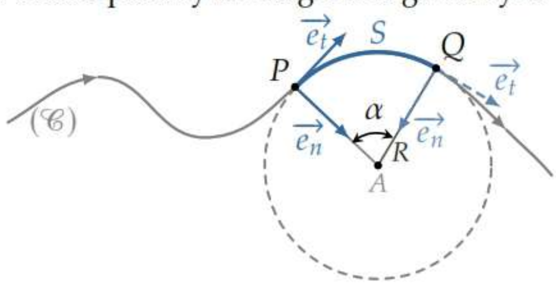


Figure 3.14: Representation of curvilinear motion

The velocity and acceleration are written as:

$$v(t) = \frac{d\,S(t)}{dt} \quad \text{and} \quad a(t) = \frac{d\,v(t)}{dt}.$$

The distance R that relates the position Q to the center of curvature A at a given instant t is called the **radius of curvature** at t.

$$\boxed{R = \frac{v^2}{a_n} = \frac{v^3}{\|\overrightarrow{a} \times \overrightarrow{v}\|}} \tag{3.39}$$

Uniform circular motion

A movement of a particle M is said to be uniform circular motion (UCM) if M moves in a **circle** (or a circular arc) of radius R with **constant speed**.

In this type of motion, the velocity changes direction continuously and is therefore not constant. This explains the existence of a non-zero normal acceleration: $a_N = \dfrac{v^2}{R}$.

Since the tangential acceleration a_t is zero, the acceleration vector can be written as: $\overrightarrow{a} = a_n\,\overrightarrow{e_n}$. This vector is normal to the trajectory at any time and is called a **centripetal acceleration** (directed towards the center of the circle).

The most suitable coordinates for describing circular motion are **polar coordinates**.

Let an object point M move from P to Q along a circumference of a circle with center C and radius R, as shown in Figure 3.15. In the polar coordinate system, the equations of motion for uniform circular motion can be written as:

$$\begin{cases} r = R \\ \theta(t) = (Ox, \overrightarrow{OM}) \end{cases}$$

where θ represents the **angular displacement**.

The position vector is expressed as: $\overrightarrow{OM} = R\,\overrightarrow{e_r}$

The velocity is obtained by differentiating the position vector with respect to time, that is:

$$\overrightarrow{v} = \frac{d\,\overrightarrow{OM}}{dt} = R\,\frac{d\,\overrightarrow{e_r}}{dt} = R\,\frac{d\,\overrightarrow{e_r}}{d\theta}\,\frac{d\theta}{dt} = R\,\dot{\theta}\,\overrightarrow{e_\theta}$$

$$\|\overrightarrow{v}\| = v = R\,\dot{\theta} = R\,\omega$$

where $\dot{\theta} = \dfrac{d\theta}{dt} = \omega$ represents the instantaneous **angular velocity** in radians per second (rad/s).
The relation between **linear velocity** v and **angular velocity** ω is given by $v = R\,\omega$.

The use of the curvilinear abscissa S yields the same result, that is:

$$S = R\,\theta \Rightarrow \frac{dS}{dt} = R\frac{d\theta}{dt} = R\,\dot{\theta} = v$$

$$\boxed{\overrightarrow{v} = R\,\dot{\theta}\,\overrightarrow{e_\theta}} \tag{3.40}$$

If the angular velocity is constant, the movement is said to be **uniform**.

To obtain the linear acceleration vector, we differentiate the previously found velocity, so:

$$\begin{aligned}\vec{a} = \frac{d\vec{v}}{dt} = R\,\frac{d\left(\dot{\theta}\,\vec{e_\theta}\right)}{dt} &= R\left(\ddot{\theta}\,\vec{e_\theta} + \dot{\theta}\,\frac{d\vec{e_\theta}}{dt}\right)\\ &= R\left(\ddot{\theta}\,\vec{e_\theta} + \dot{\theta}\left(\frac{d\vec{e_\theta}}{d\theta}\frac{d\theta}{dt}\right)\right)\\ &= R\left(\ddot{\theta}\,\vec{e_\theta} + \dot{\theta}\left(-\vec{e_r}\,\dot{\theta}\right)\right)\end{aligned}$$

$$\boxed{\vec{a} = -R\,\dot{\theta}^2\,\vec{e_r} + R\,\ddot{\theta}\,\vec{e_\theta}} \tag{3.41}$$

Here, $\ddot{\theta} = \dfrac{d\omega}{dt} = \alpha$ is called the instantaneous **angular acceleration** in units of radians per second squared (rad/s^2). It is related to the linear acceleration as: $a = R\,\alpha$.

In uniform circular motion, the angular velocity $\dot{\theta}$ is constant, therefore, the angular acceleration is zero ($\ddot{\theta} = 0$), so the acceleration vector is written as: $\vec{a} = -R\,\dot{\theta}^2\,\vec{e_r}$

This acceleration is **radial** and also called **centripetal acceleration**.

The kinematic equations that describe the uniform circular motion can be given in terms of angular variables as follows:

$$\boxed{\theta(t) = \omega_0\,(t - t_0) + \theta_0} \tag{3.42}$$

where $\theta_0 = \theta(t_0)$ and $\omega(t_0) = \omega_0$ are, respectively, the **initial position** and the **initial velocity** (position and velocity at the initial time t_0).

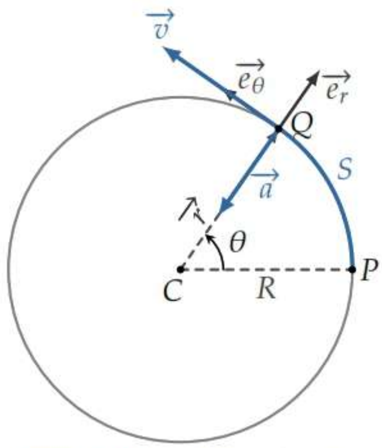


Figure 3.15: Velocity and acceleration for uniform circular motion

Non-uniform circular motion

A circular motion is said to be non-uniform if its speed is not constant. In non-uniform circular motion, the acceleration has both **radial** a_r (also called normal acceleration a_n) and **tangential** a_t components:

$$a_r = a_n = R\,\omega^2 = \frac{v^2}{R} \quad \text{and} \quad a_t = R\,\alpha = \frac{dv}{dt}$$

Uniformly accelerated circular motion

If the angular acceleration of non-uniform circular motion is constant ($\alpha = \ddot{\theta} = Cte$), the motion is said to be **Uniformly Accelerated Circular Motion** (UACM). It can be accelerated or decelerated:

- **Uniformly accelerated circular motion:**
  1. Circular trajectory,
  2. Constant angular acceleration,
  3. $\omega \cdot \alpha > 0$.
- **Uniformly decelerated circular motion:**
  1. Circular trajectory,
  2. Constant angular acceleration,
  3. $\omega \cdot \alpha < 0$.

The kinematic equations of the uniformly accelerated circular motion can be written as:

$$\boxed{\omega(t) = \alpha\,(t - t_0) + \omega_0} \tag{3.43}$$

$$\boxed{\theta(t) = \frac{1}{2}\,\alpha\,(t - t_0)^2 + \omega_0\,(t - t_0) + \theta_0} \tag{3.44}$$

$$\boxed{\omega^2 - \omega_0^2 = 2\,\alpha\,(\theta - \theta_0)} \tag{3.45}$$

Example 3.7

Study of a simple pendulum motion:

Consider a simple pendulum consisting of an object M attached to a string of length ℓ, with the other end fixed at point O. The position of M in the reference frame $\mathcal{R}$ is then determined by its angular displacement θ as shown in Figure 3.16. When the oscillations have small amplitudes, we observe that the polar angle θ is such that $\theta(t) = \theta_0 \sin \omega t$, with the choice of the time origin being when M passes through point P.

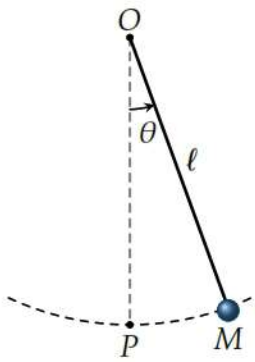
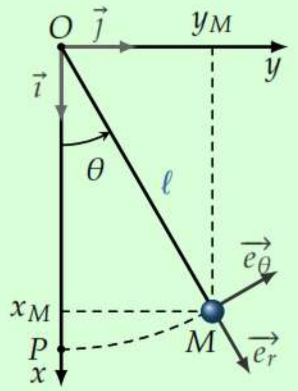


Figure 3.16: Simple pendulum

1. Define on a diagram the local polar base. Give the expression of the position, velocity, and acceleration vectors on this base.
2. Define the Cartesian base and give the expression of the position vector on this base.

Solution

1. The polar base is shown in the figure below.
   We have: $\overrightarrow{OM} = \ell\, \overrightarrow{e_r}$
   $\overrightarrow{v} = \ell\, \dot{\theta}\, \overrightarrow{e_\theta} = (\ell\, \theta_0\, \omega\, cos\omega t)\, \overrightarrow{e_\theta}$
   $\overrightarrow{a} = -\ell\, \dot{\theta}^2\, \overrightarrow{e_r} + \ell\, \ddot{\theta}\, \overrightarrow{e_\theta}$
   $= (-\ell\, \theta_0^2\, \omega^2\, cos^2\omega t)\, \overrightarrow{e_r} - (\ell\, \theta_0\, \omega^2\, sin\omega t)\, \overrightarrow{e_\theta}$
2. On the Cartesian basis defined below, we have the following:
   $\overrightarrow{OM} = x_M \vec{\imath} + y_M \vec{\jmath} = (\ell\, cos\theta)\vec{\imath} + (\ell\, sin\theta)\vec{\jmath}$
   $\theta = \theta_0\, sin\omega t$, therefore:
   $\overrightarrow{OM} = [\ell\, cos(\theta_0\, sin\omega t)]\vec{\imath} + [\ell\, sin(\theta_0\, sin\omega t)]\vec{\jmath}$.



3.7 Relative motion

Introduction

The nature of an object's motion always depends on the chosen frame of reference. For example, the trajectory of a ball in free fall inside a train moving in a straight line with a constant speed is a vertical straight line with respect to the reference frame attached to the train, while its trajectory is parabolic with respect to the frame of reference attached to the Earth.

Definitions

Let $\mathcal{R}$ be a fixed reference frame associated with the $(\vec{\imath}, \vec{\jmath}, \vec{k})$ basis, and let $\mathcal{R}_1$ be a reference frame in motion relative to $\mathcal{R}$, associated with the $(\vec{\imath}_1, \vec{\jmath}_1, \vec{k}_1)$ basis.

The fixed frame of reference $\mathcal{R}$ is called the **absolute reference frame**, and x, y, and z are the components of the object point studied in this frame of reference.

The moving frame of reference $\mathcal{R}_1$ is called the **relative reference frame**, and x_1, y_1, and z_1 are the components of the object point studied in this frame of reference.

Let M be a particle in arbitrary motion. The motion of M relative to the fixed frame $\mathcal{R}$ is called **absolute motion**.

The motion of M relative to the moving frame $\mathcal{R}_1$ is called **relative motion**.

The motion of the frame $\mathcal{R}_1$ relative to $\mathcal{R}$ is called **portable motion**.

Position vectors

The position vector in the absolute reference frame $\mathcal{R}$ and in the relative reference frame $\mathcal{R}_1$ can be written as follows:

$$\overrightarrow{OM} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}$$
$$\overrightarrow{O_1M} = x_1\,\overrightarrow{i_1} + y_1\,\overrightarrow{j_1} + z_1\,\overrightarrow{k_1}$$

To find the relationship between these two vectors, it is sufficient to apply the Chasles' relation (see Figure 3.17):

$$\boxed{\overrightarrow{OM} = \overrightarrow{OO_1} + \overrightarrow{O_1M}} \tag{3.46}$$

This is the relation of change of basis (of reference frame).

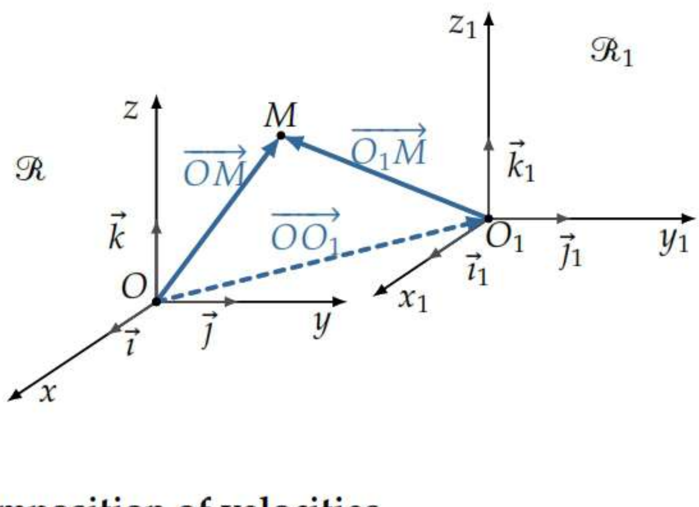


Figure 3.17: Link between position vectors

Composition of velocities

To obtain the expression of the velocity vector of point M in the absolute reference frame $\mathcal{R}$ (**law of composition of velocities**), we must differentiate the previously found position vector with respect to time in the absolute reference frame $\mathcal{R}$, yielding:

$$\begin{aligned}
\vec{v}_{(M/\mathcal{R})} &= \left.\frac{d\,\overrightarrow{OM}}{dt}\right|_{\mathcal{R}} = \left.\frac{d\,\overrightarrow{OO_1}}{dt}\right|_{\mathcal{R}} + \left.\frac{d\,\overrightarrow{O_1M}}{dt}\right|_{\mathcal{R}} \\
&= \left.\frac{d\,\overrightarrow{OO_1}}{dt}\right|_{\mathcal{R}} + \frac{dx_1}{dt}\vec{i_1} + x_1\frac{d\vec{i_1}}{dt} + \frac{dy_1}{dt}\vec{j_1} \\
&\qquad + y_1\frac{d\vec{j_1}}{dt} + \frac{dz_1}{dt}\vec{k_1} + z_1\frac{d\vec{k_1}}{dt} \\
&= \underbrace{\left.\frac{d\,\overrightarrow{OO_1}}{dt}\right|_{\mathcal{R}} + x_1\frac{d\vec{i_1}}{dt} + y_1\frac{d\vec{j_1}}{dt} + z_1\frac{d\vec{k_1}}{dt}}_{\vec{v_e}} \\
&\qquad + \underbrace{\frac{dx_1}{dt}\vec{i_1} + \frac{dy_1}{dt}\vec{j_1} + \frac{dz_1}{dt}\vec{k_1}}_{\vec{v_r}}
\end{aligned}$$

where:
$\vec{v}_{(M/\mathcal{R})} = \vec{v}_a$ is the **Absolute velocity**.
$\vec{v}_r$ is the **Relative velocity**.
$\vec{v}_e$ is the **Portable velocity**.

The law of composition of velocities is then written as:

$$\boxed{\vec{v_a} = \vec{v_e} + \vec{v_r}} \tag{3.47}$$

Special cases:

- If $\mathcal{R}_1$ is fixed relative to $\mathcal{R}$: $\vec{v_e} = \vec{0} \Rightarrow \vec{v_a} = \vec{v_r}$.
- If $\mathcal{R}_1$ is in translational motion relative to $\mathcal{R}$:

$$\vec{v_e} = \frac{d\overrightarrow{OO_1}}{dt}.$$

- If $\mathcal{R}_1$ is only in rotational motion relative to $\mathcal{R}$ with an angular velocity ω:

$$\vec{v_e} = \vec{\omega} \wedge \overrightarrow{O_1M}$$

.

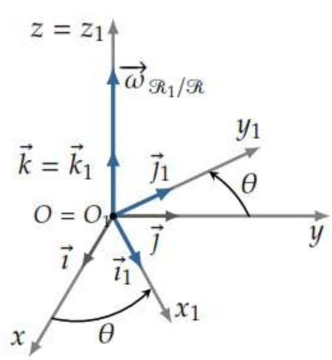


Figure 3.18: Rotational motion of $\mathcal{R}_1$ relative to $\mathcal{R}$

General case

The reference frame $\mathscr{R}_1$ is in both translational and rotational motion with respect to $\mathscr{R}$. The absolute velocity is then expressed as:

$$\boxed{\overrightarrow{v_a} = \underbrace{\dot{x}_1\,\overrightarrow{i_1} + \dot{y}_1\,\overrightarrow{j_1} + \dot{z}_1\,\overrightarrow{k_1}}_{\overrightarrow{v_r}} + \underbrace{\left.\frac{d\,\overrightarrow{OO_1}}{dt}\right|_{\mathscr{R}} + \overrightarrow{\omega}_{(\mathscr{R}_1/\mathscr{R})} \wedge \overrightarrow{O_1M}}_{\overrightarrow{v_e}}} \tag{3.48}$$

Composition of accelerations

Acceleration is obtained by taking the derivative of velocity with respect to time.

$$\overrightarrow{\gamma}_{(M/\mathscr{R})} = \overrightarrow{\gamma}_a = \left.\frac{d\,\overrightarrow{v_a}}{dt}\right|_{\mathscr{R}}.$$

$$\begin{aligned}
\overrightarrow{\gamma}_{(M/\mathscr{R})} &= \frac{d}{dt}\left(\frac{d\,\overrightarrow{OO_1}}{dt} + x_1\frac{d\,\overrightarrow{i_1}}{dt} + y_1\frac{d\,\overrightarrow{j_1}}{dt} + z_1\frac{d\overrightarrow{k_1}}{dt}\right.\\
&\qquad\left.\left. + \frac{dx_1}{dt}\overrightarrow{i_1} + \frac{dy_1}{dt}\overrightarrow{j_1} + \frac{dz_1}{dt}\overrightarrow{k_1}\right)\right|_{\mathscr{R}}\\
&= \underbrace{\frac{d^2\,\overrightarrow{OO_1}}{dt^2} + x_1\frac{d^2\,\overrightarrow{i_1}}{dt^2} + y_1\frac{d^2\,\overrightarrow{j_1}}{dt^2} + z_1\frac{d^2\overrightarrow{k_1}}{dt^2}}_{\overrightarrow{\gamma_e}}\\
&\quad + \underbrace{\frac{d^2x_1}{dt^2}\overrightarrow{i_1} + \frac{d^2y_1}{dt^2}\overrightarrow{j_1} + \frac{d^2z_1}{dt^2}\overrightarrow{k_1}}_{\overrightarrow{\gamma_r}}\\
&\quad \underbrace{+\,2\left(\frac{dx_1}{dt}\frac{d\,\overrightarrow{i_1}}{dt} + \frac{dy_1}{dt}\frac{d\,\overrightarrow{j_1}}{dt} + \frac{dz_1}{dt}\frac{d\overrightarrow{k_1}}{dt}\right)}_{\overrightarrow{\gamma_c}}
\end{aligned}$$

where: $\overrightarrow{\gamma}_{(M/\mathscr{R})} = \overrightarrow{\gamma}_a$ is the **Absolute acceleration**.

$\overrightarrow{\gamma}_e$ is the **Portable acceleration**.

$\overrightarrow{\gamma}_r$ is the **Relative acceleration**

$\overrightarrow{\gamma}_c$ is called **Coriolis acceleration**.

The law of composition of accelerations is then written as:

$$\boxed{\overrightarrow{\gamma_a} = \overrightarrow{\gamma_e} + \overrightarrow{\gamma_r} + \overrightarrow{\gamma_c}} \tag{3.49}$$

Special cases:

- If $\mathcal{R}_1$ is fixed relative to $\mathcal{R}$: $\overrightarrow{v_e} = \overrightarrow{0} \Rightarrow \overrightarrow{v_a} = \overrightarrow{v_r}$.
- If $\mathcal{R}_1$ is in translational motion relative to $\mathcal{R}$: $\overrightarrow{v_e} = \dfrac{d\overrightarrow{OO_1}}{dt}$.
- If $\mathcal{R}_1$ is only in rotational motion relative to $\mathcal{R}$ with an angular velocity ω: $\overrightarrow{v_e} = \overrightarrow{\omega} \wedge \overrightarrow{O_1M}$.
- If $\mathcal{R}_1$ is in translational motion relative to $\mathcal{R}$: $\overrightarrow{\gamma}_a = \dfrac{d^2\overrightarrow{OO_1}}{dt^2} + \overrightarrow{\gamma_r}$.
- The absolute acceleration in the case of rotational motion of $\mathcal{R}_1$ relative to $\mathcal{R}$ with an angular velocity ω is written as follows:

$$\overrightarrow{\gamma_a} = \underbrace{\ddot{x}_1\overrightarrow{i_1} + \ddot{y}_1\overrightarrow{j_1} + \ddot{z}_1\overrightarrow{k_1}}_{\overrightarrow{\gamma_r}} + \underbrace{\dot{\overrightarrow{\omega}} \wedge \overrightarrow{O_1M} + \overrightarrow{\omega} \wedge \left(\overrightarrow{\omega} \wedge \overrightarrow{O_1M}\right)}_{\overrightarrow{\gamma_e}} + \underbrace{2\overrightarrow{\omega} \wedge \overrightarrow{v_r}}_{\overrightarrow{\gamma_c}}$$

General case

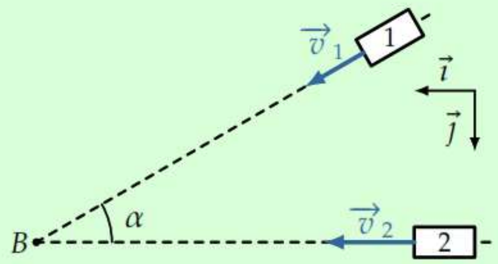
This is the case where the reference frame $\mathcal{R}_1$ is in any kind of motion. The absolute acceleration is then written as follows:

$$\boxed{\overrightarrow{\gamma_a} = \underbrace{\ddot{x}_1\overrightarrow{i_1} + \ddot{y}_1\overrightarrow{j_1} + \ddot{z}_1\overrightarrow{k_1}}_{\overrightarrow{\gamma_r}} + \underbrace{\frac{d^2\overrightarrow{OO_1}}{dt^2} + \dot{\overrightarrow{\omega}} \wedge \overrightarrow{O_1M} + \overrightarrow{\omega} \wedge \left(\overrightarrow{\omega} \wedge \overrightarrow{O_1M}\right)}_{\overrightarrow{\gamma_e}} + \underbrace{2\overrightarrow{\omega} \wedge \overrightarrow{v_r}}_{\overrightarrow{\gamma_c}}} \tag{3.50}$$

Example 3.8

Two cars, labeled 1 and 2, are moving on two roads that form an angle of $60°$ between them. Car 1 is traveling at a speed of $70km/h$ and car 2 is traveling at a speed of

$90 km/h$ in order to reach the same point B (see the Figure below).



1- Find the speed of car 2 relative to car 1.

Solution

- 1^{st} method:
  We have: $\vec{v}_{(2/1)} = \vec{v}_{(2)} - \vec{v}_{(1)}$.
  To calculate $\vec{v}_{(2)} - \vec{v}_{(1)}$, we need to determine the components of each velocity vector in the reference frame (xOy) (see the previous Figure).
  We denote the velocity of car 1 relative to the ground as $\vec{v}_{(1/gnd)}$ and the velocity of car 2 relative to the ground as $\vec{v}_{(2/gnd)}$.

  $\vec{v}_{(1/gnd)} = 90\,\vec{i}$,
  $\vec{v}_{(2/gnd)} = (70\ cos60)\vec{i} + (70\ sin60)\vec{j} = 35\,\vec{i} + 60.6\,\vec{j}$,
  then:

$$\vec{v}_{(2/1)} = 55\,\vec{i} - 60,6\,\vec{j}$$
$$\Rightarrow \|\vec{v}_{(2/1)}\| = 81,85\ km/h$$

- 2^{nd} method:

$$\|\vec{v}_{(2/1)}\| = \sqrt{\begin{array}{l}\|\vec{v}_{(2/gnd)}\|^2 + \|\vec{v}_{(1/gnd)}\|^2 \\ -2\|\vec{v}_{(2/gnd)}\|\|\vec{v}_{(1/gnd)}\|cos\alpha\end{array}}$$

N.A:

$$\begin{aligned}\|\vec{v}_{(2/1)}\| &= \sqrt{(90)^2 + (70)^2 - 2\,(90)\,(70)cos\alpha} \\ &= 81,85\ km/h.\end{aligned}$$

CHAPTER 3 SUMMARY

Coordinate systems

Cartesian coordinates
$\overrightarrow{OM} = x\,\vec{\imath} + y\,\vec{\jmath} + z\,\vec{k}, \quad \vec{v} = \dot{x}\,\vec{\imath} + \dot{y}\,\vec{\jmath} + \dot{z}\,\vec{k}, \quad \vec{a} = \ddot{x}\,\vec{\imath} + \ddot{y}\,\vec{\jmath} + \ddot{z}\,\vec{k}$
Polar coordinates:
$\overrightarrow{OM} = r\,\overrightarrow{e_r}, \quad \vec{v} = \dot{r}\,\overrightarrow{e_r} + r\,\dot{\theta}\,\overrightarrow{e_\theta}, \quad \vec{a} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\overrightarrow{e_r} + \left(2\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\overrightarrow{e_\theta}$
Cylindrical coordinates
$\overrightarrow{OM} = \rho\,\overrightarrow{e_\rho} + z\,\overrightarrow{e_z}, \quad \vec{v} = \dot{\rho}\,\overrightarrow{e_\rho} + \rho\,\dot{\theta}\,\overrightarrow{e_\theta} + \dot{z}\,\overrightarrow{e_z}, \quad \vec{a} = \left(\ddot{\rho} - \rho\,\dot{\theta}^2\right)\overrightarrow{e_\rho} + \left(2\dot{\rho}\,\dot{\theta} + \rho\,\ddot{\theta}\right)\overrightarrow{e_\theta} + \ddot{z}\,\overrightarrow{e_z}$
Spherical coordinates
$\overrightarrow{OM} = r\,\overrightarrow{e_r}, \quad \vec{v} = \dot{r}\,\overrightarrow{e_r} + r\,\dot{\theta}\,\overrightarrow{e_\theta} + r\,\sin\theta\,\dot{\varphi}\,\overrightarrow{e_\varphi},$
$\vec{a} = \left(\ddot{r} - r\,\dot{\theta}^2 - r\,\sin^2\theta\,\dot{\varphi}^2\right)\overrightarrow{e_r} + \left(2\dot{r}\,\dot{\theta} + r\,\ddot{\theta} - r\,\sin\theta\cos\theta\,\dot{\varphi}^2\right)\overrightarrow{e_\theta} + \left(2\dot{r}\dot{\varphi}\sin\theta + r\ddot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta\right)\overrightarrow{e_\varphi}$
Frenet basis
$\vec{a} = \left(\dfrac{dv}{dt}\right)\overrightarrow{e_t} + \left(\dfrac{v^2}{R}\right)\overrightarrow{e_n} = \overrightarrow{a_t} + \overrightarrow{a_n}$

Velocity and acceleration

$$\vec{v}_{avg} = \frac{\Delta\overrightarrow{OM}}{\Delta t}, \quad v_{avg} = \frac{D}{\Delta t} > 0,$$
$$\vec{a}_{avg} = \frac{\overrightarrow{v_2} - \overrightarrow{v_1}}{t_2 - t_1} = \frac{\Delta\vec{v}}{\Delta t}, \quad \vec{a}\,(t) = \frac{d\,\vec{v}}{dt} = \frac{d^2\,\overrightarrow{OM}}{dt^2}$$

Equations of motion

Uniform rectilinear motion
$x(t) = v_0\,(t - t_0) + x_0$
Uniformly variable rectilinear motion
The first equation $v(t) = a(t - t_0) + v_0$
The second equation $x(t) = \dfrac{1}{2}a(t - t_0)^2 + v_0(t - t_0) + x_0$
The third equation $(v^2 - v_0^2) = 2a(x - x_0)$
Distance covered during the n^{th} second
$x_{n^{th}} = \dfrac{1}{2}a(2n - 1) + v_0$
Uniformly accelerated rectilinear motion (UARM)
Straight line trajectory and $\vec{a} = cst$ and $\vec{v}\cdot\vec{a} > 0$ ($\|\vec{v}\| \nearrow$)
Uniformly decelerated rectilinear motion (UDRM)
Straight line trajectory and $\vec{a} = cst$ and $\vec{v}\cdot\vec{a} < 0$ ($\|\vec{v}\| \searrow$)
Uniform circular motion
$\theta(t) = \omega_0\,(t - t_0) + \theta_0$
Non-uniform circular motion
$\theta(t) = \dfrac{1}{2}\,\alpha\,(t - t_0)^2 + \omega_0\,(t - t_0) + \theta_0, \quad \omega(t) = \alpha\,(t - t_0) + \omega_0, \quad \omega^2 - \omega_0^2 = 2\,\alpha\,(\theta - \theta_0)$

Relative motion

Composition of velocities
$\overrightarrow{v_a} = \overrightarrow{v_e} + \overrightarrow{v_r}$
Composition of accelerations
$\overrightarrow{\gamma_a} = \overrightarrow{\gamma_e} + \overrightarrow{\gamma_r} + \overrightarrow{\gamma_c}$
Relative velocity of A with respect to B ($\overrightarrow{v_{AB}}$)
$\overrightarrow{v_{AB}} = \overrightarrow{v_A} - \overrightarrow{v_B}$
Relative acceleration of A with respect to B ($\overrightarrow{\gamma_{AB}}$)
$\overrightarrow{\gamma_{AB}} = \overrightarrow{\gamma_A} - \overrightarrow{\gamma_B}$

3.8 Exercises

Exercise 3.1

A cat moves 600 m to the south and then 380 m to the east. What is the displacement of the cat?

Sol : page 81

Exercise 3.2

A supercar goes from 0 to 100 km/h in only 2.3 seconds. Determine its average acceleration.

Sol : page 81

Exercise 3.3

An object point is traveling at a velocity of 6 m/s. After three seconds, its velocity increases to 9 m/s. Calculate its acceleration.

Sol : page 81

Exercise 3.4

The rectilinear motion of a particle is described by the equation:

$$x(t) = t^3 - 4t^2 - 1$$

Find:

1. The instantaneous acceleration of the particle at $t = 2\ s$.
2. The average acceleration in the time interval from $t = 2\ s$ to $t = 4\ s$.

Sol : page 81

Exercise 3.5

The cartesian coordinates of a moving point are given by $x(t) = t^2 + 2$ and $y(t) = 3t^2$. Calculate the speed of the point at $t = 4\ s$.

Sol : page 81

Exercise 3.6

A student runs at a constant speed of $10\,\mathrm{m/s}$ to catch a bus that is initially at rest at a bus stop. When the bus begins to move with a constant acceleration of $1\,\mathrm{m/s^2}$, the student is 50 m behind the bus. Is it possible for the student to catch the bus?

Sol : page 81

Exercise 3.7

A body covers a distance of $20m$ in the $7th$ second and $24m$ in the $9th$ second moving with uniform acceleration. How much distance shall it cover in $15th$ second.

Sol : page 82

Exercise 3.8

A body is moving along the negative direction of the x-axis with a constant speed of $2\ m/s$. At time $t = 0$, the particle is located at $x = 7\ m$. Determine the position of the body at $t = 2\ s$.

Sol : page 82

Exercise 3.9

An electron moving at a speed of $5 \times 10^3\ m/s$ enters an electric field that exerts an acceleration of $10^{12}\ m/s^2$.

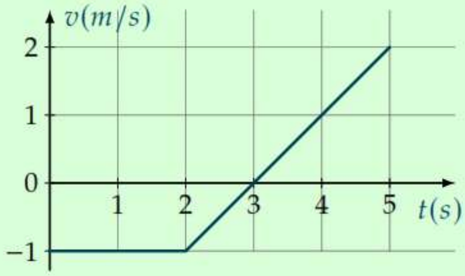
1. Determine the time required for the electron to double its initial speed.
2. Find the distance traveled during this time.

Sol : page 82

Exercise 3.10

Figure below shows the speed-time graph of a mobile moving in a straight line. $x(0) = 0\ m$.

1. Find the position of the mobile at $t = 5\,s$.
2. How many phases are there in its motion.
3. Determine the distance traveled from the origin to $t = 5\,s$.
4. Represent the velocity and acceleration vectors at $t = 5\,s$.

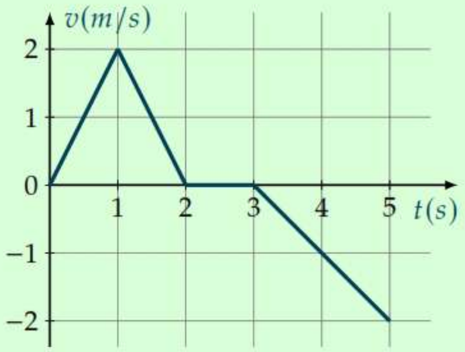


Sol : page 82

Exercise 3.11

A particle is moving along the $x'Ox$ axis. Its velocity-time graph is shown below.

1. Plot the acceleration-time graph between 0 and 5s.
2. Provide the different phases of the movement and their natures. Justify.
3. Provide the first and second equations of motion for the final phase.

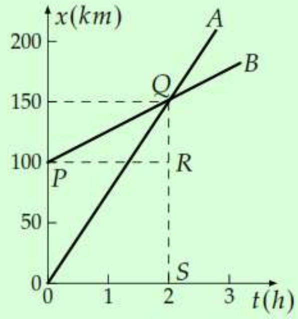


Sol : page 82

Exercise 3.12

The position-time graphs of two trains are shown in the figure below. If the two trains start moving simultaneously in the same direction, determine

1. How much ahead of A and B when the motion starts?
2. What is the speed of B?
3. When and where will A catch B?
4. What is the difference between the speeds of A and B?



Sol : page 83

Exercise 3.13

The motion of an object moving along a circular path with a radius $R = 20\,m$ is described by the equation $S(t) = t^3 + 5$. Determine the acceleration of the object using both the polar and Frenet bases.

Sol : page 84

Exercise 3.14

An aircraft flies at $400\,km/h$ in still air. A wind of $200\sqrt{2}\,km/h$ is blowing from the south.

The pilot's objective is to navigate from point A to a destination point B, which is located to the north-east of point A. The distance between points A and B is $1000\,km$. Determine the direction he must steer and time of his journey.

Sol : page 84

3.9 Solutions

Solution of exercise 3.1 (Page 79)

Suppose that the cat starts from the point O, moves to A to the south and then to B to the east. Therefore, the magnitude of the displacement is:

$$OB = \sqrt{600^2 + 380^2} = 710.2\ m.$$

Solution of exercise 3.2 (Page 79)

As : $v_f = 100\ km/h = 27.77\ m/s$,

$$a_{avg} = \frac{\Delta v}{\Delta t} = \frac{27.77 - 0}{2.3 - 0} = 12\ m/s^2.$$

Solution of exercise 3.3 (Page 79)

$$a = \frac{\Delta v}{\Delta t} = \frac{9-6}{3} = 1\ m/s^2.$$

Solution of exercise 3.4 (Page 79)

1. $a(t) = \dfrac{d^2x}{dt^2} = 6t - 8$

$$\therefore\ a(2) = 4\ m/s^2.$$

2. $a_{avg} = \dfrac{\Delta v}{\Delta t} = \dfrac{v(4) - v(2)}{4-2}$

$$v(t) = \frac{dx}{dt} = 3t^2 - 8t$$

Thus,
$$v(2) = -4\ m/s,$$
$$v(4) = 16\ m/s.$$

Therefore,
$$a_{avg} = \frac{16 - (-4)}{2} = 10\ m/s^2.$$

Solution of exercise 3.5 (Page 79)

We have: $v_x(t) = 2t$ and $v_y(t) = 6t$

Therefore,

$$v(t) = \sqrt{(2t)^2 + (6t)^2} = 2t\sqrt{10}$$

So, the speed at $t = 4$ seconds:

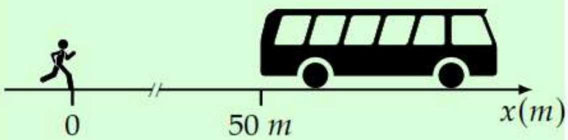
$$v(4) = 2 \cdot 4 \cdot \sqrt{10} = 25.3\ m/s.$$

Solution of exercise 3.6 (Page 79)

The student catches the bus when their positions are equal:

$$x_S(t) = x_B(t)$$

Let us consider the moment when the bus begins to move as the initial time (t=0), with the position of the student at this moment marking the start of the motion (see Figure bellow).



Let's write the equations of motion for both the student and the bus. As:

- The initial velocity of the bus is $v_B(0) = 0\ m/s$.
- The initial distance of the bus is $x_B(0) = 50\ m$.
- The initial velocity of the student is $v_S(0) = 10\ m/s$.
- The initial distance the student is $x_S(0) = 0\ m$.

Thus,

$$x_S(t) = 10\ t$$

$$x_B(t) = \frac{1}{2}t^2 + 50$$

$$x_S(t) = x_B(t) \Leftrightarrow 10\ t = \frac{1}{2}t^2 + 50$$

This quadratic equation has a double root.

$$t = 10\ s.$$

Therefore, the student can catch the bus (after 10s).

Solution of exercise 3.7 (Page 79)

$$x_{7^{th}} = \frac{a}{2}(2\times 7-1)+v_0 = \frac{13a}{2}+v_0 = 20 \qquad (1)$$

Similarly,

$$x_{9^{th}} = \frac{17a}{2}+v_0 = 24 \qquad (2)$$

Solving the system of equations (1) and (2) yields:
$a = 2\,m/s^2$ and $v_0 = 7\,m/s$

$$x_{15^{th}} = \frac{a}{2}(2\times 15-1)+v_0 = 36\,m.$$

Solution of exercise 3.8 (Page 79)

As $v = cst$, the body is in uniform motion.

$$\therefore\ x(t) = vt + x_0$$

The velocity $v = -2\,m/s$ (since the body is moving along the negative direction). Thus,

$$x(2) = (-2)\cdot 2 + 7 = 3\,m.$$

Solution of exercise 3.9 (Page 79)

1. The electron is in uniformly accelerated motion, so:

   $$v = at + v_0$$

   Where: v and v_0 are the final and initial velocities, respectively.
   Thus, $v = 2\,v_0 = 10^4\,m/s$
   Substituting the known values into the equation above, we find:

   $$t_d = 5\times 10^{-9}\,s.$$

2. Suppose that the motion of the electron begins the moment it enters the electric field; consequently, $x(0) = 0\,m$.

   Thus,

   $$D = \frac{1}{2}a\,t_d^2 + v_0\,t_d$$

   Substituting the known values, we find:

   $$D = 3.75\times 10^{-5}\,m.$$

Solution of exercise 3.10 (Page 79)

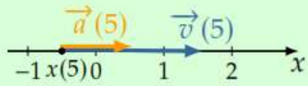
1. Using the area method:

   $$\begin{aligned} x(5) &= \text{trapeze area} + \text{triangle area} \\ &= \frac{(3+2)\times(-1)}{2} + \frac{2\times 2}{2} \\ &= -0.5\,m. \end{aligned}$$

2. We have three phases: $[0,2s]$, $[2,3s]$, and $[3,5s]$.
3. Using the area method:

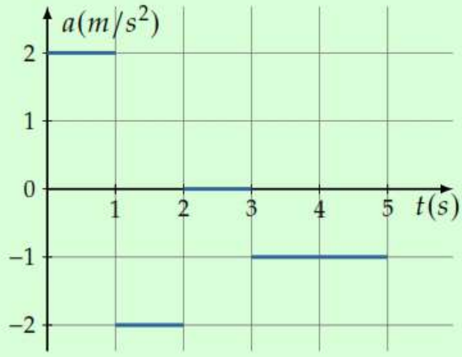
   $$\begin{aligned} D_{0\to 5} &= \left|\text{trapeze area}\right| + \left|\text{triangle area}\right| \\ &= \left|\frac{(3+2)\times(-1)}{2}\right| + \left|\frac{2\times 2}{2}\right| \\ &= 4.5\,m. \end{aligned}$$

4. To represent the velocity and acceleration vectors at $t = 5\,s$, we need: $x(5) = -0.5\,m$, $v(5) = 2\,m/s$, and $a(5) = 1\,m/s^2$.



Solution of exercise 3.11 (Page 80)

1. To plot the acceleration-time graph, we first calculate the acceleration during the various phases of the movement.



2. * $\mathbf{0\,s \leqslant t \leqslant 1\,s:}$
   In this interval, the motion is Uniformly Accelerated Rectilinear (UARM) in the positive direction.
   $v \cdot a > 0 \Rightarrow$ accelerated motion
   $a = cst \ \Rightarrow$ uniformly
   $v > 0 \Rightarrow$ positive direction

   * $\mathbf{1\,s \leqslant t \leqslant 2\,s:}$
   In this interval, the motion is Uniform Decelerated Rectilinear (UDRM) in the positive direction.
   $v \cdot a < 0 \Rightarrow$ accelerated motion
   $a = cst \ \Rightarrow$ uniformly
   $v > 0 \Rightarrow$ positive direction

   * $\mathbf{2\,s \leqslant t \leqslant 3\,s:}$
   In this interval, the mobile is stationary.
   * $\mathbf{3\,s \leqslant t \leqslant 5\,s:}$
   In this interval, the is Uniformly Decelerated Rectilinear Motion (UARM) in the positive direction.
   $v \cdot a > 0 \Rightarrow$ accelerated motion
   $a = cst \ \Rightarrow$ uniformly
   $v > 0 \Rightarrow$ negative direction
3. In the final phase ($[3, 5s]$), we are given:
   - $a = -1\,m/s^2$,
   - $v(3) = 0\,m/s$,
   - $x(3) = 2\,m$.

   First Equation of Motion (Velocity-time relation):

$$a = \frac{dv}{dt} \Rightarrow \int_{v(3)}^{v(t)} dv = \int_3^t a\,dt$$
$$\Leftrightarrow v(t) - v(3) = \int_3^t (-1)\,dt$$
$$\Leftrightarrow v(t) = -t + 3$$

Thus, the first equation of motion is:

$$v(t) = -t + 3.$$

Second Equation of Motion (Position-time relation):

$$v(t) = \frac{dx}{dt} \Rightarrow \int_{x(3)}^{x(t)} dx = \int_3^t v(t)\,dt$$
$$\Leftrightarrow x(t) - x(3) = \int_3^t (-t+3)\,dt$$
$$\Leftrightarrow x(t) - 2 = \int_3^t (-t+3)\,dt$$

This integral give:

$$x(t) = -\frac{1}{2}t^2 + 3t - \frac{5}{2}$$

Thus, the second equation of motion is:

$$x(t) = -\frac{1}{2}t^2 + 3t - \frac{5}{2}.$$

Solution of exercise 3.12 (Page 80)

1. B is ahead of A by the distance $OP = 100\ km$, when the motion starts.
2. $v_B = \dfrac{QR}{PR} = \dfrac{150 - 100}{2 - 0} = 25\ km/h$
3. Since, the two graphs intersect at point Q, so A will catch B after 2 h and at a distance of 150 km from the origin.
4. $v_A = \dfrac{QS}{OS} = \dfrac{150 - 0}{2 - 0} = 75\ km/h$
   Difference in speeds $= 75 - 25 = 50\ km/h$.

Solution of exercise 3.13 (Page 80)

- Using Frenet basis:

$$a(t) = \sqrt{a_t(t)^2 + a_n(t)^2}$$

We know that:

$$a_t = \frac{dv}{dt},\ a_n = \frac{v^2}{R}, \text{and } v = \frac{dS}{dt}$$

Thus,

$$v(t) = 3\,t^2,\ a_t = 6\,t,\ \text{and } a_n = \frac{9}{20}t^4$$

$$\therefore\ a(t) = \sqrt{(6\,t)^2 + \left(\frac{9}{20}\,t^4\right)^2}$$

Thus, $a(2) = 13.99\ m/s^2$.

- Using the polar coordinates: For a circular path, the arc length S is related to the angle θ (in radians) by:

$$S = R\theta$$

Given $R = 20\ m$, we have:

$$\theta(t) = \frac{S(t)}{R} = \frac{t^3+5}{20}$$

Also,

$$r(t) = R$$

We have:

$$\overrightarrow{a(t)} = \left(\ddot{r} - r\,\dot{\theta}^2\right)\overrightarrow{e_r} + \left(2\dot{r}\,\dot{\theta} + r\,\ddot{\theta}\right)\overrightarrow{e_\theta}$$

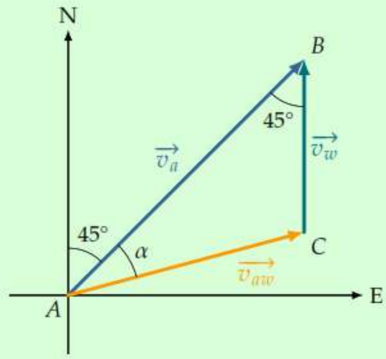
Substituting the derivatives of $r(t)$ and $\theta(t)$ into the relation above:

$$\overrightarrow{a(t)} = \left(\frac{-9}{20}\,t^4\right)\overrightarrow{e_r} + (6\,t)\,\overrightarrow{e_\theta}$$

$\therefore\ a(2) = 13.99\ m/s^2$.

Solution of exercise 3.14 (Page 80)

Given that $v_w = 200\sqrt{2}\,km/h$. $v_{aw} = 400\,km/h$ and v_a should be along AB or in north-east direction. Thus, the direction of v_{aw} should be such as the resultant of v_w and v_{aw} is along AB or in north-east direction.
Let v_{aw} makes an angle α with AB as shown in figure above.



Applying sine law in triangle ABC,

$$\frac{AC}{\sin 45^\circ} = \frac{BC}{\sin\alpha}$$

or

$$\sin\alpha = \left(\frac{BC}{AC}\right)\sin 45^\circ = \left(\frac{200\sqrt{2}}{400}\right)\frac{1}{\sqrt{2}} = \frac{1}{2}$$

$\therefore\quad \alpha = 30^\circ$
Therefore, the pilot should steer in a direction at an angle of $(45^\circ + \alpha)$ or 75° from north towards east.
Further,

$$|v_a| = \frac{\sin 105^\circ}{\sin 45^\circ} \times (400)\,\text{km/h}$$

$$= \left(\frac{\cos 15^\circ}{\sin 45^\circ}\right)(400)\,\text{km/h}$$

$$= \left(\frac{0.9659}{0.707}\right)(400)\,\text{km/h}$$

$$= 546.47\,\text{km/h}$$

$\therefore$ The time of journey from A to B is

$$t = \frac{AB}{|v_a|} = \frac{1000}{546.47}\,\text{h}$$

$$t = 1.83\,\text{h}$$

DYNAMICS OF A PARTICLE 4

4.1 Introduction

In the case of kinematics, we have limited ourselves to the study of kinematic elements, that is, the simple description of a motion (position, velocity, acceleration) in different coordinate systems. We have not previously addressed the question of what causes motion or its origins. This aspect will now be considered. This will lead to introducing the concept of force, along with several laws and principles that relate forces and kinematic elements. Laws or principles are relationships that are not proven; rather, they are validated through experimentation and remain accepted until disproven by subsequent experiments.

4.2 Newton's first law of motion (the principle of inertia)

The inertia

The inertia of a body is its tendency to resist any change in its state of motion or rest. Bodies show the same resistance to slowing down as they do to speeding up. It should be noted that the concept of inertia is related to the concept of mass, which is the measure of a body's inertia.

Isolated system and pseudo-isolated system

A system, such as an object point M, is considered isolated when it does not experience any external actions (forces). In practice, achieving an isolated system is very difficult. This would necessitate placing the object in space, at a considerable distance from any other objects. When conducting experiments on or near Earth, researchers often have to settle for **pseudo-isolated systems**.
A system is said to be pseudo-isolated if all external actions

cancel each other out (the vector sum of the forces acting on the system is zero).

The principle of inertia

Building on the work of Galileo and the French philosopher René Descartes, Newton formulated his first law in 1687.

Principle of inertia statement

There are privileged reference frames called Galilean reference frames (inertial reference frames), in which an isolated system is either at rest or in uniform rectilinear motion.

This law can be interpreted as:

$$\boxed{\Sigma \overrightarrow{F}_{ext} = \overrightarrow{0}} \tag{4.1}$$

Galilean reference frame

A **Galilean reference frame** or **inertial reference frame** is defined as a reference frame in which the principle of inertia is valid. In other words, the motion of any isolated object point in this reference frame is rectilinear and uniform, and therefore its acceleration is zero. However, this is not always the case. As a result, such a reference frame does not exist, and the usual reference frames are only approximate and assumed to be Galilean.

If we are looking for a Galilean reference frame, the closest reference frame to the Galilean character is the **Copernicus** reference frame $\mathcal{R}_C$, to which a coordinate system is associated with its origin at the center of mass of the solar system, and its three axes are directed towards three distant stars that are assumed to be fixed.

Example of Galilean reference frames

Under certain conditions, the following reference frames can be considered Galilean:

- **Heliocentric or Keplerian reference frame ($\mathcal{R}_K$):** This reference frame is used for studying the movement of planets. It is Galilean if the study only lasts a few hours, allowing us to neglect the Earth's motion around the Sun.
- **Geocentric reference frame ($\mathcal{R}_G$):** This frame of reference is suitable for studying the motion of satellites that lasts only a few hours.
- **Earth's or laboratory reference frame ($\mathcal{R}_T$):** This is a Galilean reference frame if the duration of the study is on the order of a few minutes to neglect the Earth's own rotation. It is used to study the motion of objects on Earth.

i Any reference frame in uniform rectilinear motion relative to the Galilean reference frame is also considered Galilean.

4.3 Newton's second law of motion

Concept of force

We call a mechanical action any action exerted by an object A on another object B, which can manifest itself through one of the following effects:

- Movement of the initially stationary object B.
- Modification of the velocity vector (magnitude or direction) of the object B if it is initially in motion.
- Deformation of the object B.

A mechanical action can be modeled (or represented) by a vector $\overrightarrow{F}$ called **force**, exerted by object A on object B (pull or push). The magnitude of a force is measured in **Newtons** (N) using a **dynamometer**.

Mass

In a simple way, we can define mass as the quantity of matter contained in an object, it is often denoted as m and is measured in **kilograms** in the international system (SI).

The mass of an object is a positive scalar that indicates the extent to which it resists changes in velocity, whether in magnitude or direction. Mass serves as a measure of an object's inertia and is commonly referred to as **inertial mass**.

In Newtonian mechanics (classical mechanics), mass is considered to be constant and does not change with time or position, thus remaining independent of reference frames.

Linear momentum

Let M be a particle with mass m, moving with velocity v relative to a reference frame $\mathcal{R}$. The **linear momentum** $\overrightarrow{P}$ or simply **momentum** in the reference frame $\mathcal{R}$ of M is equal to the product of its mass and its velocity vector $\overrightarrow{v}$. It is a vector quantity that has the same direction as the velocity vector.

$$\boxed{\overrightarrow{P}(M)_{/\mathcal{R}} = m\,\overrightarrow{v}(M)_{/\mathcal{R}}} \tag{4.2}$$

Principle of conservation of momentum

The total momentum of a system of particles is equal to the sum of the momentum of all the individual particles within the system.

$$\boxed{\overrightarrow{P}_{Totale} = \overrightarrow{P}_1 + \overrightarrow{P}_2 + \ldots + \overrightarrow{P}_n} \tag{4.3}$$

Law of conservation of momentum

The total momentum of an isolated system of interacting particles is constant, meaning it remains the same in the final state as it is in the initial state:

$$\overrightarrow{P}_{initial} = \overrightarrow{P}_{finale}$$

Example 4.1

A hunter fires a bullet of mass $m = 7.3\,g$ from a shotgun of mass $M = 3.4\,kg$. The bullet leaves the shotgun with a velocity of $v = 340\,m/s$ (in magnitude). What is the magnitude of the recoil velocity V of the shotgun?

Solution

The system under study, which includes a shotgun and a bullet, is pseudo-isolated. It is studied in the Earth's frame of reference, assuming Galilean. Therefore, momentum is conserved, so:

$$\overrightarrow{P}_{\textit{before shot}} = \overrightarrow{P}_{\textit{after shot}}$$

So : $\overrightarrow{P}_{\textit{before shot}} = \overrightarrow{0}$ (The shotgun and the bullet are at rest before firing, so:

$$\overrightarrow{P}_{\textit{after shot}} = m\,\overrightarrow{v} + M\,\overrightarrow{V} = \overrightarrow{0}$$

Projecting onto the axis of movement, we obtain:

$$m\,v - M\,V = 0 \Rightarrow V = \frac{m\,v}{M}$$
$$V = 0.73\ m/s.$$

Fundamental principle of dynamics

In Newtonian mechanics, where the mass of an object point is constant, we have:

$$\overrightarrow{F} = \frac{d\,\overrightarrow{P}}{dt} = \frac{d(m\,\overrightarrow{v})}{dt} = m\frac{d\,\overrightarrow{v}}{dt} = m\,\overrightarrow{a} \tag{4.4}$$

$$\boxed{\overrightarrow{F}_{\text{net}} = m\,\overrightarrow{a}} \tag{4.5}$$

Statement of Newton's second law

In a Galilean reference frame, the vector sum of the external forces (**net external force**) acting on an object is equal to the product of its mass and its acceleration vector, that is:

$$\Sigma\,\overrightarrow{F}_{ext} = m\,\overrightarrow{a} \tag{4.6}$$

This principle provides a clear relationship between the motion of an object and the causes of this motion.

Impulse

The application of force to an object necessarily alters its velocity, which consequently leads to a change in its momentum.

This correlation between momentum and force can be elucidated through the concept of **impulse**. For a constant force, impulse $\overrightarrow{J}$ (also denoted as $\overrightarrow{I}$), is defined as the product of the force and the time interval during which the force is applied.

$$\boxed{\overrightarrow{J} = \overrightarrow{F}\,\Delta t} \tag{4.7}$$

Instantaneous impulse

We have: $\overrightarrow{F} = \dfrac{d\overrightarrow{P}}{dt} \Rightarrow d\overrightarrow{P} = \overrightarrow{F}\,dt$

$$\int_{P_i}^{P_f} d\overrightarrow{P} = \int_{t_i}^{t_f} \overrightarrow{F}\,dt$$

$$\overrightarrow{P}_f - \overrightarrow{P}_i = \Delta\overrightarrow{P} = \int_{t_i}^{t_f} \overrightarrow{F}\,dt$$

As: $\overrightarrow{J} = \int_{t_i}^{t_f} \overrightarrow{F}\,dt$

Therefore,

$$\boxed{\overrightarrow{J} = \Delta\overrightarrow{P} = m\Delta\overrightarrow{v}} \tag{4.8}$$

It should be noted that the impulse (change in momentum) applied to an object over a specific time interval can also be determined by calculating the area under the force-time graph for that same interval.

Example 4.2

Calculate the average force required to stop a petanque ball with a momentum of 15 $kg.m.s^{-1}$ in 0.5 s.

Solution

We have: $\overrightarrow{J} = \overrightarrow{F}_{\text{avg}}\,\Delta t$ and $\overrightarrow{J} = \Delta\overrightarrow{P}$.

Thus, $\overrightarrow{F}_{\text{avg}} = \dfrac{\Delta\overrightarrow{P}}{\Delta t}$.

Given:

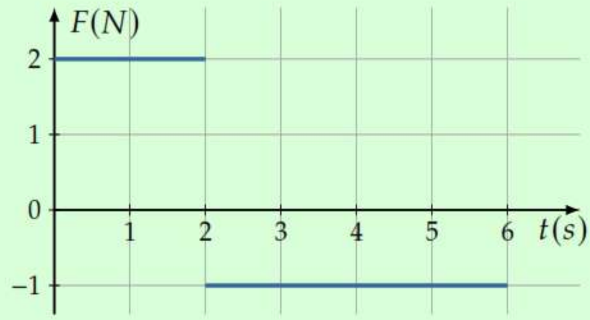
$P_i = 15\,kg.m.s^{-1}$.

$F_{\text{avg}} = \dfrac{P_f - P_i}{\Delta t} = \dfrac{0-15}{0.5} = -30\,\text{N}$.

The minus sign indicates the opposite direction of the force.

Example 4.3

The figure below illustrates the force-time graph for the motion of a particle. Calculate the change in linear momentum between 0 and 6 s.



Solution

The change in linear momentum (ΔP) is equal to the force multiplied by the time, which corresponds to the area enclosed by the force-time curve and the time axis. Mathematically, this can be expressed as:
$\Delta P = F\,\Delta t$ =Area under the force-time curve.
Thus, $\Delta P = 2 \times 2 + 4 \times (-1) = 0\ N.s$.

4.4 Newton's third law of motion

When object A exerts a force $\overrightarrow{F}$ on object B, then object B acts in the same way on object A with the same force $\overrightarrow{F}$ but in the opposite direction, that is:

$$\boxed{\overrightarrow{F}_{A \to B} = -\overrightarrow{F}_{B \to A}} \tag{4.9}$$

This is **the principle of reciprocal actions**

Statement of the law of reciprocal actions

The action is always equal to the reaction, meaning that the actions of two bodies on each other are always equal and opposite in direction.

This law is valid regardless of the study frame of reference (independent of the reference frame).

4.5 Types of forces

There exist two main categories of forces:

- **Non-contact forces:** these forces, also known as **forces at a distance** or **field forces**, exert themselves even when there is no physical contact between the two interacting bodies. This interaction is due to specific properties of matter and occurs as long as the distance between the bodies is not significantly larger than their dimensions.
- **Contact forces:** these are the forces that can only manifest themselves during contact between the two interacting bodies.

Non-contact forces (Field forces)

Gravitational force

This force, also known as the **gravitational interaction force**, acts over an infinite range and describes the attraction between two masses. It is given by **Newton's Universal Law of Gravitation**.

The expression of the gravitational force exerted mutually between two objects of mass m_A and m_B at a distance of r is given by **the universal law of gravitation** stated by Newton in 1687:

$$\boxed{\overrightarrow{F}_{A\to B} = -\overrightarrow{F}_{B\to A} = -G\frac{m_A . m_B}{r^2}\overrightarrow{u}_{AB}} \tag{4.10}$$

where $\overrightarrow{u}_{AB}$ is a unit vector directed from A to B and $G = 6.67 \times 10^{-11} N.m^2.kg^{-2}$ is the **universal gravitational constant**.

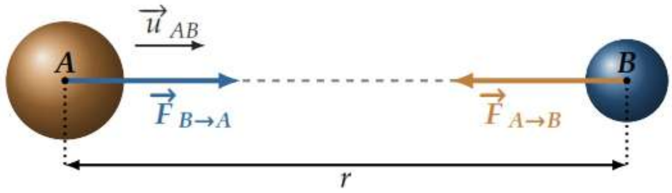


Figure 4.1: Gravitational interaction between two objects

The gravitational interaction force is of great importance, as it is what keeps us anchored to the Earth and is responsible for maintaining the planets in their orbits around the sun.

> Near Earth, the gravitational force is equivalent to weight. The law of attraction is universal and applicable at every point in space, regardless of whether the bodies are point-like or not.

Example 4.4

Calculate the weight of a box on Earth having a mass of 120 kg. Calculate the magnitude of the gravitational force between the Earth and the box.
We give: $g = 9.81\,N.kg^{-1}$, the Earth's mass $M_E = 5.972 \times 10^{24}\,kg$; the earth's radius $R_E = 6371\,km$; and $G = 6.67 \times 10^{-11} N.m^2.kg^{-2}$.

Solution

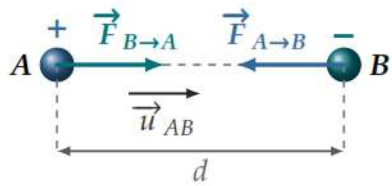
$W_B = mg = 120 \times 9.81 = 1177.2\,N.$

$$F_{B\to E} = F_{E\to B} = G\,\frac{m \times M_E}{R_E^2}$$
$$= \frac{6.67 \times 10^{-11} \times 120 \times 5.972 \times 10^{24}}{(6371.10^3)^2} = 1177.6\,N.$$

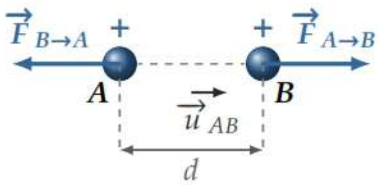
Electrostatic force

The electrostatic force, also known as **electric force** or **Coulomb force**, is a force that applies between electrically charged particles. It is described by the **Coulomb's law**. According to this law, two particles A and B, at a distance d from each other, carrying electric charges q_A and q_B, exert on each other forces $F_{A\to B}$ and $F_{B\to A}$ of the magnitude F, repulsive if q_A and q_B have the same charges, attractive if q_A and q_B have opposite charges:

$$\boxed{\overrightarrow{F}_{A\to B} = -\overrightarrow{F}_{B\to A} = \frac{1}{4\pi\varepsilon_0}\frac{q_A\ q_B}{d^2}\,\overrightarrow{u}_{AB}} \quad (4.11)$$



(a) Attractive force



(b) Repulsive force

Figure 4.2: Electrostatic force between two charged particles

By denoting $\overrightarrow{u}_{AB}$ as the unit vector directed from A to B and $\varepsilon_0 = 8.85 \times 10^{-12}\,F.m^{-1}$ as the **electrical permittivity of vacuum**, we can also express Coulomb's law in terms of the electric field as follows:

$$\boxed{\overrightarrow{F}_{A\to B} = q_B \left(\frac{1}{4\pi\varepsilon_0} \frac{q_A}{d^2}\ \vec{u}_{AB} \right) = q_B\ \overrightarrow{E}_A} \tag{4.12}$$

$\overrightarrow{E}_A$ represents the electric field created by the charge q_A at point B.

Electric interaction is responsible for the structure of atoms and molecules, as well as the properties of materials.

Electromagnetic force

In the presence of an electromagnetic field, an electric charge q moving with a velocity $\overrightarrow{v}$ experiences a force called the **electromagnetic force** or **Lorentz force**, which is expressed as:

$$\boxed{\overrightarrow{F} = q\,(\overrightarrow{E} + \overrightarrow{v} \wedge \overrightarrow{B})} \tag{4.13}$$

where $\overrightarrow{E}$ is the electric field and $\overrightarrow{B}$ is the electromagnetic field.

Strong nuclear force

This refers to the attractive forces that are responsible for binding of quarks together in the proton, as well as the cohesion between protons and neutrons together inside the atomic nucleus.

Weak nuclear force

The magnitude of this force is approximately 10^5 times weaker than that of the strong nuclear force, which is why it is named as such. It can induce a transformation of a neutron into a proton, with the emission of an electron and an anti-neutrino. This phenomenon is commonly known as **beta decay** and is responsible for radioactive processes.

Table 4.1: The four fundamental forces in nature

Force	Range	Effect	Action range
Gravitational	Infinite	Attractive	Astronomic
Electromagnetic	Infinite	Attractive or repulsive	Atom, molecule, humane
Strong nuclear	$10^{-15}\,m$	Attractive	Atomic nucleus
Weak nuclear	$10^{-18}\,m$	Transmuting	Inside nucleus

Contact forces

Normal reaction force

Let's consider a block (A) placed on a support (B). The force exerted by the support on this block, which acts to prevent its penetration into the support and is always perpendicular to it, is called the **normal reaction force**. This force, also known as the **normal reaction force** or simply the **normal force**, is distributed over the entire contact surface support-block and is applied to the block at the point of contact (see Figure 4.3).

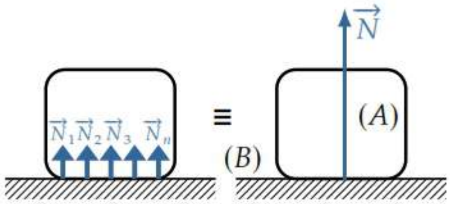


Figure 4.3: Normal reaction forces

Frictional force

If we launch the block (A) to make it slide on the support (B), we notice, after some time, that the block (A) comes to a stop due to a force that opposes its motion (see Figure 4.4). This force is known as the **frictional force**, or simply **friction**. It is always parallel to the surface of contact. Therefore, it is perpendicular to the normal force.

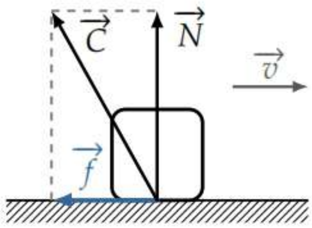


Figure 4.4: Frictional force $(\vec{f})$, normal force $(\vec{N})$, and the contact force $(\vec{C})$

> The normal and frictional forces can be considered as the normal and tangential components of the so-called **Contact force**. Therefore, the magnitude of this force can be calculated as follows: $C = \sqrt{f^2 + N^2}$.

There exist two categories of frictional forces:

- **Static friction** $\vec{f_s}$, which occurs when the block (A) is at rest (in the previous example).
- **Kinetic friction** $\vec{f_k}$, also referred to as dynamic friction or sliding friction, which arises when the block (A) is in motion (A slides on support B).

Let us consider a gradually increasing horizontal force $\overrightarrow{F}$ that is applied to a block initially at rest on a rough horizontal surface in order to move it. Initially, we observe that the block does not move. This demonstrates that the static friction force $\overrightarrow{f_s}$ is created to counteract the force $\overrightarrow{F}$, i.e. $\overrightarrow{F} = \overrightarrow{f_s}$. As the applied force $\overrightarrow{F}$ is further increased progressively, $\overrightarrow{f_s}$ also increases until it reaches the maximum (critical) value $\overrightarrow{f}_{s(max)}$. If $\overrightarrow{F}$ continues to increase, the block will start to slide and will then experience kinetic friction forces $\overrightarrow{f_k}$. In this case, the force $\overrightarrow{f_s}$ decreases from its maximum value $\overrightarrow{f}_{s(max)}$ to $\overrightarrow{f_k}$.
If we plot the applied force F as a function of the friction force f, we obtain the curve known as the **Coulomb model of friction** shown in Figure 4.5.

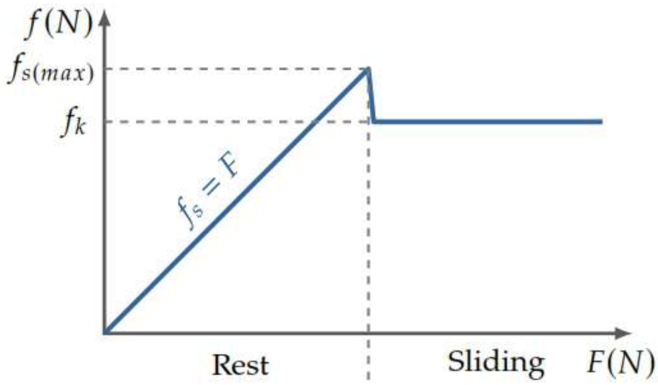


Figure 4.5: Coulomb model of friction

The experimental studies have shown that the static and kinetic friction forces are proportional to the normal reaction force $\overrightarrow{N}$. This proportionality is given by **Coulomb's friction laws**:

- If the block is immobile and subjected to static friction forces, we have:

$$\boxed{0 \leqslant f_s \leqslant f_{s(max)}} \tag{4.14}$$

  In the case where the block is on the verge of moving, i.e. the critical case, the Coulomb' friction law is expressed as follow:

$$\boxed{f_{s(max)} = \mu_s.N} \tag{4.15}$$

  where: μ_s is the **coefficient of static friction**.

- When the block is in motion, static friction becomes kinetic, and Coulomb's friction law can be written in the following manner:

$$\boxed{f_k = \mu_k.N} \tag{4.16}$$

where: μ_c is the **kinetic coefficient of friction**.

> The friction coefficients μ_s and μ_k are dimensionless and unitless numbers that depend only on the nature of the surfaces in contact.
> Generally, $\mu_k < \mu_s < 1$.

Example 4.5

To determine the coefficients of static and kinetic friction, a student starts by gradually lifting a horizontal wooden plank on which a block is resting. When the plank is inclined at an angle of 30° with respect to the horizontal, the block starts to slide with a constant acceleration and travels a distance of 3 m in 3 s. Find the static and kinetic coefficients of friction between the block and wooden plank.

Solution

We know that when the angle reaches the value of 30°, the friction force reaches its maximum value $f_{s(max)} = \mu_s.N$, and beyond this limit, it cannot increase further.
The application of Newton's second law and the projection onto the axes parallel and perpendicular to the inclined plane yields the following results:

$$N = mg\ cos30.$$
$$f_{s(max)} = mg\ sin30.$$
$$f_{s(max)} = \mu_s.N$$
$$\Rightarrow \mu_s = \left|\frac{m\ g\ sin30}{m\ g\ cos30}\right| = |tan30| = 0.58.$$

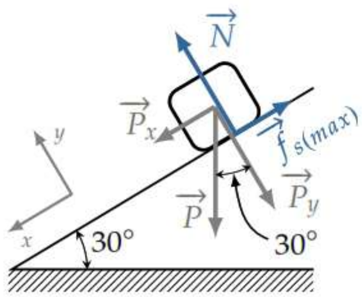


Figure 4.6

In the case where the block is in motion with constant acceleration, we have: $f_k = \mu_k.N$, and Newton's second law can be written as: $\Sigma\overrightarrow{F_{ext}} = m\overrightarrow{a}$.
Let's first calculate a. Since the motion is uniformly accelerated, therefore:
$x - x_0 = \frac{1}{2}a\ t^2 + v_0\ t$.
$x - x_0 = 3m$, $v_0 = 0$ and $t = 3s$, then: $a = \frac{2(x - x_0)}{t^2} = 0.67\ m/s^2$.

$$f_k = m(g\ sin30 - a) \Rightarrow \mu_k = \left|\frac{g\ sin30 - a}{g\ cos30}\right| = 0.5.$$

Drag Force and terminal velocity

The force exerted on a solid body moving in a fluid medium (liquid or gas) is called the **Drag Force**[1] ($\overrightarrow{F}_D$). This force is caused by the collisions between the molecules of the fluid and the solid body. Like frictional forces, it always acts in the opposite direction of the body's motion.

1: Also known as viscous drag or viscous friction force

For large objects (such as cyclists, automobiles, and others) moving at relatively high speeds, the magnitude of the drag force is directly proportional to the square of the object's speed $F_D \propto v^2$. The force F_D is influenced by several factors, including the geometric shape of the object, its dimensions, and the properties of the surrounding fluid.

$$\boxed{\overrightarrow{F}_D = -\frac{1}{2} A\, c\, \rho\, v^2 \,\overrightarrow{u}_t} \tag{4.17}$$

where, A refers to the area of the object facing the fluid, c represents the drag coefficient, ρ is the fluid density, and $\overrightarrow{u}_t$ is a unit vector that has the same direction of velocity.

For small objects with relatively low velocities, the viscous friction force can be expressed as:

$$\boxed{\overrightarrow{F}_D = -k\, \eta\, v\, \overrightarrow{u}_t} \tag{4.18}$$

where, η represents the viscosity of the fluid and k is a coefficient that depends on the shape of the body, and is expressed in meters. For example, for a sphere of radius r, we have: $k = 6\pi r$. In this case:

$$\boxed{\overrightarrow{F}_D = \overrightarrow{F}_s = -6\pi\, r\, \eta\, v\, \overrightarrow{u}_t} \tag{4.19}$$

This relation is known as **Stokes' law**.

In a vacuum, where air resistance is negligible ($F_D = 0$), objects in free fall are subject solely to the force of gravity, resulting in constant acceleration and increasing velocity. However, when drag force is present, this opposing force increases with velocity until its magnitude equals the gravitational force. At this point, the net force acting on the object becomes zero, which means that acceleration also becomes zero, in accordance with Newton's second law. Consequently,

the object continues to move at a constant velocity, known as **terminal velocity** v_T.
Applying Newton's second law, when the falling object reaches its terminal velocity,

$$F_{\text{net}} = mg - F_D = 0.$$

Thus,

$$mg = F_D = \frac{1}{2}Ac\rho v_T^2.$$

Solving for the velocity, we arrive at the following result

$$\boxed{v_T = \sqrt{\frac{2mg}{c\rho A}}} \tag{4.20}$$

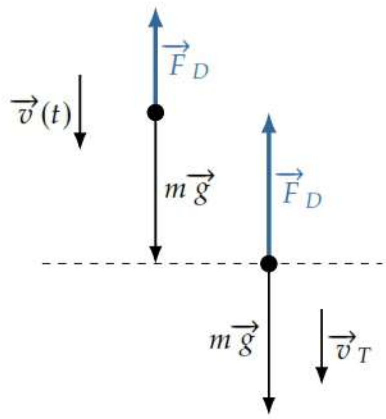


Figure 4.7: Terminal velocity

Example 4.6

A ball with a mass of 5 kg is launched vertically upward from the ground with an initial speed of 25 m/s. Given that atmospheric drag is $F_D = 0.01\ v^2\ N$, what is the maximum altitude that the ball will reach?

Solution

Initially, $y_0 = 0$ and $v_0 = 25m/s$. At the maximum height $y = h, v = 0$.
Applying Newton's second law to the ball gives:

$$\sum F_y = -w - F_D = -mg - 0.01v^2 = ma$$

Thus,

$$a = -g - \frac{0.01v^2}{m} = -9.81 - 0.002v^2 = \frac{dv}{dt}$$

Since $\frac{dy}{dt} = v = v\frac{dv}{dv}$, therefore: $\frac{dv}{dt} = v\frac{dv}{dy}$ Thus,

$$v\frac{dv}{dy} = -\left(9.81 + 0.002v^2\right)$$

Rearranging:

$$dy = \frac{-v\,dv}{9.8 + 0.002v^2}$$

$$\int_0^h dy = \int_{25}^0 \frac{-v\,dv}{9.8 + 0.002v^2}$$

Let $u = 9.81 + 0.002v^2$, so $du = 0.004v\,dv$ or $v\,dv = \dfrac{du}{0.004}$
therefore: $u_0 = u(25) = 11.06, \quad u_1 = u(0) = 9.81.$
Substituting into the integral:

$$h = \int_{25}^{0} \frac{v\,dv}{9.8 + 0.002v^2} = -\int_{11.06}^{9.81} \frac{1}{u} \cdot \frac{du}{0.004}$$
$$= \frac{-1}{0.004} \ln|u| \Big|_{11.06}^{9.81} = \frac{-1}{0.004}\Big(\ln 9.81 - \ln 11.06\Big)$$
$$h \approx 30\,m.$$

Central force

A Central force is a force whose line of action always passes through a fixed point C, called the force center. In other words, it is a force that is collinear with the position vector at all times. Examples of central forces include gravitational force and electrostatic force. The central force can be expressed as follows:

$$\boxed{\overrightarrow{F} = F(r)\overrightarrow{e}_r} \tag{4.21}$$

Spring force

Consider a mass attached to the end of a spring as shown in Figure 4.8. When the spring undergoes deformation (either extended or compressed), it exerts a force on the mass that is proportional to its stretch (or compression) at the attachment point and acts in the opposite direction of the displacement. This force is called the **spring force** or **restoring force**, and it is denoted by Hooke's law as follows:

$$\boxed{\overrightarrow{F_s} = -k.\Delta\ell\,\overrightarrow{u}} \tag{4.22}$$

Where:

ℓ_0 is the length of the spring in its relaxed state,

ℓ is the length of the spring (extended or compressed),

$\Delta\ell = (\ell - \ell_0)$ is the elongation of the spring,

k is the **spring constant** or **stiffness** expressed in $N.m^{-1}$,

$\overrightarrow{u}$ is a unit vector parallel to the spring and directed outward from the spring.

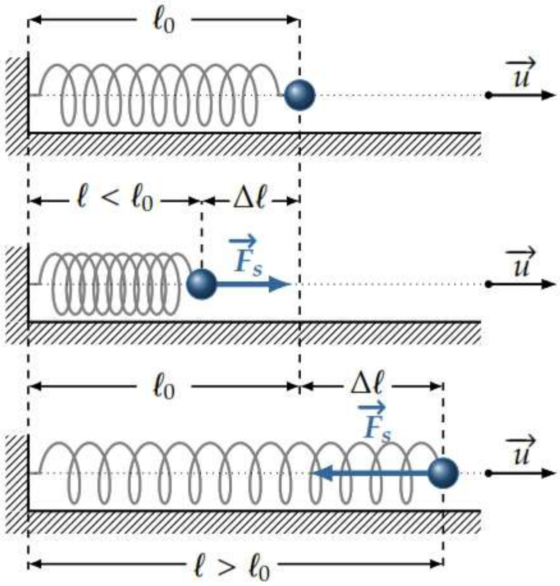


Figure 4.8: The spring force

Centripetal force

In the previous chapter, it was noted that objects undergoing uniform circular motion exhibit a centripetal acceleration, which can be expressed as $a_N = \frac{v^2}{r} = r\dot{\theta}^2$. Here, v and $\dot{\theta}$ represent the linear and angular velocities, respectively, while r denotes the radius of the circular path.

The acceleration is a result of the application of force. Any individual force or net force can induce centripetal or radial acceleration. The net force responsible for uniform circular motion is called **centripetal force**. The direction of centripetal force is always toward the center of curvature. Illustrative examples include the gravitational force exerted by the Earth on the Moon, the frictional force between roller skates and the surface of a rink, the force exerted by a banked roadway on a vehicle, and the forces acting on the tube of a spinning centrifuge.

The application of Newton's second law in that situation gives:

$$F_{net} = F_c = ma_N = m\frac{v^2}{r} = mr\dot{\theta}^2 = mr\omega^2$$

Thus, the magnitude of centripetal force can expressed as:

$$\boxed{F_c = m\frac{v^2}{r} = mr\omega^2} \tag{4.23}$$

The most commonly observed example of static friction acting as a centripetal force in daily life is the motion of vehicles on horizontal curves. This force helps keep vehicles on the road and prevents them from skidding outward. Without sufficient static friction, a vehicle would lose traction and slide off the curve. To avoid this scenario, roads are typically banked at curves. This banking effectively enhances the centripetal force, allowing curves to be negotiated safely at high speeds.

Let us consider a car traveling at a constant speed v. it takes a banked curve characterized by an angle θ and a radius of curvature r.

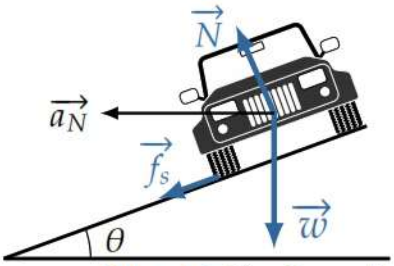


Figure 4.9: Circular motion of a car on banked road

The forces acting on the car include its weight $\overrightarrow{w}$, the normal force $\overrightarrow{N}$, and the static frictional force $\overrightarrow{f}_s$ (see Figure 4.9). In this case, the centripetal force is produced by the horizontal components of both the normal force $\overrightarrow{N}$ and the static frictional force $\overrightarrow{f}_s$.

The acceleration is present only along the x-axis. Therefore,

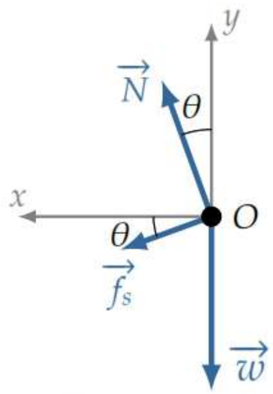


Figure 4.10: Free-body diagram

$$Ox: N\ sin\theta + f_s\ cos\theta = a_N = \frac{v^2}{r} \qquad \text{(i)}$$

$$Oy: N\ cos\theta - f_s\ sin\theta = w = mg \qquad \text{(ii)}$$

To obtain the maximum speed v_{max} at which the car can safely negotiate the curve, the static friction force must be at its maximum ($f_{s(max)}$). So,

$$f_{s(max)} = \mu_s.N \qquad \text{(iii)}$$

The equations (i) and (ii) become

$$N\ sin\theta + f_{s(max)}\ cos\theta = \frac{v_{max}^2}{r} \qquad \text{(iv)}$$

$$N\ cos\theta - \mu_s\ N\ sin\theta = mg \qquad \text{(v)}$$

Substituting (iii) in (iv) and (v), we get

$$N\ sin\theta + \mu_s\ N\ cos\theta = \frac{v_{max}^2}{r} \qquad \text{(vi)}$$

$$N\ cos\theta - \mu_s\ N\ sin\theta = mg \qquad \text{(vii)}$$

Solving this system of equations yields

$$\boxed{v_{max} = \sqrt{rg\left(\frac{\mu_s + tan\theta}{1 - \mu_s\, tan\theta}\right)}} \tag{4.24}$$

It is important to highlight that this relationship is independent of the vehicle's mass.

For a frictionless banked roadway,

$$\boxed{v_{max} = \sqrt{rg\; tan\theta}} \tag{4.25}$$

For a horizontal roadway (level roadway),

$$\boxed{v_{max} = \sqrt{rg\; \mu_s}} \tag{4.26}$$

Example 4.7

What is the ideal banking angle for a curve with a radius of curvature $r = 100\ m$ to be safely navigated at a speed of 60 km/h without relying on frictional forces?

Solution

Using the relation above,

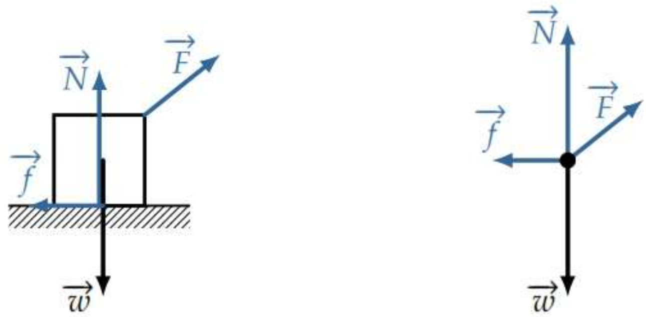
$$\theta = tan^{-1}\left(\frac{v^2}{rg}\right) = tan^{-1}\left(\frac{(16.67)^2}{100 \times 9.81}\right)$$

$$\theta \approx 15.82°.$$

Free-body diagram (FBD)

A free-body diagram is a graphical representation that illustrates all external forces acting on an object or system. In this diagram, the object or system is represented as a single isolated point (also known as a material point). The forces are depicted as vectors extending outward from this point.

Figure 4.11 illustrates the free-body diagram of a box subjected to a set of forces (the weight of the box $\overrightarrow{w}$, the normal force $\overrightarrow{N}$, the frictional force $\overrightarrow{f}$, and an applied force $\overrightarrow{F}$).



(a) System or object (b) Free-body diagram

Figure 4.11: Representation of free-body diagram

4.6 Moment of a force (Torque)

Moment of a force about a point

The moment of a force about a point (also known as **torque**) measures the force's ability to rotate a mechanical system around that point (**turning effect**).

The moment of a force $\overrightarrow{F}$ about any point O is a vector that can be expressed as:

$$\boxed{\overrightarrow{\tau}_O(\overrightarrow{F}) = \overrightarrow{OM} \times \overrightarrow{F}} \tag{4.27}$$

where M represents the object on which the force $\overrightarrow{F}$ is applied (see Figure 4.12).

The moment of a force $\overrightarrow{\tau}$ (also denotes as $\overrightarrow{\mathcal{M}}$) is a vector perpendicular to the plane formed by $\overrightarrow{OM}$ and $\overrightarrow{F}$. Its SI unit is **joule** or **newton meter** ($J = N.m$).

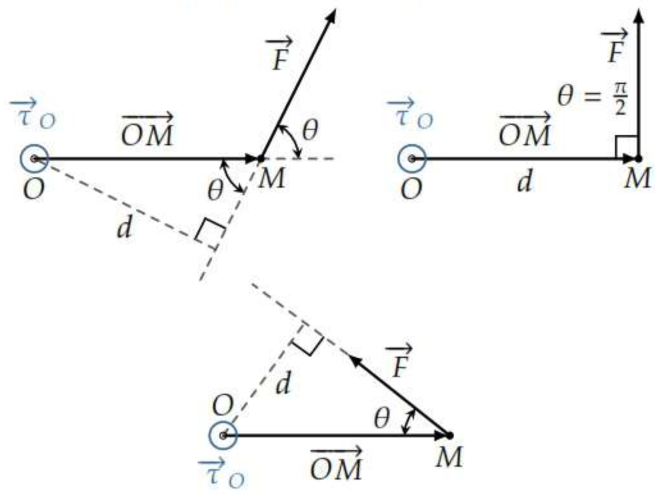


Figure 4.12: Moment of a force about a point

The magnitude of the moment of a force $\overrightarrow{F}$ about a point O $\tau_O(F)$ is given by:

$$\boxed{\|\overrightarrow{\tau}_O(\overrightarrow{F})\| = \|\overrightarrow{OM}\|.\|\overrightarrow{F}\|.|sin\,\theta| = d\,.\|\overrightarrow{F}\|} \tag{4.28}$$

As shown in Figure 4.12, d represents the perpendicular distance between the line of action of the force and the point O (pivot). This distance is commonly referred to as the **moment arm** or **lever arm**.

Moment of a force about an axis

Let us consider the moment of the force $\overrightarrow{F}$ about the point O. Let (Δ) be an axis passing through point O and $\overrightarrow{u}_\Delta$ be a unit directional vector of this axis as shown in Figure 4.13. The moment of the force $\overrightarrow{F}$ acting on a point M about the axis (Δ) is defined as the scalar quantity:

$$\boxed{\tau_\Delta(\overrightarrow{F}) = \overrightarrow{\tau}_O(\overrightarrow{F}) \,.\, \overrightarrow{u}_\Delta} \tag{4.29}$$

The magnitude of the moment of a force about an axis (Δ) is expressed as follows:

$$\boxed{\tau_\Delta(\overrightarrow{F}) = \pm \|\overrightarrow{F}\| \,.\, d} \tag{4.30}$$

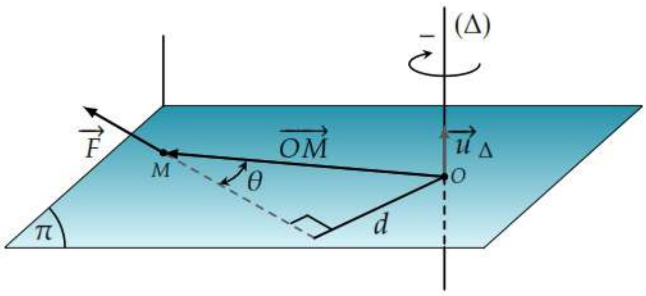
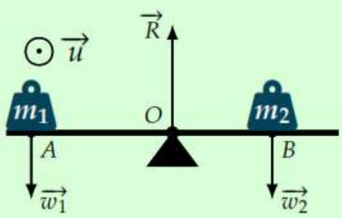


Figure 4.13: Moment of a force about an axis

Example 4.8

Two masses, $m_1 = 1\ kg$ and $m_2 = 3\ kg$, are positioned on opposite sides of a balance. Let $\overrightarrow{u}$ be a unit vector perpendicular to the plan of the figure directed from back to front (see figure below).

1. Given that $OA = 1.2\ m$ and $OB = 0.4\ m$, determine the torque of the forces $\overrightarrow{R}$, $\overrightarrow{w_1}$, and $\overrightarrow{w_2}$ about point O.
2. Deduce the total torque (τ_{net}).



Solution

1. $\vec{\tau}(\vec{F}) = \vec{r} \times \vec{F} = rF|\sin\theta|\vec{u}$
Since the forces and the distances are orthogonal, the sine term becomes 1.

$$\vec{\tau}(\vec{R}) = \overrightarrow{OO} \times \vec{R} = \vec{0}.$$

Thus, $\tau(\vec{R}) = 0\ N.m.$

$$\vec{\tau}(\vec{w_1}) = \overrightarrow{OA} \times \vec{w_1} = (1.2 \times 10 \times 1)\vec{u} = 12\vec{u}\ N.m.$$

Thus, $\tau(\vec{R}) = 12\ N.m.$

$$\vec{\tau}(\vec{w_2}) = \overrightarrow{OB} \times \vec{w_1} = (0.4 \times 30 \times 1)\overrightarrow{-u} = -12\vec{u}\ N.m.$$

Thus, $\tau(\vec{R}) = 12\ N.m.$
2. $\vec{\tau}_{net} = \vec{\tau}_{w_1} + \vec{\tau}_{w_2} + \vec{\tau}_R = (12 - 12 + 0)\vec{u} = \vec{0}\ N.m.$
Thus, $\tau_{net} = 0\ N.m.$

4.7 Angular momentum of a particle

For translational motions, we use the concepts of force and momentum, related to each other by Newton's laws. For rotational motions, we introduce the concept of torque, and for momentum, we introduce the notion of the **angular momentum** of a particle M at a given point O, in a Galilean reference frame.

In rotational motion, it is preferable to use a different theorem than the kinetic energy theorem or Newton's laws. This theorem is called the **angular momentum theorem**.

Angular momentum about a point for a particle

Let M be a particle with mass m in motion with velocity $\vec{v}$ in a Galilean reference frame $\mathcal{R}$. The particle M is subjected to a set of forces $\vec{F_i}$. Let O be a fixed point and Δ be a fixed directed line containing O. Let $\overrightarrow{OM}$ and $\vec{P}$ be the position vector and the momentum of M in $\mathcal{R}$.

We call the **angular momentum** or **kinetic momentum** of M relative to O the vector $\overrightarrow{L}_O(M)$ defined by:

$$\boxed{\overrightarrow{L}_O(M) = \overrightarrow{OM} \times \overrightarrow{P} = \overrightarrow{OM} \times m\,\vec{v}} \qquad (4.31)$$

The angular momentum vector $\overrightarrow{L}_O(M)$ is orthogonal to the plane containing the vectors $\overrightarrow{OM}$ and $\overrightarrow{v}$, its direction is given such that the basis $(\overrightarrow{OM}, \overrightarrow{v}, \overrightarrow{L}_O)$ is right-handed, and its magnitude is given by:

$$\boxed{L_O(M) = \|\overrightarrow{L}_O(M)\| = m.\|\overrightarrow{OM}\|.\|\overrightarrow{v}\||sin\alpha|} \qquad (4.32)$$

where α represents the angle formed by the vectors $\overrightarrow{OM}$ and $\overrightarrow{v}$ (see Figure 4.14).

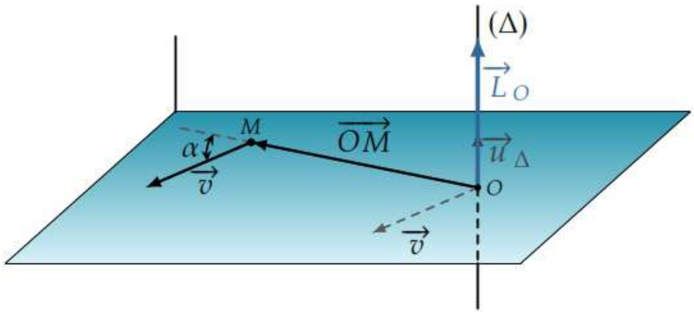


Figure 4.14: The angular momentum

Theorem

In a Galilean reference frame $\mathcal{R}$, the time derivative of the angular momentum of a particle M relative to a fixed point O is equal to the sum of the moments of the forces applied to M with respect to O, or:

$$\frac{d\overrightarrow{L}_O(M)}{dt} = \sum_i \overrightarrow{\tau}_O(\overrightarrow{F_i})$$

Angular momentum about an axis for a particle

The angular momentum of M with respect to the axis Δ passing through a point O, as shown in Figure 4.14, is given by the following dot product:

$$\boxed{L_\Delta = \overrightarrow{L}_O(M).\overrightarrow{u}_\Delta} \qquad (4.33)$$

We thus have the theorem of angular momentum with respect to a fixed axis (Δ) in a Galilean reference frame:

$$\boxed{\frac{d\,L_{\Delta}}{dt} = \sum_i \tau_{\Delta}(\overrightarrow{F_i})} \tag{4.34}$$

Example 4.9

The position and velocity vectors of a mobile object with a mass of $m = 2\,\text{kg}$ are given at a particular instant as: $\overrightarrow{r} = \overrightarrow{i} - \overrightarrow{j}$ and $\overrightarrow{v} = (-2\overrightarrow{i} + \overrightarrow{j}$. Find the object's angular momentum about the origin.

Solution

We have:

$$\overrightarrow{L} = \overrightarrow{r} \times \overrightarrow{p} = \overrightarrow{r} \times (m\,\overrightarrow{v})$$

Thus,

$$\overrightarrow{L} = m \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -1 & 0 \\ -2 & 1 & 0 \end{vmatrix} = 2 \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & -1 & 0 \\ -2 & 1 & 0 \end{vmatrix} = -2\,\overrightarrow{k}.$$

Therefore:

$$L = 2\,kg.m^2/s.$$

Conservation of angular momentum

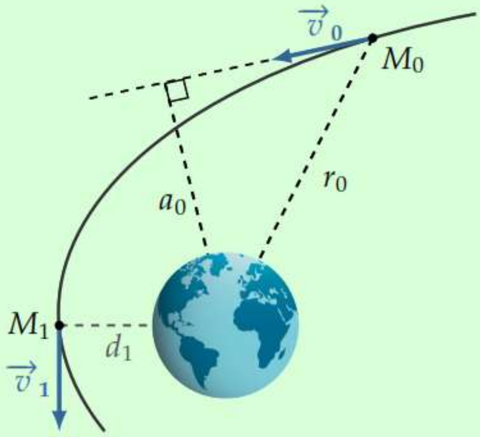
Statement of the law

The angular momentum of a particle subject to a central force or isolated is constant, or

$$\overrightarrow{L}_O(M) = \overrightarrow{Cte}$$

Example 4.10

A comet, coming from interplanetary space, enters the Earth's gravitational field and then moves away. Its trajectory is shown in the figure below. If v_0 is its speed at position M_0, determine in terms of v_0, a_0, and d_1 its speed v_1 when it is at the distance d_1 closest to the earth.



Solution

We have:
$\vec{L}(M) = \overrightarrow{OM} \wedge m\,\vec{v} \Rightarrow L = m\|\overrightarrow{OM}\|\;\|\vec{v}\|\;sin(\overrightarrow{OM}, \vec{v})$.

$$L_0 = m\ v_0\ r_0\ sin(\overrightarrow{OM}_0, \overrightarrow{v_0}) = m.v_0.a_0.$$

$$L_1 = m\ v_1\ d_1\ sin(\overrightarrow{OM}_1, \overrightarrow{v_1}) = m.v_1.d_1.sin\frac{\pi}{2} = m.v_1.d_1.$$

The force applied to the comet is the force of gravity, it is a central force, so the angular momentum is constant, that is: $\vec{L}_0 = \vec{L}_1$. So:

$$m.v_0.a_0 = m.v_1.d_1 \Rightarrow v_1 = \frac{a_0}{d_1}v_0.$$

4.8 Differential equation of motion

A differential equation is an equation involving one or more derivatives of an unknown function. If all the derivatives are taken with respect to a single variable, it is called an **ordinary differential equation**.

The study of particle motion aims to determine the equations that describe the position of the particle's components as a function of time. The time equations can only be obtained if one knows the particle's acceleration in advance.

The application of Newton's second law allows us to determine the acceleration of a particle. We then obtain the differential equation of motion for the particle. This equation relates the acceleration, velocity, and position of the particle to the variable t. We distinguish several types of differential equations based on their forms.

Periodic motion of a mass-spring system

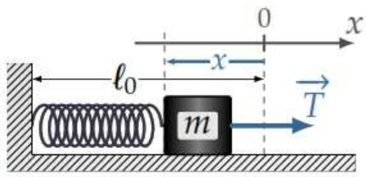


Figure 4.15: Mass-spring system

We propose to study the motion of a mass m attached to a spring of stiffness k. This mass is constrained to move without friction on the Ox axis under the effect of the spring's restoring force. We denote x as the elongation of the spring ($\Delta l = x$).

Along the Ox axis, the equation of motion is:

$$\sum \overrightarrow{F}_x = \overrightarrow{T} = m\overrightarrow{a}_x$$

$$-k\,x = m\,\frac{d^2x}{dt^2} \Rightarrow \frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

If we set $\frac{m}{k} = \omega_0^2$, the previous equation becomes:

$$\frac{d^2x}{dt^2} + \omega_0^2\,x = 0 \Leftrightarrow \ddot{x} + \omega_0^2\,x = 0$$

It is a **second order differential equation** whose solution is of the form:

$$x(t) = A\cos\left(\omega_0 t + \varphi\right)$$

$\omega_0 = \sqrt{\frac{k}{m}} = 2\pi f_0\,(rad\,.\,s^{-1})$ represents the angular frequency and A its amplitude.

CHAPTER 4 SUMMARY

Newton's first law of motion (the principle of inertia) $\Sigma \overrightarrow{F}_{ext} = \overrightarrow{0}$.

Newton's second law of motion (Fundamental principle of dynamics) $\Sigma \overrightarrow{F}_{ext} = m\,\overrightarrow{a}$.

Newton's third law of motion $\overrightarrow{F}_{A \to B} = -\overrightarrow{F}_{B \to A}$.

Linear momentum $\overrightarrow{P} = m\,\overrightarrow{v}$, $\overrightarrow{P}_{Totale} = \overrightarrow{P}_1 + \overrightarrow{P}_2 + ... + \overrightarrow{P}_n$.

Principle of conservation of momentum $\overrightarrow{P}_{initial} = \overrightarrow{P}_{finale}$.

Impulse $\overrightarrow{J} = \overrightarrow{F}\,\Delta t$

Instantaneous impulse $\overrightarrow{J} = \Delta \overrightarrow{P} = m\Delta \overrightarrow{v}$.

Gravitational force $\overrightarrow{F}_{A \to B} = -\overrightarrow{F}_{B \to A} = -G\dfrac{m_A.m_B}{r^2}\overrightarrow{u}_{AB}$.

Electrostatic force $\overrightarrow{F}_{A \to B} = -\overrightarrow{F}_{B \to A} = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q_A\ q_B}{d^2}\ \overrightarrow{u}_{AB}$.

Electromagnetic force $\overrightarrow{F} = q\,(\overrightarrow{E} + \overrightarrow{v} \wedge \overrightarrow{B})$.

The four fundamental forces in nature

Force	Range	Effect	Action range
Gravitational	Infinite	Attractive	Astronomic
Electromagnetic	Infinite	Attractive or repulsive	Atom, molecule, humane
Strong nuclear	$10^{-15}\,m$	Attractive	Atomic nucleus
Weak nuclear	$10^{-18}\,m$	Transmuting	Inside nucleus

Coulomb's friction laws $f_{s(max)} = \mu_s.N$, $\quad f_k = \mu_k.N$.

Spring force $\overrightarrow{F_s} = -k.\Delta\ell\,\overrightarrow{u}$.

Drag Force $\overrightarrow{F}_D = -\dfrac{1}{2}A\,c\,\rho\,v^2\,\overrightarrow{u}_t$.

Drag Force for small objects with relatively low velocities $\overrightarrow{F}_D = -k\ \eta\,v\,\overrightarrow{u}_t$.

Drag Force for a sphere of radius r, $k = 6\pi r$ **(Stokes' law)** $\overrightarrow{F}_D = \overrightarrow{F}_s = -6\pi\,r\,\eta\,v\,\overrightarrow{u}_t$.

In free fall when drag force is present, the terminal velocity v_T **is** $v_T = \sqrt{\dfrac{2mg}{c\rho A}}$.

Central force $\overrightarrow{F} = F(r)\overrightarrow{e}_r$.

Centripetal force $F_c = m\dfrac{v^2}{r} = mr\dot{\theta}^2 = mr\omega^2$.

The maximum speed v_{max} **to safely negotiate curves** $v_{max} = \sqrt{rg\left(\dfrac{\mu_s + tan\theta}{1 - \mu_s\,tan\theta}\right)}$.

For a frictionless banked roadway $v_{max} = \sqrt{rg\ tan\theta}$.

For level road $v_{max} = \sqrt{rg\ \mu_s}$.

Moment of a force (Torque) $\overrightarrow{\tau}_O(\overrightarrow{F}) = \overrightarrow{OM} \times \overrightarrow{F}$ and $\tau_O = OM.F.|sin\ \theta| = d\,.F$.

Angular momentum of a particle $\overrightarrow{L}_O = \overrightarrow{OM} \times \overrightarrow{P} = \overrightarrow{OM} \times m\,\vec{v}$, $L_O = m.\|\overrightarrow{OM}\|.\|\overrightarrow{v}\||sin\alpha|$ and $\dfrac{d\,\overrightarrow{L}_O(M)}{dt} = \sum_i \overrightarrow{\tau}_O(\overrightarrow{F_i})$.

4.9 Exercises

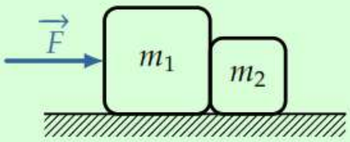
Exercise 4.1

At the time of takeoff, a rocket possesses a mass of 3×10^5 kg, while its engines generate a thrust of 9×10^6 N.
1- Determine the initial acceleration of the rocket when it ascends vertically.
2- Calculate the duration required for the rocket to attain a velocity of 100 km/h under the assumption of constant mass and thrust.

Sol : page 115

Exercise 4.2

Two blocks, m_1 and m_2, are positioned adjacent to one another on a frictionless horizontal surface, as illustrated in the accompanying figure. When a force of $F = 10$ N is applied, the two blocks move together as a single system with an acceleration of the $1.25\, m/s^2$.

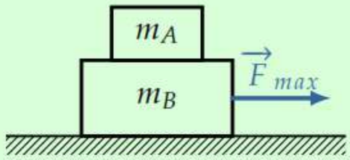


1- Draw the free-body diagram of the forces acting on m_1 and m_2 individually.
2-Determine the m_2 value if $m_1 = 6$ kg.
3- Calculate the force $F_{1/2}$ exerted by m_1 on m_2 and the force $F_{2/1}$ exerted by m_2 on m_1.

Sol : page 115

Exercise 4.3

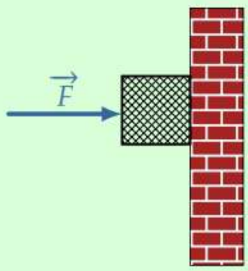
Consider two blocks, A and B, with respective masses $m_A = 3\ kg$ and $m_B = 5.5\ kg$, positioned in a stacked arrangement on a frictionless horizontal surface (see figure below). If the coefficient of friction between the two blocks is 0.3, what maximum force F_{max} that can be exerted on block B in order for both blocks to move in together.



Sol : page 115

Exercise 4.4

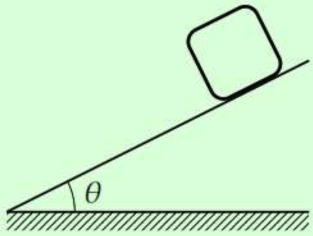
A 1 kg block is pushed against a vertical wall with a horizontal force F (see figure below). The coefficient of static friction between the block and the wall is 0.49. Calculate the force F required to keep the block stationary against the wall without it sliding.



Sol : page 116

Exercise 4.5

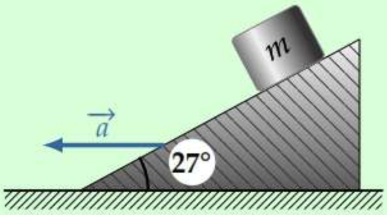
1- If the coefficient of kinetic friction is 0.4, calculate the acceleration of a 1 kg object sliding down an inclined plane at a 30° angle.
2- What is the magnitude of the kinetic friction force?



Sol : page 116

Exercise 4.6

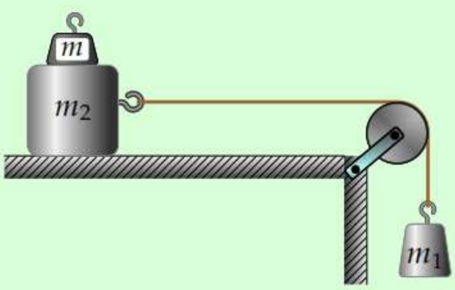
Consider a box with mass m placed on a frictionless triangular support that is accelerating horizontally with an acceleration a. What value of a will allow the block to remain at rest relative to the support?



Sol : page 116

Exercise 4.7

Consider the system illustrated in the figure below, where two blocks are connected by an ideal rope and pulley. If $m_1 = 4\,\text{kg}$, $m_2 = 8\,\text{kg}$, and the coefficient of kinetic friction between the table and m_2 is $\mu_k = 0.2$, what is the minimum additional mass m that must be placed on top of m_2 to stop the system from moving?



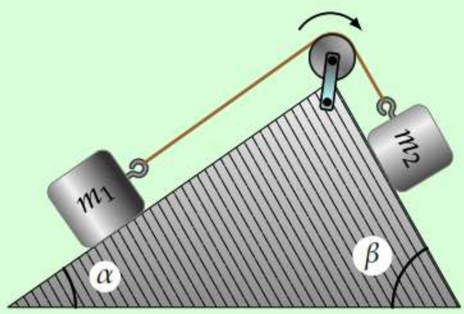
Sol : page 117

Exercise 4.8

Consider the system illustrated in the figure below. All surfaces are perfectly smooth. Given $m_1 = 1.6\,\text{kg}$, $m_2 = 1.2\,\text{kg}$, $\alpha = 20°$, $\beta = 58°$, and $g = 9.81\,\text{m/s}^2$, we will analyze the system assuming an ideal rope and pulley.

1- What is the acceleration of the system?

2- Determine the tension in the rope.



Sol : page 117

Exercise 4.9

A block A, with a mass of $2\ kg$, is fixed to a spring of negligible mass. Another block B, with a mass of $1\ kg$, is attached to block A by an ideal wire, as illustrated in the figure. Calculate the acceleration of each block immediately after the wire is cut.

Sol : page 117

Exercise 4.10

A racing car negotiates a curve that is steeply banked at an angle of $\theta = 31°$. The coefficient of friction between the road and the tires is $\mu = 0.5$. At a speed of $147\ km/h$:

1- What must the radius of curvature r be to navigate the curve without sideslip?

2- If the road is frictionless, what must the value of the angle θ be to negotiate

the bend safely?
3- On a level road (with $\mu_s = 0.5$), find the maximum speed at which the car can round the curve safely.

Sol : page 117

Exercise 4.11

A 3 kg ball travels along a vertical circular path with a radius of $r = 2\ m$. It reaches the summit (point A) at a speed of 4.8 m/s and at point B, which is located 60° from point A, at a speed of 5.1 m/s. What is the force exerted by the surface on the ball at points A and B?

Sol : page 118

Exercise 4.12

2 kg object is suspended from a fixed point O by a wire of length $\ell = 1$ m. The object can rotate in a horizontal circle with center C and radius r around the vertical with an angular velocity ω (see the figure below "conical pendulum"). At this angular velocity, the wire forms an angle $\theta = 36.8°$ with the vertical.
1- Find the angular velocity of the object.
2- What is the tension in the wire?

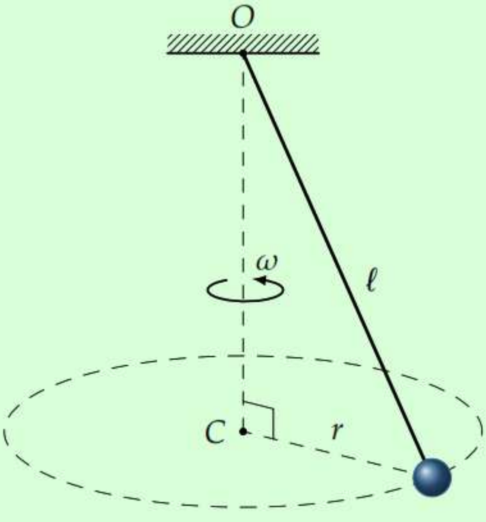


Figure 4.16: Conical Pendulum

Sol : page 118

Exercise 4.13

A variable force acts in one direction on a 3 kg object that is initially at rest. The force-time graph is shown in the figure below. Determine the velocity of the object at 12 seconds.

Sol : page 119

Exercise 4.14

A ball with a mass of 0.2 kg is released from a height and subsequently impacts the ground. The speed of the ball just prior to contact with the ground is $v_b = 6\,\mathrm{m/s}$, while immediately after the impact is $v_a = 5\,\mathrm{m/s}$. The duration of the collision between the ball and the ground is measured to be 0.14 s. Determine the average force exerted by the ground on the ball during this collision.

Sol : page 119

Exercise 4.15

A small ball with a mass of 18 g falls into a swimming pool and moves through the water, reaching a terminal speed of 3 m/s.
1- Assuming that the frictional force acting on the ball is described by the equation $f = -\lambda v$, determine the value of λ.
2- Establish the differential equations for the position, and then determine how far the ball travels before it reaches 82% of its terminal speed (use $g = 10\ m/s^2$).

Sol : page 119

4.10 Solutions

Solution of exercise 4.1 (Page 112)

1- Newton's second law along the axis of motion gives:

$$F_{\text{th}} - w = m \cdot a$$

or

$$a = \frac{F_{\text{th}} - w}{m} = \frac{9 \times 10^6 - 2.94 \times 10^6}{3 \times 10^5}$$
$$= 20.2\,\text{m/s}^2.$$

2- Assuming constant acceleration,

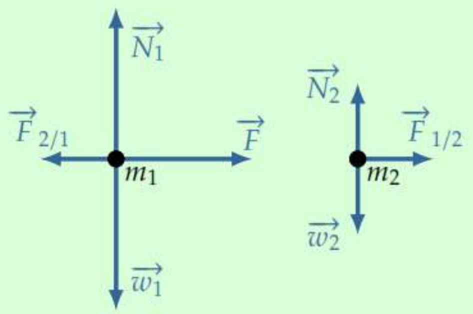
$$v = a \cdot t$$

thus,

$$t = \frac{v}{a} = \frac{27.78}{20.2} \approx 1.38\,\text{s}.$$

Solution of exercise 4.2 (Page 112)

1-



2- Using the Newton's second law along direction of motion:

$$F = (m_1 + m_2)\,a$$

Solving for m_2

$$m_2 = \frac{F}{a} - m_1 = 2\,kg.$$

3- Using Newton's second law for block m_2 :

$$F_{1/2} = m_2 \times a = 2 \times 1.25 = 2.5\,N.$$

Using Newton's second law for block m_1 :

$$F - F_{2/1} = m_1 \times a.$$

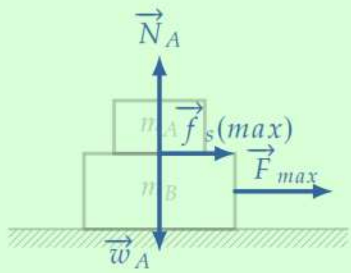
Therefore,

$$F_{2/1} = F - (m_1 \times a) = 10 - (6 \times 1.25) = 2.5\,N.$$

Or simply using the Newton's third law,

$$F_{2/1} = F_{1/2} = 2.5\,N.$$

Solution of exercise 4.3 (Page 112)



For the block "B":

Using the Newton's second law along direction of motion:

$$F_{max} = (m_A + m_B)a$$

Thus: $$a = \frac{F_{max}}{m_A + m_B}$$

For the block "B":

The force F that must be applied for the two blocks to move together is maximum when static friction force reaches its maximum value $f_{s(max)}$.

Apply the second law to the block "B":

$f_{s(max)} = m_A a$ and $m_A g = N_A$

but

$$f_{s(max)} = \mu_s N_A$$

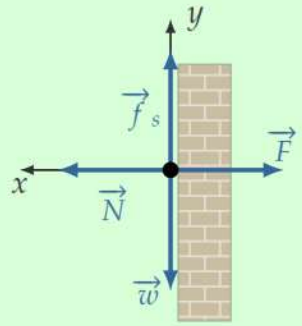
Thus:

$$\mu_s g = a = \frac{F_{max}}{m_A + m_B}$$

Finally:

$$F_{max} = \mu_s g(m_A + m_B) \approx 25\,N.$$

Solution of exercise 4.4 (Page 112)



For holding the block (not slide), the upward frictional force must balance the downward weight:

$$f_{\mathrm{s(max)}} = w.$$

As: $F = N$ and $f_{\mathrm{s(max)}} = \mu_s\, N$, thus:

$$f_{\mathrm{s(max)}} = \mu_s\, F = w = mg.$$

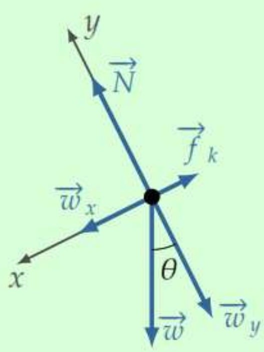
Therefore,

$$F = \frac{mg}{\mu_s} = \frac{9.81}{0.49} \approx 20\ N.$$

Solution of exercise 4.5 (Page 112)

1- Calculate the acceleration

We first draw the free-body diagram:



The forces acting on the object are:

- The two components of weight: $w_x = mg\sin\theta$ and $w_y = mg\cos\theta$
- The normal force: $N = mg\cos\theta$
- The kinetic friction force:

$$f_k = \mu_k N = \mu_k mg\cos\theta$$

The application of Newton's second law along the incline (x-axis) gives:

$$F_{\mathrm{net}} = w_x - f_k = ma$$

Thus,

$$ma = mg\sin\theta - \mu_k mg\cos\theta$$

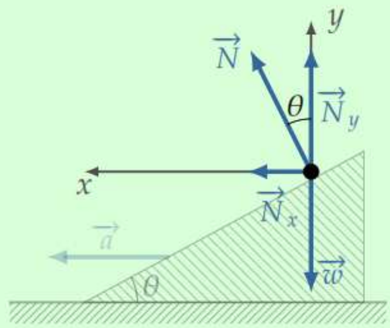
Solving for a:

$$a = g(\sin\theta - \mu_k\cos\theta) \approx 1.5\,\mathrm{m/s^2}.$$

2- Calculate the kinetic friction force

$$f_k = \mu_k(mg\cos\theta) \approx 3.4\,\mathrm{N}.$$

Solution of exercise 4.6 (Page 113)



The application of Newton's second law along the x-axis gives:

$$N_x = N\sin\theta = ma$$

There is no motion along the y-axis, therefore:

$$N_y = N\cos\theta = w = mg$$

Thus,

$$N = \frac{mg}{\cos\theta}$$

Substitute N into the first equation:

$$a = g\tan\theta = 9.81 \times \tan 27^\circ \approx 5\ m/s^2.$$

Solution of exercise 4.7 (Page 113)

For m_1, Newton's second law gives:

$$T = m_1 g \qquad \text{(i)}$$

For m_2:

$$T = f_k = \mu_k\, N \;\text{ and }\; N = g(m_2 + m)$$

Thus,

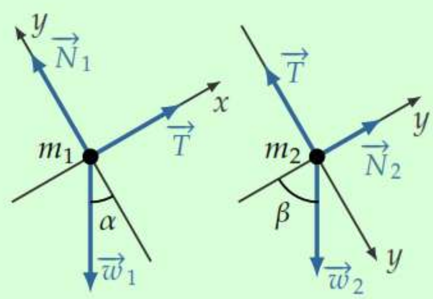
$$T = \mu_k\, g(m_2 + m) \qquad \text{(ii)}$$

Substitute (i) into (ii) and solve for m:

$$m = \frac{m_1}{\mu_k} - m_2 = 12\,kg.$$

Solution of exercise 4.8 (Page 113)

Free-Body Diagram for m_1 and m_2:



1- For m_1, Newton's second law along the x-axis gives:

$$T - m_1 g \sin\alpha = m_1 a \qquad \text{(i)}$$

For m_2, Newton's second law along the x-axis gives:

$$m_2 g \sin\beta - T = m_2 a \qquad \text{(ii)}$$

Add the two equations (i) and (ii) and solve for a:

$$a = \frac{g(m_2 \sin\beta - m_1 \sin\alpha)}{m_1 + m_2} \approx 1.65\,m/s^2.$$

2- Using the equation (i)

$$T = m_1(g \sin\alpha + a) \approx 8\,\mathrm{N}.$$

Solution of exercise 4.9 (Page 113)

For the block B:

Immediately after cutting the wire:

$$F_{\text{net}} = m_B a_B = w_B = m_B g$$

Thus,

$$a_B = g = 9.81\,m/s^2.$$

For the block A:

Immediately after cutting the wire, Newton's second law along the axis of motion (upward) gives:

$$F_{\text{net}} = T_{\text{spring}} - w_A = m_A a_A$$

Before the wire is cut:

$$T_{\text{spring}} = (m_A + m_B)g$$

Thus,

$$(m_A + m_B)g - m_A g = m_A a_A$$

Solving for a_A

$$a_A = \frac{g}{2} = 4.9\,m/s^2.$$

Solution of exercise 4.10 (Page 113)

1- On a banked road and the presence of friction:

$$v_{max} = \sqrt{rg\left(\frac{\mu_s + tan\theta}{1 - \mu_s\, tan\theta}\right)}$$

or

$$r = \frac{v_{max}^2(1 - \mu_s\, tan\theta)}{g(\mu_s + tan\theta)}$$

Substituting the known values,

$$r = 108\,m.$$

2- On a banked road and the absence of friction:

$$v_{max} = \sqrt{rg\, tan\theta} \text{ or } \theta = tan^{-1}\left(\frac{v_{max}^2}{g\,r}\right)$$

Substituting the known values,

$$\theta = 57.57^\circ.$$

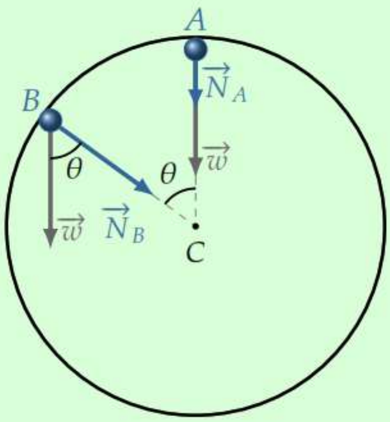
3- On a unbanked road and the presence of friction:

$$v_{max} = \sqrt{rg\,\mu_s}$$

Substituting the known values,

$$v_{max} = 23\ m/s = 82.8\ km/h.$$

Solution of exercise 4.11 (Page 114)



At point A:
The force required to keep the ball moving in a circular path is the centripetal force F_c:

$$F_c = \frac{mv_A^2}{r}.$$

At point A, F_c is provided by the sum of the normal force and the weight:

$$N_A + mg = \frac{mv_A^2}{r}.$$

Solving for N_A:

$$N_A = m\left(\frac{v_A^2}{r} - g\right) = 5.13\ N.$$

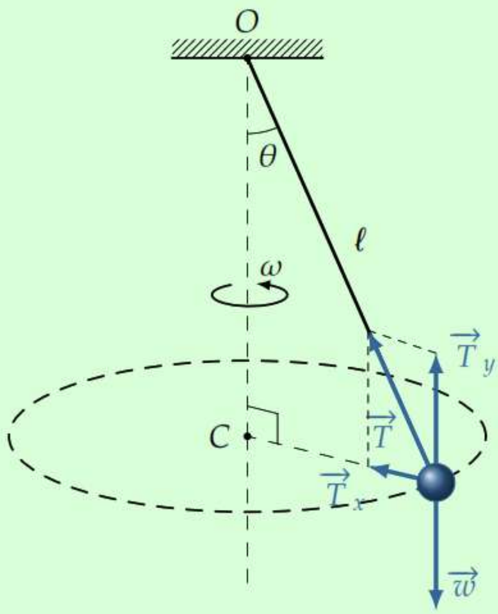
At point B:
F_c is provided by the sum of the normal force and the normal component of weight:

$$N_B + mg\cos\theta = \frac{mv_A^2}{r}.$$

Solving for N_B:

$$N_B = m\left(\frac{v_A^2}{r} - g\cos\theta\right) = 24.3\ N.$$

Solution of exercise 4.12 (Page 114)



1- T_x is the centripetal force required to describe the circle. Thus, Newton's second law along the radial direction gives:

$$T_x = T\sin\theta = \frac{mv^2}{r} = mr\omega^2 \qquad \text{(i)}$$

As:

$$\sin\theta = \frac{r}{\ell} \text{ or } r = \ell\sin\theta$$

Substituting in (i),

$$T = m\ell\omega^2 \qquad \text{(ii)}$$

Applying Newton's first law in the

vertical direction:

$$T_y = T\cos\theta = mg \text{ or } T = \frac{mg}{\cos\theta} \quad \text{(iii)}$$

From equations (ii) and (iii):

$$\omega = \sqrt{\frac{g}{\ell\cos\theta}} = 3.5\, rad/s.$$

2- Using equation (ii) or (iii):

$$T = 24.5\, N.$$

Solution of exercise 4.13 (Page 114)

Apply the impulse momentum relation:

$$\overrightarrow{J} = \Delta\overrightarrow{P} = m\Delta\overrightarrow{v} = m(\overrightarrow{v}_f - \overrightarrow{v}_i).$$

Since the force acts in one direction and $v_i = 0$. Thus:

$$J = mv_f = \mathrm{AREA}_{0\to12}.$$

Solve for v_f:

$$v_f = \frac{\mathrm{AREA}_{0\to12}}{m} = \frac{72}{3} = 24\,\mathrm{m/s}.$$

Solution of exercise 4.14 (Page 114)

Let us define the positive sense of the y-axis as upward. Hence, the velocity before the collision is $\overrightarrow{v}_b = -6\overrightarrow{j}$, and the velocity after the collision is $\overrightarrow{v}_a = 5\overrightarrow{j}$. Using the impulse-momentum theorem:

$$\overrightarrow{F}_{\text{avg}} = \frac{\Delta\overrightarrow{p}}{\Delta t} = \frac{m(\overrightarrow{v}_a - \overrightarrow{v}_b)}{\Delta t} = 11\overrightarrow{j}$$

The average force exerted on the ball by the ground has a magnitude of $F_{\text{avg}} =$ 11 N acting upward.

Solution of exercise 4.15 (Page 114)

1- At terminal velocity, the net force on the ball is zero. So,

$$mg - \lambda v_t = 0 \text{ or } \lambda = \frac{mg}{v_t} = 0.075\, kg/s.$$

2- At 82% v_t, the Newton's second law gives:

$$mg - \lambda v_t = m\,a \Leftrightarrow\ mg - \lambda\dot{y} = m\,\ddot{y}.$$

Rearranging:

$$\ddot{y} + \frac{\lambda}{m}\dot{y} - g = 0 \text{ or } \ddot{y} + 5\dot{y} - 10 = 0.$$

The solution to this differential equation is:

$$y(t) = 2t - \frac{2}{5}\left(1 - e^{-5t}\right).$$

Therefore,

$$v(t) = 2\left(1 - e^{-5t}\right).$$

Time to Reach 82% of v_t

$$0.82\cdot v_t = 2\left(1 - e^{-kt}\right)$$

Substitute $v_t = 2$ and solve for t,

$$t_{82\%} = 0.34\, s.$$

Thus, $y(t_{82\%}) \approx 0.36\, m$.

WORK AND ENERGY 5

5.1 Introduction

In the previous chapter, we saw that the utilization of Newton's laws enables us to solve problems in classical mechanics. In this chapter, we will introduce the important concepts of work done by a force and mechanical energy. We will see that these two quantities and the relationships between them can be used to solve mechanical problems easily.

5.2 Work

When applying a force to move an object, the effort required is greater when the displacement distance is large and the applied force is intense. The work of the force is a measure that takes into account this effort.

Work done by a constant force

Let us consider a moving body along a straight path from point A to point B and subjected to a constant force $\overrightarrow{F}$ during its displacement $S = AB$.

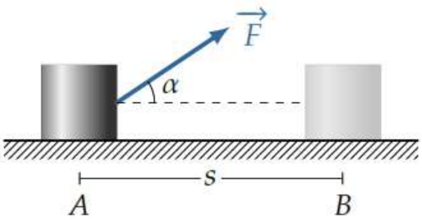


Figure 5.1: Work done by a constant force

The work done by this force on the body, denoted as W and expressed in joules, is defined, in this case, as follows:

$$\boxed{W\left(\overrightarrow{F}\right)_{A\longrightarrow B} = \overrightarrow{F}\cdot\overrightarrow{AB} = F\,S\,\cos\alpha} \tag{5.1}$$

where α represents the angle between the two vectors $\overrightarrow{F}$ and $\overrightarrow{AB}$. According to the value of this angle, three important cases can be distinguished:

- When $0 \leqslant \alpha < \frac{\pi}{2} : W > 0$, work done is said to be **positive**.

- When $\alpha = \frac{\pi}{2}$: $W = 0$, work done is said to be **zero**.
- When $\frac{\pi}{2} < \alpha \leqslant \pi$: $W < 0$, work done is said to be **negative**.

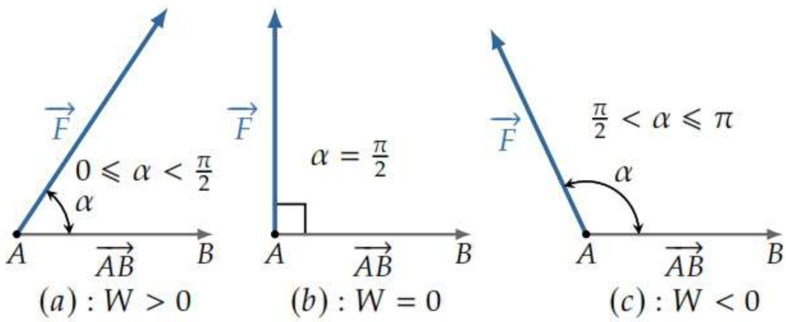
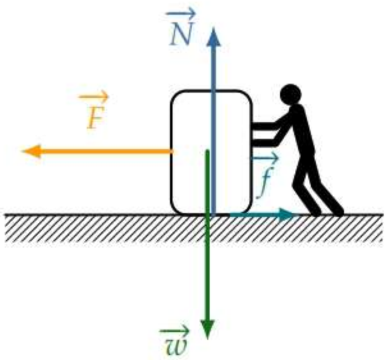


Figure 5.2: Positive, zero and negative work done by a force



Example 5.1

A box is pushed by a person with a constant force $F = 20N$ over a distance $S = 14\ m$. Knowing that the frictional force is $f = 6\ N$, find the work done by all the forces acting on the box.

Solution

As shown in the figure below, there are four forces acting on the box during the displacement S: the pushing force $\overrightarrow{F}$, the normal reaction $\overrightarrow{N}$, the weight $\overrightarrow{w}$, and the frictional force $\overrightarrow{f}$.

- $W\left(\overrightarrow{F}\right)_{A\longrightarrow B} = F\ S\ \cos\alpha = 20 \times 14 \times cos0 = 280$ J.
- $W\left(\overrightarrow{f}\right)_{A\longrightarrow B} = f\ S\ \cos\theta = 6 \times 14 \times cos180 = -84$ J.
- Since $\overrightarrow{N}$ and $\overrightarrow{w}$ are perpendicular to $\overrightarrow{S}$,
  $\therefore W\left(\overrightarrow{N}\right)_{A\longrightarrow B} = W\left(\overrightarrow{w}\right)_{A\longrightarrow B} = 0$ J.

Work done by a variable force

In cases where the force $\overrightarrow{F}$ varies in magnitude, direction, or both during the displacement, which can be arbitrary, it is no longer possible to use the Equation 5.1. In this case, the calculation of work consists of dividing the path AB into an infinite number of **elementary displacements** dS that are infinitesimally small and therefore rectilinear (see Figure 5.3).

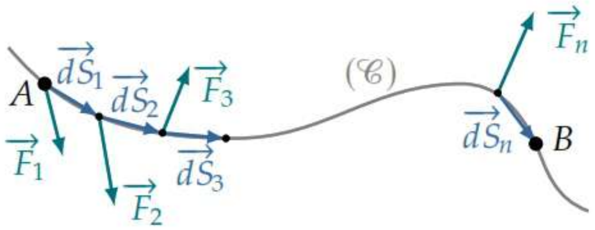


Figure 5.3: Work done by a force in general case

For each displacement, the force can be considered constant in both magnitude and direction, and the previous definition of work can be applied. The expression for the work done in such an elementary displacement (**elemental work** or **infinitesimal work**) can therefore be written as follows:

$$\boxed{dW = \overrightarrow{F} \cdot d\overrightarrow{S} = F\,dS\,\cos\alpha} \tag{5.2}$$

Now, the total work done by the force along the path $(\mathscr{C})$ corresponds to **the sum of all infinitesimal works**, which is denoted by **the integral** as follows:

$$\boxed{W\left(\overrightarrow{F}\right)_{A\longrightarrow B} = \int_{\mathscr{C}} dW = \int_{\mathscr{C}} \overrightarrow{F} \,.\, d\overrightarrow{S} = \int_{A}^{B} \overrightarrow{F} \,.\, d\overrightarrow{S}} \tag{5.3}$$

Example 5.2

In order to transport an object from the origin O to point $A(12{,}0)$, a force $\overrightarrow{F} = 2x\,\overrightarrow{i}$ is exerted in a direction that is parallel to the displacement. What is the work done by this force along the path OA?

Solution

Since the force varies, the work can be calculated using:

$$W = \int_{\text{path}} \overrightarrow{F} \cdot d\overrightarrow{x} = \int_{O}^{A} \overrightarrow{F} \cdot d\overrightarrow{x}$$
$$= \int_{0}^{12} 2x\,dx = x^2\Big|_{0}^{12} = 144\ \text{J}.$$

Example 5.3

An object traverses a parabolic trajectory described by the equation $y = \frac{1}{2}x^2$. This motion occurs under the influence of the force $\overrightarrow{F} = 6\,y\,\overrightarrow{i} + 10\,x\,\overrightarrow{j}$. Determine the work done by $\overrightarrow{F}$ from the point $A = (1, 0.5)$ to the point $B = (2, 2)$.

Solution

In this case, the work done by $\overrightarrow{F}$ is given by:

$$W = \int_A^B \overrightarrow{F} \cdot d\vec{r}$$

where:

$$d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath}$$

$$y = (0.5)x^2 \Leftrightarrow dy = x\,dx$$

thus,

$$\overrightarrow{F} = 3\,x^2\vec{\imath} + 9\,x\vec{\jmath} \quad \text{and} \quad d\vec{r} = dx\,\vec{\imath} + x\,dx\,\vec{\jmath}$$

therefore,

$$\overrightarrow{F} \cdot d\vec{r} = (3x^2 + 9x^2)\,dx = 12x^2\,dx$$

$$W = \int_{x=1}^{x=2} 12x^2\,dx = 28\,\text{J}.$$

Graphical representation of work

The work done by a force can be also evaluated graphically using the method of areas. Calculating the work to move an object from point A to point B involves determining the area enclosed by the force-displacement graph, the s-axis, and the two lines with equations $s = s_A$ and $s = s_B$. (see Figure 5.4).

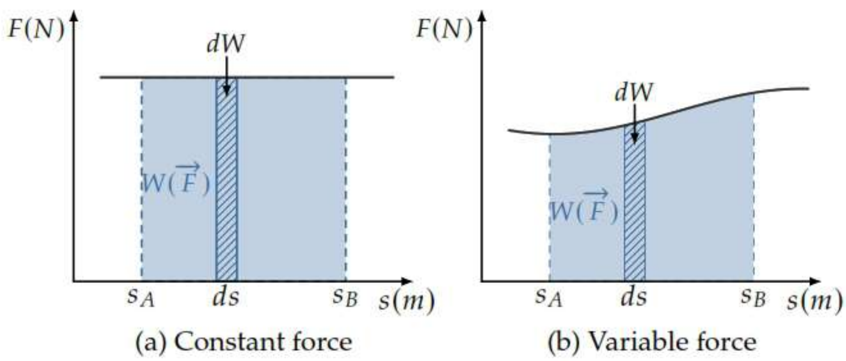


Figure 5.4: Geometrical interpretation of work

Examples of calculating work

Work done by the gravitational force

Let us consider an object of mass $m = 200\,g$ moving under the effect of the gravitational force along a trajectory AB as shown in Figure 5.5. z_A is the height of the object at point A and z_B is the height at point B ($z_A - z_B = h = 5\,m$).
Let us now calculate the work done by the weight during the displacement AB of this object.

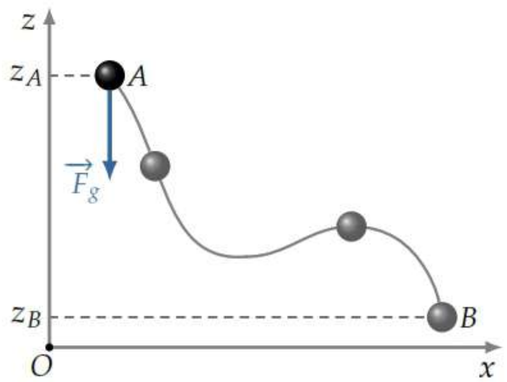


Figure 5.5: Work done by the gravitational force

As weight is a constant force in magnitude and direction, we can therefore write:

$$W\left(\vec{F}_g\right)_{A\to B} = \vec{F}_g \cdot \overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} \cdot \begin{pmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{pmatrix}$$
$$= -mg(z_B - z_A) = -mg \times -h = mgh$$

$$W\left(\vec{F}_g\right)_{A\to B} = mgh = 10\,\text{J}.$$

Therefore, we can express the work done by the weight as:

$$\boxed{W\left(\vec{F}_g\right)_{A\to B} = mg\,(z_A - z_B)} \tag{5.4}$$

Work done by spring force

Let us consider a spring-mass system as shown in Figure 5.6. The mass is pulled in order to stretch the spring from x_1 to x_2.

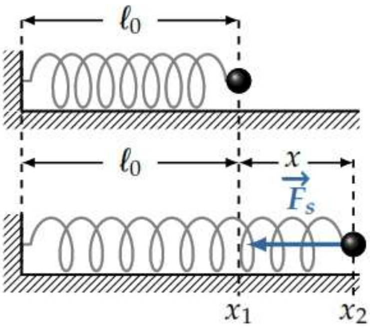


Figure 5.6: Work done by spring force

The work done by the spring force on the mass can be calculated as follows:

$$W\left(\vec{F}_s\right)_{x_1\to x_2} = \int_{x_1}^{x_2} \vec{F}_s \cdot \vec{dx} = \int_{x_1}^{x_2} -k\,x\,dx = -\frac{1}{2}k(x_2^2 - x_1^2)$$

The work done by spring force can expressed as follows:

$$\boxed{W\left(\vec{F}_s\right)_{x_1\to x_2} = -\frac{1}{2}k(x_2^2 - x_1^2)} \tag{5.5}$$

Example 5.4

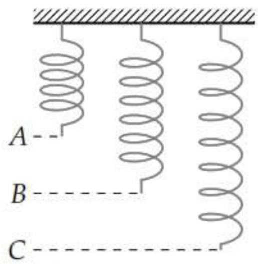


Figure 5.7

A perfectly elastic spring necessitates 0.25 J of work to be extended by 10 cm from its equilibrium position (point A), as illustrated in the figure Figure 5.7.

1- What is the value of its spring constant k?

2- How much additional work is needed to extend the spring by another 10 cm?

Solution

1- The work done to stretch a spring is given by:

$$W = \frac{1}{2}k(x_B^2 - x_A^2)$$

As point A represents the position of equilibrium, we have $x_A = 0$,

Thus,

$$W = \frac{1}{2}kx_B^2$$

Solving for k and substituting the known values

$$k = \frac{2W}{x_B^2} = \frac{2\times 0.25}{(10\times 10^{-2})^2} = 50\,\mathrm{N/m}.$$

2- The work required to stretch the spring from B to C:

$$W = \frac{1}{2}k(x_C^2 - x_B^2)$$

Substitute the known values

$$W = \frac{1}{2}k\left[(20\times 10^{-2})^2 - (10\times 10^{-2})^2\right] = 0.75\,\mathrm{J}.$$

5.3 Power

Average power

The average power over a specified time interval t is calculated by dividing the total work W by the time t, resulting in the formula:

$$\boxed{P_{\text{ave}} = \frac{W}{t}} \tag{5.6}$$

From an engineering perspective, the notion of power holds significant importance, as it is the rate at which a machine can perform work that is critical during the design process, rather than the cumulative amount of work the machine is capable of executing.

Instantaneous power

By introducing the definition of the infinitesimal work done by a force between the instants t and $t + dt$, it is possible to define an **instantaneous power** $P(t)$ as:

$$\boxed{P(t) = \frac{dW}{dt} = \vec{F} \,.\, \frac{d\vec{\ell}}{dt} = \vec{F} \,.\, \vec{v} = F\, v \cos\theta} \tag{5.7}$$

It is therefore clear that infinitesimal work can also be expressed through the power and written as:

$$dW = \vec{F} \,.\, \vec{v}\, dt = P\, dt$$

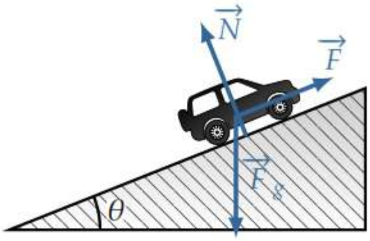
Which leads to the following expression for the work done by a force:

$$\boxed{W\left(\vec{F}\right)_{A \longrightarrow B} = \int_A^B \vec{F} \,.\, \vec{v}\, dt = \int_A^B P\, dt} \tag{5.8}$$

The power is expressed in the SI in **Watts** and it depends on the chosen frame of reference.

Example 5.5

A 1400 kg car ascends a prolonged incline at an angle of $\theta = 13.3°$ with a constant speed of 45 km/h. Assuming that all frictional forces are negligible, calculate the work



performed by the engine over a duration of 6 minutes, as well as the power generated by the engine.

Solution

Work done by the engine:
Apply the Newton's second law to calculate the force required to move the automobile up the incline:

$$F = mg\sin\theta = 3159.5\,\mathrm{N}$$

$$W = F \cdot s$$

where:

$$s = v \cdot t = 12.5 \cdot 360 = 4500\,\mathrm{m}$$

therefore

$$W = 3159.5 \cdot 4500 \approx 14.22 \times 10^6\,\mathrm{J}.$$

Power developed by the engine:

$$P = \frac{W}{t} = \frac{14.22 \times 10^6}{360} = 39.5 \times 10^3\,\mathrm{W} = 39.5\,\mathrm{kW}.$$

or

$$P = F\,v\,\cos\alpha$$

where $\alpha = 0$ is the angle formed between F and v.
Thus,

$$P = 3159.5 \times 12.5 \approx 39.5 \times 10^3\,\mathrm{W}.$$

5.4 Kinetic energy

Kinetic energy can be defined as the ability of an object to perform work by virtue of its motion.

5.5 The work-kinetic energy theorem

Let us consider an object of mass m moving in a Galilean reference frame $\mathcal{R}$ under the action of a set of external forces (net external force $\vec{F}_{net}$). The Newton's second law is then written as:

$$\vec{F}_{net} = \sum \vec{F}_{ext} = m\,\vec{a} = m\,\frac{d\vec{v}}{dt}$$

By multiplying this relation by $d\vec{x}$ (infinitesimal change in position), we obtain:

$$\begin{aligned}\sum \vec{F}_{ext} \,.\, d\vec{\ell} &= m\,\frac{d\vec{v}}{dt} \,.\, d\vec{\ell} = m\,\frac{d\vec{\ell}}{dt} \,.\, d\vec{v} \\ &= m\,\vec{v} \,.\, d\vec{v}\end{aligned}$$

The term $\sum \vec{F}_{ext} \,.\, d\vec{\ell}$ represents the sum of the elemental works done by the external forces. If we denote v_A and v_B as the speeds corresponding to the initial position A and the final position B, respectively, the total work (net work) done by the net force during the displacement AB can be written as:

$$\begin{aligned}\sum W\left(\vec{F}_{ext}\right)_{A\to B} &= \sum \int_A^B \vec{F}_{ext} \,.\, d\vec{\ell} = \int_A^B m\,\vec{v} \,.\, d\vec{v} \\ &= \frac{1}{2} m \left(v_B^2 - v_A^2\right) \\ &= \frac{1}{2} m\, v_B^2 - \frac{1}{2} m\, v_A^2\end{aligned}$$

The quantity $\frac{1}{2} m\, v^2$, denoted as KE (E_K or simply K) and expressed in **Joule**, is called **kinetic energy**.

$$\boxed{K = \frac{1}{2} m\, v^2} \tag{5.9}$$

It is important to recognize that, similar to the formulation of Newton's second law, which can be articulated in terms of either the rate of change of momentum (the product of mass and the rate of change of velocity), the kinetic energy of a particle can also be represented using its mass and momentum, rather than solely its mass and velocity.

$$\boxed{K = \frac{1}{2} m \left(\frac{p}{m}\right)^2 = \frac{p^2}{2m}} \tag{5.10}$$

In certain contexts, this formulation may prove to be more advantageous than Equation 5.9.

Kinetic energy depends on the chosen frame of reference. It is a scalar quantity and is always greater than or equal to zero.

Work-energy theorem

In a Galilean reference frame, the change in kinetic energy of a particle between two positions A and B is equal to the net work done by **all external forces** applied to this particle, that is:

$$\Delta K\Big|_A^B = K(B) - K(A) = \sum W\left(\overrightarrow{F}_{ext}\right)_{A\to B}$$

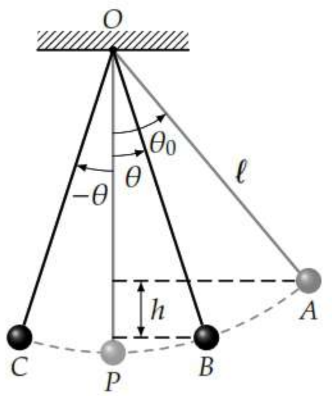


Figure 5.8: Simple pendulum

Example 5.6

Consider a simple pendulum consisting of a mass attached to an ideal rope of length $\ell = 1.28$ m, with the other end fixed at a point O. The mass is displaced from its vertical equilibrium position (point P) by an angle of $\theta_0 = 33.9°$ (point A) and then released. Determine the velocity of the mass when the rope makes an angle of 8.1° with the vertical, considering both the same side (point B) and the opposite side (point C) of the initial displacement.

Solution

Using the work-kinetic energy theorem:

$$K(B) - K(A) = \sum W\left(\overrightarrow{F}_{ext}\right)_{A\to B}$$

In our case:

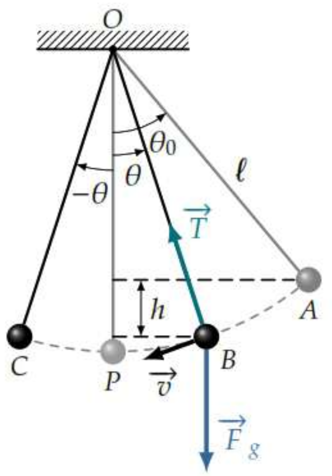
$$\frac{1}{2}m\,v_B^2 - \frac{1}{2}m\,v_A^2 = W\left(\overrightarrow{F}_g\right)_{A\to B} + W\left(\overrightarrow{T}\right)_{A\to B}$$

As $v_A = 0$ and $\overrightarrow{T}$ is perpendicular to the displacement, thus:

$$\frac{1}{2}m\,v_B^2 = W\left(\overrightarrow{F}_g\right)_{A\to B}$$

Since:

$$W\left(\overrightarrow{F}_g\right)_{A\to B} = mgh$$



where: $h = \ell(\cos\theta - \cos\theta_0)$. Thus,

$$v_B = \sqrt{2g\ell(\cos\theta - \cos\theta_0)} \approx 2\,\text{m/s}.$$

Since $\cos(-\theta) = \cos\theta$, the calculation is identical on the opposite side. Therefore,

$$v_C \approx 2\,\text{m/s}.$$

5.6 Conservative and non-conservative forces

A force is said to be **conservative** if work done between two positions A and B by this force does not depend on the path taken but only on the starting and ending points A and B. Otherwise, the force is **non-conservative**.

> The work done by a **conservative force** on a closed path (a round trip) is **zero**.

Example of conservative forces

Among conservative forces, we can mention some common forces such as weight in a uniform gravitational field, electrostatic force, gravitational force, and the restoring force of a spring.

> Any central force is conservative.

The weight

A mass m moves in the vertical plane (xOz) from point A to point B along two different paths $\mathscr{C}_1$ and $\mathscr{C}_2$. (see Figure 5.9). The work done by the weight along the two paths is given by:

$$W\left(\overrightarrow{w}\right)_{\mathscr{C}_1} = -mg(z_B - z_A)$$
$$W\left(\overrightarrow{w}\right)_{\mathscr{C}_2} = -mg(z_B - z_A)$$

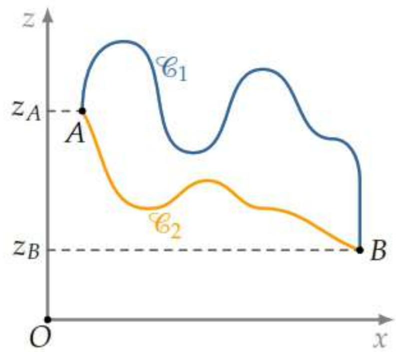


Figure 5.9: Work done by weight

We notice that the work done by the weight along the two paths is the same ($W_{\mathscr{C}_1} = W_{\mathscr{C}_2}$), it does not depend on the path taken. Therefore, weight is a conservative force.

Example of non-conservative forces

Let's now calculate the work done by the viscous friction force along the two previous paths. The velocity, assumed to be constant, is directed along a unit vector tangent to the trajectory $\overrightarrow{u}_T$. In this case, the work done by the viscous friction force $\overrightarrow{f}_v = -\alpha\, v\, \overrightarrow{u}_T$ can be calculated as:

$$W\left(\overrightarrow{f}_v\right)_{A\rightarrow B} = \int_A^B \overrightarrow{f}_v\,.\,d\overrightarrow{s} = \int_A^B -\alpha\, v\, \overrightarrow{u}_T\,.\,ds\,\overrightarrow{u}_T$$
$$= -\alpha\, v \int_A^B ds = -\alpha\, v\, s$$

where s represents the distance traveled between A and B. According to Figure 5.9, the distance traveled along path $\mathscr{C}_1$ is greater than that of path $\mathscr{C}_2$ ($s_{\mathscr{C}_1} > s_{\mathscr{C}_2}$), which shows that the work done along $\mathscr{C}_1$ is different from that of $\mathscr{C}_2$ (work depends on the path taken), therefore the viscous friction forces are **non-conservative forces**.

Potential Energy

Since the work of conservative forces $\left(\overrightarrow{F}^c\right)$ depends only on the initial and final positions of the moving body, we can express this work using a function called **potential energy**, denoted as E_p or simply U, which **depends only on the position of the body**. In other words, the potential energy is the energy that a body possesses by virtue of its position or configuration. The relationship between the work done by conservative forces and the change in the potential energy between two positions can be expressed as follows:

$$\boxed{\Delta U = U(B) - U(A) = -W\left(\overrightarrow{F}^c_{ext}\right)_{A\rightarrow B}} \qquad (5.11)$$

The potential energy is always defined through its **change** between two given positions.

It is also said that a force is conservative if it **derives from a potential or potential energy**. We can then write the potential energy as a function of the conservative force by introducing the mathematical operator **gradient** in the following way:

$$\boxed{\overrightarrow{F} = -\overrightarrow{grad}\, U} \qquad (5.12)$$

Examples of potential energy

Gravitational potential energy

As already mentioned by Equation 5.4, the work done by the gravitational force from a position z_0 to a position z_1 can be expressed as follows:

$$W\left(\overrightarrow{F_g}\right)_{z_0 \to z_1} = m\,g\,(z_0 - z_1)$$

Since $\overrightarrow{F_g}$ is a conservative force, the work done between z_0 and z_1 can be expressed as follows:

$$W\left(\overrightarrow{F_g}\right)_{z_0 \to z_1} = -\Delta U_g = U_g\,(z_0) - U_g\,(z_1) = m\,g\,(z_0 - z_1)$$

Then, we can define the potential energy function due to gravity $U_g(z)$ at a height z as:

$$\boxed{U_g\,(z) = m\,g\,z + c} \tag{5.13}$$

The potential energy is only determined up to a constant. This constant is determined by the arbitrary choice of the zero of potential energy. In general, the gravitational potential energy is taken as zero at $z = 0$ ($U_g(0) = 0$), which implies $c = 0$. This choice gives:

$$\boxed{U_g = m\,g\,z} \tag{5.14}$$

> If the Oz axis is directed downwards, the expression for the gravitational potential energy becomes:
>
> $$\boxed{U_g = -m\,g\,z}$$

Elastic potential energy

When a body undergoes a temporary change in its shape or configuration, such as stretching, compression, twisting, or bending, it acquires potential energy known as **elastic potential energy**. This energy is stored in the deformed body and is equal to the work done to change its shape. For example, the work done in stretching spring is equal to the energy stored as elastic potential energy in this stretched spring.

Upon release, the deformed body returns to its original shape, and the potential energy is transformed into another form, typically kinetic energy.

As we have mentioned in Equation 5.5, the work done by the spring force when moving the mass from x_0 to x can be expressed as follows:

$$W\left(\overrightarrow{F_s}\right)_{x_0 \longrightarrow x} = -\frac{1}{2}k\left(x^2 - x_0^2\right)$$

The spring force is a conservative force, so:

$$W\left(\overrightarrow{F_s}\right)_{x_0 \to x} = \frac{1}{2}k\left(x^2 - x_0^2\right) = -\Delta U_s$$

The elastic potential energy U_s can therefore be written as:

$$\boxed{U_s = \frac{1}{2}k\,x^2 + c} \tag{5.15}$$

If we consider the elastic potential energy to be zero when the spring is at rest (equilibrium position $x = 0$), the constant c becomes zero and therefore we will have:

$$\boxed{U_s = \frac{1}{2}k\,x^2} \tag{5.16}$$

Types of equilibrium

A body is said to be in equilibrium when both its velocity and acceleration are zero. According to Newton's second law, the net force acting on this body is zero, i.e

$$\vec{F}_{\text{net}} = \vec{0}.$$

When all the forces acting on a body in equilibrium are conservative, the work done by these forces on the body can be expressed as follows:

$$\delta W = -dU = \vec{F} \cdot \vec{dr} = F_r dr,$$

thus,

$$F_r = -\frac{dU}{dr}.$$

For a body in equilibrium (where $F = 0$),

$$-\frac{dU}{dr} = 0, \text{ or } \frac{dU}{dr} = 0.$$

At the equilibrium position, the slope of the potential energy (U) versus position (r) graph is zero, or potential energy reaches an extremum, which can be classified as a maximum, minimum, or constant value. Consequently, there are three types of equilibrium: **stable equilibrium**, **unstable equilibrium**, and **neutral equilibrium**.

The three types of equilibrium

- An equilibrium is considered stable if, when displaced from its equilibrium position, the body returns to its original equilibrium position (see Figure 5.10a).

  The potential energy at the equilibrium position is minimum as compared to its adjacent points. So:

  $$\frac{d^2U}{dr^2} > 0.$$

- An equilibrium is considered unstable if displacing it from its equilibrium position permanently prevents it from returning to that position (see Figure 5.10b).

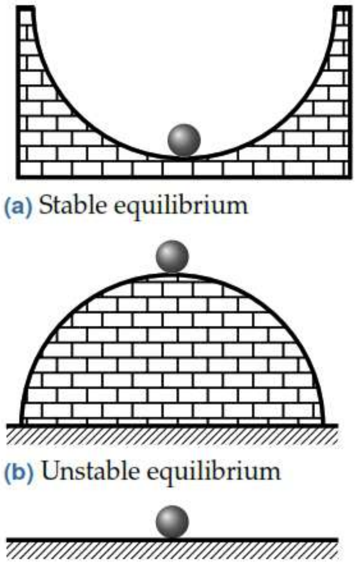
  The potential energy at the equilibrium position is maximum as compared to its adjacent points. So:

  $$\frac{d^2U}{dr^2} < 0.$$

- An equilibrium is considered neutral if, when displaced from its equilibrium position the body has neither the tendency to come back nor to move away from the original position (see Figure 5.10c).

  Potential energy remains constant even when a body is displaced from its equilibrium position. So:

  $$\frac{d^2U}{dr^2} = 0.$$



(a) Stable equilibrium

(b) Unstable equilibrium

(c) Neutral equilibrium

Figure 5.10: Graphical presentation of the three types of equilibrium

Example 5.7

The potential energy of a conservative system is expressed as: $U = 2x^2 - 8x$. Determine the equilibrium position and analyze the type of equilibrium.

Solution

For a conservative system, we have:

$$F = -\frac{dU}{dx} = -4x + 8$$

For equilibrium $F = 0$ thus, the equilibrium position is:

$$x = 2$$

$$\frac{d^2U}{dr^2} = 4 > 0.$$

A positive value, i.e. U is minimum, which indicates that $x = 2$ is a stable equilibrium position.

5.7 Total mechanical energy

The total mechanical energy or simply mechanical energy E of a body or a system is defined as the sum of its kinetic and potential energies, so that:

$$\boxed{E = K + U} \tag{5.17}$$

The work-mechanical energy theorem

This theorem is often known as the **modified form of the work-energy theorem**.

Let us consider the simultaneous action of conservative forces $\overrightarrow{F}^c$ and non-conservative forces $\overrightarrow{F}^{nc}$ on an object, which moves it from position A to position B. In that case, and according to the work-energy theorem, the net work done on the object by all the external forces can be written as:

$$\begin{aligned}\sum W\left(\overrightarrow{F}_{\text{ext}}\right)_{A\to B} &= \sum W\left(\overrightarrow{F}^{\text{nc}}_{\text{ext}}\right)_{A\to B} + \sum W\left(\overrightarrow{F}^{\text{c}}_{\text{ext}}\right)_{A\to B} \\ &= K(B) - K(A)\end{aligned}$$

We know that the net work done by all the conservative forces can be expressed as:

$$\sum W\left(\overrightarrow{F}^{c}_{ext}\right)_{A\to B} = -\left[U(B) - U(A)\right]$$

Substituting this equation into the previous one yields:

$$\Sigma W\left(\overrightarrow{F}^{nc}_{ext}\right)_{A\to B} - \left[U(B) - U(A)\right] = K(B) - K(A)$$

By rearranging, the work-energy theorem can be expressed in terms of the total mechanical energy as follows:

$$\left[K(B) + U(B)\right] - \left[K(A) + U(A)\right] = \sum W\left(\overrightarrow{F}^{nc}_{ext}\right)_{A\to B}$$

$$\boxed{E(B) - E(A) = \Delta E\Big|_{A}^{B} = \Sigma W\left(\overrightarrow{F}^{nc}_{ext}\right)_{A\to B}} \tag{5.18}$$

This last relation is called the **modified form of the work-energy theorem**, which can be stated as follows:

Statement of the theorem

In a Galilean reference frame, the change in total mechanical energy of an object between two arbitrary positions A and B is equal to the work done on this object by all the external **non-conservative** forces.

Conservation of mechanical energy

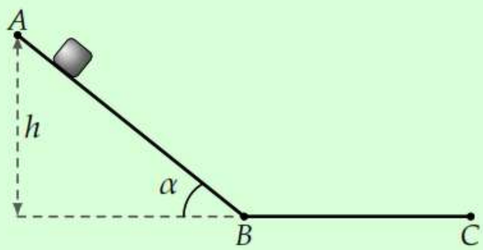
The change in total mechanical energy of an object or a system subject only to conservative forces is zero ($\Delta E = 0$), in other words, the total mechanical energy of this system is conserved during its motion ($E = Cte$).
So we obtain a conservation law when the forces derive from a potential.

i The potential energy can vary by transforming into kinetic energy, and conversely, the kinetic energy can transform into potential energy, with the sum of the two remaining constant.

Example 5.8

Suppose a block of mass m is released without initial velocity from point A on a track ABC consisting of an inclined plane AB which makes an angle α with the horizontal plane BC, as shown in figure. Give the expression for the velocity at point C in the following cases:

1. If the track ABC is perfectly smooth.
2. If the friction forces exist on the track ABC.



Solution

1. If the track is perfectly smooth, the frictional forces are zero. Then the forces acting on m are the weight $\overrightarrow{w}$ and the normal reaction $\overrightarrow{N}$. In that case, the total mechanical energy is conserved, so:

$$E(C) - E(A) = \sum W\left(\vec{F}^{nc}_{ext}\right)_{A\to C} = 0$$

By choosing a zero potential energy at the level of the plane BC, then:

$$E(C) = \frac{1}{2}m\, v_C^2 \text{ and } E(A) = mgh$$

$$\frac{1}{2}m\, v_C^2 - mgh = 0 \Rightarrow v_C = \sqrt{2gh}$$

2. In the case when the friction force $\vec{f}$ acts, $\vec{F}^{nc}_{ext} = \vec{f}$, so:

$$E(C) - E(A) = \sum W\left(\vec{F}^{nc}_{ext}\right)_{A\to B} = W\left(\vec{f}\right)_{A\to C}$$

$$W\left(\vec{f}\right)_{A\to C} = W\left(\vec{f}\right)_{A\to B} + W\left(\vec{f}\right)_{B\to C}$$

- Between A and B: $f = \mu_c mg\ cos\alpha$
  $\Rightarrow W\left(\vec{f}\right)_{A\to B} = -\mu_c mg\ AB\ cos\alpha$
- Between B and C: $f = \mu_c mg$
  $\Rightarrow W\left(\vec{f}\right)_{B\to C} = -\mu_c mg\ BC$

$$v_C = \sqrt{2g\left[h - \mu_c(BC + AB\ cos\alpha)\right]}.$$

CHAPTER 5 SUMMARY

Work done by a constant force: $W\left(\overrightarrow{F}\right)_{A\longrightarrow B} = \overrightarrow{F}\cdot\overrightarrow{AB} = F\,S\,\cos\alpha$.

Work done by a variable force: $W\left(\overrightarrow{F}\right)_{A\longrightarrow B} = \int_A^B \overrightarrow{F}\,.\,d\overrightarrow{S}$.

Work done by the gravitational force: $W\left(\overrightarrow{F_g}\right)_{A\rightarrow B} = mg\,(z_A - z_B)$.

Work done by spring force: $W\left(\overrightarrow{F_s}\right)_{x_1\rightarrow x_2} = -\frac{1}{2}k(x_2^2 - x_1^2)$.

Average power: $P_{\text{ave}} = \frac{W}{t}$.

Instantaneous power: $P(t) = \frac{dW}{dt} = \overrightarrow{F}\cdot\overrightarrow{v} = F\,v\cos\theta$.

Work-Power relation: $W\left(\overrightarrow{F}\right)_{A\longrightarrow B} = \int_A^B P(t)dt$.

Kinetic energy: $K = \frac{1}{2}mv^2$.

The work-kinetic energy theorem: $\Delta K\Big|_A^B = K(B) - K(A) = \sum W\left(\overrightarrow{F}_{ext}\right)_{A\rightarrow B}$.

Conservative and non-conservative forces: A force is said to be conservative if work done between two positions A and B by this force does not depend on the path taken but only on the starting and ending points A and B. Otherwise, the force is non-conservative.

Potential Energy:
The potential energy is the energy that a body possesses by virtue of its position or configuration. The relationship between the work done by conservative forces and the change in the potential energy between two positions can be expressed as follows: $\Delta U = U\,(B) - U\,(A) = -W\left(\overrightarrow{F}{}^{c}_{ext}\right)_{A\rightarrow B}$.

Gravitational potential energy: $U_g\,(z) = m\,g\,z + c$.
In general, the gravitational potential energy is taken as zero at $z = 0$, so $U_g = m\,g\,z$. If the Oz axis is directed downwards, the expression for the gravitational potential energy becomes: $U_g = -m\,g\,z$.

Elastic potential energy: $U_s = \frac{1}{2}k\,x^2$.

Stable equilibrium: the potential energy at the equilibrium position is **minimum** as compared to its adjacent points. So: $\frac{d^2U}{dr^2} > 0$.

Unstable equilibrium: the potential energy at the equilibrium position is **maximum** as compared to its adjacent points. So: $\frac{d^2U}{dr^2} < 0$.

Neutral equilibrium: Potential energy remains **constant** even when a body is displaced from its equilibrium position. So: $\frac{d^2U}{dr^2} = 0$.

Total mechanical energy: $E = K + U$

The work-mechanical energy theorem: $E(B) - E(A) = \sum W\left(\overrightarrow{F}{}^{nc}_{ext}\right)_{A\rightarrow B}$.

5.8 Exercises

Exercise 5.1

The potential energy used to describe the interaction between two atoms within a molecule is expressed by the equation:

$$U(r) = \frac{a}{r^{12}} - \frac{b}{r^6}$$

where, a and b are positive constants, and r represents the interatomic distance. Find the equilibrium distance between two atoms.

Sol : page 144

Exercise 5.2

A particle moves from point A(1, -2) to point B(4, 2). Calculate the total work done by the three constant forces: $\overrightarrow{F}_1 = 2\overrightarrow{\imath} - \overrightarrow{\jmath}$, $\overrightarrow{F}_2 = \overrightarrow{\imath} + \overrightarrow{\jmath}$, and $\overrightarrow{F}_3 = -3\overrightarrow{\imath} - 2\overrightarrow{\jmath}$ on this particle.

Sol : page 144

Exercise 5.3

A worker pushes a crate 35 meters at a constant speed on flat ground, applying a force at an angle of -22° relative to the horizontal. The frictional forces are represented by a force of 42 N.

1. Determine the work done on the crate by the gravitational force.
2. Calculate the work done by the frictional force on the crate.
3. What is the work done on the crate by the worker?
4. Find the force exerted by the worker using energy considerations.
5. Evaluate the total work performed on the crate.

Sol : page 144

Exercise 5.4

When brakes are applied, the minimum distance required to stop a vehicle traveling at a velocity of 50 km/h is 3 meters. Determine the minimum distance required to come to a complete stop if the vehicle's speed is increased to 100 km/h.

Sol : page 144

Exercise 5.5

An object is traveling from point A to point B. At position A, its kinetic energy is 72 J and its potential energy is -35 J. At position B, the kinetic energy increases to 112 J, and the potential energy rises to 55 J. Calculate:

1. The work done by conservative forces.
2. The work done by non-conservative forces.
3. The net work performed.

Sol : page 144

Exercise 5.6

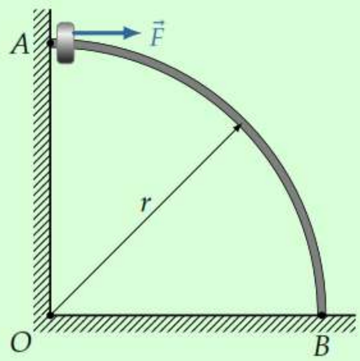
An object with a mass of 600 g is thrown downward from a height of 100 m with an initial velocity of 4 m/s. Assuming that air resistance is negligible, calculate:

1. The initial kinetic and potential energies.
2. The kinetic energy when the object hits the ground.
3. The object's speed when it hits the ground.

Sol : page 145

Exercise 5.7

A 1 kg ring begins its motion from a stationary position at point A and travels along a smooth, fixed quarter-vertical circle with a radius of $r = 4$ meters and center O. It arrives at point B with a speed of $8\sqrt{2}$ m/s. The ring is acted upon by a constant horizontal force of $F = 6$ N. Calculate the work done by the force F in two different ways, given that $g = 10\ \mathrm{m/s^2}$.



Sol : page 145

Exercise 5.8

Ignoring all frictional forces, how much time is required for a car with a mass of 1100 kg and a useful output power of 80190 W to attain a velocity of 27 m/s, when:

1. The car is traveling across a level surface.
2. The car is climbing a hill that is 4.5 meters high.

Sol : page 146

Exercise 5.9

A 3 kg object is released from rest and descends a vertical distance of 10 m, ultimately reaching a speed of 6 m/s. Determine the work done by air resistance on the object during its fall.

Sol : page 146

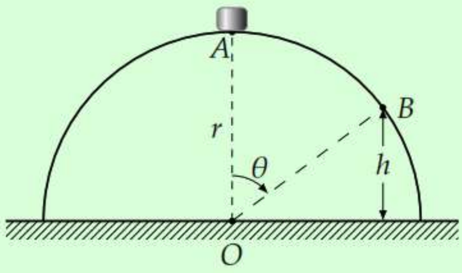
Exercise 5.10

A bullet with a mass of 3.2 grams is fired at a block of wood from very close range, with an initial velocity of 320 m/s. It penetrates the block and comes to a stop after traveling 35 cm. What is the average stopping force exerted by the wood?

Sol : page 146

Exercise 5.11

Assuming that an object with mass m begins to slide from rest at the summit of a hemispherical mound with center O and radius r, and reaches point B where it leaves the mound. What is the height of point B if all frictional forces are negligible?



Sol : page 146

Exercise 5.12

A 4 kg block is pushed against a massless horizontal spring with a spring constant of $k = 2000\ \mathrm{N/m}$. When the spring is compressed by $x = 13$ cm, the block is released and travels a distance of $d = 1.8$ m before coming to rest. Calculate the coefficient of kinetic friction μ_k between the block and the ground.

Sol : page 147

Exercise 5.13

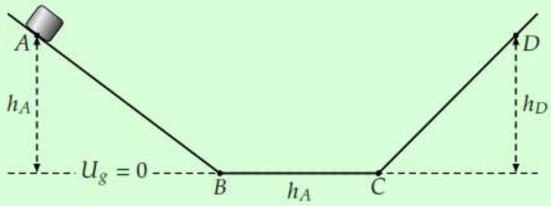
A box with an initial total energy of 30 J slides along a frictionless horizontal surface. Upon hitting a spring, the box compresses the spring by 20 cm from its equilibrium position. However, when sliding on a rough surface while maintaining the same initial energy, the box compresses the spring by only 8 cm. Determine the amount of energy dissipated due to friction while sliding on the rough surface.

Sol : page 147

Exercise 5.14

A block with a mass m slides from point A on track $ABCD$ without any initial velocity. The track consists of an inclined plane AB with a height of h_A, a horizontal section BC with a length of h_A, and another inclined plane CD with a height of h_D (see Figure below).

1. Assuming that all friction is negligible, what must the height h_D be for the block to reach point D without overshooting it?
2. Now, assuming that friction exists only in section BC and is characterized by a coefficient of kinetic friction, μ_k, what is the maximum height h reached by the block?
3. A spring with a stiffness constant k is placed on the inclined section AB. The block is pushed against the spring until it reaches point A with a compression distance x, after which the block is released. What must the value of x be for the block to reach point D without overshooting?



Sol : page 147

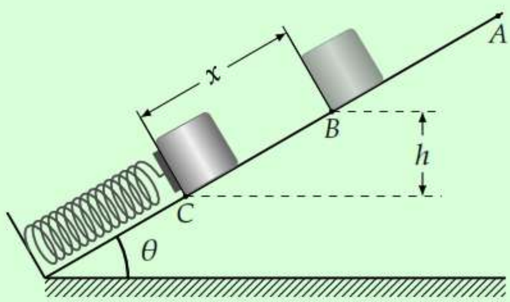
Exercise 5.15

A child with a mass of $m = 36$ kg slides down a toboggan inclined at an angle of $\theta = 45°$ to the horizontal. The vertical height difference between the top and bottom of the toboggan is $h = 5.9$ m. The toboggan has a coefficient of friction, $\mu = 0.3$. Determine the speed of the child at the bottom of the toboggan. Compare it to the speed they would have if there were no friction.

Sol : page 148

Exercise 5.16

A 7 kg block is released from rest at the top (point A) of an inclined plane that is angled at $\theta = 30°$ to the horizontal. The block slides down the plane without friction and compresses a spring with a stiffness constant of $k = 120\,\mathrm{N/m}$ placed at the bottom of the incline. Upon impact, the block comes to a momentary stop after compressing the spring by $x = 1$ m. What is the speed of the block just before it hits the spring?

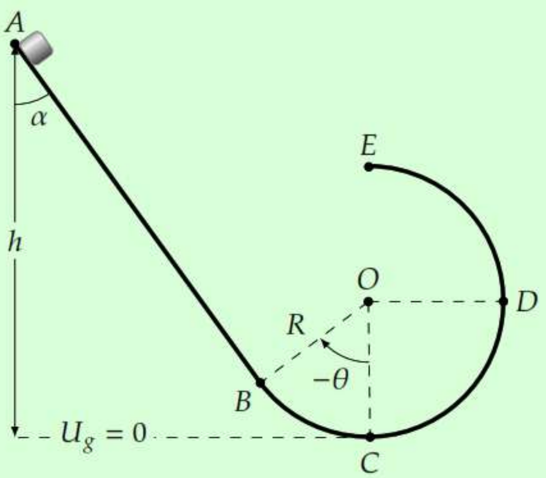


Sol : page 148

Exercise 5.17

A box is released from rest at point A on a track ABE, which consists of an inclined straight line segment AB and a circular arc segment BE with center O and radius R (refer to the accompanying Figure). If friction exists only on segment AB, where the following parameters are given: $g = 9.81\ \mathrm{m\,s^{-2}}$, $h = 2$ m, $R = 0.7$ m, $AB = 2.1$ m, $\alpha = 36°$, $\theta = 53.6°$, and $\mu_k = 0.2$.

1. Calculate the speed of the box at points B, C, D, and E.
2. Determine the required value of R for the box to come to a complete stop at point D without exceeding it.
3. What must the value of μ_k be for the box to stop at point E without surpassing it?



Sol : page 148

5.9 Solutions

Solution of exercise 5.1 (Page 140)
The force F is related to the potential energy $U(r)$ by:

$$F = -\frac{dU(r)}{dr}$$

At equilibrium, $F = 0$ thus:

$$\left.\frac{dU(r)}{dr}\right|_{r=r_{\mathrm{eq}}} = 0$$

or

$$-12ar_{\mathrm{eq}}^{-13} + 6br_{\mathrm{eq}}^{-7} = 0$$

solving for r_{eq},

$$r_{\mathrm{eq}} = \left(\frac{2a}{b}\right)^{\frac{1}{6}}.$$

Solution of exercise 5.2 (Page 140)
The work done by the three constant forces is given by:

$$\begin{aligned} W_{\mathrm{net}} &= \overrightarrow{F}_1 \cdot \overrightarrow{AB} + \overrightarrow{F}_2 \cdot \overrightarrow{AB} + \overrightarrow{F}_3 \cdot \overrightarrow{AB} \\ &= (\overrightarrow{F}_1 + \overrightarrow{F}_2 + \overrightarrow{F}_3) \cdot \overrightarrow{AB} \\ &= (-2\overrightarrow{j}) \cdot (3\overrightarrow{\imath} + 4\overrightarrow{j}) = -8\,\mathrm{J}. \end{aligned}$$

Solution of exercise 5.3 (Page 140)

1. Since the gravitational force is perpendicular to the displacement, its work is:

$$W(\overrightarrow{F}_g) = \overrightarrow{F}_g \times d \times \cos(90^\circ) = 0\,\mathrm{J}.$$

2. The work done by friction force f is given by:

$$\begin{aligned} W(\overrightarrow{f}) = \overrightarrow{f} \cdot \overrightarrow{d} &= f \times d \times \cos(180^\circ) \\ &= -fd = -1470\,\mathrm{J}. \end{aligned}$$

3. The worker's force has a horizontal component that balances the frictional force:

$$F_x = F_{\mathrm{worker}} \cos(22^\circ) = 42\,\mathrm{N}.$$

The work done by the worker can expressed as:

$$W(\overrightarrow{F}_x) = F_x \cdot d = 42 \cdot 35 = 1470\,\mathrm{J}.$$

4. Solving for F_{worker} the equation above:

$$F_{\mathrm{worker}} = \frac{42}{\cos(22^\circ)} \approx 45.3\,\mathrm{N}.$$

5. The total work done on the crate is the sum of the work done by all forces:

$$W_{\mathrm{net}} = W(\overrightarrow{F}_g) + W(\overrightarrow{f}) + W(\overrightarrow{F}_x) = 0\,\mathrm{J}.$$

Solution of exercise 5.4 (Page 140)
The work done by the braking force is given by:

$$W(\overrightarrow{F}) = \overrightarrow{F} \cdot \overrightarrow{d} = -Fd = -\frac{1}{2}mv_i^2$$

Solving for F in the two cases:

$$F = \frac{mv_1^2}{2d_1} \qquad \text{(i)}$$

$$F = \frac{mv_2^2}{2d_2} \qquad \text{(ii)}$$

Solving for d_2,

$$d_2 = d_1\frac{v_2^2}{v_1^2} = 12\,\mathrm{m}.$$

Solution of exercise 5.5 (Page 140)

1. Work done by conservative forces:

$$W(\vec{F}^c) = -\Delta U = -(U_B - U_A) = -90\,\mathrm{J}.$$

2. Work done by non-conservative

forces:

$$W(\vec{F}^{\mathrm{nc}}) = E(B) - E(A) = 130\,\mathrm{J}.$$

3. Net work done:

$$W_{\mathrm{net}} = K(B) - K(A) = 40\,\mathrm{J}.$$

Remarks:
- The work done on the object by conservative forces can also be determined as follows:

$$W(\vec{F}^{\mathrm{c}}) = W_{\mathrm{net}} - W(\vec{F}^{\mathrm{nc}}) = -90\,\mathrm{J}.$$

- The work done on the object by non-conservative forces can also be determined as follows:

$$W(\vec{F}^{\mathrm{nc}}) = W_{\mathrm{net}} - W(\vec{F}^{\mathrm{c}}) = 130\,\mathrm{J}.$$

- The net work (total) done on the object can also be determined as the sum of the work done by conservative and non-conservative forces. Therefore:

$$W_{\mathrm{net}} = W(\vec{F}^{\mathrm{c}}) + W(\vec{F}^{\mathrm{nc}}) = 40\,\mathrm{J}.$$

Solution of exercise 5.6 (Page 140)

1. The initial kinetic and potential energies:

$$K_i = \frac{1}{2}mv^2 = \frac{1}{2} \times 0.6 \times 4^2 = 4.8\,\mathrm{J}.$$

We take the zero of potential energy at ground level ($h = 0$)

$$P_i = mgh = 0.6 \times 9.81 \times 100 = 588.6\,\mathrm{J}.$$

2. The final kinetic energy:

$$\Delta K = K_f - K_i = W(\vec{F})_g = mgh$$

or

$$K_f = mgh + K_i = 593.4\,\mathrm{J}.$$

We can also use mechanical energy. Since air resistance is negligible, mechanical energy is conserved. Therefore,

$$E_f = E_i \Leftrightarrow K_f = mgh + K_i = 593.4\,\mathrm{J}.$$

3. The final speed:

$$K_f = \frac{1}{2}mv_f^2$$

or

$$v_f = \sqrt{\frac{2K_f}{m}} = 44.47\,\mathrm{m/s}.$$

Solution of exercise 5.7 (Page 141)

1. First method:

$$W(\vec{F}) = F \times r = 24\,\mathrm{J}.$$

2. Second method:
Using the work-energy theorem,

$$W_{\mathrm{net}} = K(B) - K(A)$$

or

$$W(\vec{F}) + W(\vec{F}_g) = K(B) - K(A)$$

As $K(A) = 0$ and $W(\vec{F}_g) = mgr$, thus:

$$W(\vec{F}) = \frac{1}{2}mv_B^2 + mgr$$

substituting the known values:

$$W(\vec{F}) = 24\,\mathrm{J}.$$

Solution of exercise 5.8 (Page 141)

1- We have:

$$P = \frac{W}{t} \text{ or } t = \frac{W}{P}$$

Applying the work-energy theorem gives:

$$\Delta K = W$$

Assuming that the car starts from rest ($v_0 = 0$), thus:

$$W = \frac{1}{2}mv_f^2$$

therefore,

$$t = \frac{mv_f^2}{2P} = 5 \text{ s}.$$

2- In this case, the total work done is equal to the work performed by the engine, as calculated previously, plus the work done by gravity.

$$W = \frac{1}{2}mv_f^2 + mgh$$

thus,

$$t = \frac{m(v_f^2 + 2gh)}{2P} = 5.6 \text{ s}.$$

Solution of exercise 5.9 (Page 141)

Using the work-energy theorem,

$$W_{\text{net}} = \Delta K$$

or

$$W(\vec{F}_g) + W(\vec{f}_{\text{air}}) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

or

$$mgh + W(\vec{f}_{\text{air}}) = \frac{1}{2}mv_f^2.$$

Solving for $W(\vec{f}_{\text{air}})$,

$$W(\vec{f}_{\text{air}}) = \frac{1}{2}mv_f^2 - mgh$$

Substituting the known values,

$$W(\vec{f}_{\text{air}}) = -240.3 \text{ J}.$$

Solution of exercise 5.10 (Page 141)

Since $K_f = 0$ and $W_{\text{net}} = W(\vec{F}_{\text{avg}})$, the work-energy theorem can be expressed as follows:

$$W(\vec{F}_{\text{avg}}) = -K_i$$

or

$$-F_{\text{avg}} \times d = -\frac{1}{2}mv_f^2$$

so

$$F_{\text{avg}} = \frac{mv_f^2}{2d} = 468.11 \text{ N}.$$

Solution of exercise 5.11 (Page 141)

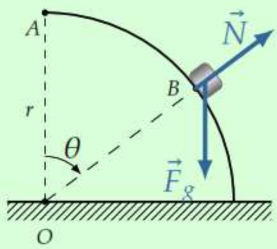
All frictional forces are negligible, thus:

$$E(A) = E(B) \Leftrightarrow mgr = \frac{1}{2}mv_B^2 + mgh$$

or

$$v_B = \sqrt{2g(r-h)}.$$

Apply Newton's second law at point B:



$$mg\cos\theta - N_B = m\frac{v_B^2}{r}$$

substituting v_B in the equation above:

$$N_B = mg\left(\cos\theta - 2 + \frac{2h}{r}\right)$$

as $\cos\theta = \frac{h}{r}$, therefore:

$$N_B = mg\left(\frac{3h}{r} - 2\right).$$

The object leaves the hemispherical mound at point B of height h, thus

$N_B = 0$. Therefore,

$$h = \frac{2r}{3}.$$

Solution of exercise 5.12 (Page 141)

We designate point A as the position of the block when the spring is compressed by $x = 13$ cm, and point B as the position of the block after it has traveled a distance of $d = 1.8$ m. Applying the work-mechanical energy theorem gives:

$$E(B) - E(A) = W\left(\overrightarrow{f}_k\right)_{A \to B}$$

as:

$$E(B) = 0,\ E(A) = \frac{1}{2}kx^2, \quad \text{and } W\left(\overrightarrow{f}_k\right)_{A \to B} = -\mu_k m g d$$

thus,

$$-\frac{1}{2}kx^2 = -\mu_k m g d$$

Solving for μ_k

$$\mu_k = \frac{kx^2}{2mgd} = 0.239.$$

Solution of exercise 5.13 (Page 141)

Let E_i, E_{f1}, and E_{f2} represent the initial energy, the final energy on the frictionless surface, and the final energy on the rough surface, respectively. Therefore,

$$E_{\text{lost}} = E_i - E_{f2} \qquad \text{(i)}.$$

$$E_{f2} = \frac{1}{2}kx_2^2 \qquad \text{(ii)}.$$

The mechanical energy is conserved on the frictionless surface. Thus:

$$E_i = E_{f1} \Leftrightarrow E_i = \frac{1}{2}kx_1^2$$

solving for k

$$k = \frac{2E_i}{x_1^2} = \frac{2 \times 30}{(0.2)^2} = 1500\,\text{N}\,\text{m}^{-1}.$$

Substituting the value of k in equ. (ii),

$$E_{f2} = \frac{1}{2}1500 \times (0.08)^2 = 4.8\,\text{J}.$$

Thus,

$$E_{\text{lost}} = 30 - 4.8 = 25.2\,\text{J}$$

Solution of exercise 5.14 (Page 142)

1. The mechanical energy is conserved, so:

   $$E_A = E_D \Leftrightarrow mgh_A = mgh_D$$

   thus,

   $$h_D = h_A.$$

2. Using the work-energy theorem:

   $$E_h - E_A = W(\vec{f}_k)$$

   or

   $$mgh - mgh_A = -\mu_k m g h_A$$

   solving for h

   $$h = h_A(1 - \mu_k).$$

   Since $\mu_k < 1$, therefore: $h < h_A$.

3. Using the work-energy theorem:

   $$mgh_D - mgh_A - \frac{1}{2}kx^2 = -\mu_k m g h_A$$

   As $h_D = h_A$, therefore:

   $$x = \sqrt{\frac{2m.g.\mu_k.h_A}{k}}.$$

Solution of exercise 5.15 (Page 142)

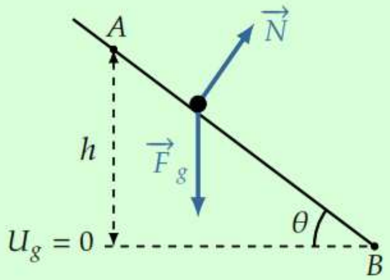
With friction, the work-energy theorem can be written:

$$E(B) - E(A) = W(\vec{f}_k)$$

or

$$mgh - \frac{1}{2}mv_B^2 = -\mu_k N AB$$

Newton's second law along the vertical axis states:



$$N = mg\cos\theta$$

thus,

$$mgh - \frac{1}{2}mv_B^2 = -\mu_k mgAB\cos\theta$$

Since $AB = \dfrac{h}{\sin\theta}$ and $\cos 45 = \sin 45$

$$v_B = \sqrt{2gh(1-\mu_k)} = 9\,\mathrm{m\,s^{-1}}.$$

Without friction, $\mu_k = 0$

$$v_B = \sqrt{2gh} = 10.76\,\mathrm{m\,s^{-1}}.$$

Therefore, the reduction in speed due to friction is:

$$\Delta v_B = 10.76 - 9 = 1.76\,\mathrm{m\,s^{-1}}.$$

Solution of exercise 5.16 (Page 142)

Let B be the point where the block makes contact with the spring. We will establish the reference height ($h = 0$) at point C, where the spring is fully compressed. Since the block slides without friction, we will apply the principle of conservation of mechanical energy, which states that:

$$E(C) = E(B) \Leftrightarrow \frac{1}{2}kx^2 = mgh + \frac{1}{2}mv_B^2,$$

the height h can be determined using the sine of the angle θ and x: $h = x\sin\theta$, substituting into the equation above and solving for v_B,

$$v_B = \sqrt{\frac{kx^2 - 2mgx\sin\theta}{m}} = 2.7\,\mathrm{m\,s^{-1}}.$$

Solution of exercise 5.17 (Page 142)

1. In the presence of friction, the work-energy theorem is expressed as follows:

$$E_f - E_i = W(\vec{f}_k).$$

In our case,

$$E(B) - E(A) = W(\vec{f}_k)_{A\to B},$$

or

$$\frac{1}{2}mv_B^2 + mgh_B - mgh = -\mu_k N AB,$$

As $h_B = R(1-\cos\theta)$ and $N = mg\sin\alpha$, therefore:

$$v_B = \sqrt{2g\left[h - \mu_k AB\sin\alpha - R(1-\cos\theta)\right]}$$
$$= 5.37\,\mathrm{m\,s^{-1}}.$$

Similarly,

$$E(C) - E(A) = W(\vec{f}_k)_{A\to B},$$

or

$$v_C = \sqrt{2g(h - \mu_k AB\sin\alpha)} = 5.86\,\mathrm{m\,s^{-1}}.$$

As no friction between B and C, thus:

$$E(C) = E(B) \Leftrightarrow \frac{1}{2}mv_C^2 = \frac{1}{2}mv_B^2 + mgh_B,$$

thus,

$$v_C = \sqrt{v_B^2 + 2gR(1-\cos\theta)} = 5.86\,\mathrm{m\,s^{-1}}.$$

Similarly,

$$v_D = \sqrt{v_C^2 - 2gR} = 4.54\,\mathrm{m\,s^{-1}}.$$

$$v_E = \sqrt{v_C^2 - 4gR} = 2.62\,\mathrm{m\,s^{-1}}.$$

It is also possible, and simpler, to calculate the speed at points C, D, and E by substituting the values of θ that correspond to these points into the formula for v_B.
At point C, $\theta = 0$, thus:

$$v_C = \sqrt{2g\left[h - \mu_k AB\sin\alpha - R(1-\cos 0)\right]}$$
$$= 5.86\,\mathrm{m\,s^{-1}}.$$

At point D, $\theta = 90$, thus:

$$v_D = \sqrt{2g\left[h - \mu_k AB\sin\alpha - R(1-\cos 90)\right]}$$
$$= 4.54\,\mathrm{m\,s^{-1}}.$$

At point E, $\theta = 180$, thus:

$$v_E = \sqrt{2g\left[h - \mu_k AB\sin\alpha - R(1-\cos 180)\right]}$$
$$= 2.63\,\mathrm{m\,s^{-1}}.$$

2. The box comes to a stop at point D without going beyond it; this signifies $v_D = 0$.

$$E(D) = E(C) \Leftrightarrow mgR = \frac{1}{2}mv_C^2,$$

thus,

$$R = \frac{v_C^2}{2g} = 1.75\,\mathrm{m}.$$

The application of the work-energy theorem between points A and D, as well as between points B and D, yields the same result. So:

$$R = h - AB\mu_k\sin\alpha = 1.75\,\mathrm{m}.$$

3. For the box to reach point E without surpassing it, the normal reaction force must be zero at that point ($\theta = \pi$).
Newton's second law at point D gives:

$$N + mg = ma_N = m\frac{v_B^2(\theta = \pi)}{R},$$

thus,

$$N(\pi) = mg\left(\frac{2h - 2AB\mu_k\sin\alpha - 3R}{R}\right)$$

Since $N(\theta = \pi) = 0$, therefore:

$$\mu_k = \frac{2h - 3R}{2AB\sin\alpha} = 0.77.$$

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