الجمهورية الجزائرية الديموقراطية الشعبية

RÉPUBLIQUE ALGÉRIENNE DÉMOCRATIQUE ET POPULAIRE

وزارة التعليم العالي والبحث العلمي

Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

جامعة عمار ثليجي الأغواط

UNIVERSITÉ AMAR THELIDJI LAGHOUAT

كـلـيـــة الـعـلـــوم

FACULTÉ DES SCIENCES

قسم الرياضيات

DÉPARTEMENT DE MATHEMATIQUES

Thèse de Doctorat troisième cycle

Spécialité : Mathématiques

Option : Analyse Mathématique

Présentée par : HAMDI Halima

Intitulé :

Propriétés de sommabilité et les opérateurs positifs

"Summability properties and the positive operators"

Soutenue publiquement le : 02/07/2022, devant le jury composé de :

Nom et Prénom	GRADE	Établissement	Qualité
Youcef BELABBACI	MCA	Université de Laghouat	Président
Amar BELACEL	Pr	Université de Laghouat	Directeur de thèse
Abdelmoumen TIAIBA	Pr	Université de M'sila	Examinateur
Abdelkader AMARA	MCA	Université de Ouargla	Examinateur
Ameur YAGOUB	MCA	Université de Laghouat	Examinateur
Amar BOUGOUTAIA	MCA	Université de Laghouat	Examinateur

Année Universitaire : 2021/2022

Abstract

The purpose of this thesis is to develop some notions and theorems of positive p-summability for multilinear operators and homogeneous polynomials. Throughout this work, first. We extend the notion of strongly p-summing multilinear operators by Dimant, in $($J. Math. Anal. Appl. $\mathbf{278}, 182-193(2003)\mathbf{)},$ to the positive framework, and prove among other results, the domination, inclusion and composition theorems. As consequence, we investigate some connections between our class and other classes of operators through, duality and linearization. Then, we compare our new class of multiple Cohen positive strongly p-summing multilinear operators along with different classes of positive multilinear p-summability and investigate a duality relationship in terms of tensor norm. Finally, we introduce the concepts of Cohen positive strongly p-summing and positive p-dominated m-homogeneous polynomials. We prove a version of Pietsch's domination theorem for the first class among other results, and a Bu-type theorem, as well as some inclusions with other known spaces. Moreover, we present a characterization of these classes in tensor terms.

Keywords and phrases: Homogeneous polynomials, Banach lattice, Positive p-summing operators, Positive strongly $p-$summing operators, Tensor norm, multiple Cohen strongly summing operators, Pietsch domination theorem, Positive strongly p-summing multilinear operators.

Résumé

Le but de cette thèse est de développer quelques notions et théorèmes de positive p-sommabilité pour les opérateurs multilinéaires et les polynômes homogènes. Dans ce travail, d'abord. Nous étendons la notion d'opérateurs multilinéaires fortement p-sommants par Dimant, dans (J. Math. Anal. Appl. 278, 182 193 (2003)), au cadre positif, et prouver entre autres résultats, les théorèmes de domination, l'inclusion et la composition. En conséquence, nous étudions des liens entre notre classe et d'autres classes d'opérateurs, a travers la dualité et la linéarisation. Ensuite, nous comparons notre nouvelle classe d'opérateurs multilinéaires dites: Multiple Cohen positif fortement p-sommant, avec différentes classes de positives p-summabilité multilinéaires et étudier une relation de dualité en termes de la norme tensorielle. Enfin, nous introduisons les concepts de Cohen positif fortement p-sommant et positif p-dominé m-homogènes polynômes. Nous prouvons une version du théorème de domination de Pietsch pour la première classe, parmi d'autres résultats. Nous prouvons aussi un théorème de Bu et quelques inclusions avec d'autres espaces connus. Et nous donnons des présentations tensorielles aux classes précédentes.

Mots clés et phrases: Polynômes homogènes, Banach réticulé, Opérateur positivement p-sommant, Opérateur positivement fortement p-sommant, Théorème de domination de Pietsch, Opérateur multilinéaire positivement fortement p-sommant.

الملخص

الهدف من هذه الاطروحة هو تطوير بعض *المفاهيم* والنظريات بالنسبة للمؤثرات الخطية p –جمعية الموجبة وكثيرات الحدود المتجانسة الموجبة. في هذا العمل، أولا قمنا بتعميم مفهوم المؤثرات المتعددة الخطية بقوة p– جمعية لديمون الى الحالة الموجبة، حيث برهنا نظرية الهيمنة لبيتش واحتواءات، إضافة الى نظرية التركيب وتحققنا من بعض العلاقات من بينها الثنوية والخطية. ثم قارنا المفهوم الجديد المسمى المؤثر المتعدد الخطية الموجب كوهين p– جمعي مع العديد من المؤثرات المتعددة الخطية الموجبة و درسنا علاقة ثنوية اعتماد على النظيم الموتري. وأخيرا قدمنا مفهومي: كثيرات الحدود المتجانسة كوهين p– جمعية وكثيرات الحدود المتجانسة الموجبة p– مهيمنة، برهنا من بين العديد من النتائج، نوع جديد لنظرية الهيمنة لبيتش بالنسبة للمفهوم الأول. وفي نفس السياق برهنا نظرية بو. بالإضافة الى بعض الاحتواءات والمساويات مع بعض الفضاءات. إضافة الى ذلك قمنا بإعطاء تمثيل موتري لهذه الفضاءات.

الكلمات والجمل المفتاحية: كثيرات الحدود المتجانسة، بناخ شبكي، نظرية الهيمنة لبيتش، مؤثر خطي موجب p–جمعي، مؤثر خطي موجب بقوة p–جمعي، مؤثر المتعدد الخطية موجب بقوة p–جمعي.

Acknowledgements

My sincere thanks go to my advisor Professor **Amar Belacel**, for his continuous guidance, support with full encouragement, and constructive feedback. I have benefited from your wealth of knowledge and meticulous remarks. Thank you very much.

My thanks also go to **Dr. Youcef Belabbaci**, **Pr. Abdelmoumen Tiaiba**, **Dr. Abdelkader Amara**, **Dr. Ameur Yagoub** and **Dr. Amar Bougoutaia**. Who have accepted to be part of the Jury.

I'm grateful to my family, whose constant love and support kept me motivated and confident.

I would like to express my thanks to every member of the **LPAM** (laboratory of pure and applied mathematics), and finally. I thank everyone who provided a shred of information for this thesis to be complete.

Contents

Dedicated to my family.

List of Symbols

$\mathbb{K}$	Field of real or complex numbers
$X, X_1, ..., X_m, Y$	Banach spaces
$E, E_1, ..., E_m, F, G$	Banach Lattices.
X^*	Topological dual of X
B_X	Unit ball of X.
E^+	Positive cone of E i.e. $\{x \in E\ , x \geq 0\}$
$B_{E^*}^+$	Is $B_{E^*} \cap E^{*+}$
$\widehat{E}$	Complete positive m-fold projective tensor product of $E_1, ..., E_m$
p^*	Conjugate of p that is $\frac{1}{p} + \frac{1}{p^*} = 1$
$L(X_1, ..., X_m; Y)$	Vector space of all multilinear operators
$\mathcal{L}(X_1, ..., X_m; Y)$	Space of all bounded multilinear operators
$\mathcal{L}_f(X_1, ..., X_m; Y)$	Vector space of all multilinear operators of finite type
$\mathcal{L}^+(E, F)$	Set of all positive operators
$\mathcal{C}(K)$	Space of all continuous real-valued functions on a topological space K
$\mathcal{M}\left(K\right)$	Banach space of all regular Borel measures on K
$L^p\left(\mu, Y\right)$	Space of measurables functions on K with $\left\Vert f\right\Vert_p = (\int_K \left\Vert f(t)\right\Vert^p d\mu)^{\frac{1}{p}}$for $1 \leq p < +\infty$
$span\left\{x_1, ..., x_n\right\}$	Vector space generated by $x_1, ..., x_n$
$\mathcal{P}\left(^m X; Y\right)$	Space of all continuous $m-$homogeneous polynomials from X to Y
δ_m	Canonical m-homogeneous polynomial $\delta_m : X \longrightarrow \widehat{\otimes}_\pi^m X$
P^*	Adjoint of the polynomial P
$\widehat{P}$	Associated symmetric m-linear operator of P
$\tilde{P}$	Linearization of P
$\mathcal{P}_{Coh,p}\left(^m X, Y\right)$	Set of all Cohen strongly p-summing m-homogeneous polynomials
$\mathcal{P}_{Coh,p}^+\left(^m X; F\right)$	Set of all Cohen positive strongly p-summing m-homogeneous polynomials
$\mathcal{P}_{d,p}^+(^m E; Y)$	Set of all positive p-dominated m-homogeneous polynomials
$\Pi_p^+(E; X)$	Set of all positive p-summing linear operators

$\mathcal{D}_p^+(X,F)$	Set of all Cohen positive strongly p-summing operators
$\Lambda_p^{mult}(E_1,...,E_m;Y)$	Set of all positive multiple $p-$summing $m-$linear operators
$\Pi_p^{mult}(E_1,...,E_m;Y)$	Set of all multiple $p-$summing $m-$linear operators
$\mathcal{C}_p^{mult}(E_1,...,E_m;Y)$	Set of all multiple p-concave m-linear operators
$\mathcal{N}_p^{mult+}(E_1,...,E_m;F)$	Set of all positive Cohen multiple p-nuclear m-linear operators
$\mathcal{N}_p^{m+}(E_1,...,E_m;F)$	Set of all positive Cohen p-nuclear m-linear operators
$\mathcal{D}_p^{m+}(X_1,...,X_m;F)$	Set of all Cohen positive strongly p-summing multilinear operators
$\mathcal{D}_p^{mult+}(X_1,...,X_m;F)$	Set of all multiple Cohen positive strongly $p-$summing operators
$\mathcal{L}_{ss,p}^m(X_1,...,X_m;Y)$	Set of all positive strongly p-summing multilinear operators

List of publications

- A. Bougoutaia, A. Belacel, and H. Hamdi. On the positive Dimant strongly p-summing multilinear operators. Carpathian Mathematical Publications, 12(2):401–411, Dec. 2020.

- H. Hamdi, A. Belacel, and A. Bougoutaia. Inclusions and coincidences for multiple Cohen positive strongly p-summing m-linear operators. (submitted).

- H. Hamdi, S. García-Hernández, A. Belacel, D. Ouchenane, and S. A. Zubair. Cohen Positive Strongly p-Summing and p-Dominated m- Homogeneous Polynomials. Journal of Mathematics,2021: 1-18, Nov. 2021. 2021.

Introduction

In 1950, A. Dvoretzky and C. A. Rogers in their famous article entitled "Absolute and unconditional convergence in normed linear spaces" [28], solved an interesting problem in Banach space theory. By showing that, in every infinite dimensional Banach space there is an unconditionally convergent series which fails to be absolutely convergent. This article was the answer to the problem raised by Stephan Banach in both books " Théorie des opérations linéaires" [10] and the Scottish book [38](Problem 122). This result caught the attention of A. Grothendieck and he presented a different proof of the Dvoretzky-Rogers theorem in [30].
Grothendieck's "Résumé de la théorie métrique des produits tensoriels topologiques" and his thesis [31, 30] can be considered as the foundation of the theory of operators ideals. Where he established the concept of absolutely 1-summing linear operators.

Followed by A. Pietsch in the 1967 who presented the concept of absolutely p-summing operators [47], and by B. Mitiagin and A. Pełczyński who introduced the notion of (p, q)-summing operators [42]. Another main development in the theory, is the work J. Lindenstrauss and A. Pełczyński [35] where they gave a new presentation as well as a new applications of the results contained in Grothendieck's paper [31], by characterizing Hilbert-Schmidt linear operators through factorization by the $\mathcal{L}_\infty$-space and $\mathcal{L}_1$-space.

In 1972, J. S. Cohen introduced the concept of Cohen strongly p-summing linear operators [23] to study the conjugate of absolutely p^*-summing linear operators, motivated by the fact that the identity from ℓ_1 into ℓ_2 is absolutely 2-summing but its adjoint is not.

The first generalization to appear for the concept of p-summing linear operators to the multilinear and homogeneous polynomials are by A. Pietsch [49], R. Alencar and M. Matos in [5]. For the positive situation, O. Blasco introduced the notion of positive p-summing linear operators in 1986 [11]. It is a class of operators from a Banach Lattice E into a Banach space X, these operators map positive sequences in weak ℓ_p-space with value in E into sequences in ℓ_p-space with value in X. In [12] Blasco characterized positive p-summing linear operators by the Radon-Nikodym property. That is, a Banach space X has the Radon-Nikodym property, if and only if every positive p-summing operator $T : L_{p^*}(\mu) \to X$ is presented by a function f in $L_p(\mu, X)$.

These first generalizations motivated in-depth studies and generalizations to the theory of ideal of linear operators to multilinear operators and homogeneous polynomials as well as the positive situation. In the last three decades, numerous generalizations viewed the light: weakly compact, integral, nuclear operators, Hilbert-Schmidt operators, unconditional converging operators, etc [7, 8, 13, 17, 22, 27, 39, 51, 52]. Many generalizations attempted to preserve the properties of the original linear ideal, and it was quite satisfactory for several multi-ideals.

The concept of Cohen strongly p-summing linear operators knew many generalizations that attempted to verify as much as possible properties of the original linear ideal. The multilinear concept of Cohen strongly summing operators was investigated in 2007 by D. Achour and L. Mezrag [2]. They proved that this new class verify a domination theorem and compare it with the class of dominated multilinear operators. Soon after concepts related to this class are being investigated [1, 3, 14, 21].

This thesis is dedicated to developing some notions and theorems of positive p-summability for multilinear and homogeneous polynomials. Our main interest was the concept of Cohen positive strongly p-summing by D. Achour and A. Belacel [1], we generalized this notion to homogeneous polynomials and multilinear operators and proved among other results: domination theorem, inclusion and duality relationships for these classes, as well as some relationships with other known spaces. We also extended the concept of strongly p-summing multilinear [26] operators in the positive multilinear framework.

This thesis is divided into four chapters

Chapter 1, this chapter includes all the preliminaries necessary for a clear understanding of this thesis. We recall essentially some results about multilinear operators and select the necessary and most used results about Banach lattices and positive operators, polynomials, and tensor product of Banach spaces. Moreover, we recall diverse inequalities, and lastly, we present some ideals of linear and multilinear operators, with proper references to each section.

In **chapter 2** we extend the notion of strongly multilinear operators [26] to the positive framework, we introduce and study a new class of operators between Banach lattices and prove a domination theorem, inclusion, and an important composition result (Proposition 2.1.4) as basic properties for this class. In the second section, we study ties with known classes of summability, including linearization of those operators and involving the positive projective tensor norm of Banach lattices. We end this chapter by giving open problems.

In **chapter 3** we set a new generalization to the concept of Cohen positive strongly p-summing [1, 14], following the steps of Campos in [21] by introducing the new class of multiple Cohen positive strongly $p-$summing operators and compare it with the class of Cohen positive strongly p-summing m-linear operators [14] and positive multiple p-summing m-linear operators [18], by giving a generalization to Cohen's theorems [23], as well as, investigating a relationship with the class multiple Cohen positive p-nuclear operators by proving that the composition of a multiple Cohen positive strongly $p-$summing operator and m positive p-summing linear operator yields a multiple Cohen positive p-nuclear operator. That is, if $S \in \mathcal{D}_p^{mult+}(E_1, ..., E_m; F)$ and $T_j \in \Pi_q^+(D_j, E_j)$ with $1 \leq j \leq m$, then $S \circ (T_1, ..., T_m) \in \mathcal{N}_r^{mult+}(D_1, ..., D_m; F)$, for all $1 \leq p, q, r \leq \infty$ with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In the last section we present a connection with tensor product.

Chapter 4 is devoted to the study of Cohen positive strongly p-summing m-homogeneous polynomials, positive p-dominated multilinear operators, and homogeneous polynomials. In Section 1 we introduce Cohen positive strongly p-summing m-homogeneous polynomials, we characterize our class by a domination theorem and prove it using the new technique of $R-S-$Abstract p-summing operators [46]. We also prove relationships with different classes of positive summability $(\Pi_{p^*}^+(E, F)$, $\mathcal{D}_p^{m+}(^mX; F)$, $\mathcal{D}_p^+(\widehat{\otimes}_{\pi,s}^m X; F))$. By means of the adjoint operator, the associated symmetric m-linear operator, and the linearization of these polynomials. Section 2 is essentially dedicated to the study of positive p-dominated m-linear operators and polynomials. This section includes our main theorem (The Bu-type theorem) which serves as a focal point in the proof of the relationship between positive p-dominated polynomials and those defined in section 1. In section 3, we present some applications based on Proposition 4.1.3.
And finally, section 4 highlights the classes defined earlier in terms of tensor product and prove a duality between these classes and tensor spaces equipped with adequate norms through a linear mapping.

Chapter 1

Preliminaries

In this chapter, we will remind the necessary notions and results used along this thesis. It consists essentially of the theory of multilinear operators and homogeneous polynomials, tensor product between Banach spaces, Banach lattices and positive p-summing linear operators. The most used references in this chapter are [6, 24, 25, 36, 53].
Unless it is mentioned otherwise, all along this chapter $X, X_1, ..., X_m, Y$ and Z are Banach spaces, $E, E_1, ..., E_m, F$ and G are Banach lattices.
A mapping $T : X_1 \times \cdots \times X_m \to Y$ between vector spaces is said to be multilinear operator or (m-linear) if

$$T(x_1, ..., (a + \lambda b), ..., x_m) = T(x_1, ..., a, ..., x_m) + \lambda T(x_1, ..., b, ..., x_m)$$

holds for all $a, b \in X_i$, $(1 \leq i \leq m)$ and $\lambda \in \mathbb{K}$. That is, T is linear separately in each variable. The vector space of all multilinear operators from $X_1 \times \cdots \times X_m$ to Y is denoted by $L(X_1, ..., X_m; Y)$.
In the case when $X_1, ..., X_m$ and Y are normed spaces, then T is said to be bounded if

$$\|T\| := sup \{\|T(x_1, ..., x_m)\| \mid x_i \in B_{X_i}\} < \infty.$$

$\mathcal{L}(X_1, ..., X_m; Y)$ denotes the Banach space of all continuous multilinear operators from $X_1 \times \cdots \times X_m$ into Y. In the case $Y = \mathbb{K}$ we simply write $L(X_1, ..., X_m)$ and $\mathcal{L}(X_1, ..., X_m)$ and their members are m-forms and continuous m-forms, respectively.

Proposition 1.0.1. *[43, Proposition 1.2] For each $T \in \mathcal{L}(X_1, ..., X_m; Y)$ the following conditions are equivalent*

(a) T is continuous.

(b) T is continuous at the origin.

(c) $\|T\| < \infty$.

A multilinear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is said to be finite type, if it is generated by mappings of the form

$$T : X_1 \times \cdots \times X_m \to Y$$
$$(x_1, ..., x_m) \mapsto \phi_1(x_1) \cdots \phi_m(x_m) b$$

for some non-zero $\phi_i \in X_i^*(1 \leq i \leq m)$ and $b \in Y$, the vector space of all multilinear operators of finite type is denoted by $\mathcal{L}_f(X_1, ..., X_m; Y)$.

The adjoint of an m-linear operator is defined as follows, if $T \in \mathcal{L}(X_1, ..., X_m; Y)$, we define the adjoint of T by

$$T^* : Y^* \to \mathcal{L}((X_1, ..., X_m), y^* \to T^*(y^*) : X_1 \times \cdots \times X_m \to \mathbb{K}$$

with $T^*(y^*)(x_1, ..., x_m) = y^*(T(x_1, ..., x_m))$.

If K is compact Hausdorff space, $\mathcal{C}(K)$ denotes the Banach space formed by all continuous functions $f : K \to \mathbb{K}$ with the usual supremum norm.

1.1 Banach Lattices and positive operators

A real vector space E is said to be ordered vector space whenever it is equipped with the order relation here $\geq$ (i.e., $\geq$ is a reflexive, antisymmetric, and transitive binary relation on E) that is compatible with the algebraic structure of E in the sense that it satisfies the following two axioms

(1) If $x \geq y$, then $x + z \geq y + z$ holds for all $z \in E$.

(2) If $x \geq y$, then $\alpha x \geq \alpha y$ holds for all $\alpha \geq 0$.

A vector x in an ordered vector space E is called positive whenever $x \geq 0$ holds. For a two-point set $\{x, y\}$ we write $x \wedge y$ or $\inf \{x, y\}$ to denote its infimum and $x \vee y$ or $\sup \{x, y\}$ to denote its supremum.

Definition 1.1.1. *We say that an ordered vector space E is a lattice(Riesz space) if for any $x, y \in E$ both $x \wedge y$ and $x \vee y$ exist in E.*

Proposition 1.1.1. *[6] For arbitrary elements x, y, z of a vector lattice, the following identities hold.*

(1) $x + y = \sup \{x, y\} + \inf \{x, y\}$;

(2) $x + \sup \{y, z\} = \sup \{x + y, x + z\}$ *and* $x + \inf \{y, z\} = \inf \{x + y, x + z\}$;

(3) $\sup \{x, y\} = - \inf \{-x, -y\}$ *and* $\inf \{x, y\} = - \sup \{-x, -y\}$;

(4) $\alpha \sup \{x, y\} = \sup \{\alpha x, \alpha y\}$ *and* $\alpha \inf \{x, y\} = \inf \{\alpha x, \alpha y\}$ *for* $\alpha \geq 0$.

(5) for any $x, y, z \in E^+$ we have

$$\inf\{x + y, z\} \leq \inf\{x, z\} + inf\{y, z\}.$$

For an element x in a vector lattice E we can define its positive, negative part and its absolute value, respectively, by

$$x^+ = \sup\{x, 0\}, \; x^- = \sup\{-x, 0\}, \; |x| = \sup\{x, -x\}.$$

The functions $(x, y) \to \sup\{x, y\}$, $(x, y) \to \inf\{x, y\}$, $x \to x^{\pm}$ and $x \to |x|$ are collectively referred to as the *lattice operations* of a vector lattice. The relation between them is given in the next proposition.

Proposition 1.1.2. *If x is an element of a vector lattice, then*

$$x = x^+ - x^-, \qquad |x| = x^+ + x^-.$$

Proof. By Proposition 1.1.1 (1) and (3) we have

$$x = x + 0 = \sup\{x, 0\} + \inf\{x, 0\} = \sup\{x, 0\} - \sup\{-x, 0\} = x^+ - x^-.$$

Furthermore, from Proposition 1.1.1 (2) and (4), and the previous result we get

$$\begin{aligned} |x| &= \sup\{x, -x\} = \sup\{2x, 0\} - x = 2\sup\{x, 0\} - x = 2x^+ - x \\ &= 2x^+ - (x^+ - x^-) = x^+ + x^-. \end{aligned}$$

□

Definition 1.1.2. *A norm on a vector lattice E is called a lattice norm if*

$$|x| \leq |y| \quad \textit{implies} \quad \|x\| \leq \|y\|. \tag{1.1.1}$$

A vector lattice E, complete under a lattice norm, is called a ***Banach lattice.***

Property 1.1.1 gives the important identity

$$\|x\| = \||x|\|, \qquad x \in E.$$

This follows by taking first x and $y = |x|$ we have $|x| \leq |(|x|)|$ and hence $\|x\| \leq \||x|\|$. Then taking $|x|$ and $y = x$, we also have $|(|x|)| \leq |x|$ and hence $\||x|\| \leq \|x\|$.

Example 1.1.1. *1. The Euclidean space $\mathbb{R}^n$ with the standard order given by $x \geq y$ if and only if $x_i \geq y_i$ for all $i = 1, ..., n$ it is a vector lattice with $x \vee y = (\max\{x_1, y_1\}, ..., \max\{x_n, y_n\})$. It is also a Banach lattice under all standard norms.*

2. *Function spaces with pointwise order are vector lattices*
   - $\mathcal{C}(K)$ *space of all continuous real-valued functions on a topological space* K.
   - $\ell_p(0 < p < \infty)$, *all real sequences* $(x_1, x_2, ...)$ *with* $\sum_{n=1}^{\infty} |x_n|^p < \infty$.
3. *The class of* L_p*-spaces under the ordering* $f \leq g$ *whenever* $f(x) \leq g(x)$ *holds for* μ*-almost for all* $x \in E$, *each* $L_p(\mu)$ *is a vector lattice.*

All the above examples are Banach lattices with standard norms.

The dual E^* of a Banach lattice E is also a Banach lattice, and its positive cone is defined by $x^* \geq 0$ in E^* if and only if $< x, x^* > \geq 0$, for every $x \geq 0$ in E. For any $x^*, y^* \in E^*$ and every $x \geq 0$, we have

$$< x, x^* \vee y^* > = \sup \{< u, x^* > + < x - u, y^* >; 0 \leq u \leq x\}$$

and

$$< x, x^* \wedge y^* > = \inf \{< v, x^* > + < x - v, y^* >; 0 \leq v \leq x\}.$$

For more details see [36].

Definition 1.1.3. *A linear operator* T *from a Banach lattice* E *into a Banach lattice* F *is called positive, denoted* $T \geq 0$, *if* $Tx \geq 0$ *for any* $x \geq 0$.

The set of all positive operators from E into F is denoted by $\mathcal{L}^+(E, F)$.

Proposition 1.1.3. *An operator* T *is positive if and only if* $|Tx| \leq T|x|$, *for any* $x \in E$.

Proof. It follows from $-|x| \leq x \leq |x|$ so, if T is positive, then $-T|x| \leq Tx \leq T|x|$. Conversely, taking $x \geq 0$, we obtain $0 \leq |Tx| \leq T|x| = Tx$. □

Proposition 1.1.4. *If* T *is positive, then*

$$\|T\| = \sup_{\substack{x \geq 0 \\ \|x\| \leq 1}} \|Tx\|.$$

The equivalence between continuity and boundedness of positive operators between normed vector lattices can be found in [9]: as the first attempt to prove this result.

Theorem 1.1.1. *[6, Theorem 4.3] Every positive operator from Banach lattice to a normed vector lattice is continuous.*

Proof. Let $T : E \to F$ be a positive operator from a Banach lattice E to a normed vector lattice F. Assume by contradiction that T is not bounded in the norm of F. Then there exists a sequence $\{x_n\}$ of E satisfying $\|x_n\| = 1$ and $\|Tx_n\| \geq n^3$ for each n. In view of $|Tx_n| \leq T|x_n|$, we can assume that $x_n \geq 0$ holds for each n. From $\sum\limits_{n=1}^{\infty} \frac{\|x_n\|}{n^2} < \infty$ and the norm completeness of E, it follows that the series $\sum\limits_{n=1}^{\infty} \frac{x_n}{n^2}$ is convergent in E. Let $x = \sum\limits_{n=1}^{\infty} \frac{x_n}{n^2}$. $0 \leq \frac{x_n}{n^2} \leq x$ holds for all n, and so

$$n \leq \|T(\frac{x_n}{n^2})\| \leq \|Tx\| < \infty$$

also holds for each n, which is impossible. Thus, T must be normed bounded hence continuous. □

1.2 Sequence spaces

Let $1 \leq p \leq \infty$ and $n \in \mathbb{N}^*$. We denote by $\ell_p^n(X)$ the Banach space of all sequences $(x_i)_{i=1}^n$ in X with the norm

$$\|(x_i)_{i=1}^n\|_p := \left(\sum_{i=1}^{n} \|x_i\|^p\right)^{\frac{1}{p}}$$
$$\|(x_i)_{i=1}^n\|_\infty := \sup_{1\leq i\leq n} \|x_i\|,$$

and by $\ell_{p,weak}^n(X)$ the Banach space of all weakly summable sequences $(x_i)_{i=1}^n$ in X with the norm

$$w_p((x_i)_{i=1}^n) := \sup_{x^* \in B_{X^*}} \left(\sum_{i=1}^{n} | < x_i, x^* > |^p\right)^{\frac{1}{p}}$$
$$w_\infty((x_i)_{i=1}^n) := \sup_{x^* \in B_{X^*}} \sup_{1\leq i\leq n} | < x_i, x^* > |.$$

The space $(c_0)_{weak}(X) := \{(x_n) : \; x_n \in X \; and \; (< x_n, x^* >) \in c_0 \; \forall x^* \in X^*\}$ is a closed subspace of $\ell_{\infty,weak}^n(X)$.
Consider the case when X is replaced by a Banach lattice E, and define([1])

$$\ell_{p,|weak|}^n(E) := \left\{(x_i)_{i=1}^n : \; (|x_i|)_{i=1}^n \in \ell_{p,weak}^n(E)\right\}$$
$$(c_0)_{|weak|}(E) := \{(x_i)_{i=1}^n : \; (|x_i|)_{i=1}^n \in (c_0)_{weak}(E)\},$$

and

$$\|(x_i)_{i=1}^n\|_{\ell_{p,|weak|}^n(E)} = w_p((|x_i|)_{i=1}^n).$$

It is a Banach lattice with the ordering induced by the pointwise ordering on $E^{\mathbb{N}}$. [34, Theorem 7.8]
Moreover, if $B_{E^*}^+ = \{\xi \in B_{E^*} \; : \; \xi \geq 0\}$ denotes the unit ball in $B_{E^*} \cap E^{*+}$.

It follows from $| < |x_i|, \xi > | \leq < |x_i|, |\xi| >$ that

$$\|(x_i)_{i=1}^n\|_{\ell^n_{1,|weak|}(E)} = \sup_{\xi \in B^+_{E^*}} \sum_{i=1}^n < |x_i|, \xi > = \| \sum_{i=1}^n x_i \|_E$$

$$\|(x_i)_{i=1}^n\|_{\ell^n_{p,|weak|}(E)} = \sup_{\xi \in B^+_{E^*}} \left(\sum_{i=1}^n < |x_i|, \xi >^p \right)^{\frac{1}{p}} \quad 1 < p < \infty$$

and

$$\|(x_i)_{i=1}^n\|_{\ell^n_{\infty,|weak|}(E)} = \sup_{\xi \in B^+_{E^*}} \sup_{1 \leq i \leq n} < |x_i|, \xi >$$

if $x_1, ..., x_n \geq 0$, we have that

$$\|(x_i)_{i=1}^n\|_{\ell^n_{p,|weak|}(E)} = \sup_{\xi \in B^+_{E^*}} \left(\sum_{i=1}^n < x_i, \xi >^p \right)^{\frac{1}{p}} = w_p((x_i)_{i=1}^n). \tag{1.2.1}$$

If $1 < p < \infty$, we denote by $\ell_p(F, \mathbb{N}^m)$ the vector space of all families $(y_j)_{j \in \mathbb{N}^m}$ of elements such that

$$\|(y_j)_{j \in \mathbb{N}^m}\|_p = \left(\sum_{j \in \mathbb{N}^m} \|y_j\|^p \right)^{\frac{1}{p}} < +\infty$$

and by $\ell_{p,weak}(F, \mathbb{N}^m)$ the vector space of all families $(y_j)_{j \in \mathbb{N}^m}$ of elements such that

$$\|(y_j)_{j \in \mathbb{N}^m}\|_{\ell_{p,weak}(F,\mathbb{N}^m)} = \sup_{\phi \in B_{F^*}} \left(\sum_{j \in \mathbb{N}^m} | < \phi, y_j > |^p \right)^{\frac{1}{p}} < +\infty$$

$\|.\|_p$ and $\|.\|_{\ell_{p,weak}(F,\mathbb{N}^m)}$ are norms on $\ell_p(F, \mathbb{N}^m)$ and $\ell_{p,weak}(F, \mathbb{N}^m)$ respectively, an element of $\mathbb{N}^m$ is represented by $(j_1, ..., j_m)$.
To avoid encumbered notations in Chapter 3 we denote $\ell_p(F)$ instead of $\ell_p(F, \mathbb{N}^m)$ and by $\ell_{p,weak}(F)$ instead of $\ell_{p,weak}(F, \mathbb{N}^m)$.

1.3 Polynomials

Definition 1.3.1. *A map $P : X \longrightarrow Y$ is an m-homogeneous polynomial if there exists a unique symetric m-linear operator $\widehat{P} : X \times \cdots \times X \longrightarrow Y$ such that $P(x) = \widehat{P}\left(x, \overset{(m)}{\ldots}, x\right)$, for all $x \in X$. We denoted by $\mathcal{P}(^m X; Y)$ the Banach space of all continuous $m-$homogeneous polynomials from X to Y with the norm*

$$\|P\| = \sup_{\|x\| \leq 1} \|P(x)\| = \inf \{C : \|P(x)\| \leq C \|x\|^m, x \in X\}.$$

When $Y = \mathbb{K}$, we write $\mathcal{P}({}^m X; \mathbb{K}) = \mathcal{P}({}^m X)$.

Theorem 1.3.1. *[43, Theorem 2.2] For each $\widehat{P} \in \mathcal{L}({}^m X; Y)$, let $P \in \mathcal{P}({}^m X; Y)$ defined by $P(x) = \widehat{P}\left(x, \overset{(m)}{\ldots}, x\right)$ for every $x \in X$. Then*

(a) The mapping $\widehat{P} \leftrightarrow P$ induces an isomorphism between $\mathcal{L}({}^m X; Y)$ and $\mathcal{P}({}^m X; Y)$.

(b) We have the inequalities

$$\|P\| \leq \left\|\widehat{P}\right\| \leq \frac{m^m}{m!} \|P\|.$$

For every $\widehat{P} \in \mathcal{L}({}^m X; Y)$.

Both P and $\widehat{P}$ are related by the polarization formula

$$\widehat{P}\left(x^1, \ldots, x^m\right) = \frac{1}{m! 2^m} \sum_{\substack{\epsilon_j = \pm 1 \\ 1 \leq j \leq m}} \epsilon_1 \ldots \epsilon_m P\left(\sum_{j=1}^{m} \epsilon_j x^j\right). \tag{1.3.1}$$

Proposition 1.3.1. *[43, Proposition 2.4] For each $P \in \mathcal{P}({}^m X; Y)$ the following conditions are equivalent:*

(a) P is continuous.

(b) P is bounded on every ball with finite radius.

(c) P is bounded on some open ball.

Definition 1.3.2. *[8] Given a continuous $m-$homogeneous polynomial $P \in \mathcal{P}({}^m X, Y)$ between the Banach spaces X and Y, the adjoint of P is the following continuous linear operator:*

$$P^* : Y^* \longrightarrow \mathcal{P}({}^m X)$$
$$P^*(y^*)(x) = y^*(P(x)).$$

We have $\|P^*\| = \|P\|$.

1.4 Tensor product of Banach spaces

The tensor products are mainly used to linearize multilinear mappings. In this section we include the most used results about tensor product. Further details on this section can be found in [53].

First, we give the construction of tensor product. Let $m \in \mathbb{N}$, $X_1, \ldots, X_m$ be vector spaces, consider the algebraic dual $L(X_1, ..., X_m)^*$ of $L(X_1, ..., X_m)$ consisting of

$$L(X_1, ..., X_m)^* = \{\phi : (X_1, ..., X_m) \to \mathbb{K} : \phi \, is \, a \, linear \, form\}.$$

The tensor product $X_1 \otimes \cdots \otimes X_m$ of the vector spaces $X_1, ..., X_m$ can be constructed as the space of linear functionals on $L(X_1, ..., X_m)$, for $x_1 \in X_1, ..., x_m \in X_m$. Denote by $x_1 \otimes \cdots \otimes x_m$ the functional given by evaluation at $(x_1, ..., x_m)$. In other words

$$\begin{aligned} x_1 \otimes \cdots \otimes x_m : L(X_1, ..., X_m) &\to \mathbb{K} \\ A &\mapsto (x_1 \otimes \cdots \otimes x_m)(A) = A(x_1, ..., x_m). \end{aligned}$$

Let D be the formed by all these functionals

$$D := \{x_1 \otimes \cdots \otimes x_m : \quad x_1 \in X_1, ..., x_m \in X_m\} \subseteq L(X_1, ..., X_m)^*$$

Definition 1.4.1. *The subspaces of $L(X_1, ..., X_m)$ generated by D is called the tensor product of $X_1, ..., X_m$ denoted $X_1 \otimes \cdots \otimes X_m$, in symbols*

$$X_1 \otimes \cdots \otimes X_m = \left\{ \sum_{i=1}^{n} \lambda_i (x_i^1 \otimes \cdots \otimes x_i^m) \; n \in \mathbb{N}; \lambda_i \in \mathbb{K}; x_i^j \in X_j, 1 \leq j \leq m, 1 \leq i \leq n \right\}.$$

An element of $X_1 \otimes \cdots \otimes X_m$ is called a tensor, a tensor of the form $x_1 \otimes \cdots \otimes x_m$ is called an *elementary tensor.*

The projective tensor norm and linearization of continuous multilinear operators

Let $X_1, ..., X_m, Y$ be Banach spaces. The projective tensor norm is defined by

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^{n} \|x_i^1\| \cdots \|x_i^m\| \qquad u = \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m \right\}$$

for every $u \in X_1 \otimes \cdots \otimes X_m$.

Definition 1.4.2. *Let $X_1, ..., X_m$ be Banach spaces, the completion of the normed space $(X_1 \otimes \cdots \otimes X_m, \| \cdot \|_\pi)$ is defined as the projective tensor product of these spaces and is denoted by $X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m$. If $X = X_1 = \cdots = X_m$ we write $\widehat{\otimes}_\pi^m X$.*

$\otimes_s^m X := X \otimes_s \overset{(m)}{\cdots} \otimes_s X$ denotes the $m-$fold symmetric tensor product of X, that is, the set of all elements $u \in \otimes^m X$ of the form

$$u = \sum_{i=1}^{n} x_i \otimes \overset{(m)}{\cdots} \otimes x_i \quad (n \in \mathbb{N}, x_i \in X, 1 \leq i \leq n).$$

By $\widehat{\otimes}_{\pi,s}^m X$ we denote the closure of $\otimes_s^m X$ in $\widehat{\otimes}_\pi^m X$.

Theorem 1.4.1. *[53] Let $X_1, ..., X_m, Y$ be Banach spaces. For every multilinear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ there is a unique continuous linear operator $T_L \in \mathcal{L}(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m; Y)$, such that*

$$T = \widetilde{T} \circ \otimes$$

and $\otimes$ be the canonical operator

$$\otimes : X_1 \times \cdots \times X_m \to X_1 \widehat{\otimes}_\pi ... \widehat{\otimes}_\pi X_m, \otimes(x_1, ..., x_m) = x_1 \otimes \cdots \otimes x_m.$$

The correspondence $T \leftrightarrow T_L$ establish an isometric isomorphism between $\mathcal{L}(X_1, ..., X_m; Y)$ and $\mathcal{L}(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m; Y)$.

Remark 1.4.1. *The previous theorem gives rise to*

$$(X_1 \widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi X_m)^* = \mathcal{L}(X_1, ..., X_m).$$

This duality yields a new formula for the projective norm

$$\|u\|_\pi = \sup \{ | < u, T > | \quad T \in \mathcal{L}(X_1, ..., X_m) \; and \; \|T\| \leq 1 \}.$$

1.5 Inequalities

Hölder's inequality

Let $1 \leq p, p^* \leq \infty$ two conjugates exponents. Then if $f \in L_p(\mu)$, $g \in L_{p^*}(\mu)$, we have $f \cdot g \in L_1(\mu)$ and

$$\|f \cdot g\|_{L_1} \leq \|f\|_{L_p} \cdot \|g\|_{L_{p^*}}.$$

Minkowski's inequality

Let $1 \leq p \leq \infty$, then for $f, g \in L_p(\mu)$, we have

$$\|f + g\|_{L_p} \leq \|f\|_{L_p} + \|g\|_{L_p}.$$

Multiple Khintchine's inequality [2]
If $0 < p < \infty$ then

$$A_p^m(\sum_{i_1,\dots,i_m=1}^{n_1,\dots,n_m}|a_{i_1,\dots,i_m}|^2)^{\frac{1}{2}}$$
$$\leq \left(\int_0^1 \dots \int_0^1 |r_{i_1}(t_1)\dots r_{i_m}(t_m)a_{i_1,\dots,i_m}|^p dt_1\dots dt_m\right)^{\frac{1}{p}}$$
$$\leq B_p^m\left(\sum_{i_1,\dots,i_m=1}^{n_1,\dots,n_m}|a_{i_1,\dots,i_m}|^2\right)^{\frac{1}{2}}$$

for each choice of the scalars $(a_{i_1,\dots,i_m})_{i_1,\dots,i_m=1}^{n_1,\dots,n_m}$, A_p, B_p are the Khintchine constants, and by $(r_m)_{m\in\mathbb{N}}$ we denote the sequence of Rademacher functions such that
$r_m : [0,1] \to \mathbb{R}$ as $r_m(t) = sng\,(sin(2^m\pi t))$
Multiple Kahane's inequality [2]
Let $1 \leq p < \infty$. Let $Rad_p(\mu, X)$ be the vector space of all almost unconditionally summable sequences (x_n) in the Banach space X. it is a Banach space under the norm

$$\|(x_n)\|_p = \left(\int_0^1 \|\sum_{n=1}^{\infty} r_n(t)x_n\|^p d\mu(t)\right)^{\frac{1}{p}}.$$

Let $1 \leq p, q < \infty$. Let X be a Banach space and $(x_{i_1,\dots,i_n})$ be in X . We have

$$\|\sum_{i_1,\dots,i_n} x_{i_1,\dots,i_n}\bigotimes_{j=1}^{n} r_{i_j}\|_{L_p(\bigotimes_{1\leq i\leq n}\mu_i,X)}$$
$$\leq K_{p,q}^n\|\sum_{i_1,\dots,i_n} x_{i_1,\dots,i_n}\bigotimes_{j=1}^{n} r_{i_j}\|_{L_q(\bigotimes_{1\leq i\leq n}\mu_i,X)},$$

where $K_{p,q}$ is the simple constant of Kahane's inequality $(\mu_i = \mu)$.

1.6 Some Ideals of linear operators

In this section, we give an overview of positive p-summing linear operators generalized by Blasco, and Cohen positive strongly p-summing linear operators. We expose their fundamental properties, as well as domination-factorization theorems for the first class.

1.6.1 Positive p-summing linear operators

Definition 1.6.1. *[12] Let $1 \leq p < \infty$. An operator $T : E \longrightarrow X$ is said to be positive $p-$summing, if there exists a constant $C > 0$ such that for all*

$n \in \mathbb{N}, x_1, ..., x_n \in E$, the inequality

$$\|T (x_i)_{i=1}^n\|_p \leq C \|(x_i)_{i=1}^n\|_{\ell^n_{p,|weak|}(E)} . \tag{1.6.1}$$

For $p = \infty$

$$\sup_{1 \leq i \leq n} \|T (x_i)\| \leq C \|(x_i)_{i=1}^n\|_{\ell^n_{\infty,|weak|}(E)} .$$

We denote by $\Pi_p^+(E; X)$, the space of positive $p-$summing operators from E into X. $\Pi_p^+(E, X)$ becomes a Banach space with norm $\pi_p^+(.)$ given by the infimum of the constants $C > 0$ that verify the inequality (1.6.1). We have $\Pi_\infty^+(E; X) = \mathcal{L}(E; X)$.

Proposition 1.6.1. ***(Inclusion property)***
For $1 \leq p < q < \infty$, then

$$\Pi_p^+(E; X) \subseteq \Pi_q^+(E; X).$$

Moreover, for all $T \in \Pi_q^+(E; X)$, we have that $\pi_q^+(T) \leq \pi_p^+(T)$.

Theorem 1.6.1. *[54]* ***(Domination Theorem)*** *Let $T \in \mathcal{L}(E, X)$. T is positive p-summing operator, if and only if there exist a probability measure μ on $B_{E^*}^+$, provided with the weak star topology, and a positive constant C such that*

$$\|T(x)\| \leq C \left(\int_{B_{E^*}^+} < x, x^* >^p d\mu(x^*) \right)^{\frac{1}{P}} \tag{1.6.2}$$

for every $x \in E^+$. Moreover, in this case

$$\pi_p^+(T) = \inf \{C\, verifiying\, the\, inequality\, (1.6.2)\} .$$

Next, we present the factorization theorem. Let a regular probability measure μ on the compact set $(B_{E^*}^+, w^*)$, the inclusion operator $J_{p,0} : C(B_{E^*}^+) \to L_0^p(\mu)$ such that $L_0^p(\mu)$ is the completion of the quotient space $i_E(E)/R$ with $R = \{f \in i_E(E) \quad |||f||| = 0\}$ and

$$|||f||| = \inf \left\{ \left(\int_{B_{E^*}^+} < |z|, \cdot >^p d\lambda(.) \right)^{\frac{1}{p}} :< |z - f|, . >= 0\mu - a.e \right\} .$$

With the norm $\| [f] \| = |||f|||$, where $[f]$ is the equivalent class of f. For more details see [1].

Lemma 1.6.1. *[1] The operator $J_{p,0} \circ i_E : E \to i_E(E) \to L_0^p(\mu)$ is at most norm one positive p-summing operator.*

Theorem 1.6.2. *[1]* ***(Factorization Theorem)*** *For every operator $T : E \to Y$, the following are equivalent*

(i) T is positive p-summing.

(ii) There exist a regular Borel probability measure μ on $B^+_{E^*}$, a Banach space $L_0^p(\mu)$ and an operator $u : L_0^p(\mu) \to Y$ such that the following diagram commutes

$$\begin{array}{ccc} E & \xrightarrow{\;T\;} & Y \\ \downarrow{\scriptstyle i_E} & & \uparrow{\scriptstyle u} \\ i_E(E) & \xrightarrow{J_{p,0}} & L_0^p(\mu) \\ \cap & & \\ C(B^+_{E^*}) & & \end{array}$$

Proof. $(i) \Rightarrow (ii)$ Assume that T is positive p-summing, the Domination theorem provides a regular Borel probability measure μ on $B^+_{E^*}$ for which

$$\begin{aligned} \|Tx\| &\leq \pi_p^+(T) \left(\int_{B^+_{E^*}} < |x|, \phi >^p d\mu \right)^{\frac{1}{p}} \qquad for\ all\ x \in E \\ &= \pi_p^+(T) \| \, [< x, . >] \, \|_{L_0^p(\mu)} \\ &= \pi_p^+(T) \| J_{p,0} \circ i_E(x) \|_{L_0^p(\mu)}. \end{aligned}$$

This informs us that if we denote the range of $J_{p,0} \circ i_E$ by S. By definition the closure of S is $L_0^p(\mu)$, the map $S \to Y : J_{p,0} \circ i_E(x) \mapsto Tx$ is a well-defined operator. It is continuous for the $L_0^p(\mu)$-topology with norm $\leq \pi_p^+(T)$, since $\|Tx\| \leq \pi_p^+(T) \|J_{p,0} \circ i_E(x)\|_{L_0^p(\mu)}$ for every x in E. Then the natural extension of our map to $L_0^p(\mu)$ is the operator u we are looking for.
$(ii) \Rightarrow (i)$ Form $u \circ J_{p,0} \circ i_E = T$ and the previous lemma and the ideal property we get T is positive p-summing and $\pi_p^+(T) \leq \|u\| \pi_p^+(J_{p,0} \circ i_E) \leq \|u\|$. □

1.6.2 Cohen positive strongly p-summing linear operators

Definition 1.6.2. *[1] Let $1 \leq p \leq \infty$. An operator $T : X \to F$ is Cohen positive strongly p-summing if there exists a constant $C > 0$, such that for all finite sets $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset F^*$, we have*

$$\sum_{i=1}^{n} | < T(x_i), y_i^* > | \leq C \|(x_i)_{i=1}^n\|_p \|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)}.$$

The class of all Cohen positive strongly p-summing operators between X and F is denoted $\mathcal{D}_p^+(X,F)$. The infimum of all constant C in the inequality defines the norm d_p^+ on $\mathcal{D}_p^+(X,F)$.

Proposition 1.6.2. ***(Ideal property)*** *Let X, Y be Banach spaces and E, F be two Banach lattices. Consider T in $\mathcal{L}(Y,E)$, S a positive operator in $\mathcal{L}(E,F)$ and R in $\mathcal{L}(X,Y)$. If $T \in \mathcal{D}_p^+(Y,E)$, then $STR \in \mathcal{D}_p^+(X,F)$ and $d_p^+(STR) \leq \|S\| d_p^+(T) \|R\|$.*

The next theorem shows the duality between positive p-summing operators and Cohen positive strongly p^*-summing operators.

Theorem 1.6.3. *[1] Let $1 \leq p \leq \infty$*

(i) $T \in \mathcal{L}(F,X)$ is positive p-summing if and only if T^* is Cohen positive strongly p^*-summing.

(ii) $T \in \mathcal{L}(X,F)$ is Cohen positive strongly p-summing if and only if T^* is positive p^*-summing.

1.7 Some ideals of multilinear operators

In this section, we present briefly the classes of positive multiple p-summing, positive Cohen p-nuclear and Cohen positive strongly p-summing multilinear operators. All the proofs of results below are in the references [14, 16, 18].

1.7.1 Positive multiple p-summing m-linear operators

Definition 1.7.1. *[18] Let $1 \leq p < \infty$. An $m-$linear operator $T : E_1 \times \cdots \times E_m \longrightarrow Y$ is called positive multiple $p-$summing if there exists a constant $K > 0$ such that for every choice of finite sequences $(x_i^j)_{i=1}^{n_j} \subseteq E_j^+$ $(1 \leq j \leq m)$,*

$$\left(\sum_{i_1,\ldots,i_m=1}^{n_1,\ldots,n_m} \left\| T\left(x_{i_1}^1,\ldots,x_{i_m}^m\right) \right\|^p \right)^{\frac{1}{p}} \leq K \prod_{j=1}^{m} \left\| \left(x_i^j\right)_{i=1}^{n_j} \right\|_{\ell_{p,weak}^n\left(E_j\right)}. \tag{1.7.1}$$

In this case, we define the positive multiple $p-$summing norm of T by

$$\Lambda_p(T) = \inf\{K : K \text{ verifies the inequality } (1.7.1)\}.$$

The class $\Lambda_p^{mult}(E_1,\ldots,E_m;Y)$ of positive multiple $p-$summing $m-$linear operators, with its associated norm $\Lambda_p(.)$, is a Banach space.

Proposition 1.7.1. *Let $1 \leq p < \infty$ and $T \in \mathcal{L}(E_1,\ldots,E_m;Y)$.
If $\widehat{T} \in \Lambda_p(E_1,\Lambda_p^{mult}(E_2,\ldots,E_m;Y))$, then $T \in \Lambda_p^{mult}(E_1,E_2,\ldots,E_m;Y)$ with $\Lambda_p(T) \leq \Lambda_p(\widehat{T})$. In particular, $T \in \Lambda_1^{mult}(E_1,E_2,\ldots,E_m;Y)$ if and only if $\widehat{T} \in \Lambda_1(E_1,\Lambda_1^{mult}(E_2,\ldots,E_m;Y))$. In this case $\Lambda_1(T) \leq \Lambda_1(\widehat{T}) \leq 2^n \Lambda_1(T)$.*

Corollary 1.7.1. *Let $1 \leq p < \infty$. Then*

$$\Lambda_1^{mult}(E_1, ..., E_m; Y) \subseteq \Lambda_p^{mult}(E_1, ..., E_m; Y).$$

1.7.2 Cohen positive strongly p-summing multilinear operators

Definition 1.7.2. *[14]. Let $1 \leq p \leq +\infty$. An $m-$linear operator $T : X_1 \times \cdots \times X_m \longrightarrow F$ is Cohen positive strongly $p-$summing multilinear operator, if there is a constant $C > 0$, such that for any $x_1^j, ..., x_n^j \in X_j, \;\; 1 \leq j \leq m$, and any $y_1^*, ..., y_n^* \in F^*$*

$$\left\| \left\langle T(x_i^1, ..., x_i^m), y_i^* \right\rangle \right\|_{\ell_1^n} \leq C \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \left\| x_i^j \right\|_{X_j}^p \right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)} . \quad (1.7.2)$$

Moreover, the class of all Cohen positive strongly $p-$summing $m-$linear operators from $X_1 \times \cdots \times X_m$ into F is denoted by $\mathcal{D}_p^{m+}(X_1, ..., X_m; F)$. This space is a Banach space with the norm $d_p^{m+}(.)$ which is the smallest constant C such that the inequality (1.7.2) holds. For $m = 1$ it is the space of positive strongly p-summing operators linear operators [1].

Proposition 1.7.2. *(**Ideal property**) Let $T \in \mathcal{L}(X_1, ..., X_m; F)$, R a positive operator in $\mathcal{L}(F, G)$ and S_j in $\mathcal{L}(E_j, X_j)$ $(1 \leq j \leq m)$.*

(i) If T is Cohen positive strongly p-summing, then RT is Cohen positive strongly p-summing and $d_p^{m+}(RT) \leq d_p^{m+}(T)\|R\|$.

(ii) If T is Cohen positive strongly p-summing, then $T \circ (S_1, ..., S_m)$ is Cohen positive strongly p-summing and

$$d_p^{m+}(T \circ (S_1, ..., S_m)) \leq d_p^{m+}(T) \prod_{j=1}^{m} \|S_j\|.$$

Theorem 1.7.1. *(**Domination theorem**) If an m-linear operator $T \in \mathcal{L}(X_1, ..., X_m; F)$ is Cohen positive strongly p-summing $(1 < p \leq \infty)$ then there is a Radon probability measure μ on $B_{F^{**}}^+$, with the weak star topology, such that for all $(x^1, ..., x^m) \in X_1 \times \cdots \times X_m$ and $y^* \in F^{*+}$ we have*

$$| < T(x^1, ..., x^m), y^* > | \leq d_p^{m+}(T) \prod_{j=1}^{m} \|x^j\| \|y^*\|_{L_{p^*}(B_{F^{**}}^+, \mu)}.$$

Conversely, if there is a positive constant C and a Radon probability measure μ on $B^+_{F^{**}}$ such that for all $(x^1, ..., x^m) \in X_1 \times \cdots \times X_m$ and $y^* \in F^{*+}$ we have

$$|<T(x^1, ..., x^m), y^*>| \leq C \prod_{j=1}^{m} \|x^j\| \|y^*\|_{L_{p^*}(B^+_{F^{**}}, \mu)}.$$

Then $T \in \mathcal{D}^{m+}_p(X_1, ..., X_m; F)$ and $d^{m+}_p(T) \leq C$.

The Ky-Fan lemma [25] is used to prove the Domination theorem. The class of Cohen positive strongly p-summing m-linear operators verifies an inclusion theorem [14, Theorem 2.7].
Next, we present a duality theorem between positive p-summing operators and Cohen positive strongly p-summing m-linear operators.

Theorem 1.7.2. *Let $1 < p \leq \infty$. Let $T \in \mathcal{L}(X_1, ..., X_m; F)$ and T^* its adjoint. Then T belongs to $\mathcal{D}^{m+}_p(X_1, ..., X_m; F)$, if and only if, T^* belongs to $\Pi^+_{p^*}(F^*, \mathcal{L}(X_1, ..., X_m))$ and we have $d^{m+}_p(T) = \pi^+_{p^*}(T^*)$.*

1.7.3 Positive Cohen p-nuclear m-linear operators

Definition 1.7.3. *[16] An $m-$linear operator $T : E_1 \times \cdots \times E_m \longrightarrow F$ is positive Cohen $p-$nuclear $(1 < p < \infty)$ if there is a constant $C > 0$ such that for any $x^j_1, ..., x^j_n \in E_j$ $(1 \leq j \leq m)$ and any $y^*_1, ..., y^*_n \in F^*$, we have*

$$\left|\sum_{i=1}^{n} \left\langle T(x^1_i, ..., x^m_i), y^*_i \right\rangle\right| \leq C \left(\sup_{\substack{x^{j*} \in B^+_{E^*_j} \\ 1 \leq j \leq m}} \sum_{i=1}^{n} \prod_{j=1}^{m} \left\langle \left|x^j_i\right|, x^{j*} \right\rangle^p \right)^{\frac{1}{p}} \|(y^*_i)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)} . \tag{1.7.3}$$

Moreover, the class of all positive Cohen $p-$nuclear $m-$linear operators from $E_1 \times \cdots \times E_m$ into F is denoted by $\mathcal{N}^{m+}_p(E_1, ..., E_m; F)$ (or $\mathcal{N}^+_p(E; F)$ if $m = 1$, the space of positive Cohen p-nuclear operators). Our space is a Banach space with the norm $n^{m+}_p(.)$, which is the smallest constant C such that (1.7.3) holds. For $p = \infty$, we have $\mathcal{N}^{m+}_\infty(E_1, ..., E_m; F) = \mathcal{D}^{m+}_\infty(E_1, ..., E_m; F)$.

Proposition 1.7.3. *Let $T \in \mathcal{L}(E_1, ..., E_m; F)$, R a positive operator in $\mathcal{L}(F, G)$ and S_j a positive operator in $\mathcal{L}(F_j, E_j)$ $(1 \leq j \leq m)$.*
(1) If T is positive Cohen $p-$nuclear, then RT is positive Cohen $p-$nuclear from $E_1 \times \cdots \times E_m$ to G and $n^{m+}_p(RT) \leq n^{m+}_p(T) \|R\|$.
(2) If T is positive Cohen $p-$nuclear, then $T \circ (S_1, ..., S_m)$ is positive Cohen $p-$nuclear from $F_1 \times \cdots \times F_m$

$$n^{m+}_p(T \circ (S_1, ..., S_m)) \leq n^{m+}_p(T) \prod_{j=1}^{m} \|S_j\|.$$

Theorem 1.7.3. *An $m-$linear operators $T \in \mathcal{L}(E_1, ..., E_m; F)$ is positive Cohen $p-$ nuclear $(1 < p \leq \infty)$ if, and only if, there exists a positive constant $C > 0$ and Radon probability measures μ_j on $B^{+}_{E^*_j}$ $(1 \leq j \leq m)$ and λ on $B^{+}_{F^{**}}$ such that for all $(x^1, ..., x^m) \in E^{+}_1 \times \cdots \times E^{+}_m$ and $y^* \in F^{*+}$*

$$\left\langle T(x^1, ..., x^m), y^* \right\rangle \leq C \prod_{j=1}^{m} \left\| x^j \right\|_{L_p\left(B^{+}_{E^*_j}, \mu_j\right)} \|y^*\|_{L_{p^*}\left(B^{+}_{F^{**}}, \lambda\right)}. \tag{1.7.4}$$

Theorem 1.7.4. *(**Kwapień's Factorization Theorem**) Let $1 < p \leq \infty$ and $j = 1, ..., m$. Then $T \in \mathcal{L}(E_1, ..., E_m; F)$ is positive Cohen $p-$nuclear if, and only if, there exist Banach spaces $X_1, ..., X_m$, positive $p-$summing linear operators $u_j \in \mathcal{L}(E_j, X_j)$ and a Cohen positive strongly $p-$summing multilinear operator $S \in \mathcal{L}(X_1, ..., X_m; F)$ such that $T = S(u_1, ..., u_m)$, i.e. $\mathcal{N}^{m+}_p = \mathcal{D}^{m+}_p \circ \left(\Pi^{+}_p, ..., \Pi^{+}_p\right)$, in addition*

$$n^{m+}_p(T) = \inf\left\{ d^{m+}_p(S) \prod_{j=1}^{m} \pi^{+}_p(u_j) : T = S(u_1, ..., u_m) \right\}.$$

Proof. To prove the first implication, we take $T \in \mathcal{N}^{m+}_p(E_1, ..., E_m; F)$. Then, by (1.7.4), there exist Radon probability measures μ_j *on* $B^{+}_{E^*_j}$ $(1 \leq j \leq m)$ *and* λ *on* $B^{+}_{F^{**}}$ *such that for all* $(x^1, ..., x^m) \in E^{+}_1 \times \cdots \times E^{+}_m$ and $y^* \in F^{*+}$

$$\langle T(x^1, ..., x^m), y^* \rangle \leq C \prod_{j=1}^{m} \|x^j\|_{L_p\left(B^{+}_{E^*_j}, \mu_j\right)} \|y^*\|_{L_{p^*}\left(B^{+}_{F^{**}}, \lambda\right)}.$$

Now consider the diagram

$$\begin{array}{ccccccccc}
E_1 & \times \cdots \times & E_m & \xrightarrow{T} & & & F & & \\
\downarrow\ i_{E_1} & & \downarrow\ i_{E_m} & & & & \uparrow\ S & & \\
i_{E_1}(E_1) & \times \cdots \times & i_{E_m}(E_m) & \underset{(J_{p,0}, ..., J_{p,0})}{\longrightarrow} & L^p_0(\mu_1) & & \times \cdots \times & & L^p_0(\mu_m) \\
\cap & & \cap & & & & & & \\
C\left(B^{+}_{E^*_1}\right) & & C\left(B^{+}_{E^*_j}\right) & & & & & &
\end{array}$$

where $i_{E_j} : E_j \longrightarrow C\left(B^{+}_{E^*_j}\right)$ is the canonical injection. If we denote the range of $J_{p,0} \circ i_{E_j}$ by X_j, and the closure of X_j by $L^p_0(\mu_j)$, the map $X_1 \times \cdots \times X_m \longrightarrow F : J_{p,0} \circ i_{E_j}(x^1, ..., x^m) \longmapsto T(x^1, ..., x^m)$ is well-defined operator. We now apply the Lemma 1.6.1, we find that the operators $u_j = J_{p,0} \circ i_{E_j} : E_j \longrightarrow i_{E_j}(E_j) \longrightarrow L^p_0(\mu_j)$ are positive p-summing, and $\pi^{+}_p(u_j) \leq 1$.

The operator S is defined on $u_1(E_1) \times \cdots \times u_m(E_m)$,
where $u_j(x^j) = \left(J_{p,0} \circ i_{E_j}\right)(x^j)$, by $S(u_1(x^1), ..., u_m(x^m)) := T(x^1, ..., x^m)$,

this definition makes sense because

$$\begin{aligned}
\left|\left\langle S\left(u_1\left(x^1\right),...,u_m\left(x^m\right)\right),y^*\right\rangle\right| &= \left|\left\langle S\left(u_1\left(x^1\right),...,u_m\left(x^m\right)\right),y^*\right\rangle\right| = |\langle T(x^1,...,x^m),y^*\rangle| \\
\text{By (1.7.4)} &\leq n_p^{m+}(T)\prod_{j=1}^{m}\|x^j\|\left(\int_{B_{F^{**}}^+}\langle y^*,y^{**}\rangle^{p^*}d\lambda\left(y^{**}\right)\right)^{\frac{1}{p^*}}. \\
&= n_p^{m+}(T)\prod_{j=1}^{m}\|u_j\left(x^j\right)\|_{L_0^p(\mu_j)}\left(\int_{B_{F^{**}}^+}\langle y^*,y^{**}\rangle^{p^*}d\lambda\left(y^{**}\right)\right)^{\frac{1}{p^*}}.
\end{aligned}$$

From [14, Theorem 2.5] S is a Cohen positive strongly $p-$summing $m-$linear operator and $d_p^{m+}(S) \leq n_p^{m+}(T)$. □

Chapter 2

On the positive Dimant strongly p-summing multilinear operators

This Chapter is intended to extend the concept of Dimant strongly p-summing multilinear operators [26] to positive framework, also to study its ties with other known classes of summability.

2.1 Positive strongly p-summing multilinear operators

In this section, we introduce and study a new class of operators between two Banach lattices, where we extend the notions in [26]. We prove a natural analog to the Pietsch domination theorem for this class. Before giving our definition let recall the definition of being strongly p-summing multilinear operator in sense of Dimant.

Definition 2.1.1. *[26] an m-linear operator $T \in \mathcal{L}(X_1, ..., X_m; Y)$ is strongly p-summing $(1 \leq p \leq \infty)$, if there exists a positive constant C such that for every $x_1^j, ..., x_n^j \in X_j$; $1 \leq j \leq m$, we have*

$$\left(\sum_{i=1}^{n} \left\| T\left(x_i^1, ..., x_i^m\right) \right\|^p \right)^{\frac{1}{p}} \leq C \sup_{\phi \in B_{\mathcal{L}(X_1,...,X_m)}} \left(\sum_{i=1}^{n} \left| \phi\left(x_i^1, ..., x_i^m\right) \right|^p \right)^{\frac{1}{p}}. \quad (2.1.1)$$

The class of all strongly p-summing m-linear operators from $X_1 \times \cdots \times X_m$ into Y which is denoted by $\mathcal{L}_{ss,p}^m(X_1, ..., X_m; Y)$ is a Banach space with the norm $d_{s,p}^m(.)$, which $d_{s,p}^m(T)$ is the smallest constant C such that the inequality (2.1.1) holds.

Now, we give our definition

Definition 2.1.2. *Let $1 \leq p \leq \infty$. An $m-$linear operator $T \in \mathcal{L}(E_1, ..., E_m; F)$ is said to be positive strongly p-summing if there exists a constant $C > 0$ such* *that for every* $x_1^j, ..., x_n^j \in E_j^+$ $(j = 1, ..., m)$, *we have*

$$\left(\sum_{i=1}^{n}\left\|T\left(x_i^1,...,x_i^m\right)\right\|^p\right)^{\frac{1}{p}} \leq C \sup_{\phi\in B^+_{\mathcal{L}(E_1,...,E_m)}}\left(\sum_{i=1}^{n}\left(\phi\left(x_i^1,...,x_i^m\right)\right)^p\right)^{\frac{1}{p}}. \quad (2.1.2)$$

Moreover, the class of all positive strongly p-summing $m-$*linear operators from* $E_1 \times \cdots \times E_m$ *into* F, *is denoted by* $\mathcal{L}^{m+}_{ss,p}(E_1, ..., E_m; F)$. *Our space is a Banach space with the norm* $d^{m+}_{s,p}(.)$, *which* $d^{m+}_{s,p}(T)$ *is the smallest constant* C *such that* (2.1.2) *holds.*

Basic properties and the domination theorem

Proposition 2.1.1. ***(Ideal property).*** *Let* $T \in \mathcal{L}(E_1, ..., E_m; F)$, R *a positive operator in* $\mathcal{L}(F, G)$ *and* S_j *a positive operator in* $\mathcal{L}(F_j, E_j)$ $(1 \leq j \leq m)$.

1) *If* T *is positive strongly p-summing,* $R \circ T$ *is positive strongly p-summing and* $d^{m+}_{s,p}(R \circ T) \leq \|R\| \, d^{m+}_{s,p}(T)$.

2) *If* T *is positive strongly p-summing,* $T \circ (S_1, ..., S_m)$ *is positive strongly p-summing and*

$$d^{m+}_{s,p}(T \circ (S_1, ..., S_m)) \leq d^{m+}_{s,p}(T)\prod_{j=1}^{m}\|S_j\|.$$

Proof. 1) Let $T \in \mathcal{L}^{m+}_{ss,p}(E_1, ..., E_m; F)$. Then for all $x_1^j, ..., x_n^j \in E_j^+$ $(j = 1, ..., m)$, we have

$$\left(\sum_{i=1}^{n}\left\|R\circ T\left(x_i^1,...,x_i^m\right)\right\|^p\right)^{\frac{1}{p}} \leq \|R\|\left(\sum_{i=1}^{n}\left\|T\left(x_i^1,...,x_i^m\right)\right\|^p\right)^{\frac{1}{p}}$$
$$\leq \|R\| \, d^{m+}_{s,p}(T) \sup_{\phi\in B^+_{\mathcal{L}(E_1,...,E_m)}}\left(\sum_{i=1}^{n}\left(\phi\left(x_i^1,...,x_i^m\right)\right)^p\right)^{\frac{1}{p}}.$$

So $R \circ T \in \mathcal{L}^{m+}_{ss,p}(E_1, ..., E_m; G)$ and $d^{m+}_{s,p}(R \circ T) \leq \|R\| \, d^{m+}_{s,p}(T)$.

2) Let $T \in \mathcal{L}^{m+}_{ss,p}(E_1, ..., E_m; F)$. Then

$$
\begin{aligned}
&(\sum_{i=1}^{n}\|T\circ(S_1,...,S_m)(x_i^1,...,x_i^m)\|^p)^{\frac{1}{p}} \\
&\quad\leq d_{s,p}^{m+}(T)\sup_{\phi\in B^+_{\mathcal{L}(E_1,...,E_m)}}\left(\sum_{i=1}^{n}\left(\phi\left(S_1\left(x_i^1\right),...,S_m\left(x_i^m\right)\right)\right)^p\right)^{\frac{1}{p}} \\
&\quad\leq d_{s,p}^{m+}(T)\prod_{j=1}^{m}\|S_j\|\sup_{\phi\in B^+_{\mathcal{L}(E_1,...,E_m)}}\left(\sum_{i=1}^{n}\left(\frac{1}{\prod_{j=1}^{m}\|S_j\|}\phi\left(S_1,...,S_m\right)\left(x_i^1,...,x_i^m\right)\right)^p\right)^{\frac{1}{p}} \\
&\quad\leq d_{s,p}^{m+}(T)\prod_{j=1}^{m}\|S_j\|\sup_{\psi\in B^+_{\mathcal{L}(F_1,...,F_m)}}\left(\sum_{i=1}^{n}\left(\psi\left(x_i^1,...,x_i^m\right)\right)^p\right)^{\frac{1}{p}}.
\end{aligned}
$$

We have $T\circ(S_1,...,S_m)$ is positive strongly p-summing multilinear operator and

$$d_{s,p}^{m+}\left(T\circ\left(S_1,...,S_m\right)\right)\leq d_{s,p}^{m+}\left(T\right)\prod_{j=1}^{m}\|S_j\|. \qquad \square$$

For the proof of the next theorem, we will use the full general Pietsch domination theorem recently presented by Pellegrino et al. in [46].

Theorem 2.1.1. *An $m-$linear operator $T\in\mathcal{L}\left(E_1,...,E_m;F\right)$ is a positive strongly p-summing if and only if there exist a regular probability measure μ on $B^+_{\mathcal{L}(E_1,...,E_m)}$ with the weak star topology, and a positive constant C such that*

$$\left\|T\left(x^1,...,x^m\right)\right\|\leq C\left(\int_{B^+_{\mathcal{L}(E_1,...,E_m)}}\left(\phi\left(x^1,...,x^m\right)\right)^p d\mu\left(\phi\right)\right)^{\frac{1}{p}} \tag{2.1.3}$$

for every $(x^1,...,x^m)\in E_1^+\times\cdots\times E_m^+$. Moreover, in this case

$$d_{s,p}^{m+}\left(T\right)=\inf\left\{C \text{ which satisfies the inequality } (2.1.3)\right\}.$$

Proof. Suppose that T is positive strongly p-summing. We will verify the hypotheses of the general Pietsch Domination Theorem. We put

$$
\begin{cases}
G=\mathbb{K} \\
\mathcal{H}=\mathcal{L}\left(E_1,...,E_m;F\right) \\
Z=E_1^+\times\cdots\times E_m^+ \\
K=B^+_{\left(E_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi E_m\right)^*} \\
x_0=\left(0,...,0\right).
\end{cases}
$$

The notion of positive strongly p-summing multilinear operator is precisely the concept of $R-S$-abstract p-summing, where

$$R: B^{+}_{(E_1\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi E_m)^*}\times E_1^{+}\times\cdots\times E_m^{+}\times\mathbb{K}\longrightarrow[0,\infty)$$
$$R\left(\varphi,\left(x^1,...,x^m\right),\lambda\right)=|\lambda|\left(\varphi\left(x^1,...,x^m\right)\right)$$
$$S:\mathcal{L}(E_1,...,E_m;F)\times E_1^{+}\times\cdots\times E_m^{+}\times\mathbb{K}\longrightarrow[0,\infty)$$
$$S\left(T,\left(x^1,...,x^m\right),\lambda\right)=|\lambda|\left\|T\left(x^1,...,x^m\right)\right\|,$$

here, R and S satisfy the conditions of full general Pietsch domination theorem. Then the result follows immediately. □

We need a small review about $R-S$-abstract p-summing.
Let X,Y and Z be (arbitrary) sets, $\mathcal{H}$ be a family of mappings from X to Y, G be a Banach space and K be a compact Hausdorff topological space. Suppose that the maps

$$R: K\times Z\times G\longrightarrow[0,\infty)$$
$$S:\mathcal{H}\times Z\times G\longrightarrow[0,\infty).$$

be mappings so that:

. For each $f\in\mathcal{H}$, there is $x_0\in Z$ such that

$$R(\varphi,x_0,b)=S(f,x_0,b),$$

for every $\varphi\in K$ and $b\in G$.

. The mapping $R_{x,b}: K\longrightarrow[0,\infty)$ defined by $R_{x,b}(\varphi)=R(\varphi,x,b)$ is continuous for every $x\in Z$ and $b\in G$.

. It holds that, $R(\varphi,x,\eta b)\le\eta R(\varphi,x,b)$ and $\eta S(f,x,b)\le S(f,x,\eta b)$,

for every $\varphi\in K$, $x\in Z$, $0\le\eta\le1$, $b\in G$ and $f\in\mathcal{H}$.

Definition 2.1.3. *[46] Let R and S be as above and $0<p<\infty$. A mapping $f\in\mathcal{H}$ is said to be $R-S$-abstract p-summing if there is a constant $C_1>0$ so that*

$$\left(\sum_{j=1}^{m}S\left(f,x^j,b^j\right)^p\right)^{\frac{1}{p}}\le C_1\sup_{\varphi\in K}\left(\sum_{j=1}^{m}R\left(\varphi,x^j,b^j\right)^p\right)^{\frac{1}{p}},$$

for all $x^1,...,x^m\in Z$ et $b^1,...,b^m\in G$. The infimum of such constants C_1 is denoted by $\pi_{RS,p}(f)$.

Theorem 2.1.2. *[46] Let R and S be as above, $0<p<\infty$ and $f\in\mathcal{H}$. Then f is $R-S$-abstract p-summing if and only if there is a constant $C>0$ and a regular Borel probability measure μ on K such that for all $x\in Z$ and $b\in G$,*

$$S(f,x,b) \leq C\left(\int_K (R(\varphi,x,b))^p \, d\mu(\varphi)\right)^{\frac{1}{p}},$$

Moreover, the infimum of such constants C equals $\pi_{RS,p}(f)$.

Proposition 2.1.2. *Let $T \in \mathcal{L}(E_1,...,E_m;F)$ is positive strongly p -summing if and only if there exist a regular probability measure μ on $B^+_{\mathcal{L}(E_1,...,E_m)}$ with the weak star topology, and a constant $K > 0$,such that the inequality*

$$\left\|T\left(x^1,...,x^m\right)\right\| \leq K\left(\int_{B^+_{\mathcal{L}(E_1,...,E_m)}} \left(\phi\left(\left|x^1\right|,...,\left|x^m\right|\right)\right)^p d\mu(\phi)\right)^{\frac{1}{p}},$$

for every $(x^1,...,x^m) \in E_1 \times \cdots \times E_m$.

Proof. For convenience, we prove the inequality for $m = 2$ only. Let $(x,y) \in E_1 \times E_2$, and by Theorem 2.1.1, we have

$$\begin{aligned}
\|T(x,y)\| &= \|T(x^+ - x^-, y^+ - y^-)\| \\
&= \left\|T\left(x^+,y^+\right) - T\left(x^+,y^-\right) - T\left(x^-,y^+\right) + T\left(x^-,y^-\right)\right\| \\
&\leq \left\|T\left(x^+,y^+\right)\right\| + \left\|T\left(x^+,y^-\right)\right\| + \left\|T\left(x^-,y^+\right)\right\| + \left\|T\left(x^-,y^-\right)\right\| \\
&\leq C_1\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\phi(x^+,y^+))^p d\mu(\phi)\right)^{\frac{1}{p}} + C_2\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\phi(x^+,y^-))^p d\mu(\phi)\right)^{\frac{1}{p}} \\
&\leq C_3\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\phi(x^-,y^+))^p d\mu(\phi)\right)^{\frac{1}{p}} + C_4\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\phi(x^-,y^-))^p d\mu(\phi)\right)^{\frac{1}{p}} \\
&\leq K\left(\int_{B^+_{\mathcal{L}(E_1,E_2)}} (\phi(|x|,|y|))^p d\mu(\phi)\right)^{\frac{1}{p}}.
\end{aligned}$$

To prove the converse, just apply Theorem 2.1.1 for $(x^1,...,x^m) \in E_1^+ \times \cdots \times E_m^+$. □

Proposition 2.1.3. *If $1 \leq p < q < \infty$, then*

$$\mathcal{L}^{m+}_{ss,p}(E_1,...,E_m;F) \subset \mathcal{L}^{m+}_{ss,q}(E_1,...,E_m;F) \text{ and } d^{m+}_{s,q}(T) \leq d^{m+}_{s,p}(T).$$

Proof. Let $T \in \mathcal{L}_{ss,p}^{m+}(E_1, ..., E_m; F)$. Then by (2.1.3)

$$\left\|T\left(x^1, ..., x^m\right)\right\| \leq d_{s,p}^{m+}(T)\left(\int_{B^+_{\mathcal{L}(E_1,...,E_m)}} \left(\phi\left(x^1, ..., x^m\right)\right)^p d\mu(\phi)\right)^{\frac{1}{p}}$$

$$\leq d_{s,p}^{m+}(T)\left(\int_{B^+_{\mathcal{L}(E_1,...,E_m)}} \left(\phi\left(x^1, ..., x^m\right)\right)^q d\mu(\phi)\right)^{\frac{1}{q}},$$

such that for every $(x^1, ..., x^m) \in E_1^+ \times \cdots \times E_m^+$.

This implies that $T \in \mathcal{L}_{ss,q}^{m+}(E_1, ..., E_m; F)$ and $d_{s,q}^{m+}(T) \leq d_{s,p}^{m+}(T)$. □

Here is another important result.

Proposition 2.1.4. *Let $1 \leq p < \infty$. Let $A_1 : E_1 \times \cdots \times E_{m_1} \to F_1, ..., A_k : E_{1+m_{k-1}} \times \cdots \times E_{m_k} \to F_k$ and $T : F_1 \times \cdots \times F_k \to F$ be non null continuous multilinear operators. If all $A_1, ..., A_k$ are positive strongly p-summing, then $T \circ (A_1, ..., A_k)$ is positive strongly p-summing and*

$$d_{s,p}^{m+}(T \circ (A_1, ..., A_k)) \leq \|T\| \prod_{j=1}^{k} d_{s,p}^{m+}(A_j), \text{ for } (1 \leq j \leq k).$$

Proof. From the domination theorem, there exist regular Borel probabilities μ_1 on $B_1 = B^+_{\mathcal{L}\left(E_1,...,E_{m_1}\right)}, ..., \mu_k$ on $B_k = B^+_{\mathcal{L}\left(E_{1+m_{k-1}},...,E_{m_k}\right)}$ such that for each $(x^1, ..., x^{m_1}) \in E_1^+ \times \cdots \times E_{m_1}^+, ..., (x^{1+m_{k-1}}, ..., x^{m_k}) \in E_{1+m_{k-1}}^+ \times \cdots \times E_{m_k}^+$,

$$\|A_1\left(x^1, ..., x^{m_1}\right)\| \leq d_{s,p}^{m+}(A_1)\left(\int_{B_1} \left(\phi_1\left(x^1, ..., x^{m_1}\right)\right)^p d\mu_1(\phi_1)\right)^{\frac{1}{p}}$$

$$\cdots$$

$$\|A_k\left(x^{1+m_{k-1}}, ..., x^{m_k}\right)\| \leq d_{s,p}^{m+}(A_k)\left(\int_{B_k} \left(\phi_k\left(x^{1+m_{k-1}}, ..., x^{m_k}\right)\right)^p d\mu_k(\phi_k)\right)^{\frac{1}{p}}.$$

Fubini's theorem gives that, for each $(x^1, ..., x^{m_k}) \in E_1^+ \times \cdots \times E_{m_k}^+$.

$$\left\|A_1\left(x^1, ..., x^{m_1}\right)\right\|^p \cdots \|A_k(x^{1+m_{k-1}}, ..., x^{m_k})\|^p \leq \left[\prod_{j=1}^{k} d_{s,p}^{m+}(A_j)\right]^p \tag{2.1.4}$$

$$\times \int\cdots\int_{B_1\times\cdots\times B_k} \left(\phi_1\left(x^1, ..., x^{m_1}\right)\right)^p \cdots (\phi_k(x^{1+m_{k-1}}, ..., x^{m_k}))^p d(\mu_1 \times \cdots \times \mu_k)(\phi_1, ..., \phi_k).$$

Let $1 \leq i \leq n$ and $(x_i^1, ..., x_i^{m_k}) \in E_1^+ \times \cdots \times E_{m_k}^+$. Since

$$\|T \circ (A_1, ..., A_k)(x_i^1, ..., x_i^{m_k})\| \leq \|T\| \|A_1(x_i^1, ..., x_i^{m_1})\| \cdots \left\|A_k\left(x_i^{1+m_{k-1}}, ..., x_i^{m_k}\right)\right\|.$$

By (2.1.4), we find

$$\left(\sum_{i=1}^{n}\left\|T \circ (A_1, ..., A_k)\left(x_i^1, ..., x_i^{m_k}\right)\right\|^p\right)^{1/p} \leq \|T\| \prod_{j=1}^{k} d_{s,p}^{m+}(A_j) \tag{2.1.5}$$

$$\times \left(\sum_{i=1}^{n} \int_{B_1 \times \cdots \times B_k} \cdots \int \left(\phi_1 (x_i^1, ..., x_i^m)^p \cdots \phi_k\left(x_i^{1+m_{k-1}}, ..., x_i^{m_k}\right)^p\right) d(\mu_1 \times \cdots \times \mu_k)(\phi_1, ..., \phi_k)\right)^{1/p}.$$

For $(\phi_1, ..., \phi_k) \in B_1 \times \cdots \times B_k$ we define $\phi_1 \otimes \cdots \otimes \phi_k : E_1 \times \cdots \times E_{m_k} \longrightarrow \mathbb{K}$ by

$$(\phi_1 \otimes \cdots \otimes \phi_k)(x^1, ..., x^{m_1}, ..., x^{1+m_{k-1}}, ..., x^{m_k}) = \phi_1(x^1, ..., x^{m_1}) \cdots \phi_k(x^{1+m_{k-1}}, ..., x^{m_k}).$$

Then

$$\phi_1 \otimes \cdots \otimes \phi_k \in \mathcal{L}(E_1, ..., E_{m_k}), \|\phi_1 \otimes \cdots \otimes \phi_k\| \leq \|\phi_1\| \cdots \|\phi_k\|$$

and

$$\left(\sum_{i=1}^{n} \phi_1\left(x_i^1, ..., x_i^{m_1}\right)^p \cdots \phi_k\left(x_i^{1+m_{k-1}}, ..., x_i^{m_k}\right)^p\right)^{1/p} \tag{2.1.6}$$

$$= \left(\sum_{i=1}^{n} (\phi_1 \otimes \cdots \otimes \phi_k)\left(x_i^1, ..., x_i^{m_k}\right)^p\right)^{1/p}$$

$$\leq \sup_{\phi_1 \otimes \cdots \otimes \phi_k \in B^+_{\mathcal{L}\left(E_1, ..., E_{m_k}\right)}} \left(\sum_{i=1}^{n}\left((\phi_1 \otimes \cdots \otimes \phi_k)\left(x_i^1, ..., x_i^{m_k}\right)\right)^p\right)^{\frac{1}{p}}.$$

Using (2.1.6) and the fact that $\mu_1, ..., \mu_k$ are probability measures , by (2.1.5), we get

$$\left(\sum_{i=1}^{n}\left\|T\circ(A_1,...,A_k)\left(x_i^1,...,x_i^{m_k}\right)\right\|^p\right)^{1/p}\leq\|T\|\prod_{j=1}^{k}d_{s,p}^{m+}(A_j)$$

$$\times\sup_{\phi_1\otimes\cdots\otimes\phi_k\in B^+_{\mathcal{L}\left(E_1,...,E_{m_k}\right)}}\left(\sum_{i=1}^{n}\left(\left(\phi_1\otimes\cdots\otimes\phi_k\right)\left(x_i^1,...,x_i^{m_k}\right)\right)^p\right)^{\frac{1}{p}}.$$

Then from Definition 2.1.2, $T\circ(A_1,...,A_k)$ is positive strongly p-summing operators.

□

2.2 Ties with other known classes of summability

Our main results of this section, is to analyse some connections between the different classes investigated earlier.

Theorem 2.2.1. *Let $1<p\leq\infty$. If $T\in\mathcal{L}(E_1,...,E_m;F)$ is such that T^* is a Cohen positive strongly p^*-summing linear operator, then T is positive strongly p-summing multilinear operator.*

Proof. Suppose that $T^*\in\mathcal{L}(F^*;\mathcal{L}(E_1,...,E_m))$ is Cohen positive strongly p^*-summing linear operator. We have by (1.6.2)

$$\sum_{i=1}^{n}|\langle T^*(y_i^*),z_i^*\rangle|\leq d_{p^*}^+(T^*)\left(\sum_{i=1}^{n}\|y_i^*\|^{p^*}\right)^{\frac{1}{p^*}}\sup_{\phi\in B^+_{\mathcal{L}(E_1,...,E_m)}}\left(\sum_{i=1}^{n}(z_i^*(\phi))^p\right)^{\frac{1}{p}}$$

Let now $\left(x_i^j\right)_{i=1}^n\subset E_j^+\ (1\leq j\leq m)$. We consider the linear form

$T_{\left(x_i^1,...,x_i^m\right)}:\mathcal{L}(E_1,...,E_m)\longrightarrow\mathbb{K}$ defined by $T_{\left(x_i^1,...,x_i^m\right)}(\phi)=\phi\left(x_i^1,...,x_i^m\right).$

We have

$$\begin{aligned}\sum_{i=1}^{n}|\langle T(x_i^1,...,x_i^m),y_i^*\rangle|&=\sum_{i=1}^{n}\left|\left\langle T^*(y_i^*),T_{\left(x_i^1,...,x_i^m\right)}\right\rangle\right|\\&\leq d_{p^*}^+(T^*)\left(\sum_{i=1}^{n}\|y_i^*\|^{p^*}\right)^{\frac{1}{p^*}}\sup_{\phi\in B^+_{\mathcal{L}(E_1,...,E_m)}}\left(\sum_{i=1}^{n}\left(T_{\left(x_i^1,...,x_i^m\right)}(\phi)\right)^p\right)^{\frac{1}{p}}\\&\leq d_{p^*}^+(T^*)\left(\sum_{i=1}^{n}\|y_i^*\|^{p^*}\right)^{\frac{1}{p^*}}\sup_{\phi\in B^+_{\mathcal{L}(E_1,...,E_m)}}\left(\sum_{i=1}^{n}\phi\left(x_i^1,...,x_i^m\right)^p\right)^{\frac{1}{p}}.\end{aligned}$$

Taking the supremum over all sequences $(y_i^*)_{i=1}^n$ with $(\sum_{i=1}^n \|y_i^*\|^{p^*})^{\frac{1}{p^*}} \leq 1$, we obtain

$$\left(\sum_{i=1}^n \left\|T\left(x_i^1, ..., x_i^m\right)\right\|^p\right)^{\frac{1}{p}} = \sup_{(\sum_{i=1}^n \|y_i^*\|^{p^*})^{\frac{1}{p^*}} \leq 1} \left(\sum_{i=1}^n \left|\left\langle T\left(x_i^1, ..., x_i^m\right), y_i^*\right\rangle\right|\right)$$
$$\leq d_{p^*}^+(T^*) \sup_{\phi \in B^+_{\mathcal{L}(E_1,...,E_m)}} \left(\sum_{i=1}^n \phi\left(x_i^1, ..., x_i^m\right)^p\right)^{\frac{1}{p}}.$$

Then, T is positive strongly p-summing and $d_{s,p}^{m+}(T) \leq d_{p^*}^+(T^*)$.

□

Open problem. Is the inverse true?

Proposition 2.2.1. *Let $T \in \mathcal{L}(E_1, ..., E_m; F)$ and R be a positive operator in $\mathcal{L}(F, G)$. If R is positive p-summing, then $R \circ T$ is positive strongly p-summing and*

$$d_{s,p}^{m+}(R \circ T) \leq \pi_p^+(R) \|T\|.$$

Proof. Let R be an operator in $\Pi_p^+(F, G)$, from Definition 1.6.1 we have

$$\left\|R \circ T\left(x_i^1, ..., x_i^m\right)\right\|_p \leq \pi_p^+(R) \sup_{y^* \in B^+_{F^*}} \left(\sum_{i=1}^n \left\langle T\left(x_i^1, ..., x_i^m\right), y^*\right\rangle^p\right)^{1/p}$$
$$\leq \pi_p^+(R) \sup_{y^* \in B^+_{F^*}} \left(\sum_{i=1}^n \left\langle T_{\left(x_i^1,...,x_i^m\right)}, T^*(y^*)\right\rangle^p\right)^{1/p}$$
$$\leq \pi_p^+(R) \|T\| \sup_{y^* \in B^+_{F^*}} \left(\sum_{i=1}^n \left\langle T_{\left(x_i^1,...,x_i^m\right)}, \frac{T^*(y^*)}{\|T\|}\right\rangle^p\right)^{1/p}$$
$$\leq \pi_p^+(R) \|T\| \sup_{\phi \in B^+_{\mathcal{L}(E_1,...,E_m)}} \left(\sum_{i=1}^n \left(\phi\left(x_i^1, ..., x_i^m\right)\right)^p\right)^{\frac{1}{p}}.$$

So $R \circ T \in \mathcal{L}_{ss,p}^{m+}(E_1, ..., E_m; G)$ and $d_{s,p}^{m+}(R \circ T) \leq \pi_p^+(R) \|T\|$. □

For our next result, we need a reminder on the positive projective tensor norm found in [19, 29].
The projective cone on the tensor product $E_1 \otimes \cdots \otimes E_m$ is defined as:

$$E_1^+ \otimes \cdots \otimes E_m^+ = \left\{\sum_{i=1}^n x_i^1 \otimes \cdots \otimes x_i^m : x_i^j \in E_j^+, \ j = 1, ..., m, \ n \in \mathbb{N}\right\}.$$

The positive projective tensor norm on $E_1 \otimes \cdots \otimes E_m$ is defined as:

$$\|u\|_{|\pi|} = \inf\left\{\sum_{i=1}^{n}\prod_{j=1}^{m}\left\|x_i^j\right\| : x_i^j \in E_j^+, n \in \mathbb{N}, |u| \leq \sum_{i=1}^{n} x_i^1 \otimes \cdots \otimes x_i^m\right\},$$

for every $u \in E_1 \otimes \cdots \otimes E_m$. By $E_1\widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|}E_m$ will denote the complete positive m-fold projective tensor product of $E_1, ..., E_m$ (Fremlin projective tensor product). Then $E_1\widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|}E_m$ with the above norm, is a Banach lattice; we will use the next notation $\widehat{E} = E_1\widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|}E_m$.

Every $T \in \mathcal{L}(E_1, ..., E_m; F)$ has an associated linear operator $T_L \in \mathcal{L}(\widehat{E}; F)$ defined by $\forall x^j \in E_j^+$ $(j = 1, ..., m)$, $T_L(x^1 \otimes \cdots \otimes x^m) = T(x^1, ..., x^m)$ (See [19] for details). Let $\otimes : E_1 \times \cdots \times E_m \longrightarrow \widehat{E}$ denote the canonical $m-$linear operator, that is, $\otimes(x^1, ..., x^m) = x^1 \otimes \cdots \otimes x^m$. Then,

$$T = T_L \circ \otimes. \tag{2.2.1}$$

Corollary 2.2.1. *Let* $1 \leq p \leq \infty$. *If* $T_L \in \Pi_p^+(\widehat{E}; F)$ *then* $T \in \mathcal{L}_{ss,p}^{m+}(E_1, ..., E_m; F)$.

Proof. If $T_L \in \Pi_p^+(\widehat{E}; F)$ and the fact that $T = T_L \circ \otimes$, by above proposition, $T \in \mathcal{L}_{ss,p}^{m+}(E_1, ..., E_m; F)$ and $d_{s,p}^{m+}(T) \leq \pi_p^+(T_L)$. □

Open Problem. Is the opposite implication in the last corollary true?

Chapter 3

Multiple Cohen positive strongly $p-$summing $m-$linear operators

In this chapter, we introduce the new class of multiple Cohen positive strongly $p-$summing operators and compare it with the class of Cohen positive strongly p-summing m-linear operators [14] and positive multiple p-summing m-linear operators [18], by giving a generalization to Cohen's theorems [23], as well as, investigating a relationship with the class multiple Cohen positive p-nuclear operators.

3.1 Multiple Cohen positive strongly p-summing operators

In this section, we give the new notion of multiple Cohen positive strongly $p-$summing operators, as a prototype of the multiple Cohen strongly summing operators initiated by Campos in [21] and motivated by Matos in his famous paper "Fully absolutely summing and Hilbert-Schmidt multilinear mappings" [37], as well as studying inclusions and coincidences with some known spaces.
All along this chapter the Banach space F we be finite dimensional.

Definition 3.1.1. *Let $1 \leq p \leq \infty$. An $m-$linear operator $T : X_1 \times \cdots \times X_m \longrightarrow F$ is multiple Cohen positive strongly $p-$summing $m-$linear operator, if there is a constant $C > 0$ such that for any, $n \in \mathbb{N}^*$, $y^*_{i_1,...,i_m} \in F^{*+}$ and any $x^j_{i_j} \in X_j$ such that $1 \leq j \leq m, \quad 1 \leq i \leq n \quad 1 \leq i_j \leq n$*

$$\sum_{i_1,...,i_m=1}^{n} | < T(x^1_{i_1}, ..., x^m_{i_m}), y^*_{i_1,...,i_m} > | \leq C \left(\prod_{j=1}^{m} \|(x^j_i)^n_{i=1}\|_p \right) \|(y^*_{i_1,...,i_m})^n_{i_1,...,i_m=1}\|_{\ell^n_{p^*,weak}(F^*)}.$$

The class of all multiple Cohen positive strongly $p-$summing $m-$linear operators from $X_1 \times \cdots \times X_m$ into F is a Banach space denoted by $\mathcal{D}^{mult+}_p(X_1, ..., X_m; F)$, *with the norm* $d_p^{mult+}(.)$ *given by the infimum of constants* C *verifying the above inequality .*

The next remark is equivalent to Definition 3.1.1, which we need the most in Section 3.

Remark 3.1.1. *Let* $T : X_1 \times \cdots \times X_m \to F$*, then* T *is multiple Cohen positive strongly p-summing m-linear operator if and only if, there exists a constant* $K > 0$ *such that the following inequality holds*

$$\sum_{i_1,\dots,i_m=1}^{n} | < T(x_{i_1}^1, \dots, x_{i_m}^m), y_{i_1,\dots,i_m}^* > | \leq K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p \right) \|(y_{i_1,\dots,i_m}^*)_{i_1,\dots,i_m=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)}$$

for any $n \in \mathbb{N}^*$*,* $y_{i_1,\dots,i_m}^* \in F^*$ *and any* $x_{i_j}^j \in X_j$ *such that* $1 \leq j \leq m, \quad 1 \leq i \leq n \quad 1 \leq i_j \leq n$*.*

For the first part, let , $y_{i_1,\dots,i_m}^* \in F^*$ and $x_{i_j}^j \in X_j$ such that $1 \leq j \leq m$, $1 \leq i \leq n$.

$$\begin{aligned}
\sum_{i_1,\dots,i_m=1}^{n} | < T(x_{i_1}^1, \dots, x_{i_m}^m), y_{i_1,\dots,i_m}^* > |
&= \sum_{i_1,\dots,i_m=1}^{n} | < T(x_{i_1}^1, \dots, x_{i_m}^m), y_{i_1,\dots,i_m}^{*+} - y_{i_1,\dots,i_m}^{*-} > | \\
&\leq \sum_{i_1,\dots,i_m=1}^{n} | < T(x_{i_1}^1, \dots, x_{i_m}^m), y_{i_1,\dots,i_m}^{*+} > | \\
&+ \sum_{i_1,\dots,i_m=1}^{n} | < T(x_{i_1}^1, \dots, x_{i_m}^m), y_{i_1,\dots,i_m}^{*-} > | \\
&\leq K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p \right) \|(y_{i_1,\dots,i_m}^{*+})_{i_1,\dots,i_m=1}^n\|_{\ell_{p^*,weak}^n(F^*)} \\
&+ K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p \right) \|(y_{i_1,\dots,i_m}^{*-})_{i_1,\dots,i_m=1}^n\|_{\ell_{p^*,weak}^n(F^*)} \\
&\leq 2K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p \right) \|(|y_{i_1,\dots,i_m}^*|)_{i_1,\dots,i_m=1}^n\|_{\ell_{p^*,weak}^n(F^*)} \\
&\leq 2K \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p \right) \|(y_{i_1,\dots,i_m}^*)_{i_1,\dots,i_m=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)} \\
&= C \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_p \right) \|(y_{i_1,\dots,i_m}^*)_{i_1,\dots,i_m=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)}.
\end{aligned}$$

The second part is obvious from the formula (1.2.1).

Example 3.1.1. *Every finite type $m-$linear operator from $X_1 \times \cdots \times X_m$ into the finite dimensional Banach lattice F, is multiple Cohen positive strongly $p-$summing $m-$linear operator.*
For any $y^*_{i_1,...,i_m} \in F^{*+}$ and $x^j_{i_j} \in X_j$ such that $1 \leq j \leq m, \quad 1 \leq i \leq n$ and any $\phi_j \in X^*_j$, $b \in F$.

$$\begin{aligned}
&\sum_{i_1,...,i_m=1}^{n} | < \phi_1 \otimes \cdots \otimes \phi_m \otimes b(x^1_{i_1}, ..., x^m_{i_m}), y^*_{i_1,...,i_m} > | \\
&\qquad \leq \|\phi_1 \otimes \cdots \otimes \phi_m \otimes b(x^1_{i_1}, ..., x^m_{i_m})\| \|(y^*_{i_1,...,i_m})^n_{i_1,...,i_m=1}\| \\
&\qquad \leq \|b\| \prod_{j=1}^{m} \|\phi_j(x^j_i)^n_{i=1}\|_p \|(y^*_{i_1,...,i_m})^n_{i_1,...,i_m=1}\|_{p^*} \\
&\qquad \leq \|b\| \prod_{j=1}^{m} \|\phi_j(x^j_i)^n_{i=1}\|_p \|(y^*_{i_1,...,i_m})^n_{i_1,...,i_m=1}\|_{\ell^n_{p^*,weak}(F^*)}.
\end{aligned}$$

Hence $\quad d^{mult+}_p(\phi_1 \otimes \cdots \otimes \phi_m \otimes b) \leq \|b\| \|\phi_1\| ... \|\phi_m\|.$

Inclusion and composition theorems

Theorem 3.1.1. *Every Cohen positive strongly $p-$summing $m-$linear operator is multiple Cohen positive strongly $p-$summing $m-$linear operator and*

$$d^{mult+}_p(.) \leq d^{m+}_p(.).$$

Proof. By the domination theorem of Cohen positive strongly $p-$summing $m-$linear operators. [14, Theorem 2.5] $T \in \mathcal{D}^{m+}_p(X_1, ..., X_m, F)$ if there are a constant C and Borel probability measure μ on $B^+_{F^{**}}$ such that

$$| < T(x^1, ..., x^m), y^* > | \leq C\|x^1\| ... \|x^m\| \left(\int_{B^+_{F^{**}}} < y^*, \phi >^{p^*} d\mu(\phi) \right)^{\frac{1}{p^*}}$$

for all $x^j \in X_j$ and $y^* \in F^{*+} \quad j = 1, ..., m$.
Giving $n \in \mathbb{N}^*$, we have

$$| < T(x^1_{i_1}, ..., x^m_{i_m}), y^*_{i_1,...,i_m} > | \leq C\|x^1_{i_1}\| ... \|x^m_{i_m}\| \left(\int_{B^+_{F^{**}}} < y^*_{i_1,...,i_m}, \phi >^{p^*} d\mu(\phi) \right)^{\frac{1}{p^*}}$$

for all $x^j_{i_j} \in X_j \quad 1 \leq j \leq m, \quad 1 \leq i_1, ..., i_m \leq n$ and any $y^*_{i_1,...,i_m} \in F^{*+}$, then

$$\sum_{i_1,...,i_m=1}^{n} | < T(x^1_{i_1}, ..., x^m_{i_m}), y^*_{i_1,...,i_m} > |$$
$$\leq C \sum_{i_1,...,i_m=1}^{n} \left(\|x^1_{i_1}\|...\|x^m_{i_m}\| \left(\int_{B^+_{F^{**}}} < y^*_{i_1,...,i_m}, \phi >^{p^*} d\mu(\phi) \right)^{\frac{1}{p^*}} \right).$$

Using a general form of Hölder's inequality

$$\sum_{i_1,...,i_m=1}^{n} | < T(x^1_{i_1}, ..., x^m_{i_m}), y^*_{i_1,...,i_m} > |$$
$$\leq C \left(\sum_{i_1,...,i_m=1}^{n} \left(\|x^1_{i_1}\|...\|x^m_{i_m}\| \right)^p \right)^{\frac{1}{p}} \left(\sum_{i_1,...,i_m=1}^{n} \int_{B^+_{F^{**}}} < y^*_{i_1,...,i_m}, \phi >^{p^*} d\mu(\phi) \right)^{\frac{1}{p^*}}$$
$$\leq C \left(\sum_{i_1=1}^{n} \|x^1_{i_1}\|^p \right)^{\frac{1}{p}} ... \left(\sum_{i_m=1}^{n} \|x^m_{i_m}\|^p \right)^{\frac{1}{p}} \left(\sum_{i_1,...,i_m=1}^{n} \int_{B^+_{F^{**}}} < y^*_{i_1,...,i_m}, \phi >^{p^*} d\mu(\phi) \right)^{\frac{1}{p^*}}$$
$$\leq C\|(x^1_i)^n_{i=1}\|_p...\|(x^m_i)^n_{i=1}\|_p \left(\int_{B^+_{F^{**}}} \sum_{i_1,...,i_m=1}^{n} < y^*_{i_1,...,i_m}, \phi >^{p^*} d\mu(\phi) \right)^{\frac{1}{p^*}}$$
$$\leq C\|(x^1_i)^n_{i=1}\|_p...\|(x^m_i)^n_{i=1}\|_p \left(\sup_{\phi \in B^+_{F^{**}}} \sum_{i_1,...,i_m=1}^{n} < y^*_{i_1,...,i_m}, \phi >^{p^*} \right)^{\frac{1}{p^*}}$$
$$\leq C\|(x^1_i)^n_{i=1}\|_p...\|(x^m_i)^n_{i=1}\|_p \|(y^*_{i_1,...,i_m})^n_{i_1,...,i_m=1}\|_{\ell^n_{p^*,weak}(F^*)}$$

and thus $T \in \mathcal{D}^{mult+}_p(X_1, ..., X_m; F)$. □

Next, we investigate composition and inclusion relationships for our class of operators. But first we take advantage of the definition of Cohen positive p-nuclear m-linear operators initiated by Bougoutaia et al. [16], we define similarly the multiple Cohen positive p-nuclear operators as follows.

Definition 3.1.2. *For $1 \leq p < \infty$, an $m-$linear operator $T : E_1 \times \cdots \times E_m \longrightarrow F$ is called multiple Cohen positive $p-$nuclear if there exists a constant $C > 0$ such that for any $(x^j_{i_j})^n_{i_j=1} \subset E_j$ $(1 \leq j \leq m)$ and any $y^*_{i_1,...,i_m} \in F^*$, we have*

$$\left\| \left\langle T(x^1_{i_1}, ..., x^m_{i_m}), y^*_{i_1,...,i_m} \right\rangle \right\|_{\ell^n_1} \leq C \prod_{j=1}^{m} \|(x^i_i)^n_{i=1}\|_{\ell^n_{p,|weak|}(E_j)} \|(y^*_{i_1,...,i_m})^n_{i_1,...,i_m=1}\|_{\ell^n_{p^*,|weak|}(F^*)}. \tag{3.1.1}$$

Moreover the class of all multiple Cohen positive p-nuclear operators from $E_1 \times \cdots \times E_m$ *into* F, *is denoted* $\mathcal{N}_p^{mult+}(E_1, ..., E_m; F)$. *It is a Banach space with the norm* $n_p^{mult+}(.)$, *which is the smallest constant* C *such that* (3.1.1) *holds.*

Theorem 3.1.2. *Let* $1 \leq p, q, r \leq \infty$ *with* $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, *if* $S \in \mathcal{D}_p^{mult+}(E_1, ..., E_m; F)$ *with* F *is finite dimensional, and* $T_j \in \Pi_q^+(D_j, E_j)$ *with* $1 \leq j \leq m$, *then* $S \circ (T_1, ..., T_m) \in \mathcal{N}_r^{mult+}(D_1, ..., D_m; F)$.

Proof. Take $T_j \in \Pi_q^+(D_j, E_j)$ with $1 \leq j \leq m$, from the domination theorem for positive summing operators [1] there is $\mu_j \in B_{D_j}^+$ such that for every $x \in D_j$, we have

$$\|T_j(x)\| \leq \pi_q^+(T_j) \left(\int_{B_{D_j}^+} |\phi(x)|^q d\mu_j(\phi) \right)^{\frac{1}{q}}$$

we take

$$\rho_i^j = \left(\int_{B_{D_j}^+} |\phi(x_i)|^r d\mu_j(\phi) \right)^{\frac{1}{q}}$$

for $1 \leq j \leq m$ and $1 \leq i \leq n$, without loss of generality we may consider $T_j(x_i^j) \neq 0$, for all $1 \leq j \leq m$ and $1 \leq i \leq n$. Hence $\rho_i^j > 0$ and we can define $z_i^j = \frac{x_i^j}{\rho_i^j}$. Now, for $a_1, ..., a_n \in \mathbb{K}$ with $\sum\limits_{i=1}^{n} |a_i|^{p^*} < 1$, since $\frac{1}{r^*} + \frac{1}{p} + \frac{1}{q} = 1$, we can use Hölder's inequality in order to write

$$\begin{aligned}
|\sum_{i=1}^{n} \phi(a_i z_i^j)| &\leq \sum_{i=1}^{n} |a_i|^{\frac{p^*}{r^*}} |a_i|^{\frac{p^*}{q}} \frac{1}{\rho_i^j} |\phi(x_i^j)|^{\frac{r}{q}} |\phi(x_i^j)|^{\frac{r}{p}} \\
&\leq \left(\sum_{i=1}^{n} |a_i|^{p^*} \right)^{\frac{1}{r^*}} \left(\sum_{i=1}^{n} |a_i|^{p^*} \frac{1}{(\rho_i^j)^q} |\phi(x_i^j)|^r \right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |\phi(x_i^j)|^r \right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^{n} |a_i|^{p^*} \frac{1}{(\rho_i^j)^q} |\phi(x_i^j)|^r \right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |\phi(x_i^j)|^r \right)^{\frac{1}{p}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\sum_{i=1}^{n} a_i T_j(z_i^j)\| &= \|\sum_{i=1}^{n} T_j(a_i z_i^j)\| \\
&\leq \pi_q^+(T_j)\left(\int\limits_{B_{D_j}^+} |\sum_{i=1}^{n}\phi(a_i z_i^j)|^q d\mu_j(\phi)\right)^{\frac{1}{q}} \\
&\leq \pi_q^+(T_j)\left(\sum_{i=1}^{n}|a_i|^{p^*}\frac{1}{(\rho_i^j)^q}\int\limits_{B_{D_j}^+}|\phi(x_i^j)|^r d\mu_j(\phi)\right)^{\frac{1}{q}}\left(\|(x_i^j)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}} \\
&\leq \pi_q^+(T_j)\left(\|(x_i^j)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}}.
\end{aligned}$$

Hence, by Krivine's calculus

$$\|(T_j(z_i^j))_{i=1}^n\|_p \leq \pi_q^+(T_j)\left(\|(x_i^j)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}}.$$

Now, for $y^*_{i_1,...,i_m}$ in F^{*+} we have

$$\begin{aligned}
&\sum_{i_1,...,i_m=1}^{n} |< S(T_1(x_{i_1}^1),...,T_m(x_{i_m}^m)), y^*_{i_1,...,i_m} >| \\
&\leq \|S(T_1(x_{i_1}^1),...,T_m(x_{i_m}^m))\|\|(y^*_{i_1,...,i_m})_{i_1,...,i_m=1}^n\| \\
&\leq \|(\rho_{i_1}^1...\rho_{i_m}^m)S(T_1(z_{i_1}^1),...,T_m(z_{i_m}^m))\|\|(y^*_{i_1,...,i_m})_{i_1,...,i_m=1}^n\| \\
&\leq \|(\rho_{i_1}^1...\rho_{i_m}^m)S(T_1(z_{i_1}^1),...,T_m(z_{i_m}^m))\|_r\|(y^*_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{r^*} \\
&\leq \|(\rho_{i_1}^1...\rho_{i_m}^m)S(T_1(z_{i_1}^1),...,T_m(z_{i_m}^m))\|_r\|(y^*_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)} \\
&\leq d_p^{mult+}(S)\left(\sum_{i_1,...,i_m=1}^{n}(\rho_{i_1}^1...\rho_{i_m}^m)^q\right)^{\frac{1}{q}}\prod_{j=1}^{m}\|(T_j(z_i^j))_{i=1}^n\|_p\|(y^*_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)} \\
&\leq d_p^{mult+}(S)\prod_{j=1}^{m}\|(\rho_i^j)_{i=1}^n\|_q\prod_{j=1}^{m}\pi_q(T_j)\left(\|(x_i^j)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}}\|(y^*_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)} \\
&\leq d_p^{mult+}(S)\prod_{j=1}^{m}\pi_q(T_j)\left(\|(x_i^j)_{i=1}^n\|_{\ell^n_{r,weak}}\right)^{\frac{r}{p}+\frac{r}{q}}\|(y^*_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)} \\
&\leq d_p^{mult+}(S)\prod_{j=1}^{m}\pi_q(T_j)\|(x_i^j)_{i=1}^n\|_{\ell^n_{r,weak}}\|(y^*_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{\ell^n_{r^*,weak}(F^*)}
\end{aligned}$$

Therefore, $S\circ(T_1,...,T_m)$ is multiple Cohen positive $r-$nuclear operator. □

3.2 Cohen's-type theorems for multiple positive strongly p-summing operators

In this section, we investigate inclusions between the the class of multiple positive p-summing and multiple positive strongly p-summing operators. By giving the Cohen's-type theorems [23] in the positive multilinear situation.

Theorem 3.2.1. *Let $r_1, ..., r_m \in \mathbb{N}^*$ and $1 < p \leq \infty$. Let T be a multilinear operator form $\ell_p^{r_1} \times \cdots \times \ell_p^{r_m}$ into the finite dimensional Banach lattice F. Then, T belongs to $\Lambda_{p^*}^{mult}(\ell_p^{r_1}, ..., \ell_p^{r_m}; F)$ and $\mathcal{D}_p^{mult+}(\ell_p^{r_1}, ..., \ell_p^{r_m}; F)$ with $d_p^{mult+}(T) \leq \Lambda_{p^*}(T)$.*

Proof. Let the multilinear operator $T : \ell_p^{r_1} \times \cdots \times \ell_p^{r_m} \to F$, T is finite type. Thus obviously T is in $\Lambda_{p^*}^{mult}(\ell_p^{r_1}, ..., \ell_p^{r_m}; F)$ and from Example 3.1.1, T is in $\mathcal{D}_p^{mult+}(\ell_p^{r_1}, ..., \ell_p^{r_m}; F)$. Let now $(e_{k_j})_{k_j=1}^{r_j}$ be the standard basis for $\ell_p^{r_j}$ $\quad 1 \leq j \leq m$.

Since T is positive multiple p-summing, then

$$\left(\sum_{k_1,...,k_m=1}^{r_1,...,r_m} \|T(e_{k_1}, ..., e_{k_m})\|^{p^*}\right)^{\frac{1}{p^*}} \leq \Lambda_{p^*}(T) \prod_{j=1}^{m} \|(e_{k_j})_{k_j=1}^{r_j}\|_{\ell_{p^*,weak}^n} \leq \Lambda_{p^*}(T).$$

Let $(x_{i_1}^1, ..., x_{i_m}^m) \in \ell_p^{r_1} \times \cdots \times \ell_p^{r_m}$ such that $x_{i_j}^j = \sum\limits_{k_j=1}^{r_j} a_{k_j,i_j}^j e_{k_j}$ and $y_{i_1,...,i_m}^* \in F^{*+}$

$$\begin{aligned}
&\sum_{i_1,...,i_m=1}^{n} | < T(x_{i_1}^1, ..., x_{i_m}^m), y_{i_1,...,i_m}^* > | \\
&\leq \sum_{i_1,...,i_m=1}^{n} \left(\sum_{k_1,...,k_m=1}^{r_1,...,r_m} | < T(a_{k_1,i_1}^1 e_{k_1}, ..., a_{k_m i_m}^m e_{k_m}), y_{i_1,...,i_m}^* > |\right) \\
&\leq \sum_{i_1,...,i_m=1}^{n} \sum_{k_1,...,k_m=1}^{r_1,...,r_m} \left(|a_{k_1,i_1}^1 ... a_{k_m,i_m}^m|| < T(e_{k_1}, ..., e_{k_m}), y_{i_1,...,i_m}^* > |\right).
\end{aligned}$$

If $1 < p < \infty$, by the Hölder's inequality we obtain

$$\sum_{i_1,...,i_m=1}^{n} | < T(x_{i_1}^1, ..., x_{i_m}^m), y_{i_1,...,i_m}^* > |$$
$$\leq \sum_{i_1,...,i_m=1}^{n} \left[\left(\sum_{k_1,...,k_m=1}^{r_1,...,r_m} |a_{k_1,i_1}^1 ... a_{k_m,i_m}^m|^p \right)^{\frac{1}{p}} \left(\sum_{k_1,...,k_m=1}^{r_1,...,r_m} | < T(e_{k_1}, ..., e_{k_m}), y_{i_1,...,i_m}^* > |^{p^*} \right)^{\frac{1}{p^*}} \right]$$
$$\leq \sum_{i_1,...,i_m=1}^{n} \left[\|x_{i_1}^1\| ... \|x_{i_m}^m\| \left(\sum_{k_1,...,k_m=1}^{r_1,...,r_m} | < T(e_{k_1}, ..., e_{k_m}), y_{i_1,...,i_m}^* > |^{p^*} \right)^{\frac{1}{p^*}} \right]$$
$$\leq \left(\sum_{i_1,...,i_m=1}^{n} \|x_{i_1}^1\| ... \|x_{i_m}^m\| \right) \left(\sum_{k_1,...,k_m=1}^{r_1,...,r_m} \|T(e_{k_1}, ..., e_{k_m})\|^{p^*} \right)^{\frac{1}{p^*}} \|(y_{i_1,...,i_m}^*)_{i=1}^n\|_{\ell_{p^*,weak}^n(F^*)}$$
$$\leq \Lambda_{p^*}(T) \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|^p \right)^{\frac{1}{p}} \|(y_{i_1,...,i_m}^*)_{i=1}^n\|_{\ell_{p^*,weak}^n(F^*)}.$$

This implies that $d_p^{mult+}(T) \leq \Lambda_{p^*}(T)$.
If $p = \infty$

$$\sum_{i_1,...,i_m=1}^{n} | < T(x_{i_1}^1, ..., x_{i_m}^m), y_{i_1,...,i_m}^* > |$$
$$\leq \sup_{1 \leq i_j \leq n} \sup_{1 \leq k_j \leq r_j} |a_{k_1,i_1}^1 ... a_{k_m,i_m}^m| \left(\sum_{i_1,...,i_m=1}^{n} \sum_{k_1,...,k_m=1}^{r_1,...,r_m} | < T(e_{k_1}, ..., e_{k_m}), y_{i_1,...,i_m}^* > | \right)$$
$$\leq \sup_{1 \leq i \leq n} \prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_{\ell_\infty^{r_j}} \sum_{k_1,...,k_m=1}^{r_1,...,r_m} \|T(e_{k_1}, ..., e_{k_m})\| \|(y_{i_1,...,i_m}^*)_{i_1,...,i_m=1}^n\|_{\ell_{1,weak}^n(F^*)}$$
$$\leq \Lambda_1(T) \prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_{\ell_\infty^{r_j}} \|(y_{i_1,...,i_m}^*)_{i_1,...,i_m=1}^n\|_{\ell_{1,weak}^n(F^*)}$$

we obtain $d_\infty^{mult+}(T) \leq \Lambda_1(T)$. This completes the proof. □

Let $1 \leq p \leq \infty$ and let $\lambda > 1$. The Banach space X is said to be an $\mathcal{L}_{p,\lambda}$-space if every finite dimensional subspace Y of X is contained in a finite dimensional subspace Z of X for which there is an isomorphism $v : Z \to \ell_p^{dimZ}$ with $\|v\|\|v^{-1}\| < \lambda$. We say that X is an $\mathcal{L}_p$-space if it is an $\mathcal{L}_{p,\lambda}$-space for some $\lambda > 1$(see[25]).

Theorem 3.2.2. *For $m \in \mathbb{N}^*$. Let $1 < p \leq \infty$ and let $E_j \quad (1 \leq j \leq m)$ be an $\mathcal{L}_{p,\lambda}$-space. Then*

$$\Lambda_{p^*}^{mult}(E_1, ..., E_m; F) \subset \mathcal{D}_p^{mult+}(E_1, ..., E_m; F)$$

and $d_p^{mult+}(T) \leq \prod_{j=1}^{m} \lambda_j \Lambda_{p^*}(T)$.

Proof. Let $n \in \mathbb{N}^*$, $(x^1_{i_1}, ..., x^m_{i_m})$ in $E_1 \times \cdots \times E_m$ and $T \in \Lambda^{mult}_{p^*}(E_1, ..., E_m; F)$. Since E_j $(1 \leq j \leq m)$ is an $\mathcal{L}_{p,\lambda}$-space, then there exists a finite dimensional subspace $M_j \subset E_j$ containing a finite dimensional subspace spanned by $x^1_{i_1}, ..., x^m_{i_m}$ and an invertible operator $S_j : \ell^{r_j}_p \to M_j$ $(DimM_j = r_j)$ such that $\|S_j\|\|S_j^{-1}\| < \lambda_j$.
Consider the following diagram

$$\begin{array}{ccccccc}
E_1 & \times \cdots \times & E_m & \xrightarrow{T} & F & & \\
\uparrow i_1 & & \uparrow i_m & & \uparrow \tilde{T} & & \\
M_1 & \times \cdots \times & M_m & \xleftarrow{(S_1,...,S_m)} & \ell^{r_1}_p & \times \cdots \times & \ell^{r_m}_p \\
\uparrow k_1 & & \uparrow k_m & & & & \\
span\left\{x^1_{1_1}, ..., x^1_{n_1}\right\} & \times \cdots \times & span\left\{x^m_{1_m}, ..., x^m_{n_m}\right\} & & & &
\end{array}$$

where i_j and k_j for $(1 \leq j \leq m)$ are the canonical inclusion mappings and the operator $\tilde{T}$ is defined by $\tilde{T} = T(i_1 \circ S_1, ..., i_m \circ S_m)$. Since $T \in \Lambda^{mult}_{p^*}(E_1, ..., E_m; F)$ then

$$\Lambda_{p^*}(\tilde{T}) \leq \Lambda_{p^*}(T) \prod_{j=1}^{m} \|S_j\|\|i_j\|$$

therefore, using the previous theorem, we have $\tilde{T} \in \mathcal{D}^{mult+}_p(\ell^{r_1}_p, ..., \ell^{r_m}_p; F)$ and

$$d^{mult+}_p(\tilde{T}) \leq \Lambda_{p^*}(\tilde{T}) \leq \Lambda_{p^*}(T) \prod_{j=1}^{m} \|S_j\|.$$

If we let $z^j_{i_j} = S_j^{-1} x^j_{i_j}$ in $\ell^{r_j}_p$ and $y^*_{i_1,...,i_m} \in F^{*+}$

$$\begin{aligned}
\sum_{i_1,...,i_m=1}^{n} | <T(x^1_{i_1}, ..., x^m_{i_m}), y^*_{i_1,...,i_m} > | &= \sum_{i_1,...,i_m=1}^{n} | < \tilde{T}(z^1_{i_1}, ..., z^m_{i_m}), y^*_{i_1,...,i_m} > | \\
&\leq d^{mult+}_p(\tilde{T}) \|(z^1_i)^n_{i=1}\|_p ... \|(z^m_i)^n_{i=1}\|_p \; \|(y^*_{i_1,...,i_m})^n_{i_1,...,i_m=1}\|_{\ell^n_{p^*,weak}(F^*)} \\
&\leq \Lambda_{p^*}(T) \prod_{j=1}^{m} \|S_j\| \|(z^1_i)^n_{i=1}\|_p ... \|(z^m_i)^n_{i=1}\|_p \; \|(y^*_{i_1,...,i_m})^n_{i_1,...,i_m=1}\|_{\ell^n_{p^*,weak}(F^*)}.
\end{aligned}$$

Since $z^j_{i_j} = S_j^{-1} x^j_{i_j}$, we obtain

$$\begin{aligned}
&\sum_{i_1,...,i_m=1}^{n} | < T(x^1_{i_1}, ..., x^m_{i_m}), y^*_{i_1,...,i_m} > | \leq \\
&\qquad \prod_{j=1}^{m} \lambda_j \Lambda_{p^*}(T) \|(x^1_i)^n_{i=1}\|_p ... \|(x^m_i)^n_{i=1}\|_p \; \|(y^*_{i_1,...,i_m})^n_{i_1,...,i_m=1}\|_{\ell^n_{p^*,weak}(F^*)}.
\end{aligned}$$

Therefore, T belongs to $\mathcal{D}^{mult+}_p(E_1, ..., E_m; F)$ and $d^{mult+}_p(T) \leq \prod_{j=1}^{m} \lambda_j \Lambda_{p^*}(T)$.
□

3.3 Connection with tensor product

In this section, we endow $X_1 \otimes \cdots \otimes X_m \otimes F^*$ with a norm in such way that its topological dual is isometric to the space of multiple Cohen positive strongly $p-$summing $m-$linear operators from $X_1 \times \cdots \times X_m$ into F.

Definition 3.3.1. *for each $z \in X_1 \otimes \cdots \otimes X_m \otimes F$*

$$\delta_p^+(z) := \inf \left\{ \|(\lambda_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{\ell_\infty^n} \left(\prod_{j=1}^m \|(x_i^j)_{i=1}^n\|_p \right) \ \|(y_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{\ell^n_{p^*,|weak|}(F)} \right\}$$

where the infimum is taken over all representations $z = \sum_{i_1,...,i_m=1}^n \lambda_{i_1,...,i_m} x_{i_1}^1 \otimes \cdots \otimes x_{i_m}^m \otimes y_{i_1,...,i_m}$.

Proposition 3.3.1. *The application $z \mapsto \delta_p^+(z)$ is a norm on $X_1 \otimes \cdots \otimes X_m \otimes F$.*

Proof. The proof of this proposition is similar to the proof of [53, Proposition 2.1]. □

Proposition 3.3.2. *The topological dual $\left(X_1 \otimes \cdots \otimes X_m \otimes F^*, \delta_p^+\right)^*$ of $\left(X_1 \otimes \cdots \otimes X_m \otimes F^*, \delta_p^+\right)$ is isometric to $\mathcal{D}_p^{mult+}(X_1, ..., X_m; F)$ through the mapping ϕ_T.*

Proof. For any $z = \sum_{i_1,...,i_m=1}^n \lambda_{i_1,...,i_m} x_{i_1}^1 \otimes \cdots \otimes x_{i_m}^m \otimes y_{i_1,...,i_m}^*$ in $X_1 \otimes ... \otimes X_m \otimes F^*$ we have

$$\begin{aligned} |\phi_T(z)| &= |\sum_{i_1,...,i_m=1}^n \lambda_{i_1,...,i_m} y_{i_1,...,i_m}^*(T(x_{i_1}^1, ..., x_{i_m}^m))| \\ &\leq d_p^{mult+}(T) \|(\lambda_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{\ell_\infty^n} \left(\prod_{j=1}^m \|(x_i^j)_{i=1}^n\|_p \right) \ \|(y_{i_1,...,i_m}^*)_{i_1,...,i_m=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)}. \end{aligned}$$

Hence for each z we have $|\phi_T(z)| \leq d_p^{mult+}(T)\delta_p^+(z)$.

For the other direction, let $(\lambda_{i_1,...,i_m})_{i_1,...,i_m=1}^n \subset \mathbb{R}^*$, $(x_{i_j}^j)_{i_j=1}^n \subset X_j$ $(1 \leq j \leq m)$ and $(y_{i_1,...,i_m}^*)_{i_1,...,i_m=1}^n \subset F^*$, then

$$\begin{aligned} |\sum_{i_1,...,i_m=1}^n \lambda_{i_1,...,i_m} y_{i_1,...,i_m}^*(T(x_{i_1}^1, ..., x_{i_m}^m))| &= |\phi_T(z)| \\ &\leq \|\phi_T\| \delta_p^+ (\sum_{i_1,...,i_m=1}^n \lambda_{i_1,...,i_m} x_{i_1}^1 \otimes \cdots \otimes x_{i_m}^m \otimes y_{i_1,...,i_m}^*) \\ &\leq \|\phi_T\| \|(\lambda_{i_1,...,i_m})_{i_1,...,i_m=1}^n\|_{\ell_\infty^n} \left(\prod_{j=1}^m \|(x_i^j)_{i=1}^n\|_p \right) \ \|(y_{i_1,...,i_m}^*)_{i_1,...,i_m=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)}. \end{aligned}$$

□

Chapter 4

Cohen positive strongly $p-$summing and p-dominated m-homogeneous polynomials

This Chapter includes three sections, in Section 4.1 we introduce the class of Cohen positive strongly p-summing m-homogeneous polynomials, characterize it with a Domination Theorem and prove that this class is related to different classes of p-summability ($\Pi^+_{p^*}(E,F)$, $\mathcal{D}^{m+}_p({}^mX;F)$, $\mathcal{D}^+_p(\widehat{\otimes}^m_{\pi,s}X;F)$), by means of the adjoint operator, the associated symmetric m-linear operator and the linearization of these polynomials. Section 4.2 is essentially dedicated to the study of positive p-dominated m-linear operators and polynomials. It includes our main theorem (The Bu-type theorem) which serves as a focal point in the proof of the relationship between positive p-dominated polynomials and those defined in Section 4.1. We end this chapter by giving example and some applications.

4.1 Cohen positive strongly p-summing m-homogeneous polynomials

In this section, we give a natural generalization of the notion of Cohen positive strongly $p-$summing to the category of homogeneous polynomial mappings. As in the multilinear case, there is an analogue connection between polynomial mapping and its linearization and associated symmetric $m-$linear operator. The next definition is due to D. Achour and Kh. Saadi in [3].

Definition 4.1.1. *Let $1 < p \leq +\infty$. An $m-$homogeneous polynomial $P : X \longrightarrow Y$ is Cohen strongly $p-$summing, if there is a constant $C > 0$ such that for any $x_1, ..., x_n \in X$ and $y^*_1, ..., y^*_n \in Y^*$,*

$$\sum_{i=1}^{n} |\langle P(x_i), y_i^* \rangle| \leq C \left(\sum_{i=1}^{n} \|x_i\|^{mp} \right)^{\frac{1}{p}} \sup_{y \in B_Y} \|(y_i^*(y))\|_{\ell^n_{p^*}} . \quad (4.1.1)$$

The class of such polynomials is denoted by $\mathcal{P}_{Coh,p}\left({}^{m}X,Y\right)$, it is equipped with the norm $d_{p}\left(.\right)$, i.e. the smallest constant C such that inequality (4.1.1) holds. For $p=1$ we have $\mathcal{P}_{Coh,1}\left({}^{m}X;Y\right)=\mathcal{P}\left({}^{m}X;Y\right)$.

4.1.1 Definition and characterization with Domination theorem

Now, we introduce the notion of Cohen positive strongly $p-$summing $m-$ homogeneous polynomials and we study some fundamental properties.

Definition 4.1.2. *Let $1<p\leq+\infty$ and $m\in\mathbb{N}$. An $m-$homogeneous polynomial $P:X\longrightarrow F$ is Cohen positive strongly $p-$summing, if there is a constant $C>0$, such that for any $x_{1},...,x_{n}\in X$ and any $y_{1}^{*},...,y_{n}^{*}\in F^{*}$,*

$$\sum_{i=1}^{n}\left|\langle P(x_{i}),y_{i}^{*}\rangle\right|\leq C\left(\sum_{i=1}^{n}\|x_{i}\|^{mp}\right)^{\frac{1}{p}}\|(y_{i}^{*})_{i=1}^{n}\|_{\ell_{p^{*},|weak|}^{n}(F^{*})}. \tag{4.1.2}$$

The class of such polynomials is denoted by $\mathcal{P}_{Coh,p}^{+}\left({}^{m}X;F\right)$. It is a Banach space with the norm $d_{p}^{+}\left(.\right)$ which is the smallest constant C such that the inequality (4.1.2) holds, for $p=1$ we have $\mathcal{P}_{Coh,1}^{+}\left({}^{m}X;F\right)=\mathcal{P}\left({}^{m}X;F\right)$.

Remark 4.1.1. *Fix $m\in\mathbb{N}$, an $m-$homogeneous polynomial $P:X\longrightarrow F$ is Cohen positive strongly $p-$summing, if and only if, there is a constant $C>0$ such that the inequality*

$$\sum_{i=1}^{n}\left|\langle P(x_{i}),y_{i}^{*}\rangle\right|\leq C\left(\sum_{i=1}^{n}\|x_{i}\|^{mp}\right)^{\frac{1}{p}}w_{p^{*}}\left((y_{i}^{*})_{i=1}^{n}\right), \tag{4.1.3}$$

holds for every $x_{1},...,x_{n}\in X$ and $y_{1}^{*},...,y_{n}^{*}\in F^{*+}$.

Proposition 4.1.1. ***(Ideal property)**. Let $P\in\mathcal{P}({}^{m}X;F)$, S in $L^{+}(F;G)$ and R in $\mathcal{L}(Y;X)$.*
If P is Cohen positive strongly p-summing polynomial, then SPR is Cohen positive strongly p-summing polynomial and $d_{p}^{+}(SPR)\leq d_{p}^{+}(P)\|R\|^{m}\|S\|$.

Proof. Let $(x_i)_{i=1}^n \subset E$ and $(y_i^*)_{i=1}^n \subset G^{*+}$

$$\begin{aligned}
\sum_{i=1}^{n} | <SPR(x_i), y_i^* > | &= \sum_{i=1}^{n} | < PR(x_i), S^*(y_i^*) > | \\
&\leq d_p^+(P) \left(\sum_{i=1}^{n} \|R(x_i)\|^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^{n} < S^*(y_i^*), y^{**} >^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq d_p^+(P) \|R\|^m \|S\| \left(\sum_{i=1}^{n} \|x_i\|^{mp} \right)^{\frac{1}{p}} \sup_{y^{**} \in B_{F^{**}}^+} \left(\sum_{i=1}^{n} < y_i^*, \frac{S^{**}(y^{**})}{\|S^{**}\|} >^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq d_p^+(P) \|R\|^m \|S\| \left(\sum_{i=1}^{n} \|x_i\|^{mp} \right)^{\frac{1}{p}} \sup_{\varphi \in B_{G^{**}}^+} \left(\sum_{i=1}^{n} < y_i^*, \varphi >^{p^*} \right)^{\frac{1}{p^*}} .
\end{aligned}$$

Thus SPR is Cohen positive strongly p-summing polynomial and

$$d_p^+(SPR) \leq d_p^+(P) \|R\|^m \|S\|.$$

□

Next, we present a version of *Pietsch's Domination Theorem*, for our class of polynomials.

Theorem 4.1.1. *(Domination Theorem).* *An $m-$homogeneous polynomial $P : X \longrightarrow F$ is Cohen positive strongly $p-$summing $(1 < p \leq +\infty)$, if and only if there is a Radon probability measure μ on $B_{F^{**}}^+$ with the weak star topology, such that for all $x \in X$ and $y^* \in F^*$, we have*

$$|\langle P(x), y^* \rangle| \leq C \|x\|^m \left(\int_{B_{F^{**}}^+} (|y^*| (y^{**}))^{p^*} d\mu (y^{**}) \right)^{1/p^*} . \tag{4.1.4}$$

In addition, in this case

$$d_p^+ (P) = \inf \{ C > 0, \text{ for all } C \text{ verifying the inequality } (4.1.4) \} .$$

For the proof of this theorem, we need to give a small review about $R-S$-abstract (p, p^*)-summing operators.
Let X, Y and Z be (arbitrary) sets, $\mathcal{H}$ be a family of mappings from X to Y, G be a Banach space and K be a compact Hausdorff topological space. Suppose that the maps

$$\begin{aligned}
R &: K \times Z \times G \longrightarrow [0, \infty) \\
S &: \mathcal{H} \times Z \times G \longrightarrow [0, \infty) ,
\end{aligned}$$

satisfy

- For each $f \in \mathcal{H}$, there is an $x_0 \in Z$, such that $R(\varphi, x_0, b) = S(f, x_0, b) = 0$, for every $\varphi \in K$ and $b \in G$.
- The mapping $R_{x,b} : K \longrightarrow [0, \infty)$ defined by $R_{x,b}(\varphi) = R(\varphi, x, b)$ is continuous for every $x \in Z$ and $b \in G$,
- and $R(\varphi, x, \eta b) \leq \eta R(\varphi, x, b)$, $\eta S(f, x, b) \leq S(f, x, \eta b)$, for every $\varphi \in K$, $x \in Z$, $0 \leq \eta \leq 1$, $b \in G$ and $f \in \mathcal{H}$.

Definition 4.1.3. *[46] Let R and S be as above and $0 < p < \infty$. A mapping $f \in \mathcal{H}$ is said to be $R-S$-abstract p-summing if there is a constant $C > 0$ such that*

$$\left(\sum_{j=1}^{m} S(f, x_j, b_j)^p\right)^{\frac{1}{p}} \leq C \sup_{\varphi \in K} \left(\sum_{j=1}^{m} R(\varphi, x_j, b_j)^p\right)^{\frac{1}{p}}$$

for all $x_1, ..., x_m \in Z$, $b_1, ..., b_m \in G$ and $m \in \mathbb{N}^*$.

Theorem 4.1.2. *[46]. Let R and S be as above, $0 < p < \infty$ and $f \in \mathcal{H}$. Then f is $R-S$-abstract p-summing if and only if there is a constant $C > 0$ and a Borel probability measure μ on K such that*

$$S(f, x, b) \leq C \left(\int_K R(\varphi, x, b)^p d\mu\right)^{\frac{1}{p}}$$

for all $x \in Z$ and $b \in G$. Moreover the infimum of such constant C equals $\pi_{RS,p}(f)$.

For our proof we also need to recall the Theorem 4.6 in [46].

Theorem 4.1.3. *[46]. A map $f \in \mathcal{H}$ is $R_1, ..., R_t - S$-abstract $(p_1, ..., p_t)$-summing if and only if there is a constant $C > 0$ and a Borel probability measures μ_j on K_j, such that*

$$S(f, x^{(1)}, ..., x^{(r)}, b^{(1)}, ..., b^{(t)}) \leq C \prod_{j=1}^{t} \left(\int_{K_j} R_j(\varphi, x^{(1)}, ..., x^{(r)}, b^{(j)})^{p_j} d\mu_j\right)^{\frac{1}{p_j}},$$

for all $x^{(l)} \in Z_l$, $l = 1, ..., r$ and $b^{(j)} \in G_j$ with $j = 1, ..., t$.

Proof. **(proof of the domination theorem).** Suppose that P is Cohen positive strongly p-summing polynomial, by choosing the parameters

$$\begin{cases} t = 2 \quad \text{and} \quad r = m-1 \\ Z = X \\ K_1 = B^+_{X^*} \quad \text{and} \quad K_2 = B^+_{F^{**}} \\ G = F \\ q = 1 \quad q_1 = p \quad \text{and} \quad q_2 = p^* \\ \mathcal{H} = \mathcal{P}(^m X, F) \\ S(P, x, y^*) = | < P(x), y^* > | \\ R_1(\varphi, x) = \|x\|^m \\ R_2(\varphi, x, y^*) = (\varphi|y^*|). \end{cases}$$

We can conclude that $P : X \longrightarrow F$ is Cohen positive strongly p-summing, if and only if P is $R_1 - R_2 - S$-abstract (p, p^*)-summing. The previous theorem tells us that P is $R_1 - R_2 - S$-abstract (p, p^*)-summing if and only if there is $C > 0$, and there are probability measures μ_k on K_k, $k = 1, 2$ such that

$$S(P, x, y^*) \leq C \left(\int\limits_{K_1} R_1(\varphi, x)^p d\mu_1 \right)^{\frac{1}{p}} \left(\int\limits_{K_2} R_2(\varphi, x, y^*)^{p^*} d\mu_2 \right)^{\frac{1}{p^*}}$$

i.e.,

$$\begin{aligned} | < P(x), y^* > | &\leq C \left(\int\limits_{B^+_{X^*}} \|x\|^{mp} d\mu_1 \right)^{\frac{1}{p}} \left(\int\limits_{B^+_{F^{**}}} (\varphi|y^*|)^{p^*} d\mu_2 \right)^{\frac{1}{p^*}} \\ &\leq C\|x\|^m \left(\int\limits_{B^+_{F^{**}}} (\varphi|y^*|)^{p^*} d\mu_2 \right)^{\frac{1}{p^*}}. \end{aligned}$$

□

Proposition 4.1.2. *Consider* $1 < p_1 < p_2 \leq +\infty$.

If $P \in \mathcal{P}^+_{Coh,p_2}(^m X; F)$ *then* $P \in \mathcal{P}^+_{Coh,p_1}(^m X, F)$ *and* $d^+_{p_1}(P) \leq d^+_{p_2}(P)$.

Proof. Assume that $P \in \mathcal{P}^+_{Coh,p_2}(^m X; F)$, the Pietsch domination Theorem 4.1.1 provides a Radon probability measure μ on $B^+_{F^{**}}$ such that for all $x \in X$ and $y^* \in F^*$ for which

$$|\langle P(x), y^*\rangle| \leq d^+_{p_2}(P)\|x\|^m \left(\int_{B^+_{F^{**}}} (|y^*|(y^{**}))^{p_2^*} d\mu(y^{**}) \right)^{1/p_2^*}$$
$$\leq d^+_{p_2}(P)\|x\|^m \left(\int_{B^+_{F^{**}}} (|y^*|(y^{**}))^{p_1^*} d\mu(y^{**}) \right)^{1/p_1^*}.$$

So, $P \in \mathcal{P}^+_{Coh,p_1}(^mX;F)$ and $d^+_{p_1}(P) \leq d^+_{p_2}(P)$. □

4.1.2 Associations with different classes of positive p-summability

the upcoming is the study of the adjoint operator, the associate symmetric, and the linearization of Cohen positive strongly p-summing m-homogeneous polynomials.

Proposition 4.1.3. *Let $1 < p \leq +\infty$. The $m-$homogeneous polynomial $P \in \mathcal{P}(^mX;F)$ is Cohen positive strongly $p-$summing if and only if, P^* is positive $p-$summing and we have $d^+_p(P) = \pi^+_{p^*}(P^*)$.*

Proof. Suppose that $P \in \mathcal{P}^+_{Coh,p}(^mX;F)$, we have by (4.1.4)

$$| < P(x), y^* > | \leq d^+_p(P)\|x\|^m \left(\int_{B^+_{F^{**}}} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} \quad for x \in X, y^* \in F^*.$$

Taking the supremum, we obtain

$$\sup_{\|x\|\leq 1} |P^*(y^*)(x)| \leq d^+_p(P) \left(\int_{B^+_{F^{**}}} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}.$$

Then

$$\|P^*(y^*)\| \leq d^+_p(P) \left(\int_{B^+_{F^{**}}} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}.$$

By the Pietsch domination theorem in [1, Proposition 3.4], $P^* \in \Pi^+_{p^*}(F^*, \mathcal{P}(^mX))$.

Conversely, let $P^* \in \Pi_{p^*}^+ \left(F^*, \mathcal{P}\left({}^m X\right)\right)$ we have, for $x \in X$ and $y^* \in F^*$,

$$\begin{aligned} | < P(x), y^* > | &= |P^*(y^*)(x)| \\ &\leq \|P^*(y^*)\| \|x\|^m, \end{aligned}$$

using the Pietsch's domination theorem for positive p^*-summing linear operators, we obtain

$$|\langle P(x), y^* \rangle| \leq C \, \|x\|^m \left(\int_{B_{F^{**}}^+} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}}.$$

$P \in \mathcal{P}_{coh,p}^+({}^m X, F)$. □

Proposition 4.1.4. *If* $1 < p \leq +\infty$

$$\mathcal{P}_{Coh,p}\left({}^m X; F\right) \subseteq \mathcal{P}_{Coh,p}^+\left({}^m X; F\right).$$

Proof. Take any P in $\mathcal{P}_{Coh,p}\left({}^m X; F\right)$, hence its adjoint operator P^* is in $\Pi_{p^*}\left(F^*, \mathcal{P}\left({}^m X\right)\right)$ by [50, Proposition 1], and egain by [12, Proposition 1] P^* is in $\Pi_{p^*}^+\left(F^*, \mathcal{P}\left({}^m X\right)\right)$, then $P \in \mathcal{P}_{Coh,p}^+\left({}^m X; F\right)$. □

Proposition 4.1.5. *The following properties are equivalent:*

i) the polynomial $P \in \mathcal{P}({}^m X; F)$ *is Cohen positive strongly* $p-$*summing,*

ii) its associated symmetric $m-$*linear operator* $\widehat{P} \in \mathcal{L}({}^m X; F)$ *is Cohen positive strongly* $p-$*summing.*

Proof. Suppose first that $\widehat{P}$ is Cohen positive strongly $p-$summing, let $x_1, ..., x_n \in X$ and $y_1^*, ..., y_n^* \in F^*$, then

$$\begin{aligned} \sum_{i=1}^n | < P(x_i), y_i^* > | &= \sum_{i=1}^n | < \widehat{P}(x_i, ..., x_i), y_i^* > | \\ &\leq d_p^+(\widehat{P}) \left(\sum_{i=1}^n \|x_i\|^{mp} \right)^{\frac{1}{p}} \|(y_i^*)_{i=1}^n\|_{\ell_{p^*,|weak|}^n(F^*)}. \end{aligned}$$

Therefore, P is Cohen positive strongly $p-$summing and $d_p^+\left(P\right) \leq d_p^+\left(\widehat{P}\right)$.

Conversely, suppose that $P \in \mathcal{P}^{+}_{Coh,p}(^{m}X;F)$. Let $x^{j} \in X$ such that $\|x^{j}\| \leq 1$ $(1 \leq j \leq m)$ and $y^{*} \in F^{*}$, we have by (1.3.1)

$$\begin{aligned}
\left|\left\langle \widehat{P}\left(x^{1},...,x^{m}\right),y^{*}\right\rangle\right| &= \left|\left\langle \frac{1}{m!2^{m}}\sum_{\substack{\epsilon_j=\pm1\\1\leq j\leq m}}\epsilon_1...\epsilon_m P\left(\sum_{j=1}^{m}\epsilon_j x^{j}\right),y^{*}\right\rangle\right| \\
&\leq \frac{1}{m!2^{m}}\sum_{\substack{\epsilon_j=\pm1\\1\leq j\leq m}}\left|\left\langle P\left(\sum_{j=1}^{m}\epsilon_j x^{j}\right),y^{*}\right\rangle\right|, \text{ by the Domination Theorem} \\
&\leq \frac{1}{m!2^{m}}\sum_{\substack{\epsilon_j=\pm1\\1\leq j\leq m}} d_p^{+}(P)\left\|\sum_{j=1}^{m}\epsilon_j x^{j}\right\|^{m}\left(\int_{B^{+}_{F^{**}}}\left(|y^{*}|\left(y^{**}\right)\right)^{p^{*}}d\mu\left(y^{**}\right)\right)^{1/p^{*}} \\
&\leq \frac{1}{m!2^{m}} d_p^{+}(P)\sum_{\substack{\epsilon_j=\pm1\\1\leq j\leq m}}\left(\sum_{i=1}^{m}\|x^{j}\|\right)^{m}\left(\int_{B^{+}_{F^{**}}}\left(|y^{*}|\left(y^{**}\right)\right)^{p^{*}}d\mu\left(y^{**}\right)\right)^{1/p^{*}} \\
&\leq \frac{1}{m!2^{m}} d_p^{+}(P)\, 2^{m}m^{m}\left(\int_{B^{+}_{F^{**}}}\left(|y^{*}|\left(y^{**}\right)\right)^{p^{*}}d\mu\left(y^{**}\right)\right)^{1/p^{*}} \\
&\leq \frac{m^{m}}{m!} d_p^{+}(P)\left(\int_{B^{+}_{F^{**}}}\left(|y^{*}|\left(y^{**}\right)\right)^{p^{*}}d\mu\left(y^{**}\right)\right)^{1/p^{*}}.
\end{aligned}$$

Therefore for every $x^{j} \in X$ $(1 \leq j \leq m)$ $x^{j} \neq 0$, we have

$$\left|\left\langle \widehat{P}\left(\frac{x^{1}}{\|x^{1}\|},...,\frac{x^{m}}{\|x^{m}\|}\right),y^{*}\right\rangle\right| \leq \frac{m^{m}}{m!} d_p^{+}(P)\left(\int_{B^{+}_{F^{**}}}\left(|y^{*}|\left(y^{**}\right)\right)^{p^{*}}d\mu\left(y^{**}\right)\right)^{1/p^{*}}.$$

Thus

$$\left|\left\langle \widehat{P}\left(x^{1},...,x^{m}\right),y^{*}\right\rangle\right| \leq \frac{m^{m}}{m!} d_p^{+}(P)\prod_{j=1}^{m}\left\|x^{j}\right\|\left(\int_{B^{+}_{F^{**}}}\left(|y^{*}|\left(y^{**}\right)\right)^{p^{*}}d\mu\left(y^{**}\right)\right)^{1/p^{*}}.$$

By [14] , $\widehat{P}$ is Cohen positive strongly $p-$summing, furthermore

$$d_p^{+}\left(\widehat{P}\right) \leq \frac{m^{m}}{m!} d_p^{+}(P).$$

□

Corollary 4.1.1. *Let $1 < p < \infty$. If the polynomial $P \in \mathcal{P}(^{m}X;F)$ is Cohen positive strongly $p-$summing then its associated symmetric $m-$linear operator $\widehat{P} \in \mathcal{P}(^{m}X;F)$ is multiple $p-$convex. If the polynomial $P \in \mathcal{P}(^{m}X;c_0)$ is Cohen positive strongly $p-$summing then its associated symmetric $m-$linear operator $\widehat{P}$ is in $\mathcal{L}(^{m}X;c_0)$.*

Proof. The proof of this corollary is an adaptation of the proof [14, Proposition 3.3]. □

Proposition 4.1.6. *Let $1 < p \leq \infty$. Let $P : X \longrightarrow F$ be a m-homogeneous polynomial and P_L its linearization. The following propreties are equivalent*

(i) The polynomial P belongs to $\mathcal{P}^+_{coh,p}(^mX; F)$.

(ii) The operator P_L belongs to $\mathcal{D}^+_p(\widehat{\otimes}^m_{\pi,s}X; F)$.

(iii) There exist a Banach space Z, $u \in \mathcal{D}^+_p(Z, F)$ and $Q \in \mathcal{P}(^mX; Z)$ such that $P = uQ$.

Proof. $(i) \Rightarrow (ii)$ Suppose that P is Cohen positive strongly p-summing, let $v \in \widehat{\otimes}^m_{\pi,s}X$ such that $v \neq 0$ and $y^* \in F^*$ and suppose that $v = \sum\limits_{i=1}^{n} x_i \otimes \cdots \otimes x_i$ then

$$\begin{aligned}
| < P_L(v), y^* > | &\leq \sum_{i=1}^{n} | < P(x_i, ..., x_i), y^* > | \\
&\leq \sum_{i=1}^{n} d^+_p(P)\|x_i\|^m \left(\int\limits_{B^+_{F^{**}}} (|y^*|(y^{**}))^{p^*} \right)^{\frac{1}{p^*}} \\
&= d^+_p(P) \sum_{i=1}^{n} \|x_i\|^m \left(\int\limits_{B^+_{F^{**}}} (|y^*|(y^{**}))^{p^*} \right)^{\frac{1}{p^*}} .
\end{aligned}$$

Taking the infimum over all represents of v we get

$$| < P_L(v), y^* > | \leq d^+_p(P)\|v\| \left(\int\limits_{B^+_{F^{**}}} (|y^*|(y^{**}))^{p^*} \right)^{\frac{1}{p^*}}$$

and by [1, Theorem 4.13] P_L is positive strongly p-summing.
$(ii) \Rightarrow (iii)$ We have the result directly from the factorization $P = P_L \circ \delta_m$ such that δ_m is the canonical polynomial $\delta_m : X \longrightarrow \widehat{\otimes}^m_{\pi}X$ defined by $\delta_m(x) = x \otimes \overset{(m)}{\cdots} \otimes x$.
$(iii) \Rightarrow (i)$ Assume that there exists a Banach space Z, $u \in \mathcal{D}^+_p(Z, F)$ and $Q \in \mathcal{P}(^mX; Z)$, such that $P = uQ$. Then

$$\|Q(x)\| \leq \|Q\|\|x\|^m$$

By [1, Theorem 4.13]

$$| < u(x), y^* > | \leq d^+_p(u)\|x\| \left(\int\limits_{B^+_{F^{**}}} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} .$$

Then

$$\begin{aligned} | < P(x), y^* > | &= | < u(Q(x)), y^* > | \\ &\leq d_p^+(u) \|Q(x)\| \left(\int_{B_{F^{**}}^+} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} \\ &\leq d_p^+(u) \|Q\| \|x\|^m \left(\int_{B_{F^{**}}^+} (|y^*|(y^{**}))^{p^*} d\mu(y^{**}) \right)^{\frac{1}{p^*}} \end{aligned}$$

and according to the domination theorem $P \in \mathcal{P}^+_{coh,p}(^m X; F)$. □

4.2 Positive p-dominated m-homogeneous polynomials

In this section, we give a natural generalization of the notion of positive $p-$dominated to the category of homogeneous polynomial and the m-linear operators.

4.2.1 Positive p-dominated m-linear operators

Definition 4.2.1. *An m-linear operator $T : E_1 \times \cdots \times E_m \longrightarrow Y$ is positive p-dominated if there is a positive constant $C > 0$ such that*

$$\left(\sum_{i=1}^{n} \|T(x_i^1, ..., x_i^m)\|^{\frac{p}{m}} \right)^{\frac{m}{p}} \leq C \|(x_i^1)_{i=1}^n\|_{\ell^n_{p,|weak|}(E_1)} ... \|(x_i^m)_{i=1}^n\|_{\ell^n_{p,|weak|}(E_m)} \tag{4.2.1}$$

for all $(x_i^j)_{i=1}^n \subset E_j$, $1 \leq j \leq m$ and $m \in \mathbb{N}$. We denote $\mathcal{L}_d^{p+}(E_1, ...E_m; Y)$ the space of all positive $p-$dominated $m-$linear operators and the infimum of all constants C verifying (4.2.1) is denoted $\pi_p^{d+}(T)$. For all $p \geq m$, π_p^{d+} is a norm on $\mathcal{L}_d^{p+}(E_1, ...E_m; Y)$, whereas for $p < m$ it is only a quasinorm.
For $m = 1$ the space of all positive $p-$dominated operators is nothing other than the space of positive p-summing operators.

We should stress out that the above definition makes sense for Banach spaces Y, but the more interesting case occurs when we consider a Banach lattice F. Since this assumption allows to establish an inclusion relation between the classes of positive dominated multilinear operators and Cohen positive strongly summing multilinear operators and later on for polynomials.

The next theorem is an analogous Pietsch's domination type theorem for m-linear positive p-dominated operators. The proof of this result is an adaptation of that of [46, Theorem 4.7] so we do not present it here.

Theorem 4.2.1. ***(Domination theorem for positive p-dominated multilinear operators)*** *An m-linear operator $T : E_1 \times \cdots \times E_m \longrightarrow Y$ is positive $p-$dominated, if and only if, there is Radon probability measures $\mu_j \in B^+_{E^*_j}$, $(1 \leq j \leq m)$ and $K > 0$*

$$\|T(x^1, ..., x^m)\| \leq K \prod_{j=1}^{m} \left(\int\limits_{B^+_{E^*_j}} < x^{j*}, x^j >^p d\mu_j(x^{j*}) \right)^{\frac{1}{p}}$$

for all $x^j \in E^+_j$. Moreover, the smallest constants K is $\pi^{d+}_p(T)$.

The relationship between positive dominated multilinear operators and Cohen positive strongly summing multilinear operators

The following theorem is a version of the Bu-theorem [18], in [2] Achour and Mezrag proved an inclusion theorem between p-dominated operators and Cohen strongly q-summing $m-$linear operators, in our work we prove a positive analogue of [2, Theorem 3.2]. This theorem plays an important role in the proof of the relationship between Cohen positive strongly summing and positive dominated polynomials.

Theorem 4.2.2. *Let $1 < p, q < \infty$. Let $\ell_2^{n_1}, ..., \ell_2^{n_m}$ and F be Banach lattices. Then*

$$\mathcal{L}^{p+}_d(\ell_2^{n_1}, ..., \ell_2^{n_m}; F) \subseteq \mathcal{D}^{m+}_q(\ell_2^{n_1}, ..., \ell_2^{n_m}; F) \tag{4.2.2}$$

for all $T \in \mathcal{L}^{p+}_d(\ell_2^{n_1}, ..., \ell_2^{n_m}, F)$ $\quad d^{m+}_q(T) \leq \pi^{d+}_p(T)$.

Proof. Let $T \in \mathcal{L}^{p+}_d(\ell_2^{n_1}, ..., \ell_2^{n_m}; F)$, then by the domination theorem for positive p-dominated m-linear operators, there are probability measures μ_j on $B^+_{\ell_2^{n_j}}$, such that for every $(x^1, ..., x^m) \in \ell_2^{n_1} \times \cdots \times \ell_2^{n_m}$, we have

$$\|T(x^1, ..., x^m)\| \leq \pi^{d+}_p(T) \prod_{j=1}^{m} \left(\int\limits_{B^+_{\ell_2^{n_j}}} < x^*, x^j >^p d\mu_j(x^*) \right)^{\frac{1}{p}}$$

For every $(x^j_i)_{i=1}^n \subset \ell_2^{n_j}$ for $j = 1, ..., m$ and $(y^*_i)_{i=1}^n \subset F^*$, we have

$$(x^1_i, ..., x^m_i) = (\sum_{k_1=1}^{n_1} x^1_{i,k_1} e_{k_1}, ..., \sum_{k_m=1}^{n_m} x^m_{i,k_m} e_{k_m})$$

then

$$\sum_{i=1}^{n} |< T(x_i^1, ..., x_i^m), y_i^* > | = \sum_{i=1}^{n} | \sum_{k_1,...,k_m=1}^{n_1,...,n_m} x_{i,k_1}^1 ... x_{i,k_m}^m < T(e_{k_1}, ..., e_{k_m}), y_i^* > |.$$

Using Hölder's inequality we obtain

$$\begin{aligned}
&\sum_{i=1}^{n} |< T(x_i^1, ..., x_i^m), y_i^* > | \\
&\quad \leq \sum_{i=1}^{n} \left[\left(\sum_{k_1=1}^{n_1} |x_{i,k_1}^1|^2 \right)^{\frac{1}{2}} ... \left(\sum_{k_m=1}^{n_m} |x_{i,k_m}^m|^2 \right)^{\frac{1}{2}} \sum_{k_1,...,k_m=1}^{n_1,...,n_m} \left(|< T(e_{k_1}, ..., e_{k_m}), y_i^* > |^2 \right)^{\frac{1}{2}} \right] \\
&\quad \leq \sum_{i=1}^{n} \left(\prod_{j=1}^{m} \|x_i^j\|_{\ell_2^{n_j}} \sum_{k_1,...,k_m=1}^{n_1,...,n_m} \left(|< T(e_{k_1}, ..., e_{k_m}), y_i^* > |^2 \right)^{\frac{1}{2}} \right)
\end{aligned}$$

by the multiple Khintchine inequality on the last inequality we get

$$\begin{aligned}
&\sum_{i=1}^{n} |< T(x_i^1, ..., x_i^m), y_i^* > | \\
&\quad \leq \sum_{i=1}^{n} \left(\prod_{j=1}^{m} \|x_i^j\|_{\ell_2^{n_j}} A_{q^*}^m \| \sum_{k_1,...,k_m=1}^{n_1,...,n_m} r_{k_1} ... r_{k_m} < T(e_{k_1}, ..., e_{k_m}), y_i^* > \|_{L_{q^*}([0,1])} \right) \\
&\quad \leq A_{q^*}^m \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|^q \right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} \| \sum_{k_1,...,k_m=1}^{n_1,...,n_m} r_{k_1} ... r_{k_m} < T(e_{k_1}, ..., e_{k_m}), y_i^* > \|_{L_{q^*}([0,1])}^{q^*} \right)^{\frac{1}{q^*}}
\end{aligned}$$

by Fubini's Theorem

$$\begin{aligned}
&\sum_{i=1}^{n} |< T(x_i^1, ..., x_i^m), y_i^* > | \\
&\leq A_{q^*}^m \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|^q \right)^{\frac{1}{q}} \| \sum_{k_1,...,k_m=1}^{n_1,...,n_m} r_{k_1} ... r_{k_m} < T(e_{k_1}, ..., e_{k_m}), y_i^* > \|_{L_{q^*}([0,1], \ell_{q^*}^n)} \\
&\leq A_{q^*}^m \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|^q \right)^{\frac{1}{q}} \|y_i^*\|_{\ell_{q^*,weak}^n(F^*)} \| \sum_{k_1,...,k_m=1}^{n_1,...,n_m} r_{k_1} ... r_{k_m} T(e_{k_1}, ..., e_{k_m}) \|_{L_{q^*}([0,1])}.
\end{aligned} \tag{4.2.3}$$

For the quantity

$$\| \sum_{k_1,...,k_m=1}^{n_1,...,n_m} r_{k_1} ... r_{k_m} T(e_{k_1}, ..., e_{k_m}) \|_{L_{q^*}([0,1],F)} \tag{4.2.4}$$

using the multiple Kahane's inequality

$$\begin{aligned}
&\|\sum_{k_1,...,k_m=1}^{n_1,...,n_m} r_{k_1}(t_1)...r_{k_m}(t_m)T(e_{k_1},...,e_{k_m})\|_{L_{q^*}([0,1],F)} \\
&\leq K_{p,q^*}^m \|\sum_{k_1,...,k_m=1}^{n_1,...,n_m} r_{k_1}(t_1)...r_{k_m}(t_m)T(e_{k_1},...,e_{k_m})\|_{L_p([0,1],F)} \\
&\leq K_{p,q^*}^m \|\sum_{k_1,...,k_m=1}^{n_1,...,n_m} T(r_{k_1}(t_1)e_{k_1},...,r_{k_m}(t_m)e_{k_m})\|_{L_p([0,1],F)} \\
&\leq \pi_p^{d+}(T)\left(\int_0^1\cdots\int_0^1\prod_{j=1}^m\left(\int_{B_{\ell_2^{n_j}}^+}|\sum_{k_j=1}^{n_j}x^*(e_{k_j})r_{k_j}(t_j)|^p d\mu_j(x^*)\right)^{\frac{1}{p}} dt_j\right) \\
&\leq \pi_p^{d+}(T)\left(\prod_{j=1}^m\left(\int_{B_{\ell_2^{n_j}}^+}\int_0^1\cdots\int_0^1|\sum_{k_j=1}^{n_j}x^*(e_{k_j})r_{k_j}(t_j)|^p d\mu_j(x^*)dt_j\right)^{\frac{1}{p}}\right)
\end{aligned}$$

applying the Khintchine's inequality

$$(4.2.4) \leq B_p^m \pi_p^{d+}(T)\prod_{j=1}^m\left(\int_{B_{\ell_2^{n_j}}^+}\left(\sum_{k_j=1}^{n_j}|x^*(e_{k_j})|^2\right)^{\frac{p}{2}} d\mu_j(x^*)\right)^{\frac{1}{p}}$$

$$\begin{aligned}
(4.2.4) &\leq B_p^m \pi_p^{d+}(T)\prod_{j=1}^m\left(\int_{B_{\ell_2^{n_j}}^+}\|x^*\|_{\ell_2^n}^p d\mu_j(x^*)\right)^{\frac{1}{p}} \\
&\leq B_p^m \pi_p^{d+}(T).
\end{aligned}$$

Consequently

$$\|\sum_{k_1,...,k_m=1}^{n_1,...,n_m} r_{k_1}...r_{k_m}T(e_{k_1},...,e_{k_m})\|_{L_{q^*}([0,1],F)} \leq K_{p,q^*}^m B_p^m \pi_p^{d+}(T). \tag{4.2.5}$$

Combining (4.2.3) and (4.2.5) we have

$$\sum_{i=1}^{n} | < T(x_i^1, ..., x_i^m), y_i^* > |$$
$$\leq (A_{q^*} K_{p,q^*} B_P)^m \pi_p^{d+}(T) \left(\sum_{i=1}^{n} \prod_{j=1}^{m} \|x_i^j\|^q \right)^{\frac{1}{q}} \|(y_i^*)_{i=1}^n\|_{\ell^n_{q^*,|weak|}(F^*)}$$

thus $T \in \mathcal{D}_q^{m+}(\ell_2^{n_1}, ..., \ell_2^{n_m}; F)$ and $d_q^{m+}(T) \leq (A_{q^*} K_{p,q^*} B_P)^m \pi_p^{d+}(T)$.
And this completes the proof. □

4.2.2 Positive p-dominated m-homogeneous polynomials

Definition 4.2.2. *Let $1 \leq p < \infty$, an $m-$homogeneous polynomial $P : E \to Y$ is positive $p-$dominated if there exists a constant $C > 0$ such that, for all $n \in \mathbb{N}^*$ and every $(x_i)_{i=1}^n \subset E^+$*

$$\left(\sum_{i=1}^{n} \|P(x_i)\|^{\frac{p}{m}} \right)^{\frac{m}{p}} \leq C \sup_{x^* \in B_{E^*}^+} \left(\sum_{i=1}^{n} < x^*, x_i >^p \right)^{\frac{m}{p}}$$

we denote $\mathcal{P}_{d,p}^+({}^m E; Y)$ the space of positive $p-$dominated polynomials from E into Y, and by $\delta_p^+(.)$ the norm defined by the infimum of all constants verifying the above inequality, for $p \geq m$, but for $p < m$ it is only a quasi-norm.

Theorem 4.2.3. ***(Domination theorem for positive p-dominated m-homogeneous polynomials)*** *Let $1 \leq p < \infty$, an $m-$homogeneous polynomial $P : E \to Y$ is positive $p-$dominated, if there are $K > 0$ and a probability measure μ on $B_{E^*}^+$ such that*

$$\|P(x)\| \leq K \left(\int_{B_{E^*}^+} < x^*, x >^p d\mu(x^*) \right)^{\frac{m}{p}}$$

for every $x \in E^+$. Moreover, the smallest K is $\delta_p^+(P)$.

In the next proposition we study the relationship between positive $p-$dominated $m-$linear operators and positive $p-$dominated $m-$homogeneous polynomials

Proposition 4.2.1. *Let $P : E \to Y$ be an $m-$homogeneous polynomial and $\widehat{P}$ be its corresponding symmetric $m-$linear operator such that $P(x) = \widehat{P}(x, ..., x)$.*

1. *If $\widehat{P}$ is positive $p-$dominated operator, then P is $p-$dominated and*

$$\delta_p^+(P) \leq \pi_p^{d+}(\widehat{P}).$$

2. *If P is positive $p-$dominated, then $\widehat{P}$ is positive $p-$dominated and*

(a) for $p \geq m$ $\quad \pi_p^{d+}(\widehat{P}) \leq \frac{m^m}{m!}\delta_p^+(P)$

(b) for $p < m$ $\quad \pi_p^{d+}(\widehat{P}) \leq \frac{m^m}{m!}2^{(\frac{m^2}{p}-m)}\delta_p^+(P)$.

Proof. 1) Suppose that $\widehat{P}$ is positive $p-$dominated, let $(x_i)_{i=1}^n \subset E^+$, then

$$\left(\sum_{i=1}^n \|P(x_i)\|^{\frac{p}{m}}\right)^{\frac{m}{p}} = \left(\sum_{i=1}^n \|\widehat{P}(x_i,...,x_i)\|^{\frac{p}{m}}\right)^{\frac{m}{p}}$$

$$\leq \pi_p^{d+}(\widehat{P})\|(x_i)_{i=1}^n\|^m_{\ell^n_{p,weak}(E)}.$$

Therefore, P is positive $p-$dominated and $\delta_p^+(P) \leq \pi_p^{d+}(\widehat{P})$.
To prove 2). For $1 \leq j \leq m$ and $n \in \mathbb{N}$, let $x^j \in E_j$ satisfy $\|x^j\| \leq 1$.

$$\begin{aligned}
(\|\widehat{P}(x^1,...,x^m)\|^{\frac{p}{m}})^{\frac{m}{p}} &= \left(\|\frac{1}{m!2^m}\sum_{\epsilon_j=\pm1}\epsilon_1...\epsilon_m P(\sum_{j=1}^m \epsilon_j x^j)\|^{\frac{p}{m}}\right)^{\frac{m}{p}}\\
&\leq \frac{1}{m!2^m}\left(\sum_{\epsilon_j=\pm1}\|P(\sum_{j=1}^m\epsilon_j x^j)\|^{\frac{p}{m}}\right)^{\frac{m}{p}} \text{ by Theorem 4.2.3}\\
&\leq \frac{1}{m!2^m}\delta_p^+(P)\sum_{\epsilon_j=\pm1}\left(\int_{B^+_{E^*}} < x^*, \sum_{j=1}^m \epsilon_j x^j >^p d\mu(x^*)\right)^{\frac{m}{p}}\\
&\leq \frac{1}{m!2^m}\delta_p^+(P)\sum_{\epsilon_j=\pm1}\left(\|\sum_{j=1}^m\epsilon_j x^j\|^p\int_{B^+_{E^*}} < x^*, \frac{\sum_{j=1}^m \epsilon_j x^j}{\|\sum_{j=1}^m \epsilon_j x^j\|} >^p d\mu(x^*)\right)^{\frac{m}{p}}\\
&\leq \frac{1}{m!2^m}\delta_p^+(P)\sum_{\epsilon_j=\pm1}\left(\sum_{j=1}^m\|x^j\|\right)^m\left(\int_{B^+_{E^*}} < x^*, y >^p d\mu(x^*)\right)^{\frac{m}{p}}\\
&\leq \frac{1}{m!2^m}\delta_p^+(P)2^m m^m\left(\int_{B^+_{E^*}} < x^*, y >^p d\mu(x^*)\right)^{\frac{m}{p}}\\
&\leq \frac{m^m}{m!}\delta_p^+(P)\left(\int_{B^+_{E^*}} < x^*, y >^p d\mu(x^*)\right)^{\frac{m}{p}}.
\end{aligned}$$

(b) for $p < m$, $1 \leq j \leq m$ and $n \in \mathbb{N}$, let $x_1^j, ..., x_n^j \in E_j$ satisfy $\|(x_i^j)_{i=1}^n\|_{\ell_{p,weak}^n} = 1$.

$$\left(\sum_{i=1}^{n} \|\widehat{P}(x_i^1, ..., x_i^m)\|^{\frac{p}{m}}\right)^{\frac{m}{p}} \leq \frac{1}{m!2^m}\left(\sum_{i=1}^{n}\sum_{\epsilon_j=\pm 1}\|P(\epsilon_1 x_i^1 + ... + \epsilon_m x_i^m)\|^{\frac{p}{m}}\right)^{\frac{m}{p}}$$
$$\leq \frac{1}{m!2^m}\delta_p^+(P)(\sum_{\epsilon_j=\pm 1}\|(\epsilon_1 x_i^1 + ... + \epsilon_m x_i^m)_{i=1}^n\|_{\ell_{p,weak}^n}^p)^{\frac{m}{p}}$$
$$\leq \frac{1}{m!2^m}\delta_p^+(P)m^m\left(\sum_{\epsilon_j=\pm 1} 1\right)^{\frac{m}{p}}$$
$$\leq \frac{m^m}{m!}2^{\frac{m^2}{p}-m}\delta_p^+(P)$$

□

Relationship between positive dominated and Cohen positive strongly summing homogeneous polynomials

In the next corollary, we give an inclusion between the two classes of polynomials defined earlier.

Corollary 4.2.1. *Let $1 < p, q < \infty$, and ℓ_2^n Banach lattice space, then*

$$\mathcal{P}_{d,p}^+(^m\ell_2^n; F) \subseteq \mathcal{P}_{Coh,q}^+(^m\ell_2^n; F)$$

for all $P \in \mathcal{P}_{Coh,q}^+(^m\ell_2^n; F)$ $\qquad d_q^+(P) \leq \delta_p^+(P).$

Proof. Let $P \in \mathcal{P}_{d,p}^+(^m\ell_2^n; F)$, from Proposition 4.2.1 the associate symmetric of P denoted $\widehat{P}$ belongs to $\mathcal{L}_d^{p+}(^m\ell_2^n; F)$, and by Theorem 4.2.2 $\widehat{P}$ is in $\mathcal{D}_q^{m+}(^m\ell_2^n; F)$, and so Proposition 4.1.5 finishes the proof. □

4.3 Applications and example

In this section we present an application based on the relationship between Cohen positive strongly p-summing m-homogeneous polynomials and the positive p^*-summing linear operators (Proposition 4.1.3). And give an example of being a Cohen positive strongly p-summing m-homogeneous polynomial.
et K be a compact Hausdorff space and $(K, \mathcal{A}, \mu)$ a finite measure space.

As usual $C(K)$ denotes the space of continuous functions on K, and $\mathcal{M}(K)$ the Banach space of all regular Borel measures on K, and we have $\mathcal{M}(K) \cong (C(K))^*$.

We shall write $L^p(\mu, Y)$ for the space of measurables functions on K with

$$\|f\|_p = (\int_K \|f(t)\|^p \, d\mu)^{\frac{1}{p}} \qquad \text{for } 1 \leq p < +\infty,$$

and for $p = +\infty$

$$\|f\|_\infty = \inf \{M \geq 0 : \|f(t)\| \leq M \ \mu \text{ almost every where}\}.$$

Let $1 \leq p < \infty$. An operator $u : E \longrightarrow X$ is said to be $p-$concave if there exists a constant $C > 0$ such that for all $n \in \mathbb{N}, x_1, ..., x_n \in E$, the inequality

$$\|u(x_i)_{i=1}^n\|_p \leq C\| \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} \| \tag{4.3.1}$$

holds.

We denote by $\mathcal{C}_p$ $(E; X)$, the space of $p-$concave operators from E into X. $\mathcal{C}_p$ (E, X) becomes a Banach space with norm given by the infimum of the constant $C > 0$ that verify the inequality (4.3.1).
The following results are a conclusion of the work done by Blasco in [**11**, **12**] and the Proposition 4.1.3.

Proposition 4.3.1. *Let* $1 < p < \infty$
1) $P \in \mathcal{P}^+_{coh,p}({}^mX, L^\infty(\mu))$ *if and only if* $P^* \in \mathcal{L}(L^1(\mu), \mathcal{P}({}^mX))$.
2) $P \in \mathcal{P}^+_{coh,p}({}^mX, \mathcal{M}(K))$ *if and only if* $P^* \in \mathcal{C}_p(C(K), \mathcal{P}({}^mX))$.
3) $P \in \mathcal{P}^+_{coh,p}({}^mX, L^p(\mu))$ *if and only if* $P^* \in L^{p^*}(\mu, \mathcal{P}({}^mX))$.
4) $P \in \mathcal{P}^+_{coh,p}({}^mX, \ell^{p^*})$ *if and only if* $P^* \in \ell^{p^*}(P({}^mX))$.

Example 4.3.1. *We use the example used in [3], with adjustments to meet our case.*
Let $1 < p \leq +\infty$, $m \in \mathbb{N}$ *and* $u : X \longrightarrow F$ *be a positive strongly* $p-$*summing operator. For* $\varphi \in X^*$, *the polynomial*

$$\begin{array}{rccc} P: & X & \longrightarrow & F \\ & x & \longmapsto & \varphi^{m-1}(x)\, u(x), \end{array}$$

is Cohen positive strongly $p-$*summing.*

Indeed, for $x_1, x_2, ..., x_n \in X$ *and* $y_1^*, y_2^*, ..., y_n^* \in F^*$. *Because* $u \in \mathcal{D}_p^+(X, F)$, *we have by [1]*

$$\sum_{i=1}^n |\langle u(x_i), y_i^* \rangle| \leq d_p^+(u) \|(x_i)_{i=1}^n\|_p \|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)},$$

then

$$\begin{aligned}
\sum_{i=1}^{n}|\langle P(x_i),y_i^*\rangle| &= \sum_{i=1}^{n}|\langle\varphi^{m-1}(x_i)\,u(x_i),y_i^*\rangle| \\
&= \sum_{i=1}^{n}|\langle u\left(\varphi^{m-1}(x_i)\,x_i\right),y_i^*\rangle| \\
&\leq d_p^+(u)\left\|\left(\varphi^{m-1}(x_i)\,x_i\right)_{i=1}^n\right\|_p\|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)} \\
&\leq d_p^+(u)\left(\sum_{i=1}^{n}\left\|\left(\varphi^{m-1}(x_i)\,x_i\right)_{i=1}^n\right\|^p\right)^{\frac{1}{p}}\|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)} \\
&\leq d_p^+(u)\,\|\varphi\|^{m-1}\left(\sum_{i=1}^{n}\|(x_i)_{i=1}^n\|^{mp}\right)^{\frac{1}{p}}\|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)}.
\end{aligned}$$

So, P is Cohen positive strongly $p-$summing and $d_p^+(P)\leq d_p^+(u)\,\|\varphi\|^{m-1}$.

4.4 Tensorial Approach

This section is intended to present a tensorial perspective of Cohen positive strongly p-summing and positive p-dominated polynomials and m-linear operators. For doing this, we apply the standard technique of associate a linear functional on a tensor product to each multilinear operator or polynomial. In the case of Cohen positive strongly $p-$summing operators, we pay more attention on the details for polynomials than m-linear operators. Later, we pay more attention on positive p-dominated m-linear operators than polynomials. Let m be a non-negative integer, X be a Banach space and F be a Banach lattice, each $m-$homogeneous polynomial $P: X\to F$ define a functional given by

$$\begin{aligned}
\phi_P &: (\otimes_s^m X)\otimes F^*\to\mathbb{R} \\
&x\otimes...\otimes x\otimes y^*\to y^*(P(x)).
\end{aligned}$$

The next step in our development is to define adequate norm α on the tensor space $(\otimes_s^m X)\otimes F^*$ in order that P to be Cohen positive strongly $p-$summing (positive p-dominated) exactly when ϕ_P is bounded on $(\otimes_s^m X\otimes F^*,\alpha)$.

4.4.1 Cohen positive Strongly p-summing polynomials from a tensor viewpoint

Before introducing the respecting tensor norm, let us state an equivalence of being a Cohen positive strongly p-summing polynomial. This is an immediate consequence of the duality between ℓ_∞^n and ℓ_1^n. The m-homogeneous polynomial $P: X\to F$ is Cohen positive strongly p-summing if and only if there exists a constant C such that

$$|\sum_{i=1}^{n}\lambda_i y_i^*(P(x_i))|\leq\|(\lambda_i)_{i=1}^n\|_{\ell_\infty^n}\|(\|x_i\|^m)_{i=1}^n\|_{\ell_p^n}\|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)} \qquad (4.4.1)$$

for all finite sequences $(\lambda_i)_{i=1}^n \subset \mathbb{R}$, $(x_i)_{i=1}^n \subset X$ and $(y_i^*)_{i=1}^n \subset F^*$. In this case, $d_p^+(P)$ agrees with the sharpest constant C so that the inequality holds.

Definition 4.4.1. *For each z in $(\otimes_s^m X) \otimes F$ define*

$$\lambda_p^+(z) = \inf \left\{ \|(\lambda_i)_{i=1}^n\|_{\ell_\infty^n} \|(\|x_i\|)_{i=1}^n\|_{\ell_p^n} \|(y_i)_{i=1}^n\|_{\ell_{p^*,|weak|}^n(F)} \right\}$$

where the infimum is taken over all representations $z = \sum\limits_{i=1}^n \lambda_i x_i \otimes ... \otimes x_i \otimes y_i$. For each z in $(\otimes_s^m X) \otimes F$ define

$$\Lambda_p^+(z) = \inf \left\{ \|(\lambda_i)_{i=1}^n\|_{\ell_\infty^n} \|(\|x_i\|)_{i=1}^n\|_{\ell_p^n} w_{p^*}((y_i)_{i=1}^n) \right\}.$$

where the infimum is taken over all representations $z = \sum\limits_{i=1}^n \lambda_i x_i \otimes ... \otimes x_i \otimes y_i$ with $y_i \geq 0$ for all $1 \leq i \leq n$.

The next proposition is a consequence of the fact that every $y \in F$ can be represented as $y = y^+ - y^-$. Moreover, it helps to reduce the calculations related to λ_p^+ and Λ_p^+.

Proposition 4.4.1. *For every z in $(\otimes_s^m X) \otimes F$ we have*

$$\lambda_p^+(z) \leq \Lambda_p^+(z) \leq 2\lambda_p^+(z).$$

Proof. Let z in $(\otimes_s^m X) \otimes F$. It is clear that $\lambda_p^+(z) \leq \Lambda_p^+(z)$ by (1.2.1). For the opposite inequality take $z = \sum\limits_{i=1}^n \lambda_i x_i \otimes ... \otimes x_i \otimes y_i$. Then

$$z = \sum_{i=1}^n \lambda_i x_i \otimes ... \otimes x_i \otimes y_i^+ + \sum_{i=1}^n (-\lambda_i) x_i \otimes ... \otimes x_i \otimes y_i^-.$$

Thus, we obtain a new representation for z such that

$$\begin{aligned}
\Lambda_p^+(z) &\leq 2^{\frac{1}{p}} \|(\lambda_i)_{i=1}^n\|_\infty \|(\|x_i\|)_{i=1}^n\|_{\ell_p^n} \sup_{w^* \in B_{F^*}^+} \left(\sum_{i=1}^n w^*(y_i^+)^{p^*} + w^*(y_i^-)^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq 2^{\frac{1}{p}} \|(\lambda_i)_{i=1}^n\|_\infty \|(\|x_i\|)_{i=1}^n\|_{\ell_p^n} \sup_{w^* \in B_{F^*}^+} \left(\sum_{i=1}^n 2w^*(|y_i|)^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq 2^{\frac{1}{p}+\frac{1}{p^*}} \|(\lambda_i)_{i=1}^n\|_\infty \|(\|x_i\|)_{i=1}^n\|_{\ell_p^n} \sup_{w^* \in B_{F^*}^+} \left(\sum_{i=1}^n w^*(|y_i|)^{p^*} \right)^{\frac{1}{p^*}} \\
&\leq 2\|(\lambda_i)_{i=1}^n\|_\infty \|(\|x_i\|)_{i=1}^n\|_{\ell_p^n} \|(y_i)_{i=1}^n\|_{\ell_{p^*,|weak|}^n}(F).
\end{aligned}$$

Therefore, $\Lambda_p^+(z) \leq 2\lambda_p^+(z)$ for all z. □

Proposition 4.4.2. *The application $z \mapsto \lambda_p^+(z)$ is a norm on $(\otimes_s^m X) \otimes F$.*

Proof. Standard tensorial techniques can be applied to prove that λ_p^+ is non-negative, homogeneous and verify the triangular inequality. So we only prove that $\lambda_p^+(z) = 0$ implies $z = 0$. Let ξ be a bounded linear functional on $\widehat{\otimes}^m_{\pi,s} X$ and $y^* \geq 0$ bounded on F^*. Then

$$\begin{aligned}
|\xi \otimes y^*(z)| &= |\sum_{i=1}^n \lambda_i \xi(x_i \otimes ... \otimes x_i) y^*(y_i)| \\
&\leq \|(\lambda_i)_{i=1}^n\|_\infty \sum_{i=1}^n |\xi(x_i \otimes ... \otimes x_i) y^*(y_i)| \\
&\leq \|(\lambda_i)_{i=1}^n\|_\infty \left(\sum_{i=1}^n |\xi(x_i \otimes ... \otimes x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y^*(y_i)|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq \|\xi\| \|(\lambda_i)_{i=1}^n\|_\infty \left(\sum_{i=1}^n \|x_i\|^{mp}\right)^{\frac{1}{p}} \left(\sum_{i=1}^n |y^*(y_i)|^{p^*}\right)^{\frac{1}{p^*}} \\
&\leq \|\xi\| \|y^*\| \|(\lambda_i)_i^n\|_\infty \|(\|x_i\|^m)_{i=1}^n\|_{\ell_p^n} w_{p^*}((y_i)_{i=1}^n).
\end{aligned}$$

Hence, $\lambda_p^+(z) = 0$ implies $\xi \otimes y^*(z) = 0$ for all ξ and $y^* \geq 0$. For arbitrary y^* we have that $y^* = (y^*)^+ - (y^*)^-$. Then $\lambda_p^+(z) = 0$ implies

$$\xi \otimes y^*(z) = \xi \otimes (y^*)^+(z) - \xi \otimes (y^*)^-(z) = 0.$$

Therefore, z must be zero since the set of linear functionals of the form $\xi \otimes y^*$ is a separating set for $(\widehat{\otimes}^m_{\pi,s} X) \otimes F^*$. □

Next, we present the characterization of Cohen positive strongly p-summing polynomials in tensor terms.

Proposition 4.4.3. ***(Duality)*** *Let $P : X \to F$ be an $m-$homogeneous polynomial. The following are equivalent*

(i) P is Cohen positive strongly p-summing.

(ii) ϕ_P is bounded on $\left((\otimes_s^m X) \otimes F^*, \lambda_p^+\right)$.

Under this circumstances $d_p^+(P) = \|\phi_P\|$.

Proof. Let $z = \sum\limits_{i=1}^n \lambda_i x_i \otimes ... \otimes x_i \otimes y_i^*$ be any element of $(\otimes_s^m X) \otimes F^*$, then

$$|\phi_P(z)| = |\sum_{i=1}^n \lambda_i y^*(P(x_i))| \leq d_p^+(P) \|(\lambda_i)_i^n\|_\infty \|(\|x_i\|^m)_{i=1}^n\|_{\ell_p^n} \|(y_i^*)_{i=1}^n\|_{\ell^n_{p,|weak|}(F^*)}.$$

After taking the infimum over all the representations of z we have that

$$|\phi_P(z)| \leq d_p^+(P) \lambda_p^+(z)$$

holds for all z.
Conversely, take sequences $(\lambda_i)_{i=1}^n$ in $\mathbb{R}$, $(x_i)_{i=1}^n$ in X and $(y_i^*)_{i=1}^n$ in F^*. Hence,

$$\begin{aligned}|\sum_{i=1}^n \lambda_i y^*(P(x_i))| = |\phi_P(z)| &\leq \|\phi_P\|\lambda_p^+\left(\sum_{i=1}^n \lambda_i x_i \otimes ... \otimes x_i \otimes y_i^*\right)\\ &\leq |\phi_P(z)|\|(\lambda_i)_i^n\|_\infty \|(\|x_i\|^m)_{i=1}^n\|_{\ell_p^n}\|(y_i^*)_{i=1}^n\|_{\ell^n_{p,|weak|}(F^*)}.\end{aligned}$$

□

As we saw, the norms λ_p^+ and Λ_p^+ are equivalent and one of them is more useful than the other depending on the circumstances we are interested on. The norm λ_p^+ provides an isometry between polynomials and linear functionals while Λ_p^+ reduces calculations in the next proposition.

Proposition 4.4.4. ***(Uniform property)*** *Let $R: X \to Y$ be a bounded linear operator and $S: F \to E$ a positive operator. Then, the operator defined by*

$$\begin{aligned}R\otimes S : ((\otimes_s^m X)\otimes F, \Lambda_p^+) &\to ((\otimes_s^m Y)\otimes E, \Lambda_p^+)\\ x\otimes ...\otimes x\otimes y &\longmapsto R(x)\otimes ...\otimes R(x)\otimes S(y)\end{aligned}$$

is bounded and $\|R\otimes S\| \leq \|R\|^m\|S\|$.

Proof. Let $z = \sum_{i=1}^n \lambda_i x_i \otimes ... \otimes x_i \otimes y_i$ with $y_i \geq 0$. Then

$$\begin{aligned}\Lambda_p^+(R\otimes S(z)) &\leq \|(\lambda_i)_i^n\|_\infty \|(\|Rx_i\otimes\cdots\otimes Rx_i\|)_{i=1}^n\|_{\ell_p^n} w_{p^*}(S(y_i))\\ &\leq \|R\|^m\|S\|\|(\lambda_i)_i^n\|_\infty\|(\|x_i\|^m)_{i=1}^n\|_{\ell_p^n} w_{p^*}((y_i)_{i=1}^n).\end{aligned}$$

After taking the infimum over all the representations of z we obtain $\Lambda_p^+(R\otimes S(z)) \leq \|R\|^m\|S\|\Lambda_p^+(z)$. □

As it is expected, previous proposition is also true if Λ_p^+ is remplaced by λ_p^+ but this change involves a factor of 2 in the inequality.

4.4.2 Cohen positive strongly p-summing m-linear operators from a tensor viewpoint

Bearing in mind the previous section, we are able to develop analogous ideas to characterize Cohen positive strongly p−summing m-linear operators. As before, throughout this section $X_1, ..., X_m$ are Banach spaces and F is a Banach lattice. Next, recall that each m-linear operator $T: X_1\times\cdots\times X_m \to F$ define a linear functional by

$$\begin{aligned}\phi_T : X_1\otimes ...\otimes X_m\otimes F^* &\to \mathbb{R}\\ x^1\otimes ...\otimes x^m\otimes y^* &\mapsto y^*(T(x^1, ..., x^m)).\end{aligned}$$

In this case for each z in $X_1 \otimes ... \otimes X_m \otimes F$ we define

$$\lambda_p^{m+}(z) = inf\left\{\|(\lambda_i)_i^n\|_\infty \|(\|x_i^1\|...\|x_i^m\|)_{i=1}^n\|_{\ell_p^n} \|(y_i)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F)}\right\}$$

where the infimum is taken over all representations $z = \sum_{i=1}^n \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i$. Usual tensor techniques and slight modifications of the proof of Proposition 4.4.2 let us prove that the assignment $z \mapsto \lambda_p^{m+}(z)$ is a norm on $X_1 \otimes ... \otimes X_m \otimes F$. The respective duality result is exhibited in the next proposition and requires an analogous expression to (4.4.1).

Proposition 4.4.5. ***(Duality)*** *Let $T : X_1 \times \cdots \times X_m \to F$ be an m-linear operator. Then, then following are equivalents*

(i) T is Cohen positive strongly p-summing.

(ii) ϕ_T is bounded on $\left(X_1 \otimes ... \otimes X_m \otimes F^*, \lambda_p^{m+}\right)$.

Under this circumstances $d_p^{m+}(T) = \|\phi_T\|$.

Proof. For any $z = \sum_{i=1}^n \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i^*$ in $X_1 \otimes ... \otimes X_m \otimes F^*$ we have

$$\begin{aligned}|\phi_T(z)| &= |\sum_{i=1}^n \lambda_i y_i^*(T(x_i^1, ..., x_i^m))| \\ &\leq d_p^{m+}(T)\|(\lambda_i)_i^n\|_\infty \|(\|x_i^1\|...\|x_i^m\|)_{i=1}^n\|_{\ell_p^n} \|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)}.\end{aligned}$$

Hence, for any z we have

$$|\phi_T(z)| \leq d_p^{m+}(T)\lambda_p^{m+}(z).$$

On the other direction, let $(\lambda_i)_{i=1}^n$ in $\mathbb{R}$, $(x_i^j)_{i=1}^n$ in X^j and $(y_i^*)_{i=1}^n$ in F^*. Then

$$\begin{aligned}|\sum_{i=1}^n \lambda_i y_i^*(T(x_i^1, ..., x_i^m))| &= |\phi_T(z)| \leq \|\phi_T\|\lambda_p^{m+}\left(\sum_{i=1}^n \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i^*\right) \\ &\leq \|\phi_T\|\|(\lambda_i)_i^n\|_\infty \|(\|x_i^1\|...\|x_i^m\|)_{i=1}^n\|_{\ell_p^n} \|(y_i^*)_{i=1}^n\|_{\ell^n_{p^*,|weak|}(F^*)}.\end{aligned}$$

□

For each z in $X_1 \otimes ... \otimes X_m \otimes F$, define

$$\Lambda_p^{m+}(z) = inf\left\{\|(\lambda_i)_i^n\|_\infty \|(\|x_i^1\|...\|x_i^m\|)_{i=1}^n\|_{\ell_p^n} w_{p^*}((y_i)_{i=1}^n)\right\}$$

where the infimum is taken over all representation $z = \sum_{i=1}^n \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ such that $y_i \geq 0$ for all $1 \leq i \leq n$. Then for any z in $X_1 \otimes ... \otimes X_m \otimes F$ we have

$$\lambda_p^{m+}(z) \leq \Lambda_p^{m+}(z) \leq 2\lambda_p^{m+}(z).$$

Having Λ_p^{m+} at our reach and knowing it defines a norm, we present the respective uniform property it verifies.

Proposition 4.4.6. ***(Uniform property)*** *Let $R_i : X_i \to Y_i$ be bounded linear operators and $S : F \to G$ be positive operator between Banach lattices. Then the operator*

$$R_1 \otimes ... \otimes R_m \otimes S : (X_1 \otimes ... \otimes X_m \otimes F, \Lambda_p^{m+}) \to (Y_1 \otimes ... \otimes Y_m \otimes G, \Lambda_p^{m+})$$
$$x^1 \otimes ... \otimes x^m \otimes y \mapsto R_1(x^1) \otimes ... \otimes R_m(x^m) \otimes S(y)$$

is bounded and $\|R_1 \otimes ... \otimes R_m \otimes S\| \leq \|R_1\|...\|R_m\|\|S\|$.

Proof. Given $z = \sum_{i=1}^{n} \lambda_i x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ with $y_i \geq 0$ we have

$$\begin{aligned}\Lambda_p^{m+}(R_1 \otimes ... \otimes R_m \otimes S(z)) &\leq \|(\lambda_i)_i^n\|_\infty \|(\|R_1 x_i^1\|...\|R_m x_i^m\|)_{i=1}^n\|_{\ell_p^n} w_p((y_i)_{i=1}^n) \\ &\leq \|R_1\|...\|R_m\|\|S\|\|(\lambda_i)_i^n\|_\infty \|(\|x_i^1\|...\|x_i^m\|)_{i=1}^n\|_{\ell_p^n} w_p((y_i)_{i=1}^n).\end{aligned}$$

As a consequence $\Lambda_p^{m+}(R_1 \otimes ... \otimes R_m \otimes S(z)) \leq \|R_1\|...\|R_m\|\|S\|\Lambda_p^{m+}(z)$. □

4.4.3 Positive p-dominated m-linear operators and polynomials form a tensor viewpoint

As before, throughout this section, $E, E_1, ..., E_m$ are Banach lattices and Y is a Banach space. In this case we pay more attention on m-linear operators than polynomials.

Proposition 4.4.7. *For each z in $E_1 \otimes ... \otimes E_m \otimes Y$ define*

$$\eta_p^{m+}(z) = \inf \left\{ \left(\prod_{j=1}^{m} \|(x_i^j)_{i=1}^n\|_{\ell^n_{p,|weak|}(E_j)} \right) \|(y_i)_{i=1}^n\|_{\ell^n_{\frac{p}{p-m}}} \right\}$$

where the infimum is taken over all representations $z = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i$. For each z in $(\otimes_s^m E) \otimes Y$ define

$$\eta_p^{+}(z) = \inf \left\{ w_p((x_i)_{i=1}^n)^m \|(y_i)_{i=1}^n\|_{\ell^n_{\frac{p}{p-m}}} \right\}$$

where the infimum is taken over all representations $z = \sum_{i=1}^{n} x_i \otimes ... \otimes x_i \otimes y_i$ with $x_i \geq 0$ for all $1 \leq i \leq n$. Then, the assignment $z \mapsto \eta_p^+(z)$ is a norm on $(\otimes_s^m E) \otimes Y$.

The proof of this proposition is an adaptation of Proposition 4.4.2 The next result concerns the characterization of positive p-dominated m-linear operators and polynomial in tensor terms.

Proposition 4.4.8. ***(Duality)*** *Let $T : E_1 \times \cdots \times E_m \to Y$ be an m-linear operator and $P : E \to Y$ be an m-homogeneous polynomial. Then*

(i) T is positive p-dominated if and only if, ϕ_T is bounded on $(E_1 \otimes ... \otimes E_m \otimes Y^*, \eta_p^{m+})$

(ii) P is positive p-dominated if and only if, ϕ_P is bounded on $((\otimes_s^m E) \otimes Y^*, \eta_p^+)$.

Under this circumstances $\pi_p^{d+}(T) = \|\phi_T\|$ and $\delta_p^+(P) = \|\phi_P\|$.

Proof. We only prove (i) since (ii) is a straightforward adaptation. First suppose T is positive p-dominated. Then for any $z = \sum\limits_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i^*$ in $E_1 \otimes ... \otimes E_m \otimes Y^*$ we have

$$\begin{aligned}
|\phi_T(z)| &\leq \sum_{i=1}^n \|y_i^*\| \|T(x_i^1, ..., x_i^m)\| \\
&\leq \left(\sum_{i=1}^n \|y_i^*\|^{\frac{p}{p-m}} \right)^{\frac{p-m}{p}} \left(\sum_{i=1}^n \|T(x_i^1, ..., x_i^m)\|^{\frac{p}{m}} \right)^{\frac{m}{p}} \\
&\leq \pi_p^{d+}(T) \left(\prod_{j=1}^m \|(x_i^j)_{i=1}^n\|_{\ell^n_{p,|weak|}(E_j)} \right) \|(y_i^*)_{i=1}^n\|_{\ell^n_{\frac{p}{p-m}}}.
\end{aligned}$$

Hence $|\phi_T(z)| \leq \pi_p^{d+}(T) \eta_p^{m+}(z)$ which means that ϕ_T is bounded and $\|\phi_T\| \leq \pi_p^{d+}(T)$. For the opposite direction. Let $(x_i^j)_{i=1}^n$ in E_j $1 \leq j \leq m$ and $\epsilon > 0$. Then, for each $1 \leq i \leq n$ there exists y_i^* in Y^* such that $y_i^*(T(x_i^1, ..., x_i^m)) = \|T(x_i^1, ..., x_i^m)\|^{\frac{p}{m}}$ and $\|y_i^*\| \leq (1+\epsilon)\|T(x_i^1, ..., x_i^m)\|^{\frac{p}{m}-1}$. Therefore

$$\begin{aligned}
\sum_{i=1}^n \|T(x_i^1, ..., x_i^m)\|^{\frac{p}{m}} &= \sum_{i=1}^n y_i^*(T(x_i^1, ..., x_i^m)) \\
&= \phi_T \left(\sum_{i=1}^n x_i^1 \otimes ... \otimes x_i^m \otimes y_i^* \right) \\
&\leq \|\phi_T\| \left(\prod_{j=1}^m \|(x_i^j)_{i=1}^n\|_{\ell^n_{p,|weak|}(E_j)} \right) \|(y_i)_{i=1}^n\|_{\ell^n_{\frac{p}{p-m}}} \\
&\leq \|\phi_T\|(1+\epsilon) \left(\prod_{j=1}^m \|(x_i^j)_{i=1}^n\|_{\ell^n_{p,|weak|}(E_j)} \right) \left(\sum_{i=1}^n \|T(x_i^1, ..., x_i^m)\|^{\frac{p}{m}} \right)^{\frac{p-m}{p}}.
\end{aligned}$$

As a consequence we obtain

$$\left(\sum_{i=1}^n \|T(x_i^1, ..., x_i^m)\|^{\frac{p}{m}} \right)^{\frac{m}{p}} \leq \|\phi_T\|(1+\epsilon) \prod_{j=1}^m \|(x_i^j)_{i=1}^n\|_{\ell^n_{p,|weak|}(E_j)}$$

which implies that T is positive p-dominated and $\pi_p^{d+}(T) \leq \|\phi_T\|$. □

The proof of next propositions are adaptations of those of Proposition 4.4.1 and 4.4.3.

Proposition 4.4.9. *Define for each z in $E_1 \otimes ... \otimes E_m \otimes Y$*

$$E_p^{m+} = \inf \left\{ \prod_{i=1}^{n} w_p((x_i^j)_{i=1}^n) \|(y_i)_{i=1}^n\|_{\ell^n_{\frac{p}{p-m}}} \right\}$$

where the infimum is taken over all representation $z = \sum_{i=1}^{n} x_i^1 \otimes ... \otimes x_i^m \otimes y_i$ with $x_i^j \geq 0$, for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Then

$$\eta_p^{m+}(z) \leq E_p^{m+}(z) \leq 2\eta_p^{m+}(z)$$

holds for all z in $E_1 \otimes ... \otimes E_m \otimes Y$.

Proposition 4.4.10. ***(Uniform property)***

- *Let $R_i : E_i \to M_i$ be a positive operators and $S : Y \to Z$ be a bounded linear operator . Then the operator*

$$\begin{aligned} R_1 \otimes ... \otimes R_m \otimes S : (E_1 \otimes ... \otimes E_m \otimes Y, E_p^{m+}) &\to (M_1 \otimes ... \otimes M_m \otimes Z, E_p^{m+}) \\ x^1 \otimes ... \otimes x^m \otimes y &\mapsto R_1(x^1) \otimes ... \otimes R_m(x^m) \otimes S(y) \end{aligned}$$

  is bounded and $\|R_1 \otimes ... \otimes R_m \otimes S\| \leq \|R_1\| ... \|R_m\| \|S\|$.

- *Let $R : E \to M$ be a positive operator and $S : Y \to Z$ bounded linear operator. Then, the operator defined by*

$$\begin{aligned} R \otimes S : \left((\otimes_s^m E) \otimes Y, \eta_p^+\right) &\to \left((\otimes_s^m M) \otimes Z, \eta_p^+\right) \\ x \otimes ... \otimes x \otimes y &\longmapsto R(x) \otimes ... \otimes R(x) \otimes S(y) \end{aligned}$$

  is bounded and $\|R \otimes S\| \leq \|R\|^m \|S\|$.

Bibliography

[1] D. Achour and A. Belacel. Domination and factorization theorems for positive strongly p-summing operators. *Positivity*, 18(4):785–804, Dec. 2014.

[2] D. Achour and L. Mezrag. On the Cohen strongly p-summing multilinear operators. *Journal of Mathematical Analysis and Applications*, 327:550–563, mar 2007.

[3] D. Achour and K. Saadi. A polynomial characterization of Hilbert spaces. *Collectanea mathematica*, 61(3):291–301, Oct. 2010.

[4] F. Albiac and N. J. Kalton. *Topics in Banach space theory*, volume 233. Springer, 2006.

[5] R. Alencar and M. C. Matos. Some classes of multilinear mappings between Banach spaces. *Publicaciones del Departamento de Análisis Matemático*, 12(Universidad Complutense Madrid), 1989.

[6] C. D. Aliprantis and O. Burkinshaw. *Positive operators*. Springer, Dordrecht, 2006.

[7] R. M. Aron, C. Hervés, and M. Valdivia. Weakly continuous mappings on Banach spaces. *Journal of functional analysis*, 52(2):189–204, 1983.

[8] R. M. Aron and M. Schottenloher. Compact holomorphic mappings on Banach spaces and the approximation property. *Bull. Amer. Math. Soc.*, 80(6):1245–1249, 11 1974.

[9] I. Bakhtin, M. Krasnoselskii, and V. Stetsenko. On the continuity of positive linear operators. *Sibirsk. Mat. Zh.*, 3(1):156–160, 1962.

[10] S. Banach. *Théorie des opérations linéaires*. Monografie Mat., PWN, Warszawa, 1932.

[11] O. Blasco. A class of operators from a Banach lattice into a Banach space. *Collectanea Mathematica*, 37(1):13–22, 1986.

[12] O. Blasco. Positive p-summing operators on L_p-spaces. *Proceedings of the American Mathematical Society*, 100(2):275, June 1987.

[13] F. Bombal, M. Fernández, and I. Villanueva. Unconditionally converging multilinear operators. *Mathematische Nachrichten*, 226(1):5–15, 2001.

[14] A. Bougoutaia and A. Belacel. Cohen positive strongly p-summing and p-convex multilinear operators. *Positivity*, 23(2):379–395, Apr. 2019.

[15] A. Bougoutaia, A. Belacel, and H. Hamdi. On the positive Dimant strongly p-summing multilinear operators. *Carpathian Mathematical Publications*, 12(2):401–411, Dec. 2020.

[16] A. Bougoutaia, A. Belacel, and H. Hamdi. Domination and Kwapień's factorization theorems for positive cohen p–nuclear m–linear operators. *Moroccan Journal of Pure and Applied Analysis*, 7(1):100–115, 2021.

[17] C. Boyd and R. A. Ryan. Geometric theory of spaces of integral polynomials and symmetric tensor products. *Journal of Functional Analysis*, 179(1):18–42, 2001.

[18] Q. Bu. Some mapping properties of p-summing operators with Hilbertian domain. In K. Jarosz, editor, *Contemporary Mathematics*, volume 328, pages 145–149. American Mathematical Society, Providence, Rhode Island, 2003.

[19] Q. Bu and G. Buskes. Polynomials on Banach lattices and positive tensor products. *Journal of Mathematical Analysis and Applications*, 388(2):845–862, 2012.

[20] Q. Bu and C. C. A. Labuschagne. Positive multiple summing and concave multilinear operators on Banach lattices. *Mediterranean Journal of Mathematics*, 12(1):77–87, Feb. 2015.

[21] J. R. Campos. Cohen and multiple Cohen strongly summing multilinear operators. *Linear and Multilinear Algebra*, 62(3):322–346, Mar. 2014.

[22] R. Cilia and J. M. Gutierrez. Polynomial factorization through $l_r(u)$ spaces. *Korean Math.Soc.*, 46(6):1293–1307, Nov. 2009.

[23] J. S. Cohen. Absolutely p-summing, p-nuclear operators and their conjugates. *Mathematische Annalen*, 201(3):177–200, Sept. 1973.

[24] A. Defant and K. Floret. *Tensor norms and operator ideals*. Number 176 in North-Holland mathematics studies. North-Holland, Amsterdam ; New York, 1993.

[25] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*. Number 43 in Cambridge studies in advanced mathematics. Cambridge University Press, Cambridge ; New York, 1995.

[26] V. Dimant. Strongly p-summing multilinear operators. *Journal of Mathematical Analysis and Applications*, 278(1):182–193, 2003.

[27] S. Dineen. Holomorphic functions on a Banach space. *Bulletin of the American Mathematical Society*, 76(4):883–886, 1970.

[28] A. Dvoretzky and C. A. Rogers. Absolute and unconditional convergence in normed linear spaces. *Proceedings of the National Academy of Sciences*, 36(3):192–197, 1950.

[29] D. H. Fremlin. Tensor Products of Archimedean Vector Lattices. *American Journal of Mathematics*, 94(3):777, July 1972.

[30] A. Grothendieck. *Produits tensoriels topologiques et espaces nucléaires*, volume 16. American Mathematical Society Providence, 1955.

[31] A. Grothendieck. *Résumé de la théorie métrique des produits tensoriels topologiques*. Soc. de Matemática de São Paulo, 1956.

[32] H. Hamdi, A. Belacel, and A. Bougoutaia. Inclusion and coincidences for multiple Cohen positive strongly p-summing m-linear operators. *submitted.*

[33] H. Hamdi, S. García-Hernández, A. Belacel, D. Ouchenane, and S. A. Zubair. Cohen Positive Strongly -Summing and -Dominated m-Homogeneous Polynomials. *Journal of Mathematics*, 2021:1–18, Nov. 2021.

[34] C. Labuschagne. Riesz Reasonable Cross Norms on Tensor Products of Banach Lattices. *Quaestiones Mathematicae*, 27(3):243–266, Sept. 2004.

[35] J. Lindenstrauss and A. Pełczyński. Absolutely summing operators in l_p-spaces and their applications. *Studia Mathematica*, 29(3):275–326, 1968.

[36] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces.* Classics in mathematics. Springer, Berlin Heidelberg, reprint edition, 1996.

[37] M. C. Matos. Fully absolutely summing and Hilbert-Schmidt multilinear mappings. *Collectanea Mathematica*, 54(2):111–136, 2003.

[38] R. D. Mauldin. *The Scottish Book: Mathematics from The Scottish Caf, with Selected Problems from The New Scottish Book.* Birkhäuser Basel, 2nd edition, 2015.

[39] Y. Meléndez and A. Tonge. Polynomials and the Pietsch Domination Theorem. *Mathematical Proceedings of the Royal Irish Academy*, 99A(2):195–212, 1999.

[40] P. Meyer-Nieberg. *Banach lattices.* Springer Science & Business Media, 2012.

[41] L. Mezrag and K. Saadi. Inclusion theorems for Cohen strongly summing multilinear operators. *Bulletin of the Belgian Mathematical Society - Simon Stevin*, 16(1):1–11, Feb. 2009.

[42] B. Mitiagin and A. Pełczyński. Nuclear operators and approximative dimensions. *Proceedings International Congress of Mathematics*, 1966.

[43] J. Mujica. *Complex analysis in Banach spaces: holomorphic functions and domains of holomorphy in finite and infinite dimensions.* North-Holland ; Sole distributors for the U.S.A. and Canada, Elsevier Science Pub. Co., Amsterdam; New York; New York, N.Y., U.S.A., 1986.

[44] D. Pellegrino, P. Rueda, and E. A. Sánchez-Pérez. Surveying the spirit of absolute summability on multilinear operators and homogeneous polynomials. *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, 110(1):285–302, Mar. 2016.

[45] D. Pellegrino and J. Santos. Absolutely summing multilinear operators: a panorama. *Quaestiones Mathematicae*, 34(4):447–478, Dec. 2011.

[46] D. Pellegrino, J. Santos, and J. B. Seoane-Sepúlveda. Some techniques on nonlinear analysis and applications. *Advances in Mathematics*, 229(2):1235–1265, Jan. 2012.

[47] A. Pietsch. Absolut p-summierende abbildungen in normierten räumen. *Studia Mathematica*, 28(3):333–353, 1967.

[48] A. Pietsch. *Operator Ideals.* Mathematical Studies. North-Holland Publishing Company, 1980.

[49] A. Pietsch. Ideals of multilinear functionals. *Proceedings of the Second International Conference on Operator Algebras Ideals and Their Applications in Theoretical Physics*, pages 185–199, 1983.

[50] D. Popa. Cohen-Kwapień type theorems for multiple summing operators. *Quaestiones Mathematicae*, 36(3):399–410, Sept. 2013.

[51] D. Popa. Multiple summing, dominated and summing operators on a product of ℓ_1 spaces. *Positivity*, 18(4):751–765, Dec. 2014.

[52] D. Pérez-García and I. Villanueva. Multiple summing operators on C(K) spaces. *Arkiv för Matematik*, 42(1):153–171, Apr. 2004.

[53] R. A. Ryan. *Introduction to tensor products of Banach spaces.* Springer monographs in mathematics, 2010.

[54] O. I. Zhukova. On modifications of the classes of p-nuclear ,p-summing, and p-integral operators. *Siberian Mathematical Journal*, 39(5):894–907, Oct. 1998.