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A publication intended for first-year students in the Common Core entitled:

Mathematics 2 : lectures

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Introduction:

Mathematics plays a fundamental role in the fields of Economics, Commercial Sciences, and Management Sciences, as it provides the essential tools required for analysis, modeling, and problem solving. In higher education, linear algebra constitutes one of the most important branches of mathematics because of its wide range of applications in economics, statistics, optimization, finance, and decision-making processes. For this reason, mastering the basic concepts of linear algebra is indispensable for first-year university students.

This course booklet, entitled *Mathematics 2*, has been specially designed for first-year students in Economics, Commercial Sciences, and Management Sciences. Its main objective is to provide students with the theoretical foundations and practical methods necessary to understand and apply the principal concepts of linear algebra in academic and professional contexts. The content has been organized progressively in order to facilitate comprehension and develop analytical reasoning.

The booklet covers several essential chapters, beginning with algebraic structures and vector spaces, followed by linear transformations and fundamental concepts related to matrices. It also includes determinants and matrix inversion, methods for solving systems of linear equations using matrix techniques, as well as eigenvalues and eigenvectors. Each chapter introduces the fundamental definitions, properties, and examples required for a clear understanding of the subject.

Particular attention has been given to clarity, simplicity, and methodological presentation in order to meet the educational needs of first-year students. Through this booklet, students are expected not only to acquire computational skills but also to develop logical thinking and mathematical rigor. It is hoped that this work will serve as a useful academic support and contribute to building a solid mathematical background for future studies in economics, management, and related disciplines.

Chapter 1:

Algebraic Structures and Vector Spaces

Chapter Introduction:

Chapter 1 introduces the fundamental concepts of algebraic structures and vector spaces, which constitute the theoretical foundation of linear algebra. These notions are essential for understanding the mathematical framework used in many areas of economics, management, and commercial sciences. The study of algebraic structures allows students to become familiar with sets equipped with operations and the properties governing these operations.

The chapter begins with an overview of basic algebraic structures and their principal characteristics. Particular attention is given to the notions of operations, fields, and the algebraic properties required for constructing vector spaces. These preliminary concepts are necessary for developing a rigorous understanding of linear algebra and its applications.

The second part of the chapter is devoted to vector spaces, one of the central concepts in modern mathematics. Students will learn the definitions and properties of vector spaces, subspaces, linear combinations, and generating systems. Through these concepts, the chapter aims to develop logical reasoning and provide students with the mathematical tools needed for the following chapters of this course booklet.

1. Internal Composition Laws

Definition 1:

Let E be a non-empty set. An **internal composition law** $\star$ on E is a function from $E \times E$ to E that assigns to each ordered pair $(a, b) \in E \times E$ a unique element of E, denoted by $a \star b$.

$$\star : E \times E \longrightarrow E$$

$$(a, b) \longmapsto a \star b$$

Remark: Internal laws can be represented by various symbols such as •, $\circ$, $\oplus$, etc.

Example 1

Define $\star$ on $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ (integers modulo 6) by:

$$a \star b = (a + 2b) \bmod 6$$

This is an internal composition law.

Example 2

Define $\star$ on $\mathbb{Q}$ by:

$$a \star b = \frac{2ab}{a + b + 1}$$

This is an internal law (provided the denominator is never zero for rational inputs — we can restrict domain appropriately).

Example 3

Define $\star$ on $\mathbb{R}$ by:

$$a \star b = \frac{3}{a^2 - b^2}$$

This is **not** an internal law because when $a = b$, the denominator is zero, so the result is undefined.

2. Properties of Internal Operations

2.1 Associativity

Definition 2:

An internal law $\star$ is said to be **associative** if and only if:

$$\forall (a,b,c) \in E^3 : (a \star b) \star c = a \star (b \star c)$$

Example 4

Define $\star$ on $\mathbb{R}$ by:

$$a \star b = a + b + 5$$

Check associativity:

$$(a \star b) \star c = (a + b + 5) + c + 5 = a + b + c + 10$$

$$a \star (b \star c) = a + (b + c + 5) + 5 = a + b + c + 10$$

Both are equal, so $\star$ is associative.

2.2 Commutativity

Definition 3:

An internal law $\star$ is said to be **commutative** if and only if:

$$\forall (a,b) \in E^2 : a \star b = b \star a$$

Example 5

Using the same law $a \star b = a + b + 5$:

$$a \star b = a + b + 5 = b + a + 5 = b \star a$$

Thus, $\star$ is commutative.

2.3 Neutral Element

Definition 4:

An internal law $\star$ on E admits a **neutral element** $e \in E$ if and only if:

$$\exists e \in E, \forall a \in E : a \star e = e \star a = a$$

Remark: If a neutral element exists, it is unique.

Example 6

For the law $a \star b = a + b + 5$ on $\mathbb{R}$, find e:

$$a \star e = a \implies a + e + 5 = a \implies e = -5$$

Check: $e \star a = -5 + a + 5 = a$. So $e = -5$ is the neutral element.

2.4 Symmetric Element (Inverse)

Definition 5:

Assume $(E, \star)$ has a neutral element e. An element $a' \in E$ is called the **symmetric** (or inverse) of a if:

$$a \star a' = a' \star a = e$$

Example 7

For $a \star b = a + b + 5$ with $e = -5$, find the inverse of a:

$$a \star a' = -5 \implies a + a' + 5 = -5 \implies a' = -10 - a$$

Check: $a' \star a = (-10 - a) + a + 5 = -5$. So the inverse exists for every $a \in \mathbb{R}$.

2.5 Distributivity

Definition 6:

Let $\star$ and $\otimes$ be two internal laws on E.

We say that $\otimes$ is **left distributive** over $\star$ if:

$$\forall(a,b,c)\in E^3 : a\otimes(b\star c) = (a\otimes b)\star(a\otimes c)$$

Right distributivity is defined similarly. The law $\otimes$ is **distributive** over $\star$ if it satisfies both.

Example 8

Define on $\mathbb{R}$:

$$a\star b = a+b+2$$

$$a\otimes b = a+b+ab$$

Check if $\otimes$ distributes over $\star$:

Left side:

$a\otimes(b\star c) = a\otimes(b+c+2) = a+(b+c+2)+a(b+c+2) = a+b+c+2+ab+ac+2a = 3a+b+c+ab+ac+2$

Right side:

$(a\otimes b)\star(a\otimes c) = (a+b+ab)\star(a+c+ac) = (a+b+ab)+(a+c+ac)+2 = 2a+b+c+ab+ac+2$

3. Stability

Definition 7:

Let E be equipped with an internal law $\star$, and let $F\subseteq E$. We say that F is **stable** under $\star$ if:

$$\forall a,b\in F : a\star b\in F$$

Example 9

$3\mathbb{Z} = \{\ldots,-6,-3,0,3,6,\ldots\}$ is stable under both ordinary addition and multiplication.

4. Group

Definition 8:

Let $\star$ be an internal law on a set G. The pair $(G, \star)$ is called a **group** if:

1. $\star$ is associative.
2. There exists a neutral element $e \in G$.
3. Every element of G has a symmetric element (inverse) in G.

Example 10

- $(\mathbb{N}, +)$ is **not** a group (no inverses for positive numbers within $\mathbb{N}$).
- $(\mathbb{Q} \setminus \{0\}, \times)$ is a group (neutral = 1, inverse of a is $1/a$).
- $(\mathbb{Z}, \times)$ is **not** a group (e.g., 2 has no inverse in $\mathbb{Z}$ since $1/2 \notin \mathbb{Z}$).

5. Subgroup

Definition 9:

Let $(G, \star)$ be a group and $H \subseteq G$ non-empty. We say that H is a **subgroup** of G if:

- H is stable under $\star$.
- The neutral element of G belongs to H.
- Every element of H has its inverse also in H.

Example 11

Consider $(\mathbb{Z}, +)$. Let $H = 4\mathbb{Z} = \{4k \mid k \in \mathbb{Z}\}$.

Then:

- $4k_1 + 4k_2 = 4(k_1 + k_2) \in H$ (stability).
- Neutral $0 = 4 \cdot 0 \in H$.
- Inverse of $4k$ is $-4k = 4(-k) \in H$.

Thus, $4\mathbb{Z}$ is a subgroup of $\mathbb{Z}$.

Equivalent Criterion (Theorem):

A non-empty subset H of a group G is a subgroup if and only if:

$$\forall a, b \in H : a \star b^{-1} \in H$$

(where b^{-1} is the inverse of b in G).

2 . Vector Space

1. Definition of a Vector Space

Let V be a non-empty set whose elements are called **vectors**, and let $\mathbb{K}$ be a field (usually $\mathbb{R}$ or $\mathbb{C}$) whose elements are called **scalars**.

We define two operations:

1. **Vector addition:** $\oplus : V \times V \to V$
2. **Scalar multiplication:** $\odot : \mathbb{K} \times V \to V$

The triple $(V, \oplus, \odot)$ is called a **vector space over** $\mathbb{K}$ if the following axioms hold for all $u, v, w \in V$ and all $\alpha, \beta \in \mathbb{K}$:

A. Axioms of vector addition

1. **Closure under addition:** $u \oplus v \in V$
2. **Associativity:** $(u \oplus v) \oplus w = u \oplus (v \oplus w)$
3. **Commutativity:** $u \oplus v = v \oplus u$
4. **Zero vector:** There exists $\mathbf{0}_V \in V$ such that $u \oplus \mathbf{0}_V = u$
5. **Additive inverse:** For each $u \in V$, there exists $-u \in V$ such that $u \oplus (-u) = \mathbf{0}_V$

B. Axioms of scalar multiplication

6. **Closure under scalar multiplication:** $\alpha \odot u \in V$
7. **Distributivity (scalar over vector):** $\alpha \odot (u \oplus v) = (\alpha \odot u) \oplus (\alpha \odot v)$
8. **Distributivity (vector over scalar):** $(\alpha + \beta) \odot u = (\alpha \odot u) \oplus (\beta \odot u)$
9. **Compatibility:** $(\alpha \cdot \beta) \odot u = \alpha \odot (\beta \odot u)$
10. **Identity scalar:** $1_{\mathbb{K}} \odot u = u$

2. Examples of Vector Spaces

Example 1 (Custom Set)

Let $V = \{(x, y) \in \mathbb{R}^2 \mid y = 2x\}$ (all points on the line through the origin with slope 2).
Define addition and scalar multiplication as usual in $\mathbb{R}^2$.
Then V is a vector space over $\mathbb{R}$.

Check:

- Sum of two points on the line: $(x_1, 2x_1) + (x_2, 2x_2) = (x_1 + x_2, 2(x_1 + x_2))$ still lies on the line.
- Scalar multiple: $\alpha(x, 2x) = (\alpha x, 2\alpha x)$ still lies on the line.

Example 2 (Polynomials of bounded degree)

Let $V = \{p(x) \in \mathbb{R}[x] \mid \deg(p) \leq 2 \text{ and } p(1) = 0\}$.
This is the set of quadratic polynomials that vanish at $x = 1$.
Example: $(x - 1)(x - 2) = x^2 - 3x + 2$ is in V.
With standard polynomial addition and scalar multiplication, V is a vector space over $\mathbb{R}$.

Example 3 (Matrices with trace zero)

Let $V = \{M \in M_{2\times 2}(\mathbb{R}) \mid \text{trace}(M) = 0\}$, where trace = sum of diagonal entries.
Example: $\begin{pmatrix} 3 & 1 \\ 4 & -3 \end{pmatrix}$ is in V.
With standard matrix addition and scalar multiplication, V is a vector space over $\mathbb{R}$.

Example 4

Let $V = \{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 2f(1)\}$.
With pointwise addition and scalar multiplication:
$(f+g)(x) = f(x) + g(x)$, $(\alpha f)(x) = \alpha f(x)$.
Then V is a vector space over $\mathbb{R}$.

Check the condition:
$(f+g)(0) = f(0) + g(0) = 2f(1) + 2g(1) = 2(f+g)(1)$.

3. Properties of Vector Spaces

From the axioms, we can derive:

1. The zero vector is unique.
2. The additive inverse of any vector is unique.
3. For any $\alpha \in \mathbb{K}$ and $v \in V$:
    - $0 \odot v = \mathbf{0}_V$
    - $\alpha \odot \mathbf{0}_V = \mathbf{0}_V$
    - $(-1) \odot v = -v$

4. Subspaces

Definition:
Let V be a vector space over $\mathbb{K}$. A non-empty subset $W \subseteq V$ is called a **subspace** of V if W itself is a vector space over $\mathbb{K}$ under the same operations.

Subspace criterion (simplified):
A non-empty subset $W \subseteq V$ is a subspace if and only if:

1. **Closure under addition:** $\forall u, v \in W : u + v \in W$
2. **Closure under scalar multiplication:** $\forall \alpha \in \mathbb{K}, \forall u \in W : \alpha u \in W$

(These two conditions automatically guarantee the zero vector is in W.)

Example 5 (Subspace of $\mathbb{R}^3$)

Let $V = \mathbb{R}^3$ and
$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y - z = 0\}$ (a plane through the origin).

Check:

- If (x_1, y_1, z_1) and (x_2, y_2, z_2) satisfy the equation, their sum also satisfies it.
- If (x, y, z) satisfies it, then $\alpha(x, y, z)$ also satisfies it.

Thus, W is a subspace of $\mathbb{R}^3$.

Example 6 (Not a subspace)

Let $W = \{(x, y) \in \mathbb{R}^2 \mid y = x + 1\}$ (a line that does NOT pass through the origin).
This is **not** a subspace because $(0, 0) \notin W$.

5. Linear Combinations and Span

Definition:

A **linear combination** of vectors $v_1, v_2, \ldots, v_n \in V$ is any vector of the form:

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$$

where $\alpha_i \in \mathbb{K}$.

Definition (Span):

The **span** of a set $S \subseteq V$, denoted $\mathrm{span}(S)$, is the set of all finite linear combinations of vectors in S.

$$\mathrm{span}(S) = \{\alpha_1 v_1 + \cdots + \alpha_k v_k \mid k \in \mathbb{N}, v_i \in S, \alpha_i \in \mathbb{K}\}$$

Fact: $\mathrm{span}(S)$ is always a subspace of V.

Example 7 (Span of two vectors in $\mathbb{R}^3$)

Let $v_1 = (1, 0, 1)$ and $v_2 = (0, 1, 1)$ in $\mathbb{R}^3$.
Then:

$$\text{span}\{v_1, v_2\} = \{a(1, 0, 1) + b(0, 1, 1) \mid a, b \in \mathbb{R}\} = \{(a, b, a + b) \mid a, b \in \mathbb{R}\}$$

This is a plane through the origin in $\mathbb{R}^3$.

6. Linear Independence

Definition:
A set of vectors $\{v_1, v_2, \dots, v_n\}$ is **linearly independent** if the only solution to:

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = \mathbf{0}$$

is $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$.
If a non-trivial solution exists, the set is **linearly dependent**.

Example 8 (Independent vectors in $\mathbb{R}^3$)

Let $v_1 = (2, 0, 0)$, $v_2 = (1, 3, 0)$, $v_3 = (4, 5, 6)$.
Check independence:

$$\alpha_1(2, 0, 0) + \alpha_2(1, 3, 0) + \alpha_3(4, 5, 6) = (0, 0, 0)$$

From the third coordinate: $6\alpha_3 = 0 \Rightarrow \alpha_3 = 0$
From the second: $3\alpha_2 = 0 \Rightarrow \alpha_2 = 0$
From the first: $2\alpha_1 + \alpha_2 + 4\alpha_3 = 2\alpha_1 = 0 \Rightarrow \alpha_1 = 0$

Thus, the set is linearly independent.

Example 9 (Dependent vectors)

Let $v_1 = (1, 2)$, $v_2 = (2, 4)$ in $\mathbb{R}^2$.
We have $v_2 = 2v_1$, so the set is linearly dependent.

7. Basis and Dimension

Definition (Basis):
A set $\mathcal{B} \subseteq V$ is called a **basis** of V if:

1. $\mathcal{B}$ is linearly independent.
2. $\text{span}(\mathcal{B}) = V$.

Definition (Dimension):
The **dimension** of V, denoted $\dim_{\mathbb{K}}(V)$, is the number of vectors in any basis of V.

Example 10 (Basis in $\mathbb{R}^2$)

The set $\mathcal{B} = \{(1,1),(1,-1)\}$ is a basis of $\mathbb{R}^2$ over $\mathbb{R}$:

- **Independence:** $a(1,1) + b(1,-1) = (0,0) \Rightarrow a+b=0, a-b=0 \Rightarrow a=b=0.$
- **Span:** Any $(x,y) \in \mathbb{R}^2$ can be written as $\frac{x+y}{2}(1,1) + \frac{x-y}{2}(1,-1)$.

Thus, $\dim_{\mathbb{R}}(\mathbb{R}^2) = 2$.

Example 11 (Basis in a polynomial space)

Let $V = \{p(x) \in \mathbb{R}[x] \mid \deg(p) \leq 2 \text{ and } p(0) = 0\}$.
Then any $p(x) = ax + bx^2$.
A basis: $\{x, x^2\}$.
Dimension = 2.

8. Summary Table (Vector Space Examples)

Vector Space	Over Field	Dimension
$\mathbb{R}^n$	$\mathbb{R}$	n
$M_{m\times n}(\mathbb{R})$	$\mathbb{R}$	$m \times n$
$\{p(x) \mid \deg \leq n\}$	$\mathbb{R}$	n+1
$\{f : \mathbb{R} \to \mathbb{R}\}$	$\mathbb{R}$	Infinite
$\{(x,y,z) \mid x+y+z=0\}$	$\mathbb{R}$	2

Chapter Summary

In this chapter, we introduced the fundamental notions of algebraic structures and vector spaces, which form an essential basis for the study of linear algebra. We examined the concept of algebraic operations and the main properties that characterize algebraic structures. These concepts provide the mathematical framework necessary for understanding more advanced topics in algebra and their practical applications.

The chapter also focused on vector spaces and their principal properties. Important notions such as subspaces, linear combinations, and generating systems were presented in order to develop a solid understanding of vector spaces and their role in mathematical analysis. These concepts are fundamental for studying matrices, linear transformations, and systems of linear equations.

Overall, this chapter establishes the theoretical foundations required for the remainder of the course. Mastering these notions enables students to strengthen their logical reasoning and mathematical skills, which are essential in economics, management sciences, and other quantitative disciplines.

Chapter 2:

Linear Maps

Chapter Introduction:

Linear transformations constitute one of the fundamental concepts of linear algebra and play an essential role in many scientific and applied fields. They provide a mathematical framework for describing relationships between vector spaces while preserving the algebraic structure of vectors. In economics, management sciences, and commercial studies, linear transformations are widely used in modeling, optimization, data analysis, and quantitative problem solving.

This chapter introduces the basic notions and properties of linear transformations. The study begins with the definition of a linear transformation and the conditions required for a mapping to be linear. Important concepts such as the kernel and image of a linear transformation are also presented, together with their algebraic and geometric interpretations.

The chapter also highlights the relationship between linear transformations and matrices, since matrices provide an effective computational tool for representing and analyzing linear maps. Through theoretical explanations and practical examples, students will develop a clear understanding of how linear transformations are applied in linear algebra and related disciplines.

1. Definition of a Linear Map (Linear Transformation)

Let V and W be two vector spaces over the same field $\mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$).

A function $T : V \longrightarrow W$ is called a **linear map** (or **linear transformation**) if it satisfies the following two conditions for all $u, v \in V$ and all $\alpha \in \mathbb{K}$:

1. **Additivity:**

$$T(u + v) = T(u) + T(v)$$

2. **Homogeneity (scalar multiplication):**

$$T(\alpha v) = \alpha T(v)$$

These two conditions can be combined into a single property:

$$\forall u, v \in V,\ \forall \alpha, \beta \in \mathbb{K} : \quad T(\alpha u + \beta v) = \alpha T(u) + \beta T(v)$$

Example 1 (Custom linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$)

Define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by:

$$T(x, y) = (2x + y,\ x - 3y)$$

Check linearity:

$$T(x_1 + x_2,\ y_1 + y_2) = \big(2(x_1 + x_2) + (y_1 + y_2),\ (x_1 + x_2) - 3(y_1 + y_2)\big)$$

$$= (2x_1 + y_1 + 2x_2 + y_2,\ x_1 - 3y_1 + x_2 - 3y_2) = T(x_1, y_1) + T(x_2, y_2)$$

$$T(\alpha x, \alpha y) = (2\alpha x + \alpha y,\ \alpha x - 3\alpha y) = \alpha(2x + y,\ x - 3y) = \alpha T(x, y)$$

Thus, T is linear.

Example 2 (Linear map from polynomials to polynomials)

Let $V = \{p(x) \in \mathbb{R}[x] \mid \deg(p) \leq 2\}$.
Define $T : V \to V$ by:

$$T(p)(x) = p(x+1) - p(x)$$

For example, if $p(x) = x^2$, then $T(p)(x) = (x+1)^2 - x^2 = 2x + 1$.

Check linearity:

$$T(p+q) = (p+q)(x+1) - (p+q)(x) = p(x+1) + q(x+1) - p(x) - q(x) = T(p) + T(q)$$

$$T(\alpha p) = \alpha p(x+1) - \alpha p(x) = \alpha(p(x+1) - p(x)) = \alpha T(p)$$

So T is linear.

Example 3 (NOT linear)

Define $T : \mathbb{R} \to \mathbb{R}$ by $T(x) = x^2$.
Check additivity: $T(1+1) = T(2) = 4$, but $T(1) + T(1) = 1 + 1 = 2 \neq 4$.
So T is **not linear**.

2. Composition of Two Linear Maps

Definition:
Let $T : V \to W$ and $S : W \to U$ be two linear maps, where V, W, U are vector spaces over the same field $\mathbb{K}$.
The **composition** $S \circ T : V \to U$ is defined by:

$$(S \circ T)(v) = S(T(v)) \quad \forall v \in V$$

Theorem:

The composition of two linear maps is itself a linear map.

Proof sketch:

$$(S \circ T)(\alpha u + \beta v) = S(T(\alpha u + \beta v)) = S(\alpha T(u) + \beta T(v))$$

$$= \alpha S(T(u)) + \beta S(T(v)) = \alpha (S \circ T)(u) + \beta (S \circ T)(v)$$

Example 4 (Composition of two linear maps from $\mathbb{R}^2$ to $\mathbb{R}^2$)

Define:

$$T(x, y) = (x + y,\ 2x - y)$$

$$S(x, y) = (x - y,\ x + y)$$

First compute $T(x, y)$:

$$T(x, y) = (x + y,\ 2x - y)$$

Then apply S:

$$(S \circ T)(x, y) = S(x + y,\ 2x - y) = \big((x + y) - (2x - y),\ (x + y) + (2x - y)\big)$$

$$= (x + y - 2x + y,\ x + y + 2x - y) = (-x + 2y,\ 3x)$$

So $(S \circ T)(x, y) = (-x + 2y,\ 3x)$, which is linear.

Example 5 (Composition of polynomial maps)

Let $T: V \to V$ be $T(p)(x) = p'(x)$ (derivative), and $S: V \to V$ be $S(p)(x) = \int_0^x p(t)dt$, where $V = \{\text{polynomials of degree} \leq 2\}$.

Then:

$$(S \circ T)(p)(x) = \int_0^x p'(t)dt = p(x) - p(0)$$

(by the Fundamental Theorem of Calculus). This is linear.

3. Kernel and Image of a Linear Map

Let $T: V \to W$ be a linear map.

3.1 Kernel (Null Space)

Definition:

The **kernel** of T, denoted $\ker(T)$, is the set of all vectors in V that map to the zero vector in W:

$$\ker(T) = \{v \in V \mid T(v) = \mathbf{0}_W\}$$

Properties:

- $\ker(T)$ is always a subspace of V.
- T is **injective (one-to-one)** if and only if $\ker(T) = \{\mathbf{0}_V\}$.

Example 6 (Finding kernel)

Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be defined by:

$$T(x, y, z) = (x + y - z,\ 2x - y + z)$$

Solve $T(x, y, z) = (0, 0)$:

$$\begin{cases} x + y - z = 0 \\ 2x - y + z = 0 \end{cases}$$

Add the two equations: $3x = 0 \Rightarrow x = 0$.
Then $y - z = 0 \Rightarrow y = z$.
So $\ker(T) = \{(0, t, t) \mid t \in \mathbb{R}\} = \text{span}\{(0, 1, 1)\}$.
Dimension of kernel = 1.

3.2 Image (Range)

Definition:

The **image** of T, denoted $\text{Im}(T)$, is the set of all vectors in W that are images of some vector in V
:

$$\text{Im}(T) = \{w \in W \mid \exists v \in V \text{ such that } T(v) = w\}$$

Properties:

- $\text{Im}(T)$ is always a subspace of W.
- T is **surjective (onto)** if and only if $\text{Im}(T) = W$.

Example 7 (Finding image)

Using the same T from Example 6:

$$T(x, y, z) = (x + y - z,\ 2x - y + z)$$

We can write:

$$T(x, y, z) = x(1, 2) + y(1, -1) + z(-1, 1)$$

Check dependency: $(1, -1) + (-1, 1) = (0, 0)$, and $(1, 2)$ is independent.
So $\text{Im}(T) = \text{span}\{(1, 2), (1, -1)\} = \mathbb{R}^2$ (since these two vectors are linearly independent in $\mathbb{R}^2$).

Thus, T is surjective.

Example 8 (Kernel and image of a matrix map)

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be:

$$T(x, y) = (x + 2y,\ 2x + 4y,\ 3x + 6y)$$

Kernel: Solve $(x + 2y,\ 2x + 4y,\ 3x + 6y) = (0, 0, 0)$.
All conditions reduce to $x + 2y = 0 \Rightarrow x = -2y$.
So $\ker(T) = \{(-2t, t) \mid t \in \mathbb{R}\}$, dimension 1.

Image: $T(x, y) = x(1, 2, 3) + y(2, 4, 6) = (x + 2y)(1, 2, 3)$.
So $\operatorname{Im}(T) = \operatorname{span}\{(1, 2, 3)\}$, dimension 1.

4. Rank of a Linear Map

Definition:

The **rank** of a linear map $T : V \to W$, denoted $\operatorname{rank}(T)$, is the dimension of its image:

$$\operatorname{rank}(T) = \dim_{\mathbb{K}} \left(\operatorname{Im}(T) \right)$$

Rank-Nullity Theorem (Fundamental Theorem of Linear Algebra):
For a linear map $T : V \to W$ where $\dim(V)$ is finite:

$$\dim(V) = \dim(\ker(T)) + \operatorname{rank}(T)$$

Here, $\dim(\ker(T))$ is called the **nullity** of T.

Example 9 (Applying rank-nullity)

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be defined by:

$$T(x_1, x_2, x_3, x_4) = (x_1 + x_2,\ x_2 + x_3,\ x_3 + x_4)$$

Find rank and nullity.

Step 1 – Find kernel:

Solve:

$$x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$$

$$x_2 + x_3 = 0 \Rightarrow x_3 = -x_2$$

$$x_3 + x_4 = 0 \Rightarrow x_4 = -x_3 = x_2$$

So $(x_1, x_2, x_3, x_4) = (-x_2,\ x_2,\ -x_2,\ x_2) = x_2(-1, 1, -1, 1)$.
Thus, $\ker(T) = \text{span}\{(-1, 1, -1, 1)\}$, so nullity = 1.

Step 2 – Apply rank-nullity:

$$\dim(\mathbb{R}^4) = 4 = \text{nullity} + \text{rank} \Rightarrow 4 = 1 + \text{rank} \Rightarrow \text{rank} = 3$$

Check: Image is a subspace of $\mathbb{R}^3$ of dimension 3, so $\text{Im}(T) = \mathbb{R}^3$. Thus T is surjective.

Example 10 (· Rank of a non-surjective map)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be:

$$T(x, y, z) = (x - y,\ y - z,\ 0)$$

Find kernel: $x - y = 0 \Rightarrow x = y$, $y - z = 0 \Rightarrow y = z$, so $x = y = z$.
Thus $\ker(T) = \{(t, t, t) \mid t \in \mathbb{R}\}$, nullity = 1.

Rank = $\dim(\mathbb{R}^3) - \text{nullity} = 3 - 1 = 2$.

Image: $T(x, y, z) = (x - y,\ y - z,\ 0)$ lies in the xy-plane (third coordinate 0).
Specifically, $\text{Im}(T) = \{(a, b, 0) \mid a, b \in \mathbb{R}\}$, dimension 2.

5. Summary Table

Concept	Symbol	How to Find	Subspace of
Kernel	$\ker(T)$	Solve $T(v) = 0$	Domain V
Image	$\text{Im}(T)$	Span of $T(basis)$	Codomain W
Rank	$\text{rank}(T)$	$\dim(\text{Im}(T))$	Integer
Nullity	$\text{nullity}(T)$	$\dim(\ker(T))$	Integer

Rank-Nullity Theorem:

$$\dim(V) = \text{nullity}(T) + \text{rank}(T)$$

Chapter Summary

In this chapter, we studied the concept of linear transformations and their fundamental properties. We introduced the definition of a linear transformation and examined the conditions that characterize linearity. The notions of kernel and image were also presented as essential tools for understanding the behavior of linear maps between vector spaces.

The chapter further demonstrated the close relationship between linear transformations and matrices. This relationship provides efficient methods for representing and simplifying many algebraic problems, particularly those involving systems of linear equations and vector space operations. Practical examples helped illustrate the theoretical concepts and their applications.

Overall, the study of linear transformations forms an important step in the understanding of linear algebra. The concepts developed in this chapter provide the necessary foundation for the study of matrices, determinants, eigenvalues, and other advanced topics that will be addressed in the following chapters.

Chapter 3:

Basic Concepts

About Matrices

Chapter Introduction:

discusses the definition and general structure of matrices, explaining their form and dimensions, and distinguishing between different types such as square, rectangular, zero, diagonal, and other matrices. This section aims to lay the conceptual foundation necessary for understanding how to represent matrices and read their components, as this is an essential step before moving on to dealing with them computationally.

The chapter concludes with a presentation of the basic operations on matrices, covering addition, subtraction, In this chapter, we address the basic concepts of matrices as essential tools in linear algebra, given their pivotal role in representing data and mathematical relationships in an organised and concise manner. Matrices are widely used in various fields such as applied mathematics, econometrics, engineering, and computer science, as they provide an effective mathematical framework for processing and analysing linear systems.

This chapter also multiplication, and transposition, with an explanation of the conditions for applying each operation and its characteristics. This topic is a fundamental introduction to understanding the subsequent applications of matrices in solving linear systems and more advanced mathematical models, paving the way for the study of more advanced topics in linear algebra.

1- Definition of a matrix:

A real number matrix is a set of real numbers, arranged in rows and columns, defined on both sides by brackets or parentheses. A matrix is denoted by capital letters such as: $A_{n\times m} B_{n\times m} C_{n\times m}$, etc., where n represents the number of rows, and m represents the number of columns, and (mæ$n \in \mathbb{N}^*$).

The matrix is generally written as follows:

$$A_{n\times m} = \begin{pmatrix} a_{11} a_{12} a_{13} \dots \dots a_{1m} \\ a_{21} a_{22} a_{23} \dots \dots a_{2m} \\ a_{31} a_{32} a_{33} \dots \dots a_{3m} \\ \dots \dots \\ \dots \dots \\ a_{n1} a_{n2} a_{n3} \dots \dots a_{nm} \end{pmatrix},$$

Where $\{a_{ij}/i = 1, \dots, n, \;\; j = 1, \dots, m\}$ represents the array element (real number) located in row number i and column number j . For example: a_{23} is the element located in row number 2 and column number 3.

Meanwhile, writing $n \times m$ under the array code A expresses the degree of the array, i.e. the total number of rows and columns that make up the array.

Example:

$$A_{3\times 3} = \begin{pmatrix} 1 & 2 & 0 \\ 5 & 5 & 1 \\ 3 & 4 & 3 \end{pmatrix}$$

is a matrix with a degree of 3×3 , i.e. it has 3 rows and 3 columns, while its element, for example: a_{32} , is the element located in the third row and second column, which in our example is the number 4.

2- **Properties:**

- ✓ The matrix $A_{1\times1}$ represents a real number (consisting of one element).
- ✓ If $m = n$, it is called a square array. Example: $m = n = 2 A_{2\times2} \begin{pmatrix} 2 & 1 \\ 3 & 5 \end{pmatrix}$
- ✓ If $m \neq n$, it is called a rectangular matrix. Example:

$$n = 2 æ\ m = 3 A_{3\times2} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \\ 1 & 5 \end{pmatrix}.$$

- ✓ If $m = 1$ (the number of columns is equal to 1), the matrix A is called a column matrix.

Example: $n = 5 æ\ m = 1 : A_{5\times1} = \begin{pmatrix} 1 \\ 3 \\ 5 \\ 3 \end{pmatrix}$

- ✓ If $n = 1$ (the number of rows is equal to 1), the matrix A is called a row matrix.

Example: $n = 1 æ\ m = 3 : A_{1\times3} = \begin{pmatrix} 2 & -1 & 1 \end{pmatrix}$

3- **The most important arrays:**

There are many types of arrays, the most important of which are:

3-1 Unit matrix:

The unit matrix is one of the most important types of matrices. It is a matrix whose main diagonal elements (the main diagonal elements are those for which i=j, such as $a11$, $a22$ ¡ $a33$ ...etc., and only in square matrices) equal to 1$\{a_{ij} = 1/i = j\}$ The rest of the elements are all zero$\{a_{ij} = 0/i \neq j\}$. The unit matrix is denoted by the symbol:I_n .

Examples:$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. $I_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

3-2- Diagonal matrix:

This is a square matrix in which all elements are zero except for those on the main diagonal, which are non-zero ($\{a_{ij} = k \;\; /i = j\}$,$\{a_{ij} = 0 \;\; /i \neq j\}$).

It is written in the following form:$A_{n\times m} = \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nm} \end{pmatrix}$,

Example:$A_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Special case: If the elements on the main diagonal are equal, the matrix is called a ladder matrix.

Example:$A_{3\times 3} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

3-3- Symmetric matrix:

This is a square matrixin which all elements are symmetrical in relation to the main diagonal ($\{a_{ij} = a_{ji}\}$), i.e. the elements of the row are the same as the elements of the column.

For example:$A_{3\times3} = \begin{pmatrix} 1 & 10 & 4 \\ 10 & 3 & 1 \\ 4 & 1 & 2 \end{pmatrix}$,$B_{3\times3} = \begin{pmatrix} 5 & 0 & -1 \\ 0 & 2 & 3 \\ -1 & 3 & -2 \end{pmatrix}$.

3-4- Inverse symmetric matrix:

This is a square matrix in which all elements symmetrical to the main diagonal are equal in absolute value and opposite in sign ($\{a_{ij} = -a_{ji}\}$).

Examples:$A_{2\times2} = \begin{pmatrix} 1 & -2 \\ 2 & 2 \end{pmatrix}$,$A_{3\times3} = \begin{pmatrix} 5 & 4 & -1 \\ -4 & 2 & 3 \\ 1 & -3 & -2 \end{pmatrix}$.

3-5- Zero matrix:

This is a matrix in which all elements, without exception, are zero ($\{\forall: a_{ij} = 0\}$).

Examples:$A_{2\times2} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$,$B_{3\times5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

3-6- Irregular matrix:

This is a square matrix whose determinant is zero ($\det(A_{m\times n}) = |A_{m\times n}| = 0$).

(We will discuss the concept of determinants in detail in Chapter 3).

3-7- Regular matrix:

This is a square matrix whose determinant is not zero. ($det(A_{n\times n}) = |A_{n\times n}| \neq 0$).

3-8- Transposed matrix:

A transposed matrix is a matrix obtained from another matrix by turning rows into columns or columns into rows while maintaining the order, and is denoted by: A^t .

Examples: $B_{2\times 2} = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix} \Rightarrow A^t_{2\times 2} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}^t$

$$B_{2\times 3} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 4 \end{pmatrix} \Rightarrow A^t_{3\times 2} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \\ 2 & 4 \end{pmatrix}^t .$$

$$B_{3\times 3} = \begin{pmatrix} 1 & 4 & 1 \\ 2 & 1 & 4 \\ 3 & 2 & 2 \end{pmatrix} \Rightarrow A^t_{3\times 3} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 2 \\ 1 & 4 & 2 \end{pmatrix}^t .$$

Properties:

- The transpose of the matrix A^t is A , i.e.: $(A^t)^t = A$.

-If the matrix A is symmetric, then: $A = A^t$.

3-9- Upper triangular matrix:

It is a square matrix in which all elements below the main diagonal are zero.

Example: $A_{3\times 3} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, A_{4\times 4} = \begin{pmatrix} 1 & 3 & 0 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 9 \end{pmatrix}$

3-9- Lower triangular matrix:

It is a square matrix in which all elements above the main diagonal are zero.

Example: $A_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 1 & 1 \end{pmatrix}, A_{4\times 4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 5 & 2 & 0 \\ 3 & 4 & 7 & 9 \end{pmatrix}$

4- Operations on matrices:

4-1- Equality:

We say that the two matrices $A_{n\times m}$ and $B_{n\times m}$ (which have the same degree) are equal if their corresponding elements are equal, i.e.: $\forall i \quad , \quad j: a_{ij} = b_{ij}$.

Example: $A_{2\times 2} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} B_{2\times 2} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$

$$A_{2\times 2} = B_{2\times 2} \Leftrightarrow a_{11} = b_{11}, a_{12} = b_{12}, a_{21} = b_{21}, a_{22} = b_{22}$$

4-2- Addition and subtraction:

The matrices $A_{m\times n}$ and $B_{m\times n}$ can be added (subtracted) together if they have the same rank, i.e. if the number of rows and columns in the two matrices is equal. The resulting matrix is $C_{n\times m}$, whose elements are c_{ij} , where: $c_{ij} = a_{ij} \pm b_{ij}$.

Example:

If: $A_{2\times 3} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 4 \end{pmatrix} B_{2\times 3} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -2 & 3 \end{pmatrix}$

then: $C_{2\times 3} = A_{2\times 3} +_{2\times 3} = \begin{pmatrix} 3 & 1 & 1 \\ 4 & 3 & 7 \end{pmatrix}$

Properties:

- ✓ Adding matrices is a commutative operation: $A + B = B + A$.
- ✓ Array addition is an associative operation: $A + (B + C) = (A + B) + C$.
- ✓ The zero matrix is a neutral element in matrix addition.

4-3- Multiplying a matrix by a number:

If the number λ is multiplied by the matrix $A_{n\times m}$, then all elements of $A_{n\times m}$ are multiplied by λ .

If $A_{2\times 3} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 4 \end{pmatrix}$ and $\lambda = 2$, then:

$$\lambda . A_{2\times 3} = 2 \times \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 2 & 2 \\ 6 & 10 & 8 \end{pmatrix}$$

Note: An array can be divided by a number by dividing all elements of the array by that number.

For example:

$$\frac{1}{\lambda} . A_{2\times 3} = \frac{1}{2} \times \begin{pmatrix} 2 & 4 & 2 \\ 4 & 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

4-4- Multiplying a matrix by a matrix:

In order for the matrix $A_{n\times P}$ to be accepted into the matrix $B_{P\times m}$, the number of columns in the matrix $A_{n\times P}$ must be equal to the number of rows in the matrix $B_{P\times m}$, and the resulting matrix $C_{n\times m}$ must have the same number of rows as the matrix $A_{n\times P}$ and the same number of columns as the matrix $B_{P\times m}$.

The resulting matrix $C_{n\times m}$ has elements c_{ij} , which are calculated using the following rule:

$$c_{ij} = \sum_{k=1}^{n} \left(a_{ik} \times b_{kj}\right).$$

Example 1:

$A_{2\times 3} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 4 \end{pmatrix}$ and $B_{3\times 1} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Since the number of columns in matrix A is equal to the number of rows in matrix B, the following addition can be performed: $A_{2\times 3} * B_{3\times 1} = C_{2\times 1} = \begin{pmatrix} c_{11} \\ c_{21} \end{pmatrix}$

where the resulting matrix C has the same number of rows as matrix A and the same number of columns as matrix B, and its elements are calculated as follows

$$c_{11} = (2\times 1) + (1\times 0) + (2\times 2) = 6c_{21}$$
$$= (3\times 1) + (5\times 0) + (4\times 2) = 11$$

$$\mathrm{C}_{2\times 1} = \mathrm{A}_{2\times 3} \times \mathrm{B}_{3\times 1} = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 5 & 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \end{pmatrix}$$

Properties:

- ✓ Matrix multiplication (if possible) is a non-commutative operation: $A \times B \neq B \times A$. Except in some cases, such as diagonal matrices.
- ✓ Matrix multiplication (if possible) is an associative operation: $A \times (B \times C) = (A \times B) \times C$.
- ✓ Matrix multiplication (if possible) is distributive with respect to addition $A \times (B + C) = (A \times B) + (A \times C)$.
- ✓ The transpose of two matrices is equal to the transpose of their transposes with the order reversed, i.e.: $(A \times B)^t = B^t \times A^t$.
- ✓ The identity matrix is a neutral element in matrix multiplication, i.e.: $A \times I = A$.

Chapter Summary

This chapter covered the basic concepts related to matrices as a central tool in linear algebra and an effective means of representing data and mathematical relationships in an organised manner. A comprehensive definition of a matrix was presented, explaining its structure based on rows and columns, its dimensions determined by the number of rows and columns, and how its elements are numbered and located within the matrix.

The chapter also addressed the different forms and types of matrices, such as square and rectangular matrices, zero matrices, diagonal matrices, and others, highlighting the characteristics that distinguish each type and its importance in mathematical applications. This presentation helped to consolidate the theoretical understanding of matrices and prepare the reader to deal with them correctly and systematically.

The chapter concluded with a presentation of the basic operations on matrices, namely addition, subtraction, and multiplication, in addition to the transpose operation, with an explanation of the conditions necessary for each operation and its most important characteristics. These operations formed the mathematical basis for the use of matrices in solving linear systems and advanced mathematical models, paving the way for a transition to more in-depth chapters on linear algebra.

Chapter 4:

Rank of matrices and matrix inverses.

Chapter Introduction:

This chapter is a natural extension of what was presented in the previous chapter on the basic concepts of matrices, moving from definitions and basic operations to a study of deeper properties more relevant to theoretical and practical applications in linear algebra. The rank of a matrix, its inverse, and its determinant are central concepts used to describe the structure of a matrix and understand its mathematical behaviour.

The chapter first addresses the rank of a matrix as a measure of the number of linearly independent rows or columns, given its critical importance in determining the nature of linear systems and the number of their solutions. The rank of a matrix reveals the degree of dependence or independence between its rows or columns, and is closely related to the matrix's reducibility and analysability.

An important part of this chapter is devoted to the study of the determinant and the inverse of a matrix, where the determinant is an essential tool for testing the invertibility of a square matrix, while the inverse is a means of solving linear systems and representing inverse transformations. This chapter aims to build a comprehensive understanding of these concepts and highlight the relationships between them, paving the way for their use in advanced applications in linear algebra and related sciences.

1- Matrix determinant:

1-1 Definition of determinant:

The determinant is a single real number calculated from a square matrix, which means that it may be zero if there is no linear independence between the rows (columns) of the matrix. In this case, the matrix is called an "ill-conditioned matrix," "irregular matrix," or "non-invertible matrix." If the determinant is not zero, the matrix is invertible and is called a regular matrix.

1-2 Second-order matrix determinant:

If we have a 2×2 matrix, i.e. it has two rows and two columns, the determinant is the product of the elements of the main diagonal minus the product of the elements of the opposite diagonal, meaning:

$$det(A_{2,2}) = |A_{2,2}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Example:

Calculating the determinant of a matrix $A = \begin{pmatrix} 4 & 2 \\ 3 & 5 \end{pmatrix}$

$$det(A) = |A| = \begin{vmatrix} 4 & 2 \\ 3 & 5 \end{vmatrix} = (4 * 5) - (2 * 3) = 20 - 6 = 14$$

Specific matrix calculation $B = \begin{pmatrix} -1 & 3 \\ 2 & 5 \end{pmatrix}$

$$det(B) = |B| = \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix} = (-1 * 5) - (3 * 2) = -5 - 6 = -11$$

1-3 Third-degree matrix determinant:

If we have a 3×3 matrix, i.e. one with 3 rows and 3 columns, the determinant can be calculated in several ways, the most important of which are three:

A - **Sarrus' rule:**

Let the matrix be $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

A. Calculating the determinant of the matrix according to Sarrus involves writing the matrix and then rewriting the first and second columns on the right side of the matrix as follows:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

B. We add the products of the three main diagonal elements, then subtract from the previous result the sum of the products of the opposite diagonal elements, which are also three elements.

That is:

$$|A| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

Example:

Calculate the determinant of the following matrix:

$$A_{3*3} = \begin{pmatrix} 2 & 3 & 2 \\ 3 & -2 & 6 \\ 4 & 5 & 3 \end{pmatrix}$$

$$|A| = \begin{vmatrix} 2 & 3 & 2 \\ 3 & -2 & 6 \\ 4 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 2 \\ 3 & -2 & 6 \\ 4 & 5 & 3 \end{vmatrix} \begin{matrix} 2 & 3 \\ 3 & -2 \\ 4 & 5 \end{matrix}$$

$$|A| = (2*-2*3+3*6*4+2*3*5)-(3*3*3+2*6*5 + 2*-2*4 = (-12+72+30)-(27+60-16) = -26$$

B. Calculating the determinant using the subtraction method:

Referring to Saros' rule, the determinant of the following matrix is:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$|A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

The elements in parentheses are called the complementary determinant of the element a_{ij} , i.e. a partial determinant of degree $n-1$ (i.e. 2) obtained by deleting the row i and the column j and denoted by M_{ij} (also called the reduced determinant of the element a_{ij}).

For example: M_{11} : a partial identifier resulting from deleting the first row and the first column from the array A .

And M_{21} : a partial identifier resulting from deleting the second row and first column of the array A .

We note that the three sides take a positive and then a negative sign, i.e. $(-1)^{i+j}$.

We call $(-1)^{i+j}M_{ij}$ the coefficient accompanying a_{ij} and symbolise it as C_{ij} . If the relationship can be written as follows:

$$|A| = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$$

$$= a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$$

$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

This is according to the first line, and in the same way, the determinant can be calculated according to the second line, so we find:

$$|A| = -a_{21}M_{21} + a_{22}M_{22} - a_{23}M_{23}$$

$$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

Using the same methodology, it can be calculated according to the third line.

The determinant can also be calculated according to the first column, and we find:

$$|A| = 3(-1)^{1+1}M_{11} + 2(-1)^{2+1}M_{21} + 1 * (-1)^{3+1}M_{31}$$

$$= 3M_{11} - 2M_{21} + M_{31}$$

Using the same method, we can calculate it according to the second and third columns.

<u>Example:</u>

Calculate the determinant of the matrix $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & 2 & 3 \\ 1 & 3 & -1 \end{pmatrix}$ **According to the second row and the first column.**

According to the second row:

$$|A| = 2(-1)^{2+1}M_{21} + 2(-1)^{2+2}M_{22} + 3(-1)^{2+3}M_{23}$$

$$= -2M_{21} + 2M_{22} - 3M_{23}$$

$$|A| = -2\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix}$$

$$= -2(-5) + 2(-5) - 3 * 10 = 10 - 10 - 30 = -30$$

According to the first column:

$$|A| = 3(-1)^{1+1}M_{11} + 2(-1)^{2+1}M_{21} + 1 * (-1)^{3+1}M_{31}$$
$$= 3M_{11} - 2M_{21} + M_{31}$$
$$|A| = 3\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} + \begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix}$$
$$= 3(-11) - 2(-5) - 7 = -30$$

1-4 Matrix specifier of degree $n*n$:

- Calculate the determinant by row i :

$$|A| = \sum_{j=1}^{n} a_{ij}(-1)^{i+j}M_{ij} = \sum_{j=1}^{n} a_{ij}C_{ij} \qquad \forall i = 1 \dots m$$

- Calculate the determinant by row j :

$$|A| = \sum_{i=1}^{m} a_{ij}(-1)^{i+j}M_{ij} = \sum_{i=1}^{m} a_{ij}C_{ij} \qquad \forall j = 1 \dots n$$

Note: Any row or column can be used to calculate the determinant, and we will get the same result, but it is preferable to choose the row or column that contains the largest number of zeros to facilitate the calculation.

Example: Calculate the determinant of the following matrix:

$$|A| = \begin{vmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & 2 & 1 \\ 2 & 0 & 1 & 2 \\ 3 & 0 & 2 & 1 \end{vmatrix}.$$

We choose the second column:

$$|A| = 2 * C_{12} + 1 * C_{22} + 0 * C_{32} + 0 * C_{42} = C_{12} + C_{22}$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = -(2 + 12 + 4 - 3 - 8 - 4) = -3$$

$$C_{22} = (-1)^{2+2} \begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 1 + 6 + 12 - 9 - 4 - 2 = 4$$

And from it: $|A| = 2(-3) + 4 = -2$

1-5 Properties of the determinant:

A. If there are two identical rows (or columns) in the matrix, the determinant is zero.

Example:

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 3 & -1 & 1 \end{vmatrix} = 0 \;, \quad |B| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 3 & 4 & 3 \end{vmatrix} = 0$$

We note that matrix A has two identical rows, the first and the second, while in matrix B, we also note that the first and third columns are identical. Therefore, the determinant is zero in both matrices.

B- If we have a row (or column) that is a multiple of another row (or column), i.e. it is written in the form $L_i = \alpha L_{i'}$ $(C_j = \beta C_{j'})$, then the determinant is zero.

Example:

$$|A| = \begin{vmatrix} 1 & 1 & 3 \\ 7 & 0 & 6 \\ 2 & 2 & 6 \end{vmatrix} = 0 , |B| = \begin{vmatrix} 1 & 0 & 3 \\ 2 & 2 & 6 \\ -1 & 3 & -3 \end{vmatrix} = 0$$

We note that the third row of matrix A is a multiple of the first row, while in matrix B, we also note that the third column is a multiple of the first column, and therefore the determinant is zero in both matrices.

c-The determinant of a transposed matrix is the same as the determinant of the matrix, i.e.: $|A^t| = |A|$

Example:

$$|A| = \begin{vmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = (2+12+4) - (3+8+4) = 3$$

$$|A|^t = \begin{vmatrix} 2 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = (2+4+12) - (4+8+3) = 3$$

D. If A and B are two matrices of the same degree, then: $|AB| = |A| * |B|$

Example:

$$|B| = \begin{vmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 3, |A| = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{vmatrix} = 4$$

$$|AB| = \begin{vmatrix} 13 & 9 & 6 \\ 12 & 9 & 6 \\ 13 & 10 & 8 \end{vmatrix} = 12 = |A| * |B| = 4 * 3$$

E. If the matrix is of the form $\begin{pmatrix} a_{11} & a_{12} & a_{13}+a \\ a_{21} & a_{22} & a_{23}+b \\ a_{31} & a_{32} & a_{33}+c \end{pmatrix}$ (applies to all types of square matrices), then:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13}+a \\ a_{21} & a_{22} & a_{23}+b \\ a_{31} & a_{32} & a_{33}+c \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a \\ a_{21} & a_{22} & b \\ a_{31} & a_{32} & c \end{vmatrix}$$

Example:

$$\begin{vmatrix} 13 & 9 & 6 \\ 12 & 9 & 6 \\ 13 & 10 & 8 \end{vmatrix} = \begin{vmatrix} 13 & 9 & 4 \\ 12 & 9 & 5 \\ 13 & 10 & 4 \end{vmatrix} + \begin{vmatrix} 13 & 9 & 2 \\ 12 & 9 & 1 \\ 13 & 10 & 4 \end{vmatrix}$$

where: 2912= -17+

e- $|A| = |A_1 A_2 \dots \dots \alpha A_j \dots \dots A_n| = \alpha |A|$ (same observation applies to rows)

Example:

We have: $\begin{vmatrix} 3 & 6 & 9 \\ 2 & 2 & 4 \\ 2 & -1 & 3 \end{vmatrix} = -12$ Therefore, according to the property, the number 3

can be extracted from the first row, and the determinant becomes as follows:

$$3 * \begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 2 & -1 & 3 \end{vmatrix} = 3 * (-4) = -12$$

Therefore: -4 = $\begin{vmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 2 & -1 & 3 \end{vmatrix}$

In addition to the number 3, we can also extract the numbers 2 and -1 from the second and third lines, respectively, giving us:

$$3 * 2 * (-1) * \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -2 & 1 & -3 \end{vmatrix} = 3 * 2 * (-1) * 2 = -12$$

Therefore, we conclude that: $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -2 & 1 & -3 \end{vmatrix} = 2$

And if A is a matrix of rank (n), then: $|\alpha A| = \alpha^n \, |A|$

Example:

$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -2 & 1 & -3 \end{vmatrix} = 2$, then $\begin{vmatrix} 2 & 4 & 6 \\ 2 & 2 & 4 \\ -4 & 2 & -6 \end{vmatrix} = 2^3 * 2 = 16$

The determinant of a diagonal matrix or triangular matrix (upper or lower) is equal to the product of the elements of the main diagonal.

Example: $\begin{vmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & 1 & 4 \end{vmatrix} = 2 \times 3 \times 4 = 24$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{vmatrix} = 1 \times 3 \times 4 = 12$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{vmatrix} = 2 \times 1 \times 4 = 8$$

F. Swapping two rows (columns) reverses the sign of the determinant. $L_i \leftrightarrow L_j$

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} = - \begin{vmatrix} 3 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 0 & -1 \end{vmatrix} = -2$$

1-5 Basic operations on determinants:

A- The determinant does not change if we perform the following elementary operation: $L_i = L_i + \alpha L$, or $C_j = C_j + \beta C$,

Any addition to a row (column) that is linearly independent of the other rows (columns) does not change the value of the determinant.

Example 1: Elementary operations on rows

$$\begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{vmatrix} \xrightarrow[L_3 = L_3 - 3L_1]{L_2 = L_2 - 2L_1} = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 4 \end{vmatrix} \xrightarrow{L_3 = L_3 - 2L_2}$$

$$= \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & -2 \end{vmatrix} = 1 * 1(-2) = -2$$

Example 2: Elementary operations on columns

$$\begin{vmatrix}1 & 0 & -1\\2 & 1 & 1\\3 & 2 & 1\end{vmatrix}\xrightarrow{C_3=C_3+C_1}=\begin{vmatrix}1 & 0 & 0\\2 & 1 & 3\\3 & 2 & 4\end{vmatrix}\xrightarrow{C_3=C_3-3C_2}$$

$$=\begin{vmatrix}1 & 0 & 0\\2 & 1 & 0\\3 & 2 & -2\end{vmatrix}=-2$$

2- Matrix rank:

2-1- Definition of rank:

The rank of a matrix is the number of linearly independent rows (columns) in it, where we say that the matrix $A_{m\times n}$ has rank p (($1 \le p \le Min(m,n$) if there is at least one minor of degree p that is non-zero and every minor of degree $p+1$ is zero.

2-2 Calculating rank using determinants:

This is specific to square matrices, and we distinguish between two cases here:

- **The first case (if the determinant is non-zero): The rank of the matrix is n (i.e., $p = n$).**
- **Second case (if the determinant is zero): The rank is $1 \le p \le n-1$, and to find the rank, we must calculate the partial determinants of rank $n-1$.**

If at least one non-zero minor determinant is found, then the rank is $n-1$, but if all of them are zero, then the rank of the matrix is: $1 \le p \le n-2$.

We repeat this process until we find a non-zero minor determinant.

If all partial determinants at all degrees are zero, then the rank is 1.

Note: If the matrix has a rank greater than 4 and its determinant is zero, calculating all the partial determinants takes a long time (a rank 4 matrix has 16 minors), so the reduction method is considered the best way to determine the rank.

__Example:__ $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \\ 2 & 4 & -1 \end{pmatrix}$ What is the rank of the matrix?

__First, we calculate the determinant:__

Without calculating, we can see that $_2 = 2c_1$, from which we can see that the determinant is zero, and therefore the rank of the matrix is $1 \le p \le 2$.

__Second, we calculate the partial determinants of degree 2:__

$$M_{31} = \begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} = 4 - 2 = 2 \neq 0$$

If: $p = 2$

__2-3 Calculating the rank by reduction (equivalent matrices):__

This is used to calculate the rank of all types of matrices, i.e. square and non-square, and the rank of the matrix is equal to the number of non-zero rows (columns) in the equivalent matrix.

By equivalent matrices, we mean: If one can be obtained from the other by performing a sequence of elementary operations (mentioned above) and we stop performing elementary operations when we obtain an equivalent matrix of the following forms:

$$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \begin{pmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{pmatrix}$$

__Example 1:__

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

The matrix is not square if $1 \le p \le Min(2,3)$. We perform the following elementary operations:

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 3 \end{pmatrix} \xrightarrow{L_2 = L_2 - 2L_1} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -3 & 5 \end{pmatrix}$$

Therefore, the rank is 2 because there are 2 non-zero rows.

Example 2:

$$A = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 2 & 3 & 0 & 1 \\ 3 & 1 & 1 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & -1 & 4 \\ 2 & 3 & 0 & 1 \\ 3 & 1 & 1 & 1 \end{pmatrix} \xrightarrow[L_3 = L_3 - 3L_1]{L_2 = L_2 - 2L_1}$$

$$= \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 2 & -7 \\ 0 & -2 & 4 & -11 \end{pmatrix} \xrightarrow{L_3 = L_3 + 2L_2}$$

$$= \begin{pmatrix} 1 & 1 & -1 & 4 \\ 0 & 1 & 2 & -7 \\ 0 & 0 & 8 & -25 \end{pmatrix}$$

Therefore,the rank is 3 because there are 3 non-zero rows.

Example 3:

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & 2 & 4 & 8 \\ 3 & 1 & 2 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 2 & 2 & 4 & 8 \\ 3 & 1 & 1 & 1 \end{pmatrix} \xrightarrow[L_3 = L_3 - 3L_1]{L_2 = L_2 - 2L_1} = \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & -4 & -11 \end{pmatrix}$$

Therefore, the rank is 2 because there are 2 non-zero rows.

3 - Matrix inverse:

3-1 Definition

Let us have the following square and regular matrix ($|A_{n\times n}| \neq 0$):

$$A_{n*n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

The inverse of the matrix A is the matrix B , which satisfies: $B \times A = A \times B = I$.

We denote it by the symbol: $B = A^{-1}$, where I is the identity matrix.

Note: If the inverse matrix exists, it is unique.

3-2 Methods for finding the inverse of a matrix.

3-2-1 Determinant method:

To calculate the inverse of the matrix A , we follow these steps:

First, calculate the determinant of the matrix: $det(A_{n\times n})$

Second, find the adjoint matrix:

$$adj(A_{n\times n}) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}.$$

Third, find the inverse of the matrix using the following relation: $A^{-1} = \frac{adj(A)}{det(A)}$

To calculate the conjugate, we follow these steps:

$$adj(A) = (A^{*})^{t} = \begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^{t}$$

$C_{ij} = (-1)^{i+j} M_{ij}$ (the companion coefficient).

M_{ij} : A partial (mini) determinant obtained by deleting the row i and the column j from the matrix A .

<u>Example:</u> Calculate the inverse of the following matrix: $A = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}.$

- Calculate the determinant value:

$$det(A_{3\times3}) = 2 \times \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} - 3 \times \begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix} + 5 \times \begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix}$$

$$det(A_{3\times3}) = 2 \times (6-8) - 3 \times (3+8) + 5 \times (2+4)$$

$$det(A_{3\times3}) = -4 - 33 + 30 = -7 \neq 0$$

- Find the accompanying matrix:

$$adj(A) = \begin{pmatrix} +\begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} & +\begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} \\ -\begin{vmatrix} 1 & -2 \\ 4 & 3 \end{vmatrix} & +\begin{vmatrix} 2 & -2 \\ 5 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix} \\ +\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -2 \\ 3 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 & 2 \\ -11 & 16 & -3 \\ 6 & -10 & 1 \end{pmatrix}$$

- Calculate the transpose: $\rightarrow adj(A) = \begin{pmatrix} -2 & -11 & 6 \\ 1 & 16 & -10 \\ 2 & -3 & 1 \end{pmatrix}$

- Find the inverse of the matrix:

$$A^{-1} = \frac{adj(A)}{\det(A)} = -\frac{1}{7}\begin{pmatrix} -2 & -11 & 6 \\ 1 & 16 & -10 \\ 2 & -3 & 1 \end{pmatrix} = \begin{pmatrix} \frac{2}{7} & \frac{11}{7} & -\frac{6}{7} \\ -\frac{1}{7} & -\frac{16}{7} & \frac{10}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{1}{7} \end{pmatrix}$$

<u>3-2-2 Gauss method:</u>

To find the inverse of a matrix using this method, we follow these steps:

First: We write the matrix and the unit matrix in the form of an augmented matrix (Matrice augmentée), as follows:

$$(\mathbf{A}|\mathbf{I}) = \left(\begin{array}{cccc|cccc} a_{11} & a_{12} & \cdots & a_{1n} & 1 & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{n1} & 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & 0 & 0 & \cdots & 1 \end{array}\right)$$

Second: We perform preliminary operations on the augmented matrix so that the matrix A becomes the unit matrix, and thus the unit matrix becomes the inverse of the matrix A .

$$(I|\mathbf{A}^{-1}) \left(\begin{array}{cccc|cccc} 1 & 0 & \cdots & 0 & a_{11}{}^{*} & a_{12}{}^{*} & \cdots & a_{1n}{}^{*} \\ 0 & 1 & \cdots & 0 & a_{21}{}^{*} & a_{22}{}^{*} & \cdots & a_{n1}{}^{*} \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & a_{n1}{}^{*} & a_{n2}{}^{*} & \cdots & a_{nn}{}^{*} \end{array}\right)$$

Where: $a_{ij}{}^{*}$ are the elements of the inverse of the matrix A .

That is, we start from the expanded matrix $(A|I)$, and by performing the initial operations, we obtain the expanded matrix $(I|A^{-1})$.

<u>Example:</u> Calculate the inverse of the following matrix: $A = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}$

$$(A|I)=\left(\begin{array}{ccc|ccc}2&1&-2&1&0&0\\3&2&2&0&1&0\\5&4&3&0&0&1\end{array}\right)\xrightarrow[L_1=\frac{1}{2}L_1]{}$$

$$=\left(\begin{array}{ccc|ccc}1&\frac{1}{2}&-1&\frac{1}{2}&0&0\\3&2&2&0&1&0\\5&4&3&0&0&1\end{array}\right)\xrightarrow[\substack{L_2=L_2-3L_1\\L_3=L_3-5L_1}]{}$$

$$=\left(\begin{array}{ccc|ccc}1&\frac{1}{2}&-1&\frac{1}{2}&0&0\\0&\frac{1}{2}&5&-\frac{3}{2}&1&0\\0&\frac{3}{2}&8&-\frac{5}{2}&0&1\end{array}\right)\xrightarrow[L_2=2L_2]{}$$

$$=\left(\begin{array}{ccc|ccc}1&\frac{1}{2}&-1&\frac{1}{2}&0&0\\0&1&10&-3&2&0\\0&\frac{3}{2}&8&-\frac{5}{2}&0&1\end{array}\right)\xrightarrow[L_3=L_3-\frac{3}{2}L_2]{}$$

$$=\left(\begin{array}{ccc|ccc}1&\frac{1}{2}&-1&\frac{1}{2}&0&0\\0&1&10&-3&2&0\\0&0&-7&2&-3&1\end{array}\right)\xrightarrow[L_3=-\frac{1}{7}L_3]{}$$

$$=\left(\begin{array}{ccc|ccc}1&\frac{1}{2}&-1&\frac{1}{2}&0&0\\0&1&10&-3&2&0\\0&0&1&-\frac{2}{7}&\frac{3}{7}&-\frac{1}{7}\end{array}\right)\xrightarrow[L_2=L_2-10L_3]{}$$

$$\left(\begin{array}{ccc|ccc}1&\frac{1}{2}&-1&\frac{1}{2}&0&0\\0&1&0&-\frac{1}{7}&-\frac{16}{7}&\frac{10}{7}\\0&0&1&-\frac{2}{7}&\frac{3}{7}&-\frac{1}{7}\end{array}\right)\xrightarrow[L_1=L_1+L_3]{}$$

$$= \left(\begin{array}{ccc|ccc} 1 & \frac{1}{2} & 0 & \frac{3}{14} & \frac{3}{7} & -\frac{1}{7} \\ 0 & 1 & 0 & -\frac{1}{7} & -\frac{16}{7} & \frac{10}{7} \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{3}{7} & -\frac{1}{7} \end{array}\right) \xrightarrow[L_1 = L_1 - \frac{1}{2}L_2]{}$$

$$= \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{7} & \frac{11}{7} & -\frac{6}{7} \\ 0 & 1 & 0 & -\frac{1}{7} & -\frac{16}{7} & \frac{10}{7} \\ 0 & 0 & 1 & -\frac{2}{7} & \frac{3}{7} & -\frac{1}{7} \end{array}\right) = (I|A^{-1})$$

Properties:

- The inverse of the inverse of a matrix is the same as the original matrix. $(A^{-1})^{-1} = A$

- The inverse of a matrix multiplied by a number is the inverse of the number multiplied by the inverse of the matrix $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$

Example:

$$2.A = 2.\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 2 & -4 \\ 6 & 4 & 4 \\ 10 & 8 & 6 \end{pmatrix}$$

Its inverse is:

$$A^{-1} = \frac{1}{2}\begin{pmatrix} \frac{2}{7} & \frac{11}{7} & -\frac{6}{7} \\ -\frac{1}{7} & -\frac{16}{7} & \frac{10}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{1}{7} \end{pmatrix} = \begin{pmatrix} \frac{2}{14} & \frac{11}{14} & -\frac{6}{14} \\ -\frac{1}{14} & -\frac{16}{14} & \frac{10}{14} \\ -\frac{2}{14} & \frac{3}{14} & -\frac{1}{14} \end{pmatrix}$$

- The inverse of an n-power matrix is the same as the inverse of the n-power matrix $(A^n)^{-1} = (A^{-1})^n$

Example: Let us take the previous matrix as an example

$$(A^3)^{-1} = \left(\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}^3\right)^{-1} = \begin{pmatrix} -3 & -4 & -8 \\ 22 & 15 & 4 \\ 37 & 25 & 7 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} \frac{2}{7} & \frac{11}{7} & -\frac{6}{7} \\ -\frac{1}{7} & -\frac{16}{7} & \frac{10}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{1}{7} \end{pmatrix}^3$$

$$= \begin{pmatrix} -0.76 & 9.09 & -5.04 \\ 0.12 & -14.45 & 8.6 \\ -0.07 & 3.63 & -2.17 \end{pmatrix}$$

- The inverse of two matrices is: $(AB)^{-1} = B^{-1}A^{-1}$

Example:

$$(AB)^{-1} = \left(\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix} * \begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix}\right)^{-1}$$

$$= \begin{pmatrix} \frac{-2}{10} & \frac{1}{10} & \frac{3}{10} \\ 0 & \frac{5}{10} & \frac{-5}{10} \\ \frac{6}{10} & \frac{-3}{10} & \frac{1}{10} \end{pmatrix} * \begin{pmatrix} \frac{2}{7} & \frac{11}{7} & -\frac{6}{7} \\ -\frac{1}{7} & -\frac{16}{7} & \frac{10}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{1}{7} \end{pmatrix}$$

- The inverse of the sum of two matrices is: $(A + B)^{-1} = A^{-1} + B^{-1}$.

$$(AB)^{-1} = \left(\begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 3 & 0 & 1 \end{pmatrix}\right)^{-1}$$

$$= \begin{pmatrix} 3 & 2 & 0 \\ 6 & 4 & 3 \\ 8 & 4 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{2}{7} & \frac{11}{7} & -\frac{6}{7} \\ -\frac{1}{7} & -\frac{16}{7} & \frac{10}{7} \\ -\frac{2}{7} & \frac{3}{7} & -\frac{1}{7} \end{pmatrix} + \begin{pmatrix} \frac{-2}{10} & \frac{1}{10} & \frac{3}{10} \\ 0 & \frac{5}{10} & \frac{-5}{10} \\ \frac{6}{10} & \frac{-3}{10} & \frac{1}{10} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{6}{70} & \frac{117}{70} & \frac{-39}{70} \\ \frac{-1}{7} & \frac{-125}{70} & \frac{65}{70} \\ \frac{22}{70} & \frac{9}{70} & \frac{-3}{70} \end{pmatrix}$$

- To find the inverse of a diagonal matrix, it is sufficient to flip the elements of the main diagonal:

$$D = \begin{pmatrix} d_{11} & 0 & \dots & 0 \\ 0 & d_{22} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & d_{nn} \end{pmatrix} \Rightarrow D^{-1} = \begin{pmatrix} \frac{1}{d_{11}} & 0 & \dots & 0 \\ 0 & \frac{1}{d_{22}} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & \frac{1}{d_{nn}} \end{pmatrix}$$

For example:

$$.\ A^{-1} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Chapter summary:

This chapter covered basic and advanced concepts in linear algebra, namely the rank, determinant, and inverse of a matrix, due to their pivotal role in analysing the structure of matrices and understanding their algebraic properties. The focus was on the concept of matrix rank as a measure of the number of linearly independent rows or columns, and what this reflects in terms of the degree of linear dependence between the elements of the matrix.

The chapter also highlighted the importance of matrix rank in the study of linear systems, as it is used to determine the number and nature of possible solutions, in addition to its close connection to the reducibility of the matrix and the simplicity of its representation. The relationship between rank and linear independence was emphasised as the basis for understanding many theoretical results in linear algebra.

The chapter concluded with a study of the determinant and inverse of a matrix, clarifying the role of the determinant in judging the invertibility of a square matrix and showing that the existence of an inverse is related to the non-vanishing of the determinant and the equality of the rank of the matrix with its dimension. This presentation highlighted the interconnection between the three concepts, providing an integrated framework that is later used in solving linear systems and advanced mathematical applications.

Chapter 5:

Solving a set of Linear Equations Using Matrix Systems

Chapter Introduction:

This chapter is one of the most important applied chapters in linear algebra, as it shifts the focus from studying the properties of matrices to their practical use in solving systems of linear equations. Representing a linear system in matrix form provides a unified and concise framework that allows the system to be analysed and processed in a systematic and efficient mathematical manner.

This chapter discusses various matrix methods for solving systems of linear equations, presenting the inverse method, which relies on the existence of an inverse for the coefficient matrix, Cramér's rule, which is based on determinants and is suitable for square equations with a single solution, and Gauss's method, which is one of the most general and flexible methods, as it is valid for studying compatible and incompatible equations and determining the number of solutions, if any.

This chapter aims to highlight the close relationship between previous concepts such as rank, determinant, and inverse, and the issue of solving linear equations, while clarifying the conditions for applying each method and its areas of use. This chapter thus enables the reader to choose the most appropriate method for solving linear equations and understand the results obtained, paving the way for broader applications in mathematics and applied sciences.

1- **Converting an equation from a linear system to a matrix system:**

To use the matrix system to solve linear equations, we must first convert the latter from linear to matrix form so that we can use the matrix system methods to solve these equations.

Let us have a set of linear equations for n variable as follows:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \cdot \quad \cdot \quad\quad \cdot \quad\quad &\cdot \\ \cdot \quad \cdot \quad\quad \cdot \quad\quad &\cdot \\ \cdot \quad \cdot \quad\quad \cdot \quad\quad &\cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

First, we denote X as the column of unknowns, A as the matrix of coefficients, and B as the column vector of numbers: b_1 , b_2 , … , b_n .

Therefore, we can write the previous set of linear equations in matrix form as follows:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

That is: $A_{n\times n}.X_{n\times 1} = B_{n\times 1}$

2- **Solve the equation using the inverse matrix method:**

If the matrix A is regular, i.e. its determinant is non-zero ($|\quad_{n\times n}| \neq 0$), then by multiplying both sides of the previous equation by the inverse of the matrix (A^{-1}), we obtain a unique solution to the previous set of linear equations according to the following relation:

$$A_{n\times n}.X_{n\times 1} = B_{n\times 1}$$

$$A_{n\times n}^{-1}.A_{n\times n}.X_{n\times 1}=A_{n\times n}^{-1}.B_{n\times 1}$$

We have: $$A_{n\times n}^{-1}.A_{n\times n}=I_{n\times n}$$

And from this: $$I_{n\times n}.X_{n\times 1}=A_{n\times n}^{-1}.B_{n\times 1}$$

Since the unit matrix is a neutral element in matrix multiplication, then:

$$X_{n\times 1}=A_{n\times n}^{-1}.B_{n\times 1}$$

Example: Let us have the following set of linear equations:

$$2x_1+1x_2-2x_3=10$$
$$3x_1+2x_2+2x_3=1$$
$$5x_1+4x_2+3x_3=4$$

Let us set:

$$B=\begin{pmatrix}10\\1\\4\end{pmatrix}X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},A=\begin{pmatrix}2&1&-2\\3&2&2\\5&4&3\end{pmatrix}$$

Therefore, the previous set of equations can be written as follows:

$$\begin{pmatrix}2&1&-2\\3&2&2\\5&4&3\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}10\\1\\4\end{pmatrix}$$

From this, we can solve the previous equation and find the matrix of unknowns by multiplying the inverse of the coefficient matrix by the result matrix. (The inverse of the matrix$A$ has been calculated previously):

$$A^{-1}=\frac{adj(A}{det(A)}=-\frac{1}{7}\begin{pmatrix}-2&-11&6\\1&16&-10\\2&-3&1\end{pmatrix}$$

Finding solutions to linear equations:

$$X = A^{-1}.B = -\frac{1}{7}\begin{pmatrix} -2 & -11 & 6 \\ 1 & 16 & -10 \\ 2 & -3 & 1 \end{pmatrix}\begin{pmatrix} 10 \\ 1 \\ 4 \end{pmatrix} = -\frac{1}{7}\begin{pmatrix} -7 \\ -14 \\ 21 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

Therefore, the previous set of equations has a unique solution

$$:X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix},$$

i.e. $x_1 = 1$, $x_2 = 2$, $x_3 = -3$.

Example: Let us consider the following set of linear equations:

$$\begin{aligned} x_2 + 3x_3 + 2x_4 &= 11 \\ x_1 - x_2 + 3x_3 + 2x_4 &= 14 \\ x_2 + x_4 &= 2 \\ x_1 - 3x_2 + x_3 + x_4 &= 9 \end{aligned}$$

First, write the previous form in matrix form:

$$\begin{pmatrix} 0 & 1 & 3 & 2 \\ 1 & -1 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 11 \\ 14 \\ 2 \\ 9 \end{pmatrix}$$

To calculate the matrix determinant using the subtraction method, we select the third

row and obtain:

$$\begin{vmatrix} 0 & 1 & 3 & 2 \\ 1 & -1 & 3 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 0 & 3 & 2 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 1 & 3 \\ 0 & -1 & 3 \\ 1 & -3 & 1 \end{vmatrix} = -(-1) - (-4) = 5$$

From this, the inverse is

$$A^{-1} = \frac{adj(A)}{det(A)} = \frac{1}{5}\begin{pmatrix} -9 & 11 & 2 & -6 \\ -2 & 3 & 1 & -3 \\ 1 & 1 & -3 & -1 \\ 2 & -3 & 4 & 3 \end{pmatrix}$$

From this, the solution is:

$$X = A^{-1}.B = \frac{1}{5}\begin{pmatrix} -9 & 11 & 2 & -6 \\ -2 & 3 & 1 & -3 \\ 1 & 1 & -3 & -1 \\ 2 & -3 & 4 & 3 \end{pmatrix}\begin{pmatrix} 11 \\ 14 \\ 2 \\ 9 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 \\ -5 \\ 10 \\ 15 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}$$

Therefore, the only solution to the above equations is: $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix}$,

i.e.: $x_1 = 1$, $x_2 = -1$, $x_3 = 2$, $x_4 = 3$

3- <u>The solution using Cramer's method:</u>

The solutions to the equation $A_{n\times n}.X_{n\times 1} = B_{n\times 1}$ (where $det(A) \neq 0$) according to Cramer's method are as follows: $x_j = \frac{det(A_j)}{det(A)}$, where: $j = 1, 2, \dots, n$

i.e.: $x_1 = \frac{det(A_1)}{det(A)}$, $x_2 = \frac{det(A_2)}{det(A)}$, …, $x_n = \frac{det(A_n)}{det(A)}$.

Where A_j ($j = 1, 2, \dots, n$) is a matrix obtained by replacing the column j of the matrix A with the column of constants B .

Example: $A_2 = \begin{pmatrix} a_{11} & b_1 & \cdots & a_{1n} \\ a_{21} & b_2 & \cdots & a_{n1} \\ \vdots & & \ddots & \vdots \\ a_{n1} & b_n & \cdots & a_{nn} \end{pmatrix}$

Example: In the previous example, we have:

$$B = \begin{pmatrix} 10 \\ 1 \\ 4 \end{pmatrix} X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, A = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}$$

we have: $det(A) = -7 \neq 0$

and $det(A_1) = \begin{vmatrix} 10 & 1 & -2 \\ 1 & 2 & 2 \\ 4 & 4 & 3 \end{vmatrix} = 10 \times \begin{vmatrix} 2 & 2 \\ 4 & 3 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} - 2 \times \begin{vmatrix} 1 & 2 \\ 4 & 4 \end{vmatrix} = 10(-2) - 1(-5) - 2(-4) = -7$

and $det(A_2) = \begin{vmatrix} 2 & 10 & -2 \\ 3 & 1 & 2 \\ 5 & 4 & 3 \end{vmatrix} = 2 \times \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} - 10 \times \begin{vmatrix} 3 & 2 \\ 5 & 3 \end{vmatrix} - 2 \times \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = 2(-5) - 10(-1) - 2(7) = -14$

and $det(A_3) = \begin{vmatrix} 2 & 1 & 10 \\ 3 & 2 & 1 \\ 5 & 4 & 4 \end{vmatrix} = 2 \times \begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} - 1 \times \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} + 10 \times \begin{vmatrix} 3 & 2 \\ 5 & 4 \end{vmatrix} = 2(4) - 1(7) + 10(2) = 21$

Therefore: $x_1 = \frac{-7}{-7} = 1, x_2 = \frac{-14}{-7} = 2, x_3 = \frac{21}{-7} = -3$.

Notes:

- The Cramer's method and the inverse method are only used if the coefficient matrix is square (the number of equations equals the number of unknowns) and the determinant of the coefficient matrix is not zero.

- If the determinant of the coefficient matrix is not zero ($det(A) \neq 0$), then the set of equations has a unique solution.
- If the determinant of the matrix of coefficients is zero ($det(A) = 0$), then the set of equations has either an infinite number of solutions ($0. x_j = 0$) or no solutions ($0. x_j = k$).
- If the coefficient matrix is rectangular (the number of equations does not equal the number of unknowns) or the determinant is zero, the only method that should be used is the Gauss-Jordan elimination method
- The Gauss method, which we will discuss in the next section, is suitable for all cases, whether the determinant of the coefficient matrix is zero or non-zero.

<u>4- Gauss method:</u>

To solve a set of linear equations of the form $_{n\times n}. X_{n\times 1} = B_{n\times 1}$ using the Gauss method, we follow these steps:

First: We write the set of linear equations in the form of an augmented matrix, as follows:

$$(A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{n1} & b_2 \\ \vdots & & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right)$$

Second: We perform preliminary operations on the expanded matrix so that the matrix$A$becomes the identity matrix.

$$\left(\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & b_1^* \\ 0 & 1 & \cdots & 0 & b_2^* \\ \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & b_n^* \end{array}\right)$$

Third: Solve the set of linear equations $_{n\times n}.X_{n\times 1} = B_{n\times 1}$:

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1^* \\ b_2^* \\ \vdots \\ b_n^* \end{pmatrix}.$$

That is: $x_1 = b_1^*, x_2 = b_2^*, \ldots, x_n = b_n^*$.

<u>Example:</u>

In the previous example, we have:

$$B = \begin{pmatrix} 10 \\ 1 \\ 4 \end{pmatrix} X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} A = \begin{pmatrix} 2 & 1 & -2 \\ 3 & 2 & 2 \\ 5 & 4 & 3 \end{pmatrix}$$

First: We write the set of linear equations in the form of an augmented matrix, as follows:

$$(A|B) = \left(\begin{array}{ccc|c} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{array}\right)$$

Second: We perform preliminary operations on the expanded matrix so that the matrix A **becomes the unit matrix:**

$$\left(\begin{array}{ccc|c} 2 & 1 & -2 & 10 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{array}\right) \xrightarrow{L_1 = \frac{1}{2}L_1} = \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 3 & 2 & 2 & 1 \\ 5 & 4 & 3 & 4 \end{array}\right) \xrightarrow[L_3 = L_3 - 5L_1]{L_2 = L_2 - 3L_1}$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & \frac{1}{2} & 5 & -14 \\ 0 & \frac{3}{2} & 8 & -21 \end{array}\right) \xrightarrow{L_2 = 2L_2}$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & 1 & 10 & -28 \\ 0 & \frac{3}{2} & 8 & -21 \end{array}\right) \xrightarrow{L_3 = L_3 - \frac{3}{2}L_2}$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & -7 & 21 \end{array}\right) \xrightarrow{L_3 = -\frac{1}{7}L_3}$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & 1 & 10 & -28 \\ 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow{L_2 = L_2 - 10L_3}$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & -1 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow{L_1 = L_1 + L_3}$$

$$= \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right) \xrightarrow{L_1 = L_1 - \frac{1}{2}L_2} = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Therefore, the only solution to the previous set of equations is: $X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$, i.e.: $x_1 = 1$, $x_2 = 2$, $x_3 = -3$.

Note: The diving method is the best method when the equation is large and it is difficult to use the previous two methods (the inverse and Cramer's).

Example:

$$x_1 + x_2 + 3x_3 + 2x_4 + x_5 = 8$$
$$x_1 - x_2 + 3x_3 + 2x_4 + x_5 = 4$$
$$x_2 + x_4 = 1$$
$$x_1 - 3x_2 + x_3 + x_4 + x_5 = -3$$
$$3x_1 + 2x_2 - x_3 + x_4 + 2x_5 = 6$$

If we have:

$$\begin{pmatrix} 1 & 1 & 3 & 2 & 1 \\ 1 & -1 & 3 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & -3 & 1 & 1 & 1 \\ 3 & 2 & -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 8 \\ 4 \\ 1 \\ -3 \\ 6 \end{pmatrix}$$

First: We write the set of linear equations in the form of an expanded matrix, as follows:

$$(A|B) = \left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 1 & -1 & 3 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 & 1 & -3 \\ 3 & 2 & -1 & 1 & 2 & 6 \end{array}\right)$$

Second: We perform preliminary operations on the expanded matrix so that the matrix becomes A The unit matrix:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 1 & -1 & 3 & 2 & 1 & 4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & -3 & 1 & 1 & 1 & -3 \\ 3 & 2 & -1 & 1 & 2 & 6 \end{array}\right) \xrightarrow[\substack{L_2 = L_2 - L_1 \\ L_4 = L_4 - L_1 \\ L_5 = L_5 - 3L_1}]{} \left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 0 & -2 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -4 & -2 & -1 & 0 & -11 \\ 0 & -1 & -10 & -5 & -1 & -18 \end{array}\right) \xrightarrow[L_2 = 2L_2]{}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 0 & -2 & 0 & 0 & 0 & -4 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -4 & -2 & -1 & 0 & -11 \\ 0 & -1 & -10 & -5 & -1 & -18 \end{array}\right) \xrightarrow[L_2 = -\frac{1}{2}L_2]{} \left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -4 & -2 & -1 & 0 & -11 \\ 0 & -1 & -10 & -5 & -1 & -18 \end{array}\right) \xrightarrow[\substack{L_3 = L_3 - L_2 \\ L_4 = L_4 + 4L_2 \\ L_5 = L_5 + L_2}]{}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -4 & -2 & -1 & 0 & -11 \\ 0 & -1 & -10 & -5 & -1 & -18 \end{array}\right) \xrightarrow[\substack{L_3 = L_3 - L_2 \\ L_4 = L_4 + 4L_2 \\ L_5 = L_5 + L_2}]{} \left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & -2 & -1 & 0 & -3 \\ 0 & 0 & -10 & -5 & -1 & -16 \end{array}\right) \xrightarrow[\substack{L^*_3 = L_3 + L_4 \\ L_4 = L_4 + 2L^*_3 \\ L_5 = L_5 + 10L^*_3}]{}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1.5 & 0 & 1.5 \\ 0 & 0 & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 10 & -1 & -11 \end{array}\right) \xrightarrow[\substack{L^*_4 = \frac{1}{2}L_4 \\ L_5 = L_5 - 10L^*_4}]{} \left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1.5 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{array}\right) \xrightarrow[\substack{L^*_5 = -L_5 \\ L_1 = L_1 - L^*_5}]{}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 1 & 8 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1.5 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{array}\right) \xrightarrow[\substack{L^*_5 = -L_5 \\ L_1 = L_1 - L^*_5}]{} \left(\begin{array}{ccccc|c} 1 & 1 & 3 & 2 & 0 & 7 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1.5 & 0 & 1.5 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[\substack{L_2 = L_1 - 2L_4 \\ L_3 = L_3 - \frac{3}{2}L_4}]{}$$

$$\left(\begin{array}{ccccc|c} 1 & 1 & 3 & 0 & 0 & 9 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[L_1 = L_1 - 3L_3]{} \left(\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow[L_1 = L_1 - L_2]{}$$

$$\left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

From this, the solutions to the equation are: $x_1 = 1$, $x_2 = 2$, $x_3 = 2$, $x_4 = -1$, $x_5 = 1$

Example: If the equation accepts an infinite number of solutions

$$x_1 + 3x_2 + 3x_3 = 13$$

$$2x_1 + 5x_2 + 4x_3 = 23$$

$$2x_1 + 7x_2 + 8x_3 = 29$$

Solution: We have

$$B = \begin{pmatrix} 13 \\ 23 \\ 29 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, A = \begin{pmatrix} 1 & 3 & 3 \\ 2 & 5 & 4 \\ 2 & 7 & 8 \end{pmatrix}$$

When calculating the determinant of the coefficient matrix, we find that it is zero $|A|$ =0.

From this, we can write the expanded matrix as follows

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \\ 2 & 7 & 8 & 29 \end{array}\right)$$

The initial operations can be performed as follows:

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \\ 2 & 7 & 8 & 29 \end{array}\right) \xrightarrow[\mathrm{L_3 = L_3 - 2L_1}]{\mathrm{L_2 = L_2 - 2L_1}} \left(\begin{array}{ccc|c} 1 & 3 & 3 & 13 \\ 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 \end{array}\right) \xrightarrow{\mathrm{L_3 = L_3 + L_2}}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 3 & 13 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From the third row, we find that $0 \times x_3 = 0$, i.e. $x_3 = k$, where $k \epsilon \mathbb{R}$

Substituting in the second row, we find: $-x_2 - 2x_2 = -3$, i.e. $-x_2 - 2k = -3$ and from this: $x_2 = 3 - 2k$.

From the first line, we have: $x_1 + 3x_2 + 3x_3 = 13$ By substituting the values of x_2 and x_3 , we get: $x_1 + 3(3 - 2k) + 3k = 13$ From this, the value of:

$$x_1 = 4 + 3k$$

Therefore, the equation accepts an infinite number of solutions.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 + 3k \\ 3 - 2k \\ k \end{pmatrix}$$

Example: In the case where the equation does not accept any solutions.

$$3x_1 - 6x_2 - 2x_3 = 1$$

$$2x_1 - 4x_2 + x_3 = 17$$

$$x_1 - 2x_2 - 2x_3 = -9$$

$$x_1 + 2x_2 + 3x_3 = 2$$

$$4x_1 + 5x_2 + 6x_3 = 10$$

$$5x_1 + 7x_2 + 9x_3 = 15$$

Solution: We have

$$B = \begin{pmatrix} 1 \\ 17 \\ -9 \end{pmatrix} X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, A = \begin{pmatrix} 3 & -6 & 2 \\ 2 & -4 & 1 \\ 1 & -2 & -2 \end{pmatrix}$$

When calculating the determinant of the coefficient matrix, we find that it is zero $|A|$

=0.

From this, we can write the expanded matrix as follows

$$\left(\begin{array}{ccc|c} 3 & -6 & 2 & 1 \\ 2 & -4 & 1 & 17 \\ 1 & -2 & -2 & -9 \end{array}\right)$$

The initial operations can be performed as follows:

$$\left(\begin{array}{ccc|c} 3 & -6 & 2 & 1 \\ 2 & -4 & 1 & 17 \\ 1 & -2 & -2 & -9 \end{array}\right) \xrightarrow[\substack{L^*_1 = \frac{1}{3}L_1 \\ L_2 = L_2 - 2L^*_1 \\ L_2 = L_3 - L^*_1}]{} \left(\begin{array}{ccc|c} 1 & -2 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & \frac{50}{3} \\ 0 & 0 & \frac{-8}{3} & \frac{-28}{3} \end{array}\right) \xrightarrow{\substack{L_2 = 3L_2 \\ L_3 = 3L_3}}$$

$$\left(\begin{array}{ccc|c} 3 & -2 & 2 & 1 \\ 0 & 0 & 1 & 50 \\ 0 & 0 & -8 & -28 \end{array}\right)$$

From the third line, we find $-8x_3 = -28$, i.e. $x_3 = 3.5$, where, by substitution in the second line, we find: $0x_2 + 3.5 = 50$, i.e. $0x_2 = 46.5$

which is an impossible value, and therefore the equation has no solution.

5- Solving a set of linear equations in a case where the number of equations is greater than the number of unknowns:

Let (S) be a system of m linear equations and n unknowns (where m>n). In this case, we form a partial system $(\acute{S})$ of n equations and n unknowns, then follow the same steps as Gus to solve $(\acute{S})$.

When solving the system $(\acute{S})$, we will encounter three cases:

- First case: $(\acute{S})$ does not accept solutions $\Leftarrow$ (S) does not accept solutions
- Case 2: $(\acute{S})$ accepts an infinite number of solutions. We write the solution in terms of one of the variables and substitute it into the remaining equations:
  - If this solution does not satisfy the remaining equations, it means that (S) does not accept solutions.
  - If this solution satisfies the remaining equations, then (S) :
    - ✓ accepts a single solution: If the form of the remaining equations after substitution is $\acute{}\ x_j = k$
    - ✓ Accepts an infinite number of solutions: If the form of the remaining equations after substitution is $0x_j = 0$.

– <u>Third case:</u>$(\acute{S})$ accepts a single solution. We substitute this solution into the remaining equations of the system(S) ($(m-n)$ remaining equation), and thus we have two cases:

- Solution$(\acute{S})$ satisfies the remaining equations$\Leftarrow (S)$ accepts a single solution (the same solution as (S′))
- Solution$(\acute{S})$ does not satisfy the remaining equations$\Leftarrow (S)$ does not accept solutions $\underline{0x_j = k}$

<u>Examples:</u>

<u>Example 01:</u>

$$(S')\begin{cases} x_1 + x_2 + 2x_3 = 2 \\ 2x_1 - x_2 + x_3 = 1 \\ x_1 - 2x_2 + x_3 = -2 \\ x_1 + x_2 + x_3 = {}^{5}/_{2} \end{cases}$$

$$B = \begin{pmatrix} 2 \\ 1 \\ -2 \\ 5/2 \end{pmatrix} X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

We have the system(S) , which contains four equations and three unknowns, so we know the system$(\acute{S})$, which consists of the first three equations, and we keep the last equation $x_1 + x_2 + x_3 = 5/2$

The coefficient matrix in the system$(\acute{S})$ is:

$$\acute{A} = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$$

We have:

$$\det(A') = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & -2 & 1 \end{vmatrix}$$

$$= 1 \times \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix}$$

$$= 1(1) - 1(1) + 2(-3) = -6 \neq 0$$

and $\det(A'_1) = \begin{vmatrix} 2 & 1 & 2 \\ 1 & -1 & 1 \\ -2 & -2 & 1 \end{vmatrix} = 2 \times \begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix} - 1 \times \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 1 & -1 \\ -2 & -2 \end{vmatrix} = 2(1) - 1(3) + 2(-4) = -9$

and $\det(A'_2) = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 1 \\ 1 & -2 & 1 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} - 2 \times \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} = 1(3) - 2(1) + 2(-5) = -9$

and $\det(A'_3) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & -2 & -2 \end{vmatrix} = 1 \times \begin{vmatrix} -1 & 1 \\ -2 & -2 \end{vmatrix} - 1 \times \begin{vmatrix} 2 & 1 \\ 1 & -2 \end{vmatrix} + 2 \times \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} = 1(4) - 1(-5) + 2(-3) = 3$

Therefore, the solution to the system $(\acute{S})$ is: $x_1 = \frac{-9}{-6} = \frac{3}{2}$, $x_2 = \frac{-9}{-6} = \frac{3}{2}$, $x_3 = \frac{3}{-6} = -\frac{1}{2}$.

Substituting this solution into the last equation, we find: $\frac{3}{2} + \frac{3}{2} - \frac{1}{2} = 5/2$.

Therefore, this solution satisfies the remaining equation, and thus the system (S) has a unique solution that is the same as the solution of the partial system. $(\acute{S})$

Example 02:

$$\begin{cases} 3x_1 + x_2 + 2x_3 = 11 \\ 2x_1 + 2x_2 + x_3 = 9 \\ x_1 + x_2 - 2x_3 = -3 \\ x_1 + x_2 + x_3 = 3 \end{cases}$$

$$B = \begin{pmatrix} 11 \\ 9 \\ -3 \\ 3 \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, A = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$

We have the system (S) , which contains four equations and three unknowns, so we know the system $(\acute{S})$, which consists of the first three equations, and we keep the last equation $\mathbf{x_1 + x_2 + x_3 = 3}$

The coefficient matrix in the system $(\acute{S})$ is:

$$\acute{S} = \begin{pmatrix} 3 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 1 & -2 \end{pmatrix} * \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \\ -3 \end{pmatrix}$$

Using the diving method, we find:

$$\left(\begin{array}{ccc|c} 3 & 1 & 2 & 11 \\ 2 & 2 & 1 & 9 \\ 1 & 1 & -2 & -3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 1 & 2 & 11 \\ 0 & \frac{4}{3} & \frac{-1}{3} & \frac{5}{3} \\ 0 & \frac{2}{3} & \frac{-8}{3} & \frac{-20}{3} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 2 & 11 \\ 0 & 1 & \frac{-1}{4} & \frac{5}{4} \\ 0 & 0 & \frac{-5}{2} & \frac{-15}{2} \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 3 & 1 & 0 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

From this, the solutions are: $x_1 = 1, x_2 = 2, x_3 = 3$

We change only the last equation $\mathbf{x_1 + x_2 + x_3 = 3}$

When substituting the solution into the last equation, we find $1 + 2 + 3 = 6 \neq 3$

,

Therefore, this solution does not satisfy the remaining equation, and thus the system(S) has no solution.

Example 03:$B = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}$,$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$,$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 1 & 1 \\ 2 & -1 & 1 & 1 \end{array}\right) \xrightarrow{\substack{L_2 = L_2 - 2L_1 \\ L_3 = L_3 - 2L_1}} = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -3 & -3 \\ 0 & -3 & -3 & -3 \end{array}\right) \xrightarrow{L_2 = -\frac{1}{3}L_2}$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -3 & -3 & -3 \end{array}\right) \xrightarrow{L_3 = L_3 + 3L_2} = \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore,the partial system(S') has an infinite number of solutions.

We write the solution in terms of one of the variables (in terms ofx_3):$X = \begin{pmatrix} 1 - x_3 \\ 1 - x_3 \\ x_3 \end{pmatrix}$

We substitute it into the last equation and find:

$$1 - x_3 + 1 - x_3 + x_3 = 1$$

That is:$x_3 = 1$ (from the form$\underline{\acute{k}x_j = k}$).

From this, the system(S) accepts a single solution:$X = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Example 04: Let the last equation for the same example 3 from the figure be:$x_1 + x_2 + 2x_3 = 2$

By substitution, we find the solution from the form$0x_3 = 0$ (2=2), which means that the system(S) has an infinite number of solutions.

Example 05: The last equation is$x_1 + x_2 + 2x_3 = 0$, i.e.:$0x_3 = -2$

Therefore, this solution does not satisfy the remaining equation, and thus the system (S) has no solution.

<u>6- Solving a set of linear equations in cases where the number of equations is smaller than the number of unknowns:</u>

After performing the initial operations on the lines (the same method used to obtain the rank), we say that the set has an infinite number of solutions in the case of $\underline{0x_j = 0}$ or $k''x_j + k'x_{j'} = k$, and has no solution in the case of $\underline{0x_j = k}$

<u>Examples:</u>

<u>Example 01:</u> Let us consider the following set of equations:

$$(S)\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 + 3x_2 + x_3 = 4 \end{cases}$$

$$\left(\begin{matrix} 1 & -1 & 1 \\ 2 & 3 & 1 \end{matrix}\middle| \begin{matrix} 2 \\ 4 \end{matrix}\right) \xrightarrow[L_2 = L_2 - 2L_1]{} = \left(\begin{matrix} 1 & -1 & 1 \\ 0 & 5 & -1 \end{matrix}\middle| \begin{matrix} 2 \\ 0 \end{matrix}\right)$$

From the second line, we have: $5x_2 - x_3 = 0$, i.e.: $x_3 = 5x_2$

From this, there are an infinite number of solutions written in the following form:

$$X = \begin{pmatrix} 2 - 4x_2 \\ x_2 \\ 5x_2 \end{pmatrix}$$

<u>Example 02:</u> Same as the first example, with a change in line 2: $2x_1 - 2x_2 + 2x_3 = 4$

$$\left(\begin{matrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{matrix}\middle| \begin{matrix} 2 \\ 4 \end{matrix}\right) \xrightarrow[L_2 = L_2 - 2L_1]{} = \left(\begin{matrix} 1 & -1 & 1 \\ 0 & 0 & 0 \end{matrix}\middle| \begin{matrix} 2 \\ 0 \end{matrix}\right)$$

From the second line, we have: $0x_3 = 0$

From this, there are an infinite number of solutions written in the following form:

$$X = \begin{pmatrix} 2 + x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix}$$

Example 03: **Same as the first example, with a change to line 2:** $2x_1 - 2x_2 + 2x_3 = 5$

$$\left(\begin{matrix} 1 & -1 & 1 \\ 2 & -2 & 2 \end{matrix}\middle| \begin{matrix} 2 \\ 5 \end{matrix}\right) \xrightarrow[L_2 = L_2 - 2L_1]{} = \left(\begin{matrix} 1 & -1 & 1 \\ 0 & 0 & 0 \end{matrix}\middle| \begin{matrix} 2 \\ 1 \end{matrix}\right)$$

From the second line, we have: $0x_3 = 1$ **. From this, there are no solutions.**

Chapter summary:

This chapter is devoted to studying how to solve linear equations using matrix representation, as this method provides organisation and accuracy in handling linear equations. It has been emphasised that writing a linear equation in matrix form is an essential step in understanding and analysing its structure, as it clearly highlights the relationship between matrices and linear equations.

The chapter also covered the most important matrix methods of solution, namely the inverse method, Cramer's rule, and Gauss's method, explaining the conditions for applying each method and their areas of use. The inverse method relies on the reversibility of the coefficient matrix, while Cramer's rule is used in square equations with a single solution. The Gauss method is considered the most general and flexible, as it allows for the study of various possible cases of linear equations.

The chapter concludes by highlighting the interconnection between theoretical concepts and practical application, where rank, determinant, and inverse are used to determine the nature of a linear equation and the number of its solutions. This presentation has consolidated the systematic understanding of solving linear equations and prepared the reader to move on to more advanced applications in linear algebra and related sciences.

Chapter 6:

Eigenvalues

and Eigenrays:

Introduction:

The chapter on eigenvalues and eigenvectors is one of the pivotal chapters in linear algebra, as it moves the discussion from solving linear equations and studying the properties of matrices to analysing the internal structure of linear transformations. Eigenvalues and eigenvectors are essential tools for understanding how a matrix behaves when it acts on a vector space, as they reveal directions whose trajectory is changed only by a scalar factor.

This chapter deals with the concept of eigenvalues as numbers that satisfy special conditions that cause the associated vectors to retain their direction after linear transformation, as well as eigenvectors, which represent those non-zero vectors. These concepts are used to characterise matrices and study their spectral properties, and are closely related to determinants, rank, and the solution of characteristic equations.

This introduction aims to highlight the importance of eigenvalues and eigenvectors in theoretical and practical applications, where they appear in various fields such as solving differential equations, stability analysis, econometrics, physics, and data science. This chapter thus paves the way for a deeper understanding of the properties of matrices and linear transformations and opens the door to advanced applications based on these fundamental concepts.

1-Eigenvalues:

We say that the number $\boldsymbol{\lambda}$ is an eigenvalue of the square matrix $\boldsymbol{A}$ if there exists a non-zero vector $\boldsymbol{X} \in \boldsymbol{R^n}$ that satisfies the following system: $\boldsymbol{A\vec{V} = \lambda\vec{V}}$, where $\boldsymbol{\vec{V}}$ is the eigenvector of the square matrix.

The system can be written as follows:

$$\boldsymbol{A\vec{V} = \lambda\vec{V} \Leftrightarrow (A - \lambda I)\vec{V} = 0}$$

For the system to accept a non-zero solution, it must be that $|\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}| = \mathbf{0}$

(if the determinant is non-zero, it means that the eigenvector is zero $\boldsymbol{\vec{V} = (A - \lambda I)^{-1} * 0 = 0}$).

This is in the case where the eigenvalues are given and we are asked to verify that they are indeed eigenvalues of the matrix.

However, if we want to find the eigenvalues, we can follow these steps:

1-Calculate the characteristic polynomial $\boldsymbol{p(\lambda)}$

$$\boldsymbol{p(\lambda) = |A - \lambda I| = c_0 + c_1\lambda + c_2\lambda^2 + \cdots + c_n\lambda^n}$$

2-Calculate the roots of the polynomial $\boldsymbol{p(\lambda) = 0}$, where the roots $\boldsymbol{p(\lambda)}$ are the eigenvalues of the matrix.

Example 01:

Find the eigenvalues of the following matrix:

$$\boldsymbol{A = \begin{pmatrix} 3 & 2 \\ 6 & 7 \end{pmatrix}}$$

Solution

$$|A-\lambda I| = \left|\begin{pmatrix} 3 & 2 \\ 6 & 7 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right| = \begin{vmatrix} 3-\lambda & 2 \\ 6 & 7-\lambda \end{vmatrix}$$

$$= (3-\lambda)(7-\lambda) - 12 \rightarrow p(\lambda) = \lambda^2 - 10\lambda + 9$$

$$\Delta = 100 - 4(1)(9) = 64 \Rightarrow \begin{cases} \lambda_1 = \dfrac{10-8}{2} = 1 \\ \lambda_2 = \dfrac{10+8}{2} = 9 \end{cases}$$

Therefore, the matrix has two eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 9$

Example 02:

Find the eigenvalues of the following matrix:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix}.$$

$$|A-\lambda I| = \left|\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right|$$

$$= \begin{vmatrix} 1-\lambda & 0 & -1 \\ 1 & 2-\lambda & 1 \\ 2 & 2 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)\big((2-\lambda)(3-\lambda) - 2\big) - 1\big(2 - 2(2-\lambda)\big)$$

$$p(\lambda) = (1-\lambda)(2-\lambda)(3-\lambda)$$

From this, we find that the eigenvalues are: $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

Example 03: (The eigenvalues belong to the set of complex numbers):

Let $A = \begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$|A-\lambda I| = \left|\begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right|$$

$$= \begin{vmatrix} -1-\lambda & 2 & 1 \\ 0 & -1-\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix}$$

$$= (-1-\lambda) \times \begin{vmatrix} -1-\lambda & 1 \\ 0 & -\lambda \end{vmatrix} + 1 \times \begin{vmatrix} 2 & 1 \\ -1-\lambda & 1 \end{vmatrix}$$

$$= (-1-\lambda)\big((-1-\lambda)(-\lambda)\big) + 1(2-(-1-\lambda))$$

$$= (-1-\lambda)(\lambda+\lambda^2) + 3 + \lambda$$

$$= -\lambda - \lambda^2 - \lambda^2 - \lambda^3 + 3 + \lambda = -\lambda^3 - 2\lambda^2 + 3$$

Then: $p(\lambda) = -\lambda^3 - 2\lambda^2 + 3 = (\lambda - 1)(-\lambda^2 - 3\lambda - 3)$

$$\Delta = 9 - 4(-1)(-3) = -3 \Rightarrow \begin{cases} \lambda_1 = \dfrac{3 - i\sqrt{3}}{-2} \\ \lambda_2 = \dfrac{3 + i\sqrt{3}}{-2} \end{cases}$$

From this, the three eigenvalues of matrix A are: $\lambda_1 = 1, \lambda_2 = \frac{-3+\sqrt{3}i}{2}, \lambda_3 = \frac{-3-\sqrt{3}i}{2}$

<u>Notes:</u>

1- *The eigenvalues of a triangular or diagonal matrix are the elements of its main diagonal:*

Because:

$$|A-\lambda I| = \left|\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}\right| =$$

$$\begin{vmatrix} a-\lambda & 0 & 0 \\ 0 & b-\lambda & 0 \\ 0 & 0 & c-\lambda \end{vmatrix} = (a-\lambda)(b-\lambda)(c-\lambda)$$

2- λ^{n} *The eigenvalue of the matrix* A^{n}

3- λ^{-n} *Eigenvalue of the matrix* A^{-n}

4- $1/\lambda$ *Eigenvalue of the matrix* A^{-1}

5- $|A| = c_0$

6- $p(0) \neq 0$ *Meaning the matrix is invertible*

- Calculate the inverse of the matrix based on the characteristic polynomial $p(\lambda)$ ***:***

The matrix A *is considered the root of the characteristic polynomial, i.e.:*

$$p(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_n A^n = 0$$

which is an equation from which we can derive the inverse.

Example: *Calculate the inverse of the matrix* $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

Calculating the characteristic polynomial

$$p(\lambda) = |A-\lambda I| = (1-\lambda)(2-\lambda) - 1 = \lambda^2 - 3\lambda + 1$$

We know that A *is the root of the distinguished polynomial, i.e.:*

$$p(A) = A^2 - 3A + I = 0 \Longrightarrow (-A^{-1})(A^2 - 3A) = (-A^{-1})(-I)$$

$$\Longrightarrow A^{-1} = -A + 3I = \begin{pmatrix} -1 & -1 \\ -1 & -2 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

2-Eigenrays:

The eigenvector of matrix A, denoted by $\vec{V}$ *, is a non-zero vector corresponding to the eigenvalue* (λ) *, i.e. it is a solution to the system* $(A-)\vec{V}=\vec{0}$

<u>*Example:*</u> $A=\begin{pmatrix}1&0&-1\\1&2&1\\2&2&3\end{pmatrix}$

1- *We found that the eigenvalues are equal to:*

$$p(\lambda)=0\Longrightarrow\lambda_1=1,\lambda_2=2,\lambda_3=3$$

2-Eigenrays: We have the following relationship:

$$A\vec{V}=\lambda\vec{V}\Longrightarrow$$

From this:

$$(A-\lambda I)\vec{V}=\vec{0}$$

From the previous relationship, we derive the eigenvectors for each eigenvalue:

For $\lambda_1=1$.

We have

$$\left[\begin{pmatrix}1&0&-1\\1&2&1\\2&2&3\end{pmatrix}-\begin{pmatrix}\lambda_1&0&0\\0&\lambda_1&0\\0&0&\lambda_1\end{pmatrix}\right]*\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\left[\begin{pmatrix}1&0&-1\\1&2&1\\2&2&3\end{pmatrix}-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right]*\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\begin{pmatrix}0&0&-1\\1&1&1\\2&2&2\end{pmatrix}*\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

And from this:

$$-z = 0$$

$$x + y + z = 0$$

$$2x + 2y + 2z = 0$$

From the previous set of equations, we find:

$$x = -yz = 0$$

And from this:

$$\overrightarrow{V_1} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $_1 = 1$ *is:*

$$\overrightarrow{V_1} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

The result can be verified by applying the previous relationship: $A\vec{V} = \lambda\vec{V}$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} * \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1 * \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

And from this, the relationship is verified.

For$\lambda_2 = 2$.

We have

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda_2 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}\right] * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right] * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & -1 \\ 1 & 0 & 1 \\ 2 & 2 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and from it:

$$-x - z = 0$$

$$x + z = 0$$

$$2x + 2y + z = 0$$

From the previous equations, we find:

$$x = -2yz = 2y$$

And from this:

$$\overrightarrow{V_2} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2y \\ y \\ 2y \end{pmatrix} = y \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue $_2 = 2$ *is:*

$$\overrightarrow{V_2} = \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

The result can be verified by applying the previous relationship: $A\vec{V} = \lambda\vec{V}$

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} * \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 4 \end{pmatrix} = 2 * \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}$$

From this, the relationship is verified.

For$\lambda_3 = 3$ *.*

We have

$$\left[\begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{pmatrix} - \begin{pmatrix} \lambda_3 & 0 & 0 \\ 0 & \lambda_3 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}\right] * \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left[\begin{pmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{pmatrix} - \begin{pmatrix}3 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 3\end{pmatrix}\right] * \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}-2 & 0 & -1\\ 1 & -1 & 1\\ 2 & 2 & 0\end{pmatrix} * \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

And from this:

$$-2x - z = 0$$

$$x - y + z = 0$$

$$2x + 2y = 0$$

From the previous set of equations, we find:

$$x = -yz = 2y$$

And from this:

$$\overrightarrow{V_3} = \begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}-y\\ y\\ 2y\end{pmatrix} = y\begin{pmatrix}-1\\ 1\\ 2\end{pmatrix}$$

Therefore, the eigenvector corresponding to the eigenvalue$\lambda_2 = 3$ *is:* $\overrightarrow{V_3} = \begin{pmatrix}-1\\ 1\\ 2\end{pmatrix}$

The result can be verified by applying the previous relationship: $A\vec{V} = \lambda\vec{V}$

$$\begin{pmatrix}1 & 0 & -1\\ 1 & 2 & 1\\ 2 & 2 & 3\end{pmatrix} * \begin{pmatrix}-1\\ 1\\ 2\end{pmatrix} = \begin{pmatrix}-3\\ 3\\ 6\end{pmatrix} = 3 * \begin{pmatrix}-1\\ 1\\ 2\end{pmatrix}$$

And from this, the relationship is verified.

Chapter summary:

This chapter discussed the concept of intrinsic values and intrinsic rays as two of the most important tools in linear algebra for analysing matrices and linear transformations. It was explained that eigenvalues represent numbers that only change the magnitude of a vector when a linear transformation is applied, while eigenvectors represent non-zero vectors that retain their direction under the influence of this transformation.

The chapter also focused on the mathematical methodology for extracting eigenvalues and eigenvectors through the characteristic equation and the study of determinants, highlighting the relationship between these concepts and matrix properties such as rank and diagonalizability. This provided a deeper understanding of the internal structure of matrices and their behaviour in vector spaces.

The chapter concluded by highlighting the importance of eigenvalues and eigenvectors in various applications, where they are used in stability analysis, solving differential equations, economic modelling, and data processing. This presentation enabled the connection between theory and practical application, paving the way for more advanced topics in linear algebra and its scientific applications.

Conclusion

At the end of this publication entitled "Lectures in Mathematics 2," it is clear that the concepts covered form an indispensable foundation for first-year students pursuing a bachelor's degree in economics, business, and management. The most important principles of linear algebra have been presented in a systematic and gradual manner, starting with matrices and their operations, moving on to rank, determinant and inverse, and ending with the solution of linear equations, eigenvalues and eigenvectors.

This publication aimed to combine precise theoretical understanding with practical application, highlighting the economic and managerial dimensions of these mathematical tools. Matrices, linear systems, and eigenvalues are not just abstract concepts, but essential analytical tools used in economic modelling, equilibrium analysis, data analysis, and decision-making.

In conclusion, we hope that these lectures have contributed to strengthening students' logical thinking and quantitative analysis skills, paving the way for them to pursue advanced metrics with confidence and efficiency. Mathematics is not an end in itself, but rather a precise language for understanding and analysing economic and managerial reality in a rigorous scientific manner.

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