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THEME

Exponential stability of solutions for some Transmission problems

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Abstract

The present memory is devoted to the study of well-posedness and stability of the solution for some transmission problems with a distributed delay term and infinite memory term in a bounded domain by using the semi-group theory and some energy estimates. This type of systems is very important from the point of view of application in sciences and engineering.

Key words : Transmission problem, existence and uniqueness, stability, distributed delay, infinite memory.

مُلَخَّص

تم تخصيص المذكرة الحالية لدراسة وجود، وحدانية واستقرار الحل لبعض أنظمة الإنتقال مع فترة تأخير موزعة و حد الذاكرة اللانهائية في مجال محدود باستخدام نظريات شبه الزمر وبعض تقديرات الطاقة. هذا النوع من الأنظمة مهم جدًا من وجهة نظر التطبيق في العلوم والهندسة.

الكلمات المفتاحية: نظام الإنتقال، حد التأخير الموزع، حد ذاكرة لانهائية، الاستقرار الأسي، الوجود و الوحدانية.

Résumé

Ce mémoire est consacré à l'étude d'existence, unicité et stabilité de la solution pour certaines systèmes de transmission avec un terme de retard distribué et terme de mémoire infini dans un domaine borné en utilisant la théorie de semi-groupe et quelques estimations d'énergie. Ce type de systèmes est trés important du point de vue de l'application en sciences et ingénierie .

Mots clés : Système de transmission, existence et unicité, stabilité, retard distribué, mémoire infini.

DEDICATION

It is with deep gratitude and sincere words, That I dedicate this modest work to :

My dearest parents.

Although no dedication could express my respect, consideration and deep feelings towards them.

They have sacrificed their lives for our success, my sisters, my brother and me, and have enlightened us with their wise advice.

I hope that one day, I will be able to give them back a little of what they have done for us.

May God grant them happiness and long life.

I also dedicate this work to my dearest sisters and brother: Hadjer, Soumia and Abderazzak.

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To my dear little nephews: Sami, Abderahmane and Abdelhamid.

To my beloved family TOUHAMI and SEHLAOUI .

To my friends .

To all my teachers who taught us.

And to all those who are dear to me.

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TOUHAMI Khedidja

Contents

introduction

Problems of global existence and intimate stability of partial differntial equations are subject, recently, of many works.
In this memory we are interested in the study of the global existence and stability of some transmission systems. The purpose of the stability is to attenuate the vibrations by feedback, thus consists in guaranteing the decease of energy of the solutions to 0 in a more or less fast way by a mechanism of dissipation. More precisely, the stability of problem consists in determining the asymptotic behavior of the energy by $E(t)$, to study its limits in order to determine if this limit is null or not and if this limit is null, to give an estimate of the decay rate of the energy to zero.
This problem has been studied by many authors for various systems. There are severals types of stability :

1) Strong stability

$$E(t) \to 0, \quad \text{as} \qquad t \to \infty.$$

2) Logarithmic stability

$$E(t) \leq C(log\ t)^{-\delta}, \qquad \forall t > 0\ ,\ (C, \delta) > 0.$$

3) Polynomial stability

$$E(t) \leq Ct^{-\delta}, \qquad \forall t > 0\ ,\ (C, \delta) > 0.$$

4) Exponential stability

$$E(t) \leq Ce^{-\delta t}, \qquad \forall t > 0\ ,(C, \delta) > 0.$$

Many authors have worked since then on energy decay rates. First results were obtained for linear stability, then for polynomial stability (see [13]) and then extended to arbitrary

growing feedbacks (close to 0). In the same time, geometrical aspects were considered. By combining the multiplier method with the techniques of micro-local analysis [9], have investigated different dissipative systems of partial differential equations (with Dirichlet and Neumann boundary conditions) under general geometrical conditions with nonlinear feedback without any powth restrictions near the origin or at infinity.
In this work, we treat the exponential decay.
Our main results in this memory can be summarized as follows:

Chapter 1. In this chapter, we will give some basic reminders and preliminaries on functional analysis that will be used in the following chapters of this memory.

Chapter 2. In this chapter, we considered the problem with a distributed delay

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1 u_t(x,t) + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| u_t(t-s)ds = 0, \quad x \in \Omega,\ t > 0, \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad x \in (L_1, L_2),\ t \geq 0, \end{cases}$$

under the boundary and the transmission conditions

$$\begin{cases} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2, \\ au_x(L_i,t) = bv_x(L_i,t), \quad i = 1,2, \end{cases}$$

and the initial conditions

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1,L_2), \quad , \\ u_t(x,-t) = f_0(x,-t), \quad x \in \Omega, \quad t \in (0,\tau_2). \end{cases}$$

It consists of three sections: position of the problem, existence and uniqueness result and exponential stability result.

Chapter 3. In this chapter, we considered the problem with infinite memory and distributed delay

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^{\infty} g(p)u_{xx}(x,t-p)\,dp + \mu_1 u_t(x,t) \\ + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| u_t(t-s)ds = 0, \quad x \in \Omega,\ t > 0, \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad x \in (L_1, L_2),\ t \geq 0, \end{cases}$$

under the boundary and the transmission conditions

$$\begin{cases} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2, \\ au_x(L_i,t) - \displaystyle\int_0^{\infty} g(p)u_x(L_i,t-p)\,dp = bv_x(L_i,t), \quad i = 1,2, \end{cases}$$

and the initial conditions

$$\begin{cases} u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), \quad (x,t) \in \Omega \times (0,+\infty), \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1,L_2), \\ u_t(x,-t) = f_0(x,-t), \quad x \in \Omega, \quad t \in (0,\tau_2). \end{cases}$$

Using the method of multipliers we demonstrate the exponential stability of this system. In this chapter, to ensure the well-posedness of the problems, we use the theory of semi-groups to establish the existence and uniqueness of solutions.
In the theory of semi-groups, the Hille-yosida theorem is a powerful and fundamental tool linking the energy dissipation properties of a boundless operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ the existence, uniqueness and regularity of solutions of a stationary differential equation stationary differential equation (Cauchy problem)

$$\begin{cases} U'(t) = \mathcal{A}U(t), \qquad t > 0 \\ U(0) = U_0. \end{cases}$$

For the stability results, we use the multiplier method based on the construction of a Lyapunov function $\mathcal{L}$ equivalent to the energy E of the solution. We denote by $\mathcal{L} \sim E$ the equivalence

$$\beta_1 E(t) \leq \mathcal{L} \leq \beta_2 E(t), \qquad \forall t \geq 0. \tag{0.1}$$

For two positive constants β_1 and β_2. For example, to establish an exponential stability it is sufficient to show that

$$\mathcal{L}'(t) \leq -\kappa \mathcal{L}(t), \qquad \forall t > 0. \tag{0.2}$$

For some $\kappa > 0$, a simple integration of (0.2) over $(0, t)$ with (0.1) leads to the desired result of exponential stability.

It is worth nothing that the Lyapunov theorems are only sufficient conditions for stability and the difficulty here is to to find the adequate Lyapunov function.

CHAPTER 1

PRELIMINARIES

1.1 Some basic spaces

In this section, we present the fundamental spaces (Hilbert[①],Lebesgue[②]and Sobolev[③]). For more details on the notions recalled in this paragraph see H. Brezis and L. Sonrier.

Hilbert space

Definition 1.1. *[5] A Hilbert space $\mathcal{H}$ is a vector space with scalar product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let $\mathcal{H}$ complete.*

Lebesgue space

Definition 1.2. *[5] Let Ω be a domain in $\mathbb{R}^n$, we define the space $L^p(\Omega)$ as follows*

- *For $1 \leq p < \infty$, the space $L^p(\Omega)$ is the space of real functions f on Ω such that f is measurable and*

  $$\int_\Omega |f(x)|^p dx < \infty.$$

  If $f \in L^p(\Omega)$, we define the norm

  $$\boxed{\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}.}$$

- *For $p = \infty$, the space $L^\infty(\Omega)$ is the space of measurable functions f that are essentially bounded on Ω.*

[①]David Hilbert born in 1862 in Königsbergh 1 and died in 1943 in Göttingen, is a German mathematician. He is often considered as one of the greatest mathematicians of the XX^e .

[②]Henri-Léon Lebesgue (1875-1941) : is one of the great French mathematicians. He is known for his theory of integration .

[③]Sergueï Lvovitch Sobolev (1908-1989): is a Russian mathematician and atomic physicist of the Soviet period.

If $f \in L^\infty(\Omega)$, *we have*

$$\|f\|_\infty = \operatorname{ess\,sup}_\Omega |f(x)| = \inf\{A \geq 0 : \mu\{x \in \Omega : f(x) > A\} = 0\}.$$

Sobolev space in $\mathbb{R}$ [5]

Sobolev spaces play a fundamental role in variational calculus. They are named after the Russian mathematician Sergueï Lvovitch Sobolev (1908-1989).

Sobolev space $W^{1,P}(\Omega)$

Definition 1.3. *Let* Ω *be any open in* $\mathbb{R}^n$ *and* $P \in \mathbb{R}$ *with* $1 \leq P \leq \infty$, *the Sobolev space* $W^{1,P}(\Omega)$ *is defined by*

$$W^{1,P}(\Omega) = \left\{u \in L^P(\Omega)\,, \exists g \in L^P(\Omega) \text{ such that } \int_\Omega u(x)\varphi'(x)\mathrm{d}x = -\int_\Omega g(x)\varphi(x)\mathrm{d}x\right\}.$$

We pose

$$\mathbb{H}^1(\Omega) = W^{1,2}(\Omega).$$

Sobolev space $W^{m,P}(\Omega)$

Definition 1.4. *Let* Ω *be any open in* $\mathbb{R}^n$, $m \geq 2$ *and a real* p, $1 \leq p \leq \infty$, *the Sobolev space* $W^{m,P}(\Omega)$ *is defined by*

$$W^{m,p}(\Omega) = \left\{ \begin{array}{c} u \in L^p(\Omega) \quad \forall \alpha \text{ with } \quad |\alpha| \leq m \quad \exists g_\alpha \in L^p(\Omega) \\ \text{such that } \displaystyle\int_\Omega u D^\alpha \varphi \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi \mathrm{d}x, \forall \varphi \in C_0^\infty(\Omega) \end{array} \right\},$$

where $\alpha \in N^n$, $|\alpha| = \sum_{i=1}^{N} \alpha_i$ *and* $D^\alpha \varphi = \dfrac{\partial^{|\alpha|}\varphi}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_N^{\alpha_N}}$.

We pose

$$\mathbb{H}^m(\Omega) = W^{m,2}(\Omega).$$

The space $W^{m,p}$ is supplied with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \sum_{0<\alpha\leq m} \|D^\alpha u\|_{L^p(\Omega)}.$$

And the space $\mathbb{H}^m$ is supplied with the scalar product

$$(u,v)_{H^m(\Omega)} = (u,v)_{L^2(\Omega)} + \sum_{\alpha=1}^{m} (D^\alpha u, D^\alpha v)_{L^2(\Omega)},$$

for any $u, v \in \mathbb{H}^m(\Omega)$.

Sobolev space $W_0^{1,P}(\Omega)$

Definition 1.5. *Given the real p , $1 \leq p \leq \infty$ we call Sobolev space, and we note $W_0^{1,p}(\Omega)$, the adherence of $D(\Omega)$ in $W^{1,p}(\Omega)$, (resp $\mathbb{H}_0^1(\Omega)$ if $p = 2$).*

1.2 Reminder of some inequalities

In this section, we will give a series of essential inequalities that will be of great use in the rest of the memory.

Poincaré inequality [①]

lemma 1.1. *[15] Suppose that I is a bounded interval, then there exists a constant C (depending on $|I| < \infty$) such that*

$$\|u\|_{L^p(I)} \leq C\|u'\|_{L^p(I)}, \quad u \in W_0^{1,p}(I).$$

In other words, on $W_0^{1,p}$ the quantity $\|u'\|_{L^p(I)}$ is a norm equivalent to the $W^{1,p}$ norm.

R The Poincaré inequality (named after the French mathematician Henri Poincaré) is a result of the theory of Sobolev spaces. This inequality allows to a function from an estimate of its derivatives and the geometry of its domain of definition. These estimates are of great importance for the method of calculating variations.

[①] Henri Poincaré was a French mathematician, theoretical physicist and philosopher of science, born on April 29, 1854 in Nancy and died on July 17, 1912 in Paris.

Cauchy[①]-Schwartz[②] inequality

lemma 1.2. *[5] Let $\mathbb{H}$ be a Hilbert space supplied with a scalar product $(.,.)$, then*

$$|(u,v)| \leq (u,u)^{\frac{1}{2}}(v,v)^{\frac{1}{2}}, \quad \forall u,v \in \mathbb{H}.$$

Young's Inequality [③]

Theorem 1.1. *[4] Let $a,b \geq 0$ and p,q be two conjugate real numbers in $]1,\infty[$, $\left(\frac{1}{p}+\frac{1}{q}=1\right)$, then for all $\varepsilon > 0$*

$$ab \leq \varepsilon a^p + C_\varepsilon b^q,$$

where $C_\varepsilon = \dfrac{1}{q(\varepsilon p)^{\frac{p}{q}}}$.

R For $p = q = 2$, the previous inequality can be written as

$$ab \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon}.$$

1.3 Some notions about operators

Definition 1.6. *[5] A linear operator on a space X is a linear application defined on a sub vector space $D(\mathcal{A}) \subset X$ has values in X, ($D(\mathcal{A})$ is called the domain of operator $\mathcal{A}$).*

Definition 1.7. *[5] Let $\mathcal{H}$ be a Hilbert space and $\mathcal{A}$ an unbounded operator on $\mathcal{H}$ of domain $D(\mathcal{A})$.*
*We say that $\mathcal{A}$ is **monotonic** (or accretive) if*

$$\langle \mathcal{A}v, v\rangle \geq 0, \quad \forall v \in D(\mathcal{A}).$$

[①] Augustin Louis, baron Cauchy, born in Paris on August 21, $1789 and died in Sceaux on May 23$, 1857, was a French mathematician, member of the Academy of Sciences and professor at the École Polytechnique.

[②] Laurent Schwartz was a French mathematician, born on March 5, 1915 in Paris where he died on July 4, 2002. He was the first Frenchman to win the Fields Medal, in 1950, for his work on the theory of distributions.

[③] William Henry Young (London, October 20, 1863 - Lausanne, July 7, 1942) was an English mathematician who graduated from Cambridge University and worked at the University of Liverpool and the University of Lausanne.

$\mathcal{A}$ is ***dissipative*** *if*

$$\langle \mathcal{A}u, u\rangle \leq 0, \quad \forall u \in D(\mathcal{A}).$$

We say that $\mathcal{A}$ is ***maximal monotonic*** *if moreover* $\operatorname{Im}(\operatorname{Id} + \mathcal{A}) = \mathcal{H}$, *i.e*

$$\forall f \in \mathcal{H}\ ,\ \exists u \in D(\mathcal{A}) \quad \textit{such that} \quad u + \mathcal{A}u = f.$$

Proposition 1.1. *[5] Let $\mathcal{A}$ be a maximal monotone operator, then*

- *$D(\mathcal{A})$ is dense in $\mathcal{H}$.*
- *$\mathcal{A}$ is a closed operator .*
- *For every $\lambda > 0, (I + \lambda\mathcal{A})$ is bijective of $D(\mathcal{A})$ on $\mathcal{H}, (I + \lambda\mathcal{A})^{-1}$ is a bounded operator, and $\|(I + \lambda\mathcal{A})^{-1}\|_{\mathcal{L}(H)} \leq 1$.*

1.4 Strongly continuous semigroup of linear operators

Definition 1.8. *[34] Let X be a Banach space, let $\mathcal{L}(X)$ be the set of all bounded linear operators from X to X, a family $\{S(t), t \geq 0\}$ in $\mathcal{L}(X)$ is a semi-group of bounded linear operator on X if*

1) $S(0) = I$, *(I is the identity operator on x) .*
2) $S(t_1 + t_2) = S(t_1)S(t_2) \quad \forall\ t_1, t_2 \geq 0$ *(the semi-group property) .*
3) *For every* $x \in X,\ S(t)x$ *is continuous on* $[0, \infty)$.

Definition 1.9. *[34] The infinitesimal generator, or generator of the semigroup of linear operators* $\{S(t), t \geq 0\}$, *the operator* $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ *defined by*

$$D(\mathcal{A}) = \left\{ x \in X : \quad \lim_{t\to 0} \frac{S(t)x - x}{t} \quad exist \right\},$$

and defined by

$$\mathcal{A}x = \lim_{t\to 0} \frac{S(t)x - x}{t} \quad \textit{for} \quad x \in D(A).$$

In the infinitesimal generator of the semigroup $S(t)$, $D(\mathcal{A})$ *is the domain of an operator* $\mathcal{A}$.

1.5 Existence and uniqueness of the solution

To deal with the existence and uniqueness of the solution one uses the Lumer[1]-Phillips[2]theorem or the Hille [3]-Yosida[4]theorem.

Definition 1.10. *[19] Let V be a (real) Hilbert space of scalar product noted $(.,.)v$ and the associated norm $\|.\|_V$, we propose to solve the following problem*
find $u \in V$ such that for all $v \in V$ we have : $A(u,v) = L(v)$,
we impose the following conditions

1) L is an application defined on V, with values in $\mathbb{R}$ verifying moreover :

i) L is linear.

ii) L is continuous, i.e. there exists a constant $C > 0$ such that

$$\text{for all } v \in V, \quad |L(v)| \leq C\|v\|_V .$$

2) A is an application defined on $V \times V$, with values in $\mathbb{R}$ verifying moreover :

i) A is bilinear.

ii) A is continuous, i.e. there exists a constant $M > 0$ such that

$$\text{for all } (u,v) \in V^2 , \quad |A(u,v)| \leq M\|u\|_V\|v\|_V .$$

iii) A is coercive , i.e. there exists a constant $\alpha > 0$ such that

$$\text{for all } v \in V, \quad A(v,v) \geq \alpha\|v\|_V^2 .$$

[1]Günter Lumer (1929-2005) is a mathematician known for his work in functional analysis. Born in Germany and raised in France and Uruguay.

[2]Ralph Saul Phillips (June 23, 1913-November 23, 1998) was an American mathematician and scholar known for his contributions to functional analysis.

[3]Carl Einar Hille, born June 28, 1894 and died February 12, 1980, was an American mathematics professor and researcher. He signed or co-authored twelve mathematical books and several articles.

[4]Kosaku Yosida Yoshida (1909-1990) was a Japanese mathematician, specialist in functional analysis. He is known for the Hille-Yosida theorem concerning C_0semi-groups.

lemma 1.3. *[19](Lax[1]- Milgram[2])*

Let V be a real Hilbert space, A a bilinear form, continuous and coercive on V and L a continuous linear form on F.
Then, there exists a unique element u of V solution of the variational problem .
If the bilinear form is symmetric, (i.e. if $A(u,v) = A(v,u)$ for all u, v in V), by posing

$$\text{for all } v \in V \ , \quad E(v) = \frac{1}{2}A(v,v) - L(v).$$

Such that is equivalent to a minimization problem for the quadratic functional E.

Hille-Yosida theorem

Given the following evolution problem

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} + \mathcal{A}u = 0 \quad \text{on} \quad [0,+\infty[, \\ u(0) = u_0 \quad \text{(initial condition)} \ . \end{cases} \tag{1.1}$$

Theorem 1.2. *[4] Let $\mathcal{A}$ be a monotonic maximal operator in a Hilbert space $\mathcal{H}$, then for all $u_0 \in D(\mathcal{A})$ there exists a unique function*

$$u \in \mathbf{C}^1\left([0,+\infty[\ , \ \mathcal{H}\right) \ \cap \ \mathbf{C}\left([0,+\infty[\ , \ D(\mathcal{A})\right),$$

solution of (1.1).
Furthermore we have

$$|u(t)| \leq |u_0| \quad \text{and} \quad \left|\frac{\mathrm{d}u}{\mathrm{d}t}(t)\right| = |\mathcal{A}u(t)| \leq |\mathcal{A}u_0|\,, \quad \forall t \geq 0.$$

[1] Peter Lax, born in 1926 in Budapest, is a mathematician of Hungarian origin and American nationality. He was awarded the Abel Prize in 2005.
[2] Arthur Norton Milgram, born on June 3, 1912 in Philadelphia and died on January 30, 1961 (at age 48), was an American mathematician. He worked in functional analysis.

Lumer-Phillips theorem

Theorem 1.3. *[34] Let $\mathcal{A} : D(\mathcal{A}) \subseteq X \to X$ be a linear operator defined, and $D(\mathcal{A})$ dense domain in X, then $\mathcal{A}$ generates a C_0 semi-group of contractions on X if and only if*

(i) $\mathcal{A}$ is dissipative.

(ii) There exists $\lambda > 0$ such that $\lambda I - \mathcal{A}$ is surjective.

1.6 Stability

Let's first introduce the basic concepts of stability

Definition 1.11. *[33](Autonomous and non-autonomous systems).*
The non-linear system

$$x = f(x, t), \tag{1.2}$$

is said autonomous if f does not depend explicitly on time , that is if the equation of state of the system can be written $x = f(x)$, otherwise, the system is called non autonomous.

Equilibrium point[33]
A state x^* is an equilibrium state (or equilibrium point) of the system if once $x(t)$ is equal to x^*, it remains at x^* for all future time. Mathematically, this means that the constant vector x^* satisfies $f(x^*) = 0$, the equilibrium points can be found by solving the nonlinear algebraic equations.

Definition 1.12. *[14](Stability in the sense of Lyapunov)*
The equilibrium point $x = 0$ of (1.2) is :

- *Stable if , for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that*

$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| < \varepsilon \ , \quad \forall t \geq 0.$$

- *Unstable if it's not stable.*

- *Asymptotically stable if it is stable and δ can be chosen such that*

$$\|x(0)\| \leq \delta \ \Rightarrow \lim_{t \to \infty} \|x(t)\| = 0.$$

Definition 1.13. *[33] (Exponential stability)*
An equilibrium point 0 *is exponentially stable if there exist two strictly positive numbers* α *and* λ *such that*

$$\forall t > 0, \quad \|x(t)\| \leq \alpha \|x(0)\| \mathrm{e}^{-\lambda t}.$$

Note that exponential stability implies asymptotic stability . But asymptotic stability does not guarantee exponential stability .

Definition 1.14. *[33] A a continuous scalar function* $v(x)$ *is said locally positive definite if* $v(0) = 0$ *is in a ball* B_{R_0}

$$x \neq 0 \ \Rightarrow \ v(x) > 0 \ ,$$

if $v(0) = 0$ *and the above property applies to the whole state space , then* $v(x)$ *is said globally positive definite.*

Definition 1.15. *[14] Let* $x = 0$ *be an equilibrium point for (1.2) and* $D \subset \mathbb{R}^n$ *a domain containing* $x = 0$*. Let* $V : D \rightarrow \mathbb{R}$ *be a differentiable function such that*

$$V(0) = 0 \ \textit{and} \ V(x) > 0 \ \textit{in} \ D - \{0\}.$$

$$V(x) \leq 0 \ \textit{in} \ D.$$

Then, $x = 0$ *is stable, moreover if*

$$V'(x) < 0 \ \textit{in} \ D - \{0\},$$

then $x = 0$ *is asymptotically stable.*

CHAPTER 2

WELL-POSEDNESS AND EXPONENTIAL DECAY OF SOLUTIONS FOR A TRANSMISSION PROBLEM WITH DISTRIBUTED DELAY

2.1 Position of the problem

In this chapter, we study the transmission problem with a distributed delay,

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1 u_t(x,t) + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| u_t(t-s)ds = 0, \quad x \in \Omega,\ t > 0, \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad x \in (L_1, L_2),\ t \geq 0, \end{cases} \tag{2.1}$$

under the boundary and the transmission conditions

$$\begin{cases} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2, \\ au_x(L_i,t) = bv_x(L_i,t), \quad i = 1,2, \end{cases} \tag{2.2}$$

and the initial conditions

$$\begin{cases} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1,L_2), \ , \\ u_t(x,-t) = f_0(x,-t), \quad x \in \Omega, \quad t \in (0,\tau_2). \end{cases} \tag{2.3}$$

Where $0 < L_1 < L_2 < L_3$, $\Omega = (0,L_1) \cup (L_2,L_3)$, a, b, μ_1 are positive constants, and the initial data $(u_0, u_1, v_0, v_1, f_0)$ belongs to suitable space. Moreover, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real number satisfying $0 \leq \tau_1 < \tau_2$.

It is known that transmission problems happen frequently in applications where the domain is occupied by two or several materials , whose elastic properties are different, joined together over the whole of a surface. From the mathematical point of view, a transmission problem for wave propagation consists on a hyperbolic equation for which the corresponding elliptic operator has discontinuous coefficients, see [2, 10].

In absence of delay ($\mu_2(s) = 0$), the system (2.1)-(2.3) has been investigated in [2] by Bastaos and Raposo; for $\Omega = [0, L_1]$, they showed that the well-posedness and exponential stability of the total energy. Rivera and Oquendo [25] studied the transmission problem of viscoelastic waves and established that the dissipation produced by the viscoelastic part is strong enough to produce the exponential stability, no matter small its size is. Interested readers are referred to [20, 21, 22, 24] , for more results concerning other types of transmission problems.

Introducing the delay term makes the problem different from those considered in the literatures. Delay effect arises in many applications depending not only on the present state but also on some past occurrences. It may turn a well-behaved system into a wild one. The presence of delay may be a source of instability. For example, it was shown in [8, 6, 12, 28, 29, 36] that an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used. Here we mention the some interesting results on the relation between the delay term and source term [18, 17, 11, 31].

The effect of the delay term $u_t(x, t - \tau)$ in the transmission system has been investigated by Benseghir [3]. The well-posedness and the decay of solution for a transmission problem in a bounded domain with a viscoelastic term and a delay term $u_t(x, t - \tau)$ have been studied in [16, 35].

2.2 Well-posedness of the problem

Throughout this section c and c_i are used to denote the generic positive constant. From now on, we shall omit x and t in all functions of x and t if there is no ambiguity.

As in [29], we introduce the new variable

$$z(x, \rho, t, s) = u_t(x, t - \rho s) , \quad x \in \Omega \, , \; \rho \in (0, 1) \, , \; t > 0 \, , \; s \in (\tau_1, \tau_2).$$

They are following

$$z_t(x,\rho,t,s) = \frac{\partial u_t(x,t-\rho s)}{\partial(t-\rho s)} \times \frac{\partial(t-\rho s)}{\partial t} = \frac{\partial u_t(x,t-\rho s)}{\partial(t-\rho s)}$$
$$z_\rho(x,\rho,t,s) = \frac{\partial u_t(x,t-\rho s)}{\partial(t-\rho s)} \times \frac{\partial(t-\rho s)}{\partial \rho} = \frac{\partial u_t(x,t-\rho s)}{\partial(t-\rho s)} \times (-s).$$

So it is easy to check

$$sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad x \in \Omega,\ \rho \in (0,1),\ t > 0,\ s \in (\tau_1,\tau_2). \tag{2.4}$$

Consequently, system (2.1) is equivalent to

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1 u_t(x,t) + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)ds = 0, \\ x \in \Omega,\ t > 0, \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad x \in (L_1,L_2),\ t \geq 0, \\ sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad x \in \Omega,\ \rho \in (0,1),\ t > 0,\ s \in (\tau_1,\tau_2). \end{cases} \tag{2.5}$$

Defining $U = (u,v,\varphi,\psi,z)^T$, we formally get that U satisfies

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = U_0 = (u_0,v_0,u_1,v_1,f_0), \end{cases} \tag{2.6}$$

where the operator $\mathcal{A}$ is defined as

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ au_{xx} - \mu_1\varphi - \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)ds \\ bv_{xx} \\ -\frac{1}{s} z_\rho(x,\rho,t,s) \end{pmatrix}.$$

Introducing the space

$$\left\{ \begin{array}{l} X_* = (u,v) \in H^1(\Omega) \cap H^1(L_1,L_2) : u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t),\ au_x(L_i,t) = bv_x(L_i,t),\ i = 1,2, \end{array} \right\}.$$

We define the energy space as

$$\mathcal{H} = X_* \times L^2(\Omega) \times L^2(L_1, L_2) \times L^2\big(\Omega \times (0,1) \times (\tau_1, \tau_2)\big),$$

equipped with the inner product

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}} &= \int_\Omega (\varphi \tilde{\varphi} + a u_x \tilde{u}_x) dx + \int_{L_1}^{L_2} (\psi \tilde{\psi} + b v_x \tilde{v}_x) dx \\ &+ \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z(x,\rho,t,s) \tilde{z}(x,\rho,t,s)\, ds\, d\rho\, dx. \end{aligned}$$

The domain of $\mathcal{A}$ is

$$\begin{aligned} D(\mathcal{A}) = \big\{ &(u, v, \varphi, \psi, z)^T \in \mathcal{H} : (u, v) \in (H^2(\Omega) \times H^2(L_1, L_2)) \cap X_*, \\ &\varphi \in H^1(\Omega),\ \psi \in H^1(L_1, L_2), z(x, 0, s) = \varphi, \\ &z, z_\rho \in L^2\big(\Omega \times (0,1) \times (\tau_1, \tau_2)\big) \big\}. \end{aligned}$$

$D(\mathcal{A})$ is dense in $\mathcal{H}$.

Concerning the weight of the distributed delay, we assume that

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \leq \mu_1. \tag{2.7}$$

The well-posedness of the system (2.5) , (2.2) and (2.3) is ensured by the following theorem.

Theorem 2.1. *Under the assumption* (2.7)*, for any* $U_0 \in \mathcal{H}$*, there exists a unique weak solution* $U \in C(\mathbb{R}^+, \mathcal{H})$ *of problem* (2.6)*. Moreover , if* $U_0 \in D(\mathcal{A})$*, then* $U \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C(\mathbb{R}^+, \mathcal{H})$*.*

proof. We use the semigroup approach and the Hille-Yosida theorem to prove the well-posedness of the problem. First, we prove that the operator $\mathcal{A}$ is dissipative.

Indeed, for $U=(u,v,\varphi,\psi,z)\in D(\mathcal{A})$, where $\varphi(L_i)=\psi(L_i)\ ,\ i=1,2$, we have

$$\begin{aligned}\langle \mathcal{A}U,U\rangle_{\mathcal{H}} &= \int_\Omega \big(au_{xx}-\mu_1\varphi-\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)ds\big)\varphi\,dx + a\int_\Omega u_x\varphi_x dx \\ &\quad + \int_{L_1}^{L_2} bv_{xx}\psi\,dx + \int_{L_1}^{L_2} bv_x\psi_x\,dx + \int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 |\mu_2(s)|z(x,1,\rho,s)z_\rho(x,\rho,t,s)d\rho\,ds\,dx.\end{aligned} \tag{2.8}$$

For the last term of the right hand side of (2.8), we have

$$\begin{aligned}\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 |\mu_2(s)|z(x,\rho,t,s)z_\rho(x,\rho,t,s)d\rho\,ds\,dx &= \int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 \frac{1}{2}|\mu_2(s)|\frac{d}{d\rho}z^2(x,\rho,t,s)d\rho\,ds\,dx. \\ &= \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_0^1\frac{d}{d\rho}z^2(x,\rho,t,s)d\rho\,ds\,dx. \\ &= \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|\left[z^2(x,\rho,t,s)\right]_0^1 ds\,dx. \\ &= \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|\left[z^2(x,1,t,s)-z^2(x,0,t,s)\right] ds\,dx. \\ &= \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)ds\,dx \\ &\qquad -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,0,t,s)ds\,dx.\end{aligned} \tag{2.9}$$

Integrating by parts in (2.8), and noticing the fact $z(x,0,t,s)=\varphi(x,t)$, from (2.9), we have :

$$\begin{aligned}\langle \mathcal{A}U,U\rangle_{\mathcal{H}} &= a\int_\Omega u_{xx}\varphi\,dx + b\int_{L_1}^{L_2} v_{xx}\psi\,dx + a\int_\Omega u_x\varphi_x\,dx + b\int_{L_1}^{L_2} v_x\psi_x\,dx - \mu_1\int_\Omega\varphi^2\,dx \\ &\quad - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\varphi\,ds\,dx - \int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1|\mu_2(s)|z(x,\rho,t,s)z_\rho(x,\rho,t,s)d\rho\,ds\,dx. \\ &= a\left[u_x\varphi|_\Omega - \int_\Omega u_x\varphi_x\,dx\right] + b\left[v_x\psi|_{L_1}^{L_2} - \int_{L_1}^{L_2} v_x\psi_x\,dx\right] + a\int_\Omega u_x\varphi_x\,dx + b\int_{L_1}^{L_2} v_x\psi_x\,dx \\ &\quad - \mu_1\int_\Omega\varphi^2\,dx - \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\varphi\,ds\,dx - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx \\ &\quad + \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,0,t,s)\,ds\,dx.\end{aligned}$$

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= [au_x\varphi]_{\partial\Omega} + [bv_x\psi]_{L_1}^{L_2} - \mu_1 \int_\Omega \varphi^2\,dx - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)\varphi\,ds\,dx - \\
&\quad \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx + \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \varphi^2\,ds\,dx \\
&= [au_x\varphi]_{\partial\Omega} + [bv_x\psi]_{L_1}^{L_2} - \Big(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\Big) \int_\Omega \varphi^2 dx \\
&\quad - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)\varphi\,ds\,dx.
\end{aligned}$$

Using Young's inequality, and the equality $\varphi(L_i) = \psi(L_i),\ \ i = 1,2,$ from (2.2) and (2.9). From the last term of $\langle \mathcal{A}U, U\rangle_{\mathcal{H}}$, we have :

$$-\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)\varphi\,ds\,dx.$$

We pose :

$$a = z(x,1,t,s), \qquad b = \varphi, \qquad p = q = 2.$$

So, we get :

$$-\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)\varphi\,ds\,dx \le \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx + \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \varphi^2\,ds\,dx.$$

So

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\le -(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)|) \int_\Omega \varphi^2 dx - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx \\
&\quad + \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx + \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \varphi^2\,ds\,dx \\
&\le -\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\Big) \int_\Omega \varphi^2 dx \\
&\le 0.
\end{aligned}$$

by (2.7). Hence, the operator $\mathcal{A}$ is dissipative.

Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $\lambda I - \mathcal{A}$ is surjective for a fixed $\lambda > 0$. Indeed, given $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, we prove that there exists $U = (u, v, \varphi, \psi, z) \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})U = \mathcal{F}, \tag{2.10}$$

that is

$$\begin{cases} \lambda u - \varphi = f_1, \\ \lambda v - \psi = f_2, \\ \lambda\varphi - au_{xx} + \mu_1\varphi + \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, 1, t, s)ds = f_3, \\ \lambda\psi - bv_{xx} = f_4, \\ \lambda sz + z_\rho = sf_5. \end{cases} \tag{2.11}$$

Suppose we have obtained (u, v) with the suitable regularity, then

$$\begin{cases} \varphi = \lambda u - f_1, \\ \psi = \lambda v - f_2, \end{cases} \tag{2.12}$$

so we have $\varphi \in H^1(\Omega)$ and $\psi \in H^1(L_1, L_2)$. Moreover, using the approach as in Nicaise and Pignotti [28], we obtain that the last equation in (2.11) with $z(x, 0, s)$ has a unique solution.

$$\lambda sz + z_\rho = sf_5$$

$$z = z_0 e^{-\lambda s\rho} + se^{\lambda s\rho} \int_0^\rho e^{\lambda\sigma s} f_5(x, \sigma, s)d\sigma$$

$$z(x, \rho, s) = \varphi(x)e^{-\lambda\rho s} + se^{\lambda\rho s} \int_0^\rho e^{\lambda\sigma s} f_5(x, \sigma, s)d\sigma.$$

It follows from (2.12) that

$$z(x, \rho, s) = \lambda ue^{-\lambda\rho s} - f_1 e^{-\lambda\rho s} + se^{\lambda\rho s} \int_0^\rho e^{\lambda\sigma s} f_5(x, \sigma, s)d\sigma, \tag{2.13}$$

in particular, $z(x, 1, s) = \lambda ue^{-\lambda s} + z_0(x, s)$ with $z_0 \in L^2(\Omega \times (\tau_1, \tau_2))$ defined by

$$z_0(x, s) = -f_1 e^{-\lambda s} + se^{\lambda s} \int_0^1 e^{\lambda\sigma s} f_5(x, \sigma, s)d\sigma.$$

By (2.11) and (2.12), the functions (u, v) satisfy the equations

$$\begin{cases} \tilde{k}u - au_{xx} = \tilde{f}, \\ \lambda^2 v - bv_{xx} = f_4 + \lambda f_2, \end{cases} \tag{2.14}$$

where

$$\tilde{k} = \lambda^2 + \lambda\mu_1 + \int_{\tau_1}^{\tau_2} \lambda|\mu_2(s)|e^{-\lambda s}ds \quad > 0,$$
$$\tilde{f} = f_3 + (\lambda + \lambda\mu_1)f_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|z_0(x,s)ds \quad \in L^2(\Omega),$$

we want to show that the equalities of (2.14) are true .

$$\begin{aligned} \tilde{k}u - au_{xx} &= \lambda^2 u + \lambda\mu_1 u + \int_{\tau_1}^{\tau_2} \lambda u|\mu_2(s)|e^{-\lambda s}\,ds - au_{xx} \\ &= \lambda^2 u + \lambda\mu_1 u + \int_{\tau_1}^{\tau_2} \lambda u|\mu_2(s)|e^{-\lambda s}\,ds + f_3 - \lambda\varphi - \mu_1\varphi - \int_{\tau_1}^{\tau_2} |\mu_2(s)|z(x,1,t,s)\,ds \\ &= \lambda^2 u + \lambda\mu_1 u + \int_{\tau_1}^{\tau_2} \lambda u|\mu_2(s)|e^{-\lambda s}\,ds + f_3 - \lambda(\lambda u - f_1) - \mu_1(\lambda u - f_1) \\ &\quad - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\lambda u e^{-\lambda s} + z_0(x,s)\,ds \\ &= \lambda^2 u + \lambda\mu_1 u + \int_{\tau_1}^{\tau_2} \lambda u|\mu_2(s)|e^{-\lambda s}\,ds + f_3 - \lambda^2 u + \lambda f - \lambda\mu_1 u + \mu_1 f_1 \\ &\quad - \int_{\tau_1}^{\tau_2} \lambda u|\mu_2(s)|e^{-\lambda s}\,ds - \int_{\tau_1}^{\tau_2} |\mu_2(s)|z_0(x,s)\,ds \\ &= f_3 + (\lambda + \mu_1)\,f_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|z_0(x,s)\,ds, \\ f_4 + \lambda f_2 &= \lambda\psi - bv_{xx} + \lambda(\lambda v - \psi) \\ &= \lambda\psi - bv_{xx} + \lambda^2 v - \lambda\psi \\ &= \lambda^2 v - bv_{xx}, \end{aligned}$$

which can be reformulated as

$$\begin{cases} \displaystyle\int_\Omega (\tilde{k}u - au_{xx})w_1\,dx = \int_\Omega \tilde{f}w_1\,dx, \\ \displaystyle\int_{L_1}^{L_2} (\lambda^2 v - bv_{xx})w_2\,dx = \int_{L_1}^{L_2} (f_2 + \lambda f_4)w_2\,dx, \end{cases} \tag{2.15}$$

for any $(w_1, w_2) \in X_*$.

Integrating by parts in (2.15), we obtain that the variational formulation corresponding to (2.14) takes the form

$$\Phi\big((u,v),(w_1,w_2)\big) = l(w_1,w_2), \tag{2.16}$$

where the bilinear form $\Phi : (X_*, X_*) \to \mathbb{R}$ and the linear form $l : X_* \to \mathbb{R}$ are defined by

$$\begin{aligned}\Phi\big((u,v),(w_1,w_2)\big) = &\int_\Omega \tilde{k} u w_1 dx + \int_\Omega a u_x w_{1x} - [a u_x w_1]_\Omega + \int_{L_1}^{L_2} \lambda^2 v w_2 dx \\ &+ \int_{L_1}^{L_2} v_x w_{2x} dx - [b v_x w_2]_{L_1}^{L_2},\end{aligned}$$

and

$$l(w_1,w_2) = \int_\Omega \tilde{f} w_1 dx + \int_{L_1}^{L_2} (f_2 + \lambda f_4) w_2 dx.$$

We can apply the Lax-Milgram theorem, on the space X_* for the bilinear form $\Phi\big((u,v),(w_1,w_2)\big)$ and the linear form $l(w_1,w_2)$

1. **<u>Φ is continuous :</u>**

$$\begin{aligned}|\Phi\left((u,v),(w_1,w_2)\right)| = \Big|&\tilde{k}\int_\Omega u w_1\, dx + \lambda^2 \int_{L_1}^{L_2} v w_2\, dx + a\int_\Omega u_x w_{1x}\, dx + \int_{L_1}^{L_2} v_x w_{2x}\, dx \\ &- [a u_x w_1]_{\partial\Omega} - [b v_x w_2]_{L_1}^{L_2}\Big|,\end{aligned}$$

we use the Cauchy-Shwartz inequality, we get

$$\begin{aligned}|\Phi\left((u,v),(w_1,w_2)\right)| &\le \tilde{k}\|u\|_2\|w_1\|_2 + \lambda^2\|v\|_2\|w_2\|_2 + a\|u_x\|_2\|w_{1x}\|_2 + \|v_x\|_2\|w_{2x}\|_2 \\ &\le max\left\{\tilde{k},\lambda^2,a\right\}\left(\|u\|_2 + \|v\|_2 + \|u_x\|_2 + \|v_x\|_2\right) \\ &\quad\times\left(\|w_1\|_2 + \|w_2\|_2 + \|w_{1x}\|_2 + \|w_{2x}\|_2\right) \\ &\le C\left(\|(u,v)\|_{X_*}\|(w_1,w_2)\|_{X_*}\right).\end{aligned}$$

2. **Φ is coercive :**

$$\begin{aligned}|\Phi\left((u,v),(u,v)\right)| &= \tilde{k}\int_{\Omega} u^2\,dx + a\int_{\Omega} u_x^2\,dx - [au_x^2]_{\partial\Omega} + \lambda^2\int_{L_1}^{L_2} v^2\,dx + \int_{L_1}^{L_2} v_x^2\,dx - [bv_x^2]_{L_1}^{L_2}\\ &= \tilde{k}\|u\|_2^2 + a\|u_x\|_2^2 + \lambda^2\|v\|_2^2 + \|v_x\|_2^2 - [au_x^2]_{\partial\Omega} - [bv_x^2]_{L_1}^{L_2}\\ &= \tilde{k}\|u\|_2^2 + a\|u_x\|_2^2 + \lambda^2\|v\|_2^2 + \|v_x\|_2^2\\ &\geq min\left\{\tilde{k}, a, \lambda^2\right\}\left(\|u\|_2^2 + \|u_x\|_2^2\right)\left(\|v\|_2^2 + \|v_x\|_2^2\right)\\ &\geq C\|(u,v)\|^2_{H_1(\Omega)\times H_1(L_1\times L_2)}.\end{aligned}$$

3. **l is continous :**

$$\begin{aligned}|l(w_1,w_2)| &= \Big|\int_{\Omega}\tilde{f}w_1dx + \int_{L_1}^{L_2}(f_4+\lambda f_2)w_2dx\Big|\\ &= \Big|\int_{\Omega}\left(f_3 + (\lambda+\mu)f_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|z_0(x,s)\,ds\right)w_1\,dx + \int_{L_1}^{L_2}\left(f_4+\lambda f_2\right)w_2\,dx\Big|,\end{aligned}$$

we use the Cauchy-Shwartz inequality, we get

$$\begin{aligned}|l(w_1,w_2)| &\leq \|f_3\|_2\|w_1\|_2 + (\lambda+\mu_1)\|f_1\|_2\|w_1\|_2 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|z_0(x,s)\,ds\|w_1\|_2\\ &\quad + \lambda\|f_2\|_2\|w_2\|_2 + \|f_4\|_2\|w_2\|_2\\ &\leq max\left\{\|f_3\|_2, (\lambda+\mu_1)\|f_1\|_2, -\int_{\tau_1}^{\tau_2}|\mu_2(s)|z_0(x,s)\,ds, \lambda\|f_2\|_2, \|f_4\|_2\right\}\\ &\quad \left(\|w_1\|_2 + \|w_2\|_2\right)\\ &\leq C\left(\|w_1\|_2 + \|w_2\|_2\right)\\ &\leq C\|(w_1,w_2)\|_{X_*}.\end{aligned}$$

Consequently ,

Φ is continuous and coercive, and l is continuous on the space X_* .

Applying the Lax-Milgram theorem, we deduce that problem (2.16) admits a unique solution $(u,v)\in X_*$ for all $(w_1,w_2)\in X_*$.

It follows from (2.14) that $(u,v)\in\left((H^2(\Omega)\times H^2(L_1,L_2))\right)\cap X_*$. Thus, the operator $\lambda I-\mathcal{A}$ is surjective for any $\lambda>0$. Hence the Hille-Yosida theorem guarantees the existence of a unique solution to the problem (2.10). This completes the proof. □

2.3 Exponential stability

In this section, we state and prove the stability result for the energy of the system (2.1)-(2.3). For the regular solution of the system (2.1)-(2.3), we define the energy as (see [3])

$$E_1(t) = \frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{a}{2}\int_\Omega u_x^2(x,t)dx, \tag{2.17}$$

$$E_2(t) = \frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)dx + \frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)dx. \tag{2.18}$$

And the total energy is defined as

$$E(t) = E_1(t) + E_2(t) + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\, ds\, d\rho\, dx. \tag{2.19}$$

For the energy decay result, we assume a restriction on the weight of the distribute delay and the damping as

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds < \mu_1. \tag{2.20}$$

The stability result reads as follows.

Theorem 2.2. *Let (u,v,z) be the solution of the system* (2.5), (2.2) *and* (2.3). *Assume* (2.20) *and*

$$\frac{a}{b} < \frac{L_1 + L_3 - L_2}{2(L_2 - L_1)}, \quad L_3 > 3(L_2 - L_1). \tag{2.21}$$

Then there exist two positive constants K and κ, such that

$$E(t) \le Ke^{-\kappa t}, \quad \forall t \ge 0. \tag{2.22}$$

The proof will be established through the following Lemmas.

lemma 2.1. *Let assumption* (2.20) *holds. Then the energy functional defined by* (2.19), *satisfies the estimate*

$$E'(t) \le -\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds\Big)\int_\Omega u_t^2(x,t)dx \le 0. \tag{2.23}$$

proof. By differentiating (2.17), using the first equation in (2.5), and integrating by parts, we obtain

$$\begin{aligned}
E_1'(t) &= \frac{1}{2}\int_\Omega \frac{d}{dt}\left(u_t^2(x,t)\right)\,dx + \frac{a}{2}\int_\Omega \frac{d}{dt}\left(u_x^2(x,t)\right)\,dx\\
&= \int_\Omega u_{tt}(x,t)u_t(x,t)\,dx + a\int_\Omega u_{xt}(x,t)u_x(x,t)\,dx\\
&= \int_\Omega \left(au_{xx}(x,t) - \mu_1 u_t(x,t) - \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)\,ds\right)\,dx + a\int_\Omega u_{xt}u_x(x,t)\,dx\\
&= \int_\Omega au_{xx}(x,t)u_t(x,t)\,dx - \int_\Omega \mu_1 u_t^2(x,t)\,dx - \int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)u_t(x,t)\,ds\,dx\\
&\quad + a\int_\Omega u_{xt}u_x(x,t)\,dx\\
&= [au_xu_t]_{\partial\Omega} - a\int_\Omega u_x(x,t)u_{xt}(x,t)\,dx + a\int_\Omega u_{xt}(x,t)u_x(x,t)\,dx - \mu_1\int_\Omega u_t^2(x,t)\,dx\\
&\quad - \int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)u_t(x,t)\,ds\,dx\\
&= [au_xu_t]_{\partial\Omega} - \mu_1\int_\Omega u_t^2(x,t)dx - \int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)u_t(x,t)\,ds\,dx.
\end{aligned}$$

Similarly,

$$\begin{aligned}
E_2'(t) &= \frac{1}{2}\int_{L_1}^{L_2} \frac{d}{dt}\left(v_t^2(x,t)\right)\,dx + \frac{b}{2}\int_{L_1}^{L_2} \frac{d}{dt}\left(v_x^2(x,t)\right)\\
&= \int_{L_1}^{L_2} v_{tt}(x,t)v_t(x,t)\,dx + b\int_{L_1}^{L_2} v_{xt}(x,t)v_x(x,t)\,dx\\
&= \int_{L_1}^{L_2} bv_{xx}(x,t)v_t(x,t)\,dx + b\int_{L_1}^{L_2} v_{xt}(x,t)v_x(x,t)\,dx\\
&= [bv_xv_t]_{L_1}^{L_2} - b\int_{L_1}^{L_2} v_{xt}(x,t)v_x(x,t)\,dx + b\int_{L_1}^{L_2} v_{xt}(x,t)v_x(x,t)\,dx\\
&= [bv_xv_t]_{L_1}^{L_2}.
\end{aligned}$$

Noticing that $z(x,0,t,s)=u_t(x,t)$, from (2.5), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx &= \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\int_0^1 \frac{d}{dt}z^2(x,\rho,t,s)\,d\rho\,ds\,dx\\
&= \int_\Omega\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\int_0^1 z_t(x,\rho,t,s)z(x,\rho,t,s)\,d\rho\,ds\,dx\\
&= -\int_\Omega\int_{\tau_1}^{\tau_2} s|\mu_2(s)|\int_0^1 \frac{1}{s}z_\rho(x,\rho,t,s)z(x,\rho,t,s)\,d\rho\,ds\,dx\\
&= -\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)|\int_0^1 z_\rho(x,\rho,t,s)z(x,\rho,t,s)\,d\rho\,ds\,dx\\
&= -\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)|[\frac{1}{2}z^2(x,\rho,t,s)]_0^1\,ds\,dx\\
&= -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,t,s)\,ds\,dx\\
&\quad+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,0,t,s)\,ds\,dx\\
&= -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,t,s)\,ds\,dx\\
&\quad+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)|u_t^2(x,t)\,ds\,dx.
\end{aligned}
$$

Meanwhile, using Young's inequality, we have

$$
\begin{aligned}
&-\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)|z(x,1,t,s)u_t(x,t)\,ds\,dx\\
&\le \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_\Omega u_t^2(x,t)dx+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx.
\end{aligned}
$$

Combining the above equalities and using (2.20), we show that (2.23) holds, where we also use the fact $[au_xu_t]_{\partial\Omega}=[bv_xv_t]_{L_1}^{L_2}$ from (2.2).
So,

$$
E'(t)=E_1'(t)+E_2'(t)+\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,d\rho\,ds\,dx
$$

$$
\begin{aligned}
E'(t)\le{}& [au_xu_t]_{\partial\Omega}-\mu_1\int_\Omega u_t^2(x,t)\,dx+\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_\Omega u_t^2(x,t)\,dx+[bv_xv_t]_{L_1}^{L_2}\\
&+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx\\
&+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx.
\end{aligned}
$$

$$
\begin{aligned}
E'(t) &\leq [a u_x u_t]_{\partial\Omega} + [b v_x v_t]_{L_1}^{L_2} - \mu_1 \int_\Omega u_t^2(x,t)\,dx + \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_\Omega u_t^2(x,t)\,dx\,ds \\
&\leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds\right) \int_\Omega u_t^2(x,t)\,dx \\
&\leq 0.
\end{aligned}
$$

□

As in [23], we define the functional

$$
I(t) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-\rho s} |\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx,
$$

then we have the following estimate.

lemma 2.2. *The functional $I(t)$ satisfies the estimate*

$$
\begin{aligned}
I'(t) \leq -e^{-\tau_2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\right) \int_\Omega u_t^2(x,t) dx \\
- e^{-\tau_2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s |\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx.
\end{aligned}
\tag{2.24}
$$

proof. By differentiating $I(t)$ and using the third equation in (2.5), we obtain

$$
\begin{aligned}
I'(t) &= \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-\rho s} |\mu_2(s)| \frac{d}{dt} z^2(x,\rho,t,s)\,ds\,d\rho\,dx \\
&= \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-\rho s} |\mu_2(s)| \left[2 z_t(x,\rho,t,s) z(x,\rho,t,s)\right]\,d\rho\,ds\,dx \\
&= -\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\rho s} |\mu_2(s)| \left[z_\rho(x,\rho,t,s) z(x,\rho,t,s)\right]\,d\rho\,ds\,dx \\
&= -\int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 e^{-\rho s} |\mu_2(s)| \frac{d}{d\rho} z^2(x,\rho,t,s) d\rho\,ds\,dx.
\end{aligned}
$$

We have

$$
\begin{aligned}
&\frac{d}{d\rho}\left(e^{-\rho s} z^2(x,\rho,t,s)\right) = -s e^{-\rho s} z^2(x,\rho,t,s) + \left(\frac{d}{d\rho} z^2(x,\rho,t,s)\right) e^{-\rho s} \\
\Longrightarrow &\left(\frac{d}{d\rho} z^2(x,\rho,t,s)\right) e^{-\rho s} = \frac{d}{d\rho}\left(e^{-\rho s} z^2(x,\rho,t,s)\right) + s e^{-\rho s} z^2(x,\rho,t,s).
\end{aligned}
$$

So,

$$
\begin{aligned}
I'(t) &= -\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \left(\frac{d}{d\rho} \left(e^{-\rho s} z^2(x,\rho,t,s) \right) + s e^{-\rho s} z^2(x,\rho,t,s) \right) d\rho\, ds\, dx \\
&= -\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \frac{d}{d\rho} \left(e^{-\rho s} z^2(x,\rho,t,s) \right) d\rho\, ds\, dx \\
&\quad - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| e^{-\rho s} z^2(x,\rho,t,s)\, ds\, d\rho\, dx \\
&= -\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left[e^{-\rho s} z^2(x,\rho,t,s) \right]_0^1 ds\, dx - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| e^{-\rho s} z^2(x,\rho,t,s)\, ds\, d\rho\, dx \\
&= -\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} z^2(x,1,t,s)\, ds\, dx + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,0,t,s)\, ds\, dx \\
&\quad - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| e^{-\rho s} z^2(x,\rho,t,s)\, ds\, d\rho\, dx \\
&= -\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| e^{-s} z^2(x,1,t,s)\, ds\, dx + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| u_t^2(x,t)\, ds\, dx \\
&\quad - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| e^{-\rho s} z^2(x,\rho,t,s)\, ds\, d\rho\, dx,
\end{aligned}
$$

we have

$$
e^{-s} \leqslant e^{-\rho s} \leqslant 1 \quad \Rightarrow \quad -e^{-s} \geqslant -e^{-\rho s} \qquad \text{for all } \rho \in [0,1],
$$

and

$$
s \leqslant \tau_2 \quad \Rightarrow \quad -s \geqslant -\tau_2 \quad \Rightarrow \quad e^{-s} \geqslant e^{-\tau_2} \quad \Rightarrow \quad -e^{-s} \leqslant -e^{-\tau_2} \quad \text{for all } s \in [\tau_1, \tau_2].
$$

So , we get

$$
-e^{-\rho s} \leqslant -e^{-s} \leqslant -e^{-\tau_2}.
$$

Hence

$$
\begin{aligned}
I'(t) \leqslant -\int_\Omega \int_{\tau_1}^{\tau_2} e^{-\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| u_t^2(x,t)\, ds\, dx \\
- \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-\tau_2} |\mu_2(s)| z^2(x,\rho,t,s)\, ds\, d\rho\, dx.
\end{aligned}
$$

□

Now we define the functional

$$\mathcal{D}(t) = \int_{\Omega} u u_t dx + \frac{\mu_1}{2}\int_{\Omega} u^2 dx + \int_{L_1}^{L_2} v v_t dx.$$

Then we have the following estimate.

lemma 2.3. *The functional $\mathcal{D}(t)$ satisfies*

$$\begin{aligned}\mathcal{D}'(t) \leq -(a-\varepsilon_0 C_0^2)\int_{\Omega} u_x^2 dx - b\int_{L_1}^{L_2} v_x^2 dx + \int_{\Omega} u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx \\ + \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx.\end{aligned} \tag{2.25}$$

proof. Taking the derivative of $\mathcal{D}(t)$ with respect to t, using (2.5), we obtain

$$\begin{aligned}\mathcal{D}'(t) &= \int_{\Omega} \frac{d}{dt} u u_t dx + \frac{\mu_1}{2}\int_{\Omega} \frac{d}{dt} u^2 dx + \int_{L_1}^{L_2} \frac{d}{dt} v v_t dx \\ &= \int_{\Omega} u_t^2\, dx + \int_{\Omega} u_{tt} u\, dx + \frac{\mu_1}{2}\int_{\Omega} 2u_t u\, dx + \int_{L_1}^{L_2} v_t^2\, dx + \int_{L_1}^{L_2} v_{tt} v\, dx \\ &= \int_{\Omega} u_t^2\, dx + \int_{\Omega}\left(a u_{xx} - \mu_1 u_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)\, ds\right) u\, dx + \mu_1 \int_{\Omega} u_t u\, dx + \int_{L_1}^{L_2} v_t^2\, dx \\ &\quad + \int_{L_1}^{L_2} b v_{xx} v\, dx \\ &= \int_{\Omega} u_t^2\, dx + a\int_{\Omega} u_{xx} u\, dx - \mu_1 \int_{\Omega} u_t u\, dx - \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s) u(x,t)\, ds\, dx + \mu_1 \int_{\Omega} u_t u\, dx\end{aligned}$$

$$\begin{aligned}\mathcal{D}'(t) &= + \int_{L_1}^{L_2} v_t^2\, dx + b\int_{L_1}^{L_2} v_{xx} v\, dx \\ &= [a u_x u]_{\partial\Omega} + [b v_x v]_{L_1}^{L_2} - a\int_{\Omega} u_x^2 dx - b\int_{L_1}^{L_2} v_x^2 dx + \int_{\Omega} u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx \\ &\quad - \int_{\Omega}\int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s) u(x,t)\, ds\, dx.\end{aligned} \tag{2.26}$$

It follows from the boundary condition (2.2) that

$$\begin{aligned}[a u_x u]_{\partial\Omega} + [b v_x v]_{L_1}^{L_2} &= a u_x(L_1,t) u(L_1,t) - a u_x(L_2,t) u(L_2,t) + b v_x(L_1,t) v(L_1,t) - b v_x(L_2,t) v(L_2,t) \\ &= 0.\end{aligned}$$

Using the boundary condition (2.2), we obtain

$$\begin{aligned}\int_0^x u_x(x,t)\,dx = \int_0^x \frac{d}{dt}u(x,t)\,dx = [u(x,t)]_0^x = u(x,t) - u(0,t)\\ \Longrightarrow \left(\int_0^x u_x(x,t)\,dx\right)^2 = u^2(x,t)\end{aligned}$$

$$\begin{aligned}u^2(x,t) = \left(\int_0^x u_x(x,t)dx\right)^2 \leq \int_0^{L_1} u_x^2(x,t)\,dx \leq L_1 \int_0^{L_1} u_x^2(x,t)dx, \quad x \in [0, L_1],\\ u^2(x,t) \leq (L_3 - L_2) \int_{L_2}^{L_3} u_x^2(x,t)dx, \ \ x \in [L_2, L_3],\end{aligned}$$

which imply the following Poincaré's inequality

$$\int_\Omega u^2(x,t)dx \leq C_0^2 \int_\Omega u_x^2(x,t)dx, \quad x \in \Omega, \tag{2.27}$$

$$\left(\int_\Omega u(x,t)dx\right)^2 \leq C_0 \left(\int_\Omega u_x^2(x,t)dx\right)^2,$$

where $C_0 = \max\{L_1, L_3 - L_2\}$ is the Poincaré's constant.
Using Young's inequality and (2.27), we have

$$\begin{aligned}-\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s) u(x,t)\,ds\,dx &\leq \int_\Omega \left[\varepsilon_0 u^2(x,t) + \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2} \Big(|\mu_2(s)| z(x,1,t,s)\,ds\Big)^2\right] dx\\ &\leq \int_\Omega \left[\varepsilon_0 u^2(x,t) + \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2} |\mu_2(s)|^2 z^2(x,1,t,s)\,ds\right] dx\\ &\leq \int_\Omega \left[\varepsilon_0 u^2(x,t) + \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2} |\mu_2(s)||\mu_2(s)| z^2(x,1,t,s)\,ds\right] dx\\ &\leq \int_\Omega \Bigg[\varepsilon_0 u^2(x,t) + \frac{1}{4\varepsilon_0}\\ &\qquad \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\Bigg] dx\\ &\leq \int_\Omega \varepsilon_0 C_0^2 u_x^2(x,t) + \frac{1}{4\varepsilon_0}\\ &\qquad \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx,\end{aligned}$$

for any $\varepsilon_0 > 0$. Inserting the above estimates in (2.26), we obtain

$$\begin{aligned}
\mathcal{D}'(t) \leq & \int_\Omega u_t^2(x,t)\,dx + \int_{L_1}^{L_2} v_t^2(x,t)\,dx - a\int_\Omega u_x^2(x,t)\,dx - b\int_{L_1}^{L_2} v_x^2(x,t)\,dx + \varepsilon_0 C_0^2 \int_\Omega u_x^2(x,t)\,dx \\
& + \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx \\
\leq & -(a - \varepsilon_0 C_0^2)\int_\Omega u_x^2(x,t)\,dx - b\int_{L_1}^{L_2} v_x^2(x,t)\,dx + \int_\Omega u_t^2(x,t)\,dx + \int_{L_1}^{L_2} v_t^2(x,t)\,dx \\
& \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx.
\end{aligned}$$

□

Inspired by [21], we introduce the functional

$$q(x) = \begin{cases} x - \frac{L_1}{2}, & x \in [0, L_1], \\ x - \frac{L_2+L_3}{2}, & x \in [L_2, L_3], \\ \frac{L_1}{2} + \frac{L_2-L_3-L_1}{2(L_2-L_1)}(x - L_1), & x \in [L_1, L_2]. \end{cases} \tag{2.28}$$

It is easy to see that $q(x)$ is bounded, that is $|q(x)| \leq M$, where $M = max\left\{\frac{L_1}{2}, \frac{L_3-L_2}{2}\right\}$ is a positive constant .

We define the two functionals

$$\mathcal{F}_1(t) = -\int_\Omega q(x) u_x u_t\,dx, \quad \mathcal{F}_2(t) = -\int_{L_1}^{L_2} q(x) v_x v_t\,dx.$$

Then, we have the following estimates.

lemma 2.4. *For any $\varepsilon_1 > 0$, the functionals $\mathcal{F}(t)$ and $\mathcal{F}_2(t)$ satisfy*

$$\begin{aligned}\mathcal{F}_1'(t) \leq C(\varepsilon_1)\int_\Omega u_t^2 dx + (\frac{a}{2} + \varepsilon_1)\int_\Omega u_x^2 dx \\ + C(\varepsilon_1)\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, t, s)\, ds\, dx \\ - \frac{a}{4}[(L_3 - L_2)u_x^2(L_2, t) + L_1 u_x^2(L_1, t)] - \frac{1}{4}[L_1 u_t^2(L_1, t) + (L_3 - L_2)u_t^2(L_2, t)],\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}\mathcal{F}_2'(t) = -\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\Big(\int_{L_1}^{L_2} v_t^2 dx + \int_{L_1}^{L_2} b v_x^2 dx\Big) + \frac{L_1}{4} v_t^2(L_1, t) \\ + \frac{L_3 - L_2}{4} v_t^2(L_2, t) + \frac{b}{4}[(L_3 - L_2)v_x^2(L_2, t) + L_1 v_x^2(L_1, t)].\end{aligned} \tag{2.30}$$

proof. Taking the derivative of $\mathcal{F}_1(t)$ with respect to t and using (2.5), we have

$$\begin{aligned}\mathcal{F}_1'(t) &= -\int_\Omega q(x)\frac{d}{dt}(u_x u_t)\, dx \\ &= -\int_\Omega q(x) u_x u_{tt} - \int_\Omega q(x) u_{xt} u_t\, dx \\ &= -\int_\Omega q(x) u_x\Big(a u_{xx} - \mu_1 u_t - \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, 1, t, s) ds\Big)\, dx - \int_\Omega q(x) u_{xt} u_t\, dx. \\ &= -a\int_\Omega q(x) u_x u_{xx}\, dx + \mu_1 \int_\Omega q(x) u_x u_t\, dx + \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, 1, t, s) q(x) u_x\, ds\, dx \\ &\quad - \int_\Omega q(x) u_{xt} u_t\, dx.\end{aligned} \tag{2.31}$$

Integrating by parts, we have

$$\begin{aligned}\int_\Omega q(x) u_{xt} u_t dx &= -\frac{1}{2}\int_\Omega q'(x) u_t^2 dx + \frac{1}{2}[q(x) u_t^2]_{\partial\Omega}, \\ \int_\Omega q(x) a u_x u_{xx} dx &= -\frac{1}{2}\int_\Omega a q'(x) u_x^2 dx + \frac{1}{2}[a q(x) u_x^2]_{\partial\Omega}.\end{aligned}$$

Inserting the above two equalities into (2.31), and noticing (2.28) and Young's inequality, we obtain

$$
\begin{aligned}
\mathcal{F}_1'(t) &= -\frac{1}{2}\left[aq(x)u_x^2\right]_{\partial\Omega} + \frac{1}{2}\int_\Omega aq'(x)u_x^2\,dx + \mu_1\int_\Omega q(x)u_xu_t\,dx \\
&\quad + \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)q(x)u_x\,ds\,dx - \frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} + \frac{1}{2}\int_\Omega q'(x)u_t^2\,dx \\
&= -\frac{1}{2}\left[aq(x)u_x^2\right]_{\partial\Omega} - \frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} + \frac{1}{2}\int_\Omega aq'(x)u_x^2\,dx + \frac{1}{2}\int_\Omega q'(x)u_t^2\,dx \\
&\quad + \int_\Omega q(x)u_x\left(\mu_1u_t + \int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\,ds\right)\,dx,
\end{aligned} \tag{2.32}
$$

we use Young's inequality on the last term

$$
\int_\Omega q(x)u_x\left(\mu_1u_t + \int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\,ds\right)\,dx.
$$

We obtain

$$
\begin{aligned}
\int_\Omega u_xq(x)\left(\mu_1u_t + \int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\,ds\right)\,dx &\le \int_\Omega\Bigg(\varepsilon_1u_x^2 + \frac{1}{4\varepsilon_1} \\
&\qquad q^2(x)\Big(\mu_1u_t + \int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\,ds\Big)^2\Bigg)\,dx \\
&\le \varepsilon_1\int_\Omega u_x^2\,dx + \frac{1}{4\varepsilon_1} \\
&\qquad \int_\Omega q^2(x)\Big(\mu_1^2u_t^2 + \int_{\tau_1}^{\tau_2}|\mu_2(s)|^2z^2(x,1,t,s)\,ds\Big)\,dx \\
&\le \varepsilon_1\int_\Omega u_x^2\,dx + \frac{1}{4\varepsilon_1}\int_\Omega q^2(x)\mu_1^2u_t^2\,dx + \frac{1}{4\varepsilon_1} \\
&\qquad \int_\Omega q^2(x)\int_{\tau_1}^{\tau_2}|\mu_2(s)||\mu_2(s)|z^2(x,1,t,s)\,ds\,dx
\end{aligned}
$$

$$
\begin{aligned}
\int_\Omega u_xq(x)\left(\mu_1u_t + \int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\,ds\right)\,dx &\le \varepsilon_1\int_\Omega u_x^2\,dx + \frac{M^2\mu_1^2}{4\varepsilon_1}\int_\Omega u_t^2\,dx + \frac{M^2}{4\varepsilon_1} \\
&\qquad \int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx.
\end{aligned}
$$

So,

$$\begin{aligned}
\mathcal{F}_1'(t) \leq & -\frac{1}{2}\left[aq(x)u_x^2\right]_{\partial\Omega} - \frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} + \frac{a}{2}\int_\Omega u_x^2\,dx + \frac{1}{2}\int_\Omega u_t^2\,dx + \varepsilon_1\int_\Omega u_x^2\,dx \\
& + \frac{M^2\mu_1^2}{4\varepsilon_1}\int_\Omega u_t^2\,dx + \frac{M^2}{4\varepsilon_1}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx \\
\leq & \left(\frac{1}{2} + \frac{M^2\mu_1^2}{4\varepsilon_1}\right)\int_\Omega u_t^2\,dx\left(\frac{a}{2}+\varepsilon_1\right)\int_\Omega u_x^2 - \frac{1}{2}\left[aq(x)u_x^2\right]_{\partial\Omega} - \frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} \\
& + \frac{M^2}{4\varepsilon_1}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx \\
\leq & \, C_1(\varepsilon_1)\int_\Omega u_t^2\,dx + \left(\frac{a}{2}+\varepsilon_1\right)\int_\Omega u_x^2 - \frac{1}{2}\left[aq(x)u_x^2\right]_{\partial\Omega} - \frac{1}{2}\left[q(x)u_t^2\right]_{\partial\Omega} \\
& C_2(\varepsilon_1)\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx.
\end{aligned}$$

Such that

$$C_1(\varepsilon_1) = max\left\{\frac{1}{2}\,,\frac{M^2\mu_1^2}{4\varepsilon_1}\right\}, \qquad C_2(\varepsilon_1) = \frac{M^2}{4\varepsilon_1}\,, \qquad \forall \varepsilon_1 > 0.$$

On the other hand, by the boundary conditions (2.2), we have

$$\begin{aligned}
\frac{1}{2}\left[aq(x)u_x^2\right]_{\partial\Omega} &= \frac{1}{2}\left[aq(x)u_x^2\right]_0^{L_1} + \frac{1}{2}\left[aq(x)u_x^2\right]_{L_2}^{L_3} \\
&= \frac{1}{2}\left[a\left(L_1 - \frac{L_1}{2}\right)u_x^2(L_1,t) - a\left(-\frac{L_1}{2}\right)u_x^2(0,t)\right] + a \\
&\quad \left[\left(L_3 - \frac{L_2+L_3}{2}\right)u_x^2(L_3,t) - \left(L_2 - \frac{L_2+L_3}{2}\right)u_x^2(L_2,t)\right] \\
&= \frac{1}{2}\left[a\frac{L_1}{2}u_x^2(L_1,t) + a\frac{L_3-L_2}{2}u_x^2(L_2,t)\right] \\
&= \frac{a}{4}\left[L_1u_x^2(L_1,t) + (L_3-L_2)u_x^2(L_2,t)\right] \\
&\geq 0.
\end{aligned}$$

$$
\begin{aligned}
\frac{1}{2}[q(x)u_t^2]_{\partial\Omega} &= \frac{1}{2}[q(x)u_t^2]_0^{L_1} + \frac{1}{2}[q(x)u_t^2]_{L_2}^{L_3} \\
&= \frac{1}{2}\left[\left(L_1 - \frac{L_1}{2}\right)u_t^2(L_1,t) + \frac{L_1}{2}u_t^2(0,t)\right] + \frac{1}{2} \\
&\quad \left[\left(L_3 - \frac{L_2+L_3}{2}\right)u_t^2(L_3,t) - \left(L_2 - \frac{L_2+L_3}{2}\right)u_t^2(L_2,t)\right] \\
&= \frac{1}{2}\left[\frac{L_1}{2}u_t^2(L_1,t)\right] + \frac{1}{2}\left[\frac{L_2-L_3}{2}u_t^2(L_2,t)\right] \\
&= \frac{1}{4}\left[L_1u_t^2(L_1,t) + (L_3-L_2)u_t^2(L_2,t)\right] \\
&\geq 0.
\end{aligned}
$$

Inserting the above two equalities into (2.32), then (2.32) gives

$$
\begin{aligned}
\mathcal{F}_1'(t) &\leq C(\varepsilon_1)\int_\Omega u_t^2\,dx + \left(\frac{a}{2}+\varepsilon_1\right)\int_\Omega u_x^2\,dx \\
&+ C(\varepsilon_1)\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx \\
&- \frac{a}{4}\left[L_1u_x^2(L_1,t) + (L_3-L_2)u_x^2(L_2,t)\right] - \frac{1}{4}\left[L_1u_t^2(L_1,t) + (L_3-L_2)u_t^2(L_2,t)\right].
\end{aligned}
$$

By the same method, taking the derivative of $\mathcal{F}_2(t)$ with respect to t, we have

$$
\begin{aligned}
\mathcal{F}_2'(t) &= -\int_{L_1}^{L_2} q(x)\frac{d}{dt}(v_xv_t)\,dx \\
&= -\int_{L_1}^{L_2} q(x)v_{xt}v_t\,dx - b\int_{L_1}^{L_2} q(x)v_xv_{xx}\,dx \\
&= -\left[\frac{1}{2}q(x)v_t^2\right]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2\,dx - \frac{b}{2}\left[q(x)v_x^2\right]_{L_1}^{L_2} + \frac{b}{2}\int_{L_1}^{L_2} q'(x)v_x^2\,dx \\
&= -\frac{1}{2}\left[q(x)v_t^2\right]_{L_1}^{L_2} - \frac{1}{2}\left[bq(x)v_x^2\right]_{L_1}^{L_2} + \frac{1}{2}q'(x)\left[\int_{L_1}^{L_2} v_t^2\,dx + \int_{L_1}^{L_2} bv_x^2\,dx\right] \\
&= -\frac{1}{2}\left[q(x)v_t^2(L_2,t) - q(x)v_t^2(L_1,t)\right] - \frac{1}{2}\left[bq(x)v_x^2(L_2,t) - bq(x)v_x^2(L_1,t)\right] + \frac{1}{2} \\
&\quad q'(x)\left[\int_{L_1}^{L_2} v_t^2\,dx + \int_{L_1}^{L_2} bv_x^2\,dx\right] \\
&= -\frac{1}{2}\left[\frac{L_1}{2} + \frac{L_2-L_3-L_1}{2(L_2-L_1)}(L_2-L_1)v_t^2(L_2,t) - \frac{L_1}{2}v_t^2(L_1,t)\right] \\
&\quad - \frac{1}{2}\left[b\left(\frac{L_1}{2} + \frac{L_2-L_3-L_1}{2(L_2-L_1)}(L_2-L_1)\right)v_x^2(L_2,t) - b\frac{L_1}{2}v_x^2(L_1,t)\right] \\
&\quad + \frac{L_2-L_3-L_1}{4(L_2-L_1)}\left[\int_{L_1}^{L_2} v_t^2\,dx + \int_{L_1}^{L_2} bv_x^2\,dx\right]
\end{aligned}
$$

$$\mathcal{F}_2'(t) = -\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)} \left[\int_{L_1}^{L_2} v_t^2 \, dx + b \int_{L_1}^{L_2} v_x^2 \, dx \right] + \frac{L_3 - L_2}{4} v_t^2(L_2, t) + \frac{L_1}{4} v_t^2(L_1, t) + \frac{b}{4} \left[(L_3 - L_2) v_x^2(L_2, t) + L_1 v_x^2(L_1, t) \right].$$

Hence, the proof is complete. □

proof. [Proof of Theorem 2.2] We define the Lyapunov functional

$$L(t) = N_1 E(t) + N_2 I(t) + \gamma_1 \mathcal{F}_1(t) + \gamma_2 \mathcal{F}_2(t) + \gamma_3 \mathcal{D}(t), \tag{2.33}$$

where $N_1, N_2, \gamma_1, \gamma_2, \gamma_3$ are positive constants that will be chosen later. It follows from the boundary conditions (2.2) that

$$a^2 u_x^2(L_i, t) = b^2 v_x^2(L_i, t), \quad i = 1, 2. \tag{2.34}$$

Taking the derivative of (2.33) with respective to t, using the above lemmas and (2.34), we have

$$
\begin{aligned}
L'(t) \leq{} & N_1\Big[-\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\,ds\Big)\int_\Omega u_t^2(x,t)\,dx\Big] + N_2 \\
& \Big[-e^{-\tau_2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx + \Big(\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\Big)\int_\Omega u_t^2(x,t)\,dx \\
& - e^{-\tau_2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx\Big] \\
& + \gamma_1\Big[C(\varepsilon_1)\int_\Omega u_t^2\,dx + \Big(\frac{a}{2}+\varepsilon_1\Big)\int_\Omega u_x^2\,dx + C(\varepsilon_1) \\
& \int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx - \frac{a}{4}\Big[L_1u_x^2(L_1,t) + (L_3-L_2)u_x^2(L_2,t)\Big] \\
& - \frac{1}{4}\Big[L_1u_t^2(L_1,t) + (L_3-L_2)u_t^2(L_2,t)\Big]\Big] + \gamma_2\Big[-\frac{L_1+L_3-L_2}{4(L_2-L_1)}\Big[\int_{L_1}^{L_2} v_t^2\,dx + b\int_{L_1}^{L_2} v_x^2\,dx\Big] \\
& + \frac{L_3-L_2}{4}v_t^2(L_2,t) + \frac{L_1}{4}v_t^2(L_1,t) + \frac{b}{4}\Big[(L_3-L_2)v_x^2(L_2,t) + L_1v_x^2(L_1,t)\Big]\Big] \\
& + \gamma_3\Big[-(a-\varepsilon_0C_0^2)\int_\Omega u_x^2(x,t)\,dx - b\int_{L_1}^{L_2} v_x^2(x,t)\,dx + \int_\Omega u_t^2(x,t)\,dx + \int_{L_1}^{L_2} v_t^2(x,t)\,dx \\
& \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx\Big] \\
\leq{} & -\Big[N_1(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds) - N_2\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds - \gamma_1C(\varepsilon_1) - \gamma_3\Big]\int_\Omega u_t^2dx \\
& -\Big[N_2e^{-\tau_2} - \gamma_1C(\varepsilon_1)\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds - \gamma_3\frac{\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds}{4\varepsilon_0}\Big] \\
& \times\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx \\
& - \big[(a-\varepsilon_0C_0^2)\gamma_3 - (\frac{a}{2}+\varepsilon_1)\gamma_1\big]\int_\Omega u_x^2dx \\
& - [\frac{L_1+L_3-L_2}{4(L_2-L_1)}\gamma_2 + \gamma_3]\int_{L_1}^{L_2} bv_x^2dx \\
& - [\frac{L_1+L_3-L_2}{4(L_2-L_1)}\gamma_2 - \gamma_3]\int_{L_1}^{L_2} v_t^2dx \\
& - N_2e^{-\tau_2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx \\
& - \big[\gamma_1-\gamma_2\big]\big(\frac{L_1}{4}u_t^2(L_1,t) + \frac{L_3-L_2}{4}u_t^2(L_2,t)\big) \\
& - \big[\gamma_1 - \frac{a}{b}\gamma_2\big]\big(\frac{a}{4}[L_1u_x^2(L_1,t) + (L_3-L_2)u_x^2(L_2,t)]\big).
\end{aligned}
\tag{2.35}
$$

At this point we will choose all the constants, carefully, such that all the coefficients in (2.35) will be negative. In fact, it follows from the assumption (2.21) that we can always choose γ_1, γ_2 and γ_3 such that

$$\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\gamma_2 - \gamma_3 > 0, \qquad \gamma_1 > \frac{a}{b}\gamma_2, \qquad \gamma_1 > \gamma_2.$$

Once the above constants $\gamma_1, \gamma_2, \gamma_3$ are fixed, we may choose ε_0 and ε_1 sufficiently small such that

$$\begin{aligned}(a - \varepsilon_0 C_0^2)\gamma_3 - (\frac{a}{2} + \varepsilon_1)\gamma_1 > 0,\\ \gamma_3 a - \gamma_3\varepsilon_0 C_0^2 - \gamma_1\frac{a}{2} - \gamma_1\varepsilon_1 > 0,\\ a(\gamma_3 - \frac{\gamma_1}{2}) - \gamma_3\varepsilon_0 C_0^2 - \gamma_1\varepsilon_1 > 0,\\ a(\gamma_3 - \frac{\gamma_1}{2}) > \gamma_3\varepsilon_0 C_0^2 + \gamma_1\varepsilon_1.\end{aligned}$$

Then we can take N_2 sufficiently large such that

$$N_2 e^{-\tau_2} - \gamma_1 C(\varepsilon_1)\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds - \gamma_3\frac{\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds}{4\varepsilon_0} > 0.$$

Finally, noticing the assumption (2.20), we can always choose N_1 sufficiently large such that the first coefficient in (2.35) is negative. Thus, we obtain that there exists a positive constant α such that, $\alpha = max\Big\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\Big\}$, (2.35) yields

$$\begin{aligned}L'(t) \le& -\Big[\alpha_1\int_\Omega u_t^2\,dx + \alpha_2\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx + \alpha_3\int_\Omega u_x^2\,dx + \alpha_4\int_{L_1}^{L_2} bv_x^2\,dx\\ &+ \alpha_5\int_{L_1}^{L_2} v_t^2\,dx + \alpha_6\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx + \alpha_7 + \alpha_8\Big]\\ \le& -\alpha\Big[\int_\Omega u_t^2\,dx + \int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx + \int_\Omega u_x^2\,dx + \int_{L_1}^{L_2} bv_x^2\,dx + \int_{L_1}^{L_2} v_t^2\,dx\\ &+ \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx\Big]\end{aligned}$$

$$L'(t) \le -\alpha\Big[\int_\Omega u_t^2\,dx + \int_{L_1}^{L_2} v_t^2\,dx + \int_\Omega a u_x^2\,dx + \int_{L_1}^{L_2} b v_x^2\,dx$$
$$+ \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx\Big],$$

recalling (2.19), which implies

$$L'(t) \le -\frac{\alpha}{2}E(t), \quad \forall t \ge 0. \tag{2.36}$$

On the hand, it is not hard to see that $L(t) \sim E(t)$, i.e. there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \le L(t) \le \beta_2 E(t), \quad t \ge 0. \tag{2.37}$$

Combining (2.36) and (2.37), we obtain that

$$L'(t) \le -\kappa L(t), \quad t \ge 0,$$

for the positive constant $\kappa = \alpha/2\beta_2$. Integration over $(0,t)$ gives

$$\begin{gathered}
L'(t) \le -\kappa L(t)\\
\frac{L'(t)}{L(t)} \le -\kappa\\
\int_0^t \frac{L'(x)}{L(x)}\,dx \le \int_0^t -\kappa\,dx\\
ln\ L(x)|_0^t \ \le\ -\kappa x|_0^t\\
ln\ L(t) - ln\ L(0) \le -\kappa t\\
ln\ L(t) \le ln\ L(0) - \kappa t\\
L(t) \le e^{(ln\ L(0) - \kappa t)}\\
L(t) \le L(0)e^{-\kappa t}.
\end{gathered}$$

Recall (2.37) again, then (2.22) holds. Hence, the proof is complete. □

WELL-POSEDNESS AND EXPONENTIAL DECAY OF SOLUTIONS FOR A TRANSMISSION PROBLEM IN INFINITE MEMORY-TYPE THERMOELASTICITY WITH DISTRIBUTED DELAY

3.1 Position of the problem

In this chapter, we study the transmission problem in infinite memory-type thermoelasticity with a distributed delay term

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) + \displaystyle\int_0^\infty g(p)u_{xx}(x,t-p)\,dp + \mu_1 u_t(x,t) \\ + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(s)|u_t(t-s)ds = 0, \quad x \in \Omega,\ t > 0, \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad x \in (L_1, L_2),\ t \geq 0, \end{cases} \tag{3.1}$$

under the boundary and the transmission conditions

$$\begin{cases} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2, \\ au_x(L_i,t) - \displaystyle\int_0^\infty g(p)u_x(L_i,t-p)\,dp = bv_x(L_i,t), \quad i = 1,2. \end{cases} \tag{3.2}$$

And the initial conditions

$$\begin{cases} u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), \quad (x,t) \in \Omega \times (0,+\infty) \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1, L_2), \\ u_t(x,-t) = f_0(x,-t), \quad x \in \Omega, \quad t \in (0,\tau_2). \end{cases} \tag{3.3}$$

Where $0 < L_1 < L_2 < L_3$, $\Omega = (0, L_1) \cup (L_2, L_3)$, $u(x,t)$ is the displacement in Ω, $v(x,t)$ is the displacement (L_1, L_2), a, b, μ_1 are positive constants. Moreover , $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function , where τ_1 and τ_2 are two real number satisfying $0 \leq \tau_1 < \tau_2$. We present now some materials that shall be used to prove our main results. For the relaxation function g, we assume

(G_1) $\quad g : \mathbb{R}_+ \to \mathbb{R}_+$ is a $\mathbb{C}^1$ function satisfying :

$$g(0) > 0 \ , \ a - \int_0^{\infty} g(p)\, dp = a - g_0 = l > 0.$$

(G_2) $\quad$ There exists a nonincreasing differentiable function $\xi(t) : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leq -\xi(t) g(t) \ , \quad \forall t \geq 0 \quad \text{and} \quad \int_0^{\infty} \xi(t)\, dt = +\infty.$$

Concerning the weight of the distributed delay, we assume that

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \leq \mu_1.$$

3.2 Well-posedness of the problem

Throughout this section, c and c_i are used to denote the generic positive constant.
As in [29], we introduce the new variable

$$z(x, \rho, t, s) = u_t(x, t - \rho s), \quad x \in \Omega, \ \rho \in (0,1), \ t > 0, \ s \in (\tau_1, \tau_2).$$

They are following

$$z_t(x,\rho,t,s) = \frac{\partial u_t(x, t-\rho s)}{\partial (t - \rho s)} \times \frac{\partial (t-\rho s)}{\partial t} = \frac{\partial u_t(x, t-\rho s)}{\partial (t - \rho s)}$$
$$z_\rho(x,\rho,t,s) = \frac{\partial u_t(x, t-\rho s)}{\partial (t - \rho s)} \times \frac{\partial (t-\rho s)}{\partial \rho} = \frac{\partial u_t(x, t-\rho s)}{\partial (t - \rho s)} \times (-s).$$

So the variable z satisfies

$$s z_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad x \in \Omega, \ \rho \in (0,1), \ t > 0, \ s \in (\tau_1, \tau_2). \tag{3.4}$$

Following the ideal of [7], we set

$$\eta^t(x,p) = u(x,t) - u(x,t-p)\ , \quad (x,t,p) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Direct calculations show that

$$\eta_t^t(x,p) = u_t(x,t) - u(x,t-p) \quad \text{and} \quad \eta_p^t(x,p) = u(x,t-p).$$

Then

$$\eta_t^t(x,p) + \eta_p^t(x,p) = u_t(x,t), \quad (x,t,p) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Thus , system (3.1) becomes

$$\begin{cases} u_{tt}(x,t) - au_{xx}(x,t) - \displaystyle\int_0^\infty g(p)\eta_{xx}^t(x,p)\,dp + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s) ds = 0, \\ \qquad (x,t) \in \Omega \times (0,+\infty) \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad x \in (L_1, L_2),\ t \geq 0, \\ sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad x \in \Omega,\ \rho \in (0,1),\ t > 0,\ s \in (\tau_1,\tau_2), \\ \eta_t^t(x,p) + \eta_p^t(x,p) = u_t(x,t) \quad (x,p,t) \in \Omega \times (0,+\infty) \times (0,+\infty). \end{cases} \tag{3.5}$$

With the following boundary

$$\begin{cases} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2, \\ au_x(L_i,t) - \displaystyle\int_0^\infty g(p) u_x(L_i, t-p)\,dp = bv_x(L_i,t), \quad i = 1,2, \\ z(x,\rho,t,0) = f_0(x,\rho t), \quad x \in \Omega\ ,\ \rho \in (0,1)\ ,\ t \in (0,+\infty), \\ \eta^t(x,0) = 0, \quad \eta^0(x,p) = \eta_0(x,p), \quad x,p \in \Omega \times (0,+\infty). \end{cases}$$

And the initial conditions

$$\begin{cases} u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), \quad (x,t) \in \Omega \times (0,+\infty) \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1,L_2), \\ u_t(x,-t) = f_0(x,-t), \quad x \in \Omega, \quad t \in (0,\tau_2). \end{cases}$$

Meanwhile, from (3.1)and (3.2), it's follows that

$$\frac{d^2}{dt^2} \int_{L_1}^{L_2} v(x,t)\,dx = 0. \tag{3.6}$$

Therefore, by solving (3.6) and using the initial condition of v, we get :

$$\int_{L_1}^{L_2} v(x,t)\ dx = t\int_{L_1}^{L_2} v_1(x)\ dx + \int_{L_1}^{L_2} v_0(x)\ dx.$$

Consequently, if we let

$$\bar{v}(x,t) = v(x,t) - t\int_{L_1}^{L_2} v_1(x)\ dx - \int_{L_1}^{L_2} v_0(x)\ dx,$$

we get

$$\int_{L_1}^{L_2} \bar{v}(x,t)\ dx = 0 \quad ,\ \forall t \leq 0.$$

Defining $U = (u, v, \varphi, \psi, z, w)^T$, such that $w = \eta^t$, we formally get that U satisfies

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = U_0, \end{cases} \tag{3.7}$$

where the operator $\mathcal{A}$ is defined as

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \varphi \\ \psi \\ z \\ w \end{pmatrix} = \begin{pmatrix} \varphi \\ \psi \\ au_{xx} + \int_0^\infty g(p)w_{xx}(x,p)\ dp - \mu_1\varphi - \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)ds \\ bv_{xx} \\ -\frac{1}{s} z_\rho(x,\rho,t,s) \\ \varphi - w_p \end{pmatrix}.$$

Introducing the space

$$\left\{ \begin{array}{l} X_* = (u,v) \in H^1(\Omega) \cap H^1(L_1, L_2) : u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t),\ au_x(L_i,t) + \int_0^\infty g(p)\eta_x^t(L_i,p)\ dp = bv_x(L_i,t),\ i = 1,2 \end{array} \right\},$$

and

$$\left\{ \ Y_* = w \in L_g^2(\mathbb{R}_+, H^1(\Omega)) : w(0,s) = w(L_3,t) = 0 \ \right\}.$$

We define the energy space as

$$\left\{ \begin{array}{c} \mathcal{H} = (H^1(\Omega) \times H^1(L_1, L_2)) \cap X_* \times L^2(\Omega) \times L^2(L_1, L_2) \times L^2\big(\Omega \times (0,1) \times (\tau_1, \tau_2)\big) \times \\ L^2_g(\mathbb{R}_+, H^1(\Omega)) \cap Y_* \end{array} \right\},$$

where, $L^2_g(\mathbb{R}_+, H^1(\Omega))$ denotes the hilbert space of H^1, valued functions on $\mathbb{R}_+$ endowed with the inner product

$$(\Phi, v)_{L^2_g(\mathbb{R}_+, H^1(\Omega)} = \int_\Omega \int_0^\infty g(p)\Phi_x(p)v_x(p)\; dp\; dx.$$

The space $\mathcal{H}$ equipped with the inner product

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}} = a \int_\Omega u_{xx}\tilde{\varphi}\; dx - \mu_1 \int_\Omega \varphi\tilde{\varphi}\; dx - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, 1, t, s)\tilde{\varphi}\; ds\; dx \\ + a \int_\Omega u_x \tilde{u}_x\; dx + \int_{L_1}^{L_2} \left(\psi\tilde{\psi} + bv_x\tilde{v}_x \right)\; dx \\ + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z(x, \rho, t, s)\tilde{z}(x, \rho, t, s)\, ds\, d\rho\; dx + \langle w, \tilde{w} \rangle_{L^2_g(\mathbb{R}_+, H^1(\Omega))}. \end{aligned}$$

The domain of $\mathcal{A}$ is

$$\begin{aligned} D(\mathcal{A}) = \big\{ &(u, v, \varphi, \psi, z, w)^T \in \mathcal{H} : (u, v) \in (H^2(\Omega) \times H^2(L_1, L_2)) \cap X_*, \\ &\varphi \in H^1(\Omega),\; \psi \in H^1(L_1, L_2), z(x, 0, s) = \varphi, z, z_\rho \in L^2\big(\Omega \times (0,1) \times (\tau_1, \tau_2)\big), \\ &x \in L^2_g(\mathbb{R}_+, H^2(\Omega) \cap H^1(\Omega)), w_s \in L^2_g(\mathbb{R}_+, H^1(\Omega), w(x, 0) = 0 \big\}. \end{aligned}$$

$D(\mathcal{A})$ is dense in $\mathcal{H}$.
Concerning the weight of the distributed delay, we assume that

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \leq \mu_1. \tag{3.8}$$

The well-posedness of the system (3.5) , (3.2) and (3.3) is ensured by the following theorem.

Theorem 3.1. *Under the assumption* (3.8) , (G_1) *and* (G_2) *hold . For any* $U_0 \in \mathcal{H}$, *there exists a unique weak solution* $U \in C(\mathbb{R}^+, \mathcal{H})$ *of problem* (3.7). *Moreover , if* $U_0 \in D(\mathcal{A})$, *then* $U \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C(\mathbb{R}^+, \mathcal{H})$.

proof. We use the semigroup approach and the Hille-Yosida theorem to prove the well-posedness of the problem . First , we prove that the operator $\mathcal{A}$ is dissipative.
Indeed, for $U = (u, v, \varphi, \psi, z, w) \in D(\mathcal{A})$, where $\varphi(L_i) = \psi(L_i),\ i = 1, 2$, we have

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = &\int_\Omega \Big(au_{xx} + \int_\Omega \int_0^\infty g(p) w_{xx}(x,p)\varphi\, dp\, dx - \mu_1 \varphi - \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s) ds\Big)\varphi\, dx \\ &+ a\int_\Omega u_x \varphi_x dx + \int_{L_1}^{L_2} b v_{xx}\psi\, dx + \int_{L_1}^{L_2} b v_x \psi_x\, dx \\ &+ \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(s)| z(x,1,\rho,s) z_\rho(x,\rho,t,s) d\rho\, ds\, dx.\end{aligned} \tag{3.9}$$

For the last term of the right hand side of (3.9), we have

$$\begin{aligned}\int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(s)| z(x,\rho,t,s) z_\rho(x,\rho,t,s) d\rho\, ds\, dx &= \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \frac{d}{d\rho} z^2(x,\rho,t,s) d\rho\, ds\, dx \\ &= \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s) ds\, dx \\ &\quad - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,0,t,s) ds\, dx.\end{aligned} \tag{3.10}$$

Integrating by parts in (3.9), and noticing the fact $z(x,0,t,s) = \varphi(x,t)$, from (3.10), we have :

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = &[au_x\varphi]_{\partial\Omega} + [bv_x\psi]_{L_1}^{L_2} - \Big(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\Big)\int_\Omega \varphi^2 dx \\ &- \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s)\varphi\, ds\, dx. \\ &+ \frac{1}{2}\int_\Omega \int_0^\infty g'(p) w_x^2(p)\, dp\, dx.\end{aligned}$$

Using Young's inequality, and the equality $\varphi(L_i) = \psi(L_i),\ \ i = 1, 2$, from (3.2) and (3.10). we have :

$$\begin{aligned}\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\leq -(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|) \int_\Omega \varphi^2 dx - \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx \\ &\quad + \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx + \frac{1}{2}\int_\Omega \int_0^\infty g'(p) w_x^2(p)\ dp\ dx \\ &\leq -\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\Big) \int_\Omega \varphi^2 dx + \frac{1}{2}\int_\Omega \int_0^\infty g'(p) w_x^2(p)\ dp\ dx \\ &\leq 0.\end{aligned}$$

by (3.8). Hence, the operator $\mathcal{A}$ is dissipative.

Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $\lambda I - \mathcal{A}$ is surjective for a fixed $\lambda > 0$. Indeed, given $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6) \in \mathcal{H}$, we prove that there exists $U = (u, v, \varphi, \psi, z) \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})U = \mathcal{F}, \tag{3.11}$$

that is

$$\begin{cases} \lambda u - \varphi = f_1, \\ \lambda v - \psi = f_2, \\ \lambda\varphi - a u_{xx} - \int_0^\infty g(p) w_{xx}(x,p)\ dp + \mu_1\varphi + \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s) ds = f_3, \\ \lambda\psi - b v_{xx} = f_4, \\ \lambda s z + z_\rho = s f_5, \\ \lambda w - \varphi + w_p = f_6. \end{cases} \tag{3.12}$$

Suppose we have obtained (u, v) with the suitable regularity, then

$$\begin{cases} \varphi = \lambda u - f_1, \\ \psi = \lambda v - f_2, \end{cases} \tag{3.13}$$

so we have $\varphi \in H^1(\Omega)$ and $\psi \in H^1(L_1, L_2)$. Moreover, using the approach as in Nicaise and Pignotti [28], we obtain that the last equation in (3.12) with $z(x,0,s)$ has a unique solution.

$$z(x,\rho,s) = \varphi(x)e^{-\lambda\rho s} + se^{\lambda\rho s}\int_0^\rho e^{\lambda\sigma s} f_5(x,\sigma,s)d\sigma.$$

It follows from (3.13) that

$$z(x,\rho,s) = \lambda u e^{-\lambda\rho s} - f_1 e^{-\lambda\rho s} + se^{\lambda\rho s}\int_0^\rho e^{\lambda\sigma s} f_5(x,\sigma,s)d\sigma, \tag{3.14}$$

in particular, $z(x,1,s) = \lambda u e^{-\lambda s} + z_0(x,s)$ with $z_0 \in L^2(\Omega \times (\tau_1,\tau_2))$ defined by

$$z_0(x,s) = -f_1 e^{-\lambda s} + se^{\lambda s}\int_0^1 e^{\lambda\sigma s} f_5(x,\sigma,s)d\sigma.$$

By (3.12) and (3.13), the functions (u,v) satisfy the equations

$$\begin{cases} \tilde{k}u - au_{xx} = \tilde{f}, \\ \lambda^2 v - bv_{xx} = f_4 + \lambda f_2, \end{cases} \tag{3.15}$$

where

$$\tilde{k} = \lambda^2 + \lambda\mu_1 + \int_{\tau_1}^{\tau_2} \lambda|\mu_2(s)|e^{-\lambda s}ds \quad > 0,$$

$$\tilde{f} = f_3 + (\lambda + \lambda\mu_1)f_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|z_0(x,s)ds + \int_0^\infty g(p)\left[e^{\lambda p}\int_0^p e^{\tau}(\varphi + f_6(\tau))\,d\tau\right] \quad \in L^2(\Omega),$$

which can be reformulated as

$$\begin{cases} \displaystyle\int_\Omega (\tilde{k}u - au_{xx})w_1\,dx = \int_\Omega \tilde{f}w_1\,dx, \\ \displaystyle\int_{L_1}^{L_2} (\lambda^2 v - bv_{xx})w_2\,dx = \int_{L_1}^{L_2} (f_2 + \lambda f_4)w_2\,dx, \end{cases} \tag{3.16}$$

for any $(w_1,w_2) \in X_*$. Integrating by parts in (3.16), we obtain that the variational formulation corresponding to (3.15) takes the form

$$\Phi\big((u,v),(w_1,w_2)\big) = l(w_1,w_2), \tag{3.17}$$

where the bilinear form $\Phi : (X_*, X_*) \to \mathbb{R}$ and the linear form $l : X_* \to \mathbb{R}$ are defined by

$$\Phi\big((u,v),(w_1,w_2)\big) = \int_\Omega \tilde{k} u w_1 dx + \int_\Omega a u_x w_{1x} - [a u_x w_1]_\Omega + \int_{L_1}^{L_2} \lambda^2 v w_2 dx \\ + \int_{L_1}^{L_2} v_x w_{2x} dx - [b v_x w_2]_{L_1}^{L_2},$$

and

$$l(w_1,w_2) = \int_\Omega \tilde{f} w_1 dx + \int_{L_1}^{L_2} (f_2 + \lambda f_4) w_2 dx.$$

By the properties of the space X_*, it is easy to see that Φ is continuous and coercive, and l is continuous. Applying the Lax-Milgram theorem, we deduce that problem l (3.17) admits a unique solution $(u,v) \in X_*$ for all $(w_1,w_2) \in X_*$. It follows from (3.15) that $(u,v) \in \big((H^2(\Omega) \times H^2(L_1,L_2))\big) \cap X_*$. Thus, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. Hence the Hille-Yosida theorem guarantees the existence of a unique solution to the problem (3.11). This completes the proof. $\square$

3.3 Exponential stability

In this section, we state and prove the stability result for the energy of the system (3.1)-(3.3) .

Theorem 3.2. *Let* (u, v, z) *be the solution of the system* (3.5), (3.2) *and* (3.3).

$$\frac{a}{b} < \frac{L_1 + L_3 - L_2}{2(L_2 - L_1)}, \quad L_3 > 3(L_2 - L_1). \tag{3.18}$$

Then there exist two positive constants K *and* κ, *such that*

$$E(t) \leq K e^{-\kappa t}, \quad \forall t \geq 0. \tag{3.19}$$

The proof will be established through the following Lemmas.

lemma 3.1. *The energy functional*, E, *defined by*

$$E(t) = \frac{1}{2}\int_\Omega \left(u_t^2(x,t) + a u_x^2(x,t)\right) \, dx + \frac{1}{2}\int_{L_1}^{L_2} \left(v_t^2(x,t) + b v_x^2(x,t)\right) \, dx + \frac{1}{2}(g \circ u_x)(t)$$
$$+ \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s) \, ds \, d\rho \, dx. \tag{3.20}$$

Such that

$$(g \circ u)(t) = \int_\Omega \int_0^\infty g(p) \left(u(x,t) - u(x,t-p)\right)^2 \, dp \, dx.$$

Satisfies

$$E'(t) \leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\right) \int_\Omega u_t^2(x,t) dx + \frac{1}{2}(g' \circ u_x)(t) \leq 0. \tag{3.21}$$

proof. Multiplying the first equation of (3.15) by u_t, the second equation of (3.15) by v_t and integrating the result over Ω and (L_1, L_2)

$$\int_\Omega u_{tt}u_t\,dx - a\int_\Omega u_{xx}u_t\,dx - \int_\Omega\int_0^\infty g(p)w_{xx}u_t\,dp\,dx + \mu_1\int_\Omega u_t^2\,dx + \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)u_t\,ds\,dx = 0.$$

$$\int_{L_1}^{L_2} v_{tt}v_t\,dx - b\int_{L_1}^{L_2} v_t v_{xx}\,dx = 0.$$

Summing them up, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(u_t^2 + au_x^2\right)\,dx + \frac{1}{2}\frac{d}{dt}\int_{L_1}^{L_2}\left(v_t^2 + bv_x^2\right)\,dx + \mu_1\int_\Omega u_t^2\,dx + \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)u_t\,ds\,dx - \int_\Omega\int_0^\infty g(p)w_{xx}u_t\,dp = 0. \tag{3.22}$$

The last term is estimated as follows :

$$\begin{aligned}-\int_\Omega\int_0^\infty g(p)w_{xx}u_t\,dp\,dx &= -\int_\Omega u_t\int_0^\infty g(p)w_{xx}\,dp\,dx\\ &= \int_\Omega (w_t + w_p)\int_0^\infty g(p)w_{xx}\,dp\,dx\\ &= -\int_0^\infty g(p)\int_\Omega w_t w_{xx}\,dx\,dp - \int_0^\infty g(p)\int_\Omega w_p w_{xx}\,dx\,dp,\end{aligned}$$

and integrating by parts, we have

$$-\int_\Omega\int_0^\infty g(p)w_{xx}u_t\,dp\,dx = \frac{d}{2dt}(g\circ u_x)(t) - \frac{1}{2}(g'\circ u_x)(t). \tag{3.23}$$

Meanwhile, using young's inequality, we have

$$\begin{aligned}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)u_t(x,t)\,ds\,dx \le{}& \frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_\Omega u_t^2(x,t)\,dx\\ &+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx.\end{aligned} \tag{3.24}$$

Now, multiplying the third equation in (3.5) by $z|\mu_2(s)|$ and integrating the result over $\Omega \times (0,1) \times (\tau_1, \tau_2)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx &= -\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|z(x,\rho,t,s)z_\rho(x,\rho,t,s)\,ds\,d\rho\,dx\\ &= -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|\frac{d}{d\rho}z^2(x,\rho,t,s)\,ds\,dx\\ &= -\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx\\ &\quad+\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|u_t^2(x,t)\,ds\,dx.\end{aligned} \tag{3.25}$$

Now, using (3.22) , (3.23), (3.24) and (3.25), we have

$$E'(t) \le -\Big(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\Big)\int_\Omega u_t^2(x,t)dx + \frac{1}{2}(g'\circ u_x)(t) \le 0.$$

□

As in [23], we define the functional

$$I(t) = \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} se^{-\rho s}|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx,$$

then we have the following estimate.

lemma 3.2. *The functional $I(t)$ satisfies the estimate*

$$\begin{aligned}I'(t) \le -e^{-\tau_2}\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx + \Big(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\Big)\int_\Omega u_t^2(x,t)dx\\ - e^{-\tau_2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx.\end{aligned} \tag{3.26}$$

proof. By differentiating $I(t)$ and using the third equation in (3.5), we obtain

$$I'(t) = -\int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 e^{-\rho s}|\mu_2(s)|\frac{d}{d\rho}z^2(x,\rho,t,s)d\rho\, ds\, dx,$$
$$= -\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \frac{d}{d\rho}(e^{-\rho s}z^2(x,\rho,t,s))d\rho\, ds\, dx$$
$$- \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)|e^{-\rho s}z^2(x,\rho,t,s)\, ds\, d\rho\, dx.$$

Hence

$$I'(t) = -\int_\Omega \int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,t,s)\, ds\, dx + \left(\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds\right)\int_\Omega u_t^2(x,t)dx$$
$$- \int_\Omega \int_{\tau_1}^{\tau_2} s|\mu_2(s)| \int_0^1 e^{-\rho s}z^2(x,\rho,t,s)\, d\rho\, ds\, dx.$$

Recalling $e^{-s} \le e^{-\rho s} \le 1$, for all $\rho \in [0,1]$, and $-e^{-s} \le -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$, we obtain (3.26). □

Now we define the functional

$$\mathcal{D}(t) = \int_\Omega u u_t dx + \frac{\mu_1}{2}\int_\Omega u^2 dx + \int_{L_1}^{L_2} v v_t dx.$$

Then we have the following estimate.

lemma 3.3. *The functional $\mathcal{D}(t)$ satisfies*

$$\mathcal{D}'(t) \le -(a - \varepsilon_0 C_0^2)\int_\Omega u_x^2 dx - b\int_{L_1}^{L_2} v_x^2 dx + \int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx$$
$$+ \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|z^2(x,1,t,s)\, ds\, dx. \tag{3.27}$$

proof. Taking the derivative of $\mathcal{D}(t)$ with respect to t, using (3.5), we obtain

$$\mathcal{D}'(t) = \int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx - a\int_\Omega u_x^2 dx - \int_{L_1}^{L_2} v_x^2 dx$$
$$- \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)|z(x,1,t,s)u(x,t)\, ds\, dx + [a u_x u]_{\partial\Omega} + [b v_x v]_{L_1}^{L_2}. \tag{3.28}$$

It follows from the boundary condition (3.2) that

$$[au_x u]_{\partial\Omega} + [bv_x v]_{L_1}^{L_2} = 0.$$

Using the boundary condition (3.2), we obtain

$$u^2(x,t) = \Big(\int_0^x u_x(x,t)dx\Big)^2 \le L_1 \int_0^{L_1} u_x^2(x,t)dx, \quad x \in [0, L_1],$$
$$u^2(x,t) \le (L_3 - L_2)\int_{L_2}^{L_3} u_x^2(x,t)dx, \;\; x \in [L_2, L_3],$$

which imply the following Poincaré's inequality

$$\int_\Omega u^2(x,t)dx \le C_0^2 \int_\Omega u_x^2(x,t)dx, \quad x \in \Omega, \tag{3.29}$$

where $C_0 = \max\{L_1, L_3 - L_2\}$ is the Poincaré's constant. Using Young's inequality and (3.29), we have

$$\begin{aligned}
&-\int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x,1,t,s) u(x,t)\, ds\, dx \\
&\le \varepsilon_0 C_0^2 \int_\Omega u_x^2(x,t)dx + \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx,
\end{aligned}$$

for any $\varepsilon_0 > 0$. Inserting the above estimates in (3.28), then (3.27) is fulfilled. □

Inspired by [21], we introduce the functional

$$q(x) = \begin{cases} x - \frac{L_1}{2}, & x \in [0, L_1], \\ x - \frac{L_2 + L_3}{2}, & x \in [L_2, L_3], \\ \frac{L_1}{2} + \frac{L_2 - L_3 - L_1}{2(L_2 - L_1)}(x - L_1), & x \in [L_1, L_2]. \end{cases} \tag{3.30}$$

It is easy to see that $q(x)$ is bounded, that is $|q(x)| \le M$, where $M = max\left\{\frac{L_1}{2}, \frac{L_3 - L_2}{2}\right\}$ is a positive constant .

We define the two functionals

$$\mathcal{F}_1(t) = -\int_\Omega q(x) u_t \Big(a u_x + \int_0^\infty g(p) w_x(p)\; dp\Big)\; dx, \quad \mathcal{F}_2(t) = -\int_{L_1}^{L_2} q(x) v_x v_t\, dx.$$

Then, we have the following estimates.

lemma 3.4. *For any* $\varepsilon_1 > 0$*, the functionals* $\mathcal{F}(t)$ *and* $\mathcal{F}_2(t)$ *satisfy*

$$\begin{aligned}\mathcal{F}_1'(t) \leq& \left(\frac{a+g_0}{2}+\varepsilon_1 M^2\right)\int_\Omega u_t^2 dx + \left(a^2+a^2\varepsilon_1+\frac{\mu_1 a}{4\varepsilon_1}\right)\int_\Omega u_x^2 dx \\ &+\frac{M^2}{4\varepsilon_1}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx \\ &+\left(g_0+g_0\varepsilon_1+\frac{\mu_1 g_0}{4\varepsilon_2}\right)\int_\Omega\int_0^\infty g(p)w_x^2(x,p)\,dp\,dx \\ &-\frac{g(0)}{4\varepsilon_1}\int_\Omega\int_0^\infty g'(p)w_x^2(x,p)\,dp\,dx+\mu_1 M^2(a\varepsilon_1+\varepsilon_2) \\ &-\left[\frac{a+g_0}{2}q(x)u_t^2\right]_{\partial\Omega}-\left[\frac{q(x)}{2}\left(au_x+\int_0^\infty g(p)w_x(x,p)\,dp\right)^2\right]_{\partial\Omega},\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\mathcal{F}_2'(t) =& -\frac{L_1+L_3-L_2}{4(L_2-L_1)}\Big(\int_{L_1}^{L_2} v_t^2 dx+\int_{L_1}^{L_2} bv_x^2 dx\Big)+\frac{L_1}{4}v_t^2(L_1,t) \\ &+\frac{L_3-L_2}{4}v_t^2(L_2,t)+\frac{b}{4}[(L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)].\end{aligned} \tag{3.32}$$

proof. Taking the derivative of $\mathcal{F}_1(t)$ with respect to t and using (3.5), we have

$$\begin{aligned}\mathcal{F}_1'(t) =& -\int_\Omega q(x)u_{tt}\left(au_x+\int_0^\infty g(p)w_x(x,p)\,dp\right)\,dx \\ &-\int_\Omega q(x)u_t\left(au_{xt}+\int_0^\infty g(p)w_{xt}(x,p)\,dp\right)\,dx \\ =& -\int_\Omega q(x)\left(au_{xx}+\int_0^\infty g(p)w_{xx}(x,p)\,dp\right)\left(au_x+\int_0^\infty g(p)w_x(x,p)\,dp\right)\,dx \\ &+\mu_1\int_\Omega q(x)u_t\left(au_x+\int_0^\infty g(p)w_x(x,p)\,dp\right)\,dx \\ &+\int_\Omega q(x)\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\,ds\left(au_x+\int_0^\infty g(p)w_x(x,p)\,dp\right)\,dx \\ &-\int_\Omega q(x)u_t\left(au_{xt}+\int_0^\infty g(p)w_{xt}(x,p)\,dp\right)\,dx.\end{aligned} \tag{3.33}$$

We pay attention to

$$
\begin{aligned}
&-\int_\Omega q(x)\left(au_{xx}+\int_0^\infty g(p)w_{xx}(x,p)\ dp\right)\left(au_x+\int_0^\infty g(p)w_x(x,p)\ dp\right)\ dx\\
&=\frac{1}{2}\int_\Omega q'(x)\left(au_x+\int_0^\infty g(p)w_x(x,p)\ dp\right)^2\ dx-\left[\frac{q(x)}{2}\left(au_x+\int_0^\infty g(p)w_x(x,p)\ dp\right)^2\right]_{\partial\Omega}.
\end{aligned}
\tag{3.34}
$$

The last term in (3.33) can be treated

$$
\begin{aligned}
&-\int_\Omega q(x)u_t\left(au_{xt}+\int_0^\infty g(p)w_{xt}(x,p)\ dp\right)\ dx\\
&=-a\int_\Omega q(x)u_tu_{xt}\ dx-\int_\Omega q(x)u_t\int_0^\infty g(p)w_{xt}\ dp\ dx\\
&=\left[-\frac{a}{2}q(x)u_t^2\right]_{\partial\Omega}+\frac{1}{2}\int_\Omega q'(x)u_t^2\ dx-\int_\Omega q(x)u_t\int_0^\infty g(p)(u_t-w_p)_x\ dp\ dx\\
&=\left[-\frac{a}{2}q(x)u_t^2\right]_{\partial\Omega}+\frac{1}{2}\int_\Omega q'(x)u_t^2\ dx-\int_\Omega q(x)u_t\int_0^\infty g(p)u_{tx}\ dp\ dx\\
&+\int_\Omega q(x)u_t\int_0^\infty g(p)w_{px}\ dp\ dx\\
&=\left[-\frac{a+g_0}{2}q(x)u_t^2\right]_{\partial\Omega}+\frac{a+g_0}{2}\int_\Omega q'(x)u_t^2\ dx-\int_\Omega q(x)u_t\int_0^\infty g'(p)w_x\ dp\ dx,
\end{aligned}
\tag{3.35}
$$

where we have used the fact that

$$
-\left[\int_\Omega q(x)u_tg(p)w_x(x,p)\ dx\right]_0^\infty=0.
$$

Inserting (3.34) and (3.35) in (3.33), we arrive at

$$
\begin{aligned}
\mathcal{F}_1'(t)=&-\left[\frac{q(x)}{2}\left(au_x+\int_0^\infty g(p)w_x(x,p)\ dp\right)^2\right]_{\partial\Omega}+\frac{1}{2}\int_\Omega q'(x)\left(au_x+\int_0^\infty g(p)w_x(x,p)\ dp\right)^2dx\\
&+\mu_1\int_\Omega q(x)\left(au_x+\int_0^\infty g(p)w_x(x,p)\ dp\right)\ dx\\
&+\int_\Omega q(x)\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\ ds\left(au_x+\int_0^\infty g(p)w_x\ dp\right)\ dx-\left[\frac{a+g_0}{2}q(x)u_t^2\right]_{\partial\Omega}\\
&+\frac{a+g_0}{2}\int_\Omega q'(x)u_t^2\ \mathrm{d}x-\int_\Omega q(x)u_t\int_0^\infty g'(p)w_x\ \mathrm{d}p\ \mathrm{d}x.
\end{aligned}
\tag{3.36}
$$

Using Minkowski and Young's inequalities, we have

$$\frac{1}{2}\int_{\Omega}\left(au_x+\int_0^{\infty}g(p)w_x(x,p)\;dp\right)^2\;dx\leq a^2\int_{\Omega}u_x^2\;dx+g_0\int_{\Omega}\int_0^{\infty}g(p)\left|w_x(x,p)\right|^2\;dp\;dx. \tag{3.37}$$

Young's inequality gives us for any $\varepsilon_1>0$,

$$\begin{aligned}&\left|\int_{\Omega}q(x)\int_{\tau_1}^{\tau_2}|\mu_2(s)|z(x,1,t,s)\;ds\left(au_x+\int_0^{\infty}g(p)w_x(x,p)\;dp\right)\;dx\right|\\&\quad\leq\frac{M^2}{4\varepsilon_1}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_{\Omega}\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\;ds\;dx+a^2\varepsilon_1\int_{\Omega}u_x^2(x,t)\;dx\\&\quad+g_0\varepsilon_1\int_{\Omega}\int_0^{\infty}g(p)\left|w_x(x,p)\right|^2\;dp\;dx.\end{aligned} \tag{3.38}$$

And we have also ,

$$\begin{aligned}\left|\mu_1\int_{\Omega}q(x)\left(au_x+\int_0^{\infty}g(p)w_x\;dp\right)\;dx\right|&\leq a\mu_1\left[\varepsilon_1\int_{\Omega}q^2(x)\;dx+\frac{1}{4\varepsilon_1}\int_{\Omega}u_x^2\;dx\right]\\&\quad+\mu_1\int_{\Omega}\left[\varepsilon_2q^2(x)+\frac{1}{4\varepsilon_2}\int_0^{\infty}g^2(p)w_x^2\;dp\right]\;dx\\&\leq a\mu_1\left[\varepsilon_1M^2+\frac{1}{4\varepsilon_1}\int_{\Omega}u_x^2\;dx\right]+\mu_1\int_{\Omega}\Big[\varepsilon_2M^2\\&\qquad+\frac{g_0}{4\varepsilon_2}\int_0^{\infty}g(p)w_x^2\;dp\Big]\;dx\\&\leq\mu_1M^2(a\varepsilon_1+\varepsilon_2)+\frac{\mu_1a}{4\varepsilon_1}\int_{\Omega}u_x^2\;dx\\&\qquad+\frac{\mu_1g_0}{4\varepsilon_2}\int_{\Omega}\int_0^{\infty}g(p)w_x^2\;dp\;dx.\end{aligned} \tag{3.39}$$

It is clear that

$$\left|\int_{\Omega}q(x)u_t\int_0^{\infty}g'(p)w_x\;\mathrm{dp}\;\mathrm{dx}\right|\leq\varepsilon_1M^2\int_{\Omega}u_t^2\;\mathrm{dx}-\frac{g(0)}{4\varepsilon_1}\int_{\Omega}\int_0^{\infty}g'(p)\left|w_x(x,p)\right|^2\;\mathrm{dp}\;\mathrm{dx}. \tag{3.40}$$

Inserting (3.37)-(3.40) into (3.36), we get (3.31).

By the same method, taking the derivative of $\mathcal{F}_2(t)$ with respect to t,we have

$$
\begin{aligned}
\mathcal{F}_2'(t) &= -\int_{L_1}^{L_2} q(x)v_{xt}v_t dx - \int_{L_1}^{L_2} q(x)v_x v_{tt} dx \\
&= \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2 dx - \frac{1}{2}[q(x)v_t^2]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2 dx - \frac{1}{2}[bq(x)v_x^2]_{L_1}^{L_2} \\
&= -\frac{L_1+L_3-L_2}{4(L_2-L_1)}\Big(\int_{L_1}^{L_2} v_t^2 dx + \int_{L_1}^{L_2} bv_x^2 dx\Big) + \frac{L_1}{4}v_t^2(L_1,t) \\
&+ \frac{L_3-L_2}{4}v_t^2(L_2,t) + \frac{b}{4}[(L_3-L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)].
\end{aligned}
$$

Hence, the proof is complete. □

proof. [Proof of Theorem 3.2] We define the Lyapunov functional

$$L(t) = N_1 E(t) + N_2 I(t) + \gamma_1 \mathcal{F}_1(t) + \gamma_2 \mathcal{F}_2(t) + \gamma_3 \mathcal{D}(t), \tag{3.41}$$

where $N_1, N_2, \gamma_1, \gamma_2, \gamma_3$ are positive constants that will be chosen later.

It follows from the boundary conditions (3.2) that

$$a^2 u_x^2(L_i,t) = b^2 v_x^2(L_i,t), \quad i = 1, 2. \tag{3.42}$$

Taking the derivative of (3.41) with respective to t, using the above lemmas and (3.42), we have

$$
\begin{aligned}
&L'(t)\\
&\leq -\Big\{N_1(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds) - N_2\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds - \gamma_1\Big(\frac{a+g_0}{2}+\varepsilon_1 M^2\Big) - \gamma_3\Big\}\int_\Omega u_t^2 dx\\
&\quad - \Big\{N_2 e^{-\tau_2} - \gamma_1\frac{M^2}{4\varepsilon_1}\int_{\tau_1}^{\tau_2}|\mu_2(s)| - \frac{\gamma_3}{4\varepsilon_0}\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\Big\}\times\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx\\
&\quad - \big\{(a-\varepsilon_0 C_0^2)\gamma_3 - (a^2 + a^2\varepsilon_1 + \frac{\mu_1 a}{4\varepsilon_1})\gamma_1\big\}\int_\Omega u_x^2 dx\\
&\quad - \big\{\frac{L_1+L_3-L_2}{4(L_2-L_1)}\gamma_2 + \gamma_3\big\}\int_{L_1}^{L_2} bv_x^2 dx\\
&- \big\{\frac{L_1+L_3-L_2}{4(L_2-L_1)}\gamma_2 - \gamma_3\big\}\int_{L_1}^{L_2} v_t^2 dx\\
&\quad - N_2 e^{-\tau_2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx\\
&\quad - \big\{\gamma_1(a+g_0)-\gamma_2\big\}\big(\frac{L_1}{4}v_t^2(L_1,t) + \frac{L_3-L_2}{4}v_t^2(L_2,t)\big)\\
&\quad - \big\{\frac{b}{4}(b\gamma_1-\gamma_2)\big\}\big([L_1 v_x^2(L_1,t) + (L_3-L_2)v_x^2(L_2,t)]\big)\\
&\quad + \Big\{\gamma_1\Big(g_0 + g_0\varepsilon_1 + \frac{\mu_1 g_0}{4\varepsilon_2}\Big)\Big\}(g\circ u_x)(t)\\
&\quad + \Big\{\frac{N_1}{2} - \frac{g(0)}{4\varepsilon_1}\gamma_1\Big\}(g'\circ u_x)(t).
\end{aligned}
\tag{3.43}
$$

At this moment, we wish all coeffcients except the last two in (3.43) would be negative. We want to choose γ_1, γ_2 and γ_3 to ensure that

$$
\begin{cases}
\gamma_1(a+g_0)-\gamma_2 \geq 0\\
\frac{b}{4}(b\gamma_1-\gamma_2)\geq 0\\
\frac{L_1+L_3-L_2}{4(L_2-L_1)}\gamma_2 - \gamma_3 \geq 0.
\end{cases}
\tag{3.44}
$$

Once the above constants $\gamma_1,\gamma_2,\gamma_3$ are fixed, we may choose ε_0 and ε_1 sufficiently small such that

$$\gamma_3\varepsilon_0 C_0^2 < a\left(\gamma_3 - a(\gamma_1+\varepsilon_1\gamma_1) - \frac{\mu_1}{4\varepsilon_1}\right).$$

Then we can take N_2 sufficiently large such that

$$N_2 e^{-\tau_2} - \gamma_1\frac{M^2}{4\varepsilon_1}\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds - \frac{\gamma_3}{4\varepsilon_0}\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds > 0.$$

Finally, we can always choose N_1 sufficiently large such that the first coefficient in (3.43) is negative.

Thus, we obtain that there exists a positive constant α such that (3.43) yields

$$\begin{aligned} L'(t) \leq -\alpha\Big(&\int_\Omega u_t^2 dx + \int_\Omega a u_x^2 dx + \int_{L_1}^{L_2} v_t^2 dx \\ &+ \int_{L_1}^{L_2} b v_x^2 dx + \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s)\, ds\, d\rho\, dx + (g \circ u_x)(t)\Big), \end{aligned}$$

recalling (3.20), which implies

$$L'(t) \leq -\frac{\alpha}{2} E(t), \quad \forall \geq 0. \tag{3.45}$$

On the hand, it is not hard to see that $L(t) \sim E(t)$, i.e. there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad t \geq 0. \tag{3.46}$$

Combining (3.45) and (3.46), we obtain that

$$L'(t) \leq -\kappa L(t), \quad t \geq 0,$$

for the positive constant $\kappa = \alpha/\beta_2$. Integration over $(0,t)$ gives

$$L(t) \leq L(0) e^{-\kappa t}, \quad t \geq 0,$$

recall (3.46) again, then (3.19) holds. Hence, the proof is complete.

□

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