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Thème

Numerical Solutions of Stochastic Differential Equations.

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Dedication

You who have showered me with your sincere prayers, encouraged me with your warm words, and surrounded me with your boundless affection, you who have been my aid and support in every trial and joy. I dedicate to you the fruits of my labor and effort, for you taught me that success is only achieved through perseverance and determination, and that dreams can become reality with hard work and dedication. Dear mother, you have all my love and appreciation. Every moment of happiness and joy I experience today is because of you and your prayers. I hope I have achieved some of what you wish for me, and that I have made you proud.

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In the name of God, the Most Gracious, the Most Merciful. All praise is due to God, by whose grace good deeds are completed. Praise be to God who taught by the pen, taught man what he did not know. Praise be to God for His countless blessings.

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I cannot forget to thank all my honorable professors who played a role in my academic formation, and my friends and colleagues who were always by my side. Lastly, thank you to everyone who supported and encouraged me with a kind word or a sincere prayer.

I ask God to grant us all success in all that is good and righteous. Peace and blessings be upon you.

ملخص

"الحلول العددية للمعادلات التفاضلية العشوائية" تستخدم لتقريب حلول هذه المعادلات التي تتضمن عوامل عشوائية. يستخدم هذا النهج الرياضيات التحليلية والاحتمالية لإيجاد حلول رقمية لمعادلات معقدة تتغير بشكل عشوائي مع مرور الوقت. الكلمات المفتاحية : الطرق العددية، المعادلات التفاضلية العشوائية (ضزس)،المحاكاة،بواسطة مونتي كارلو، طريقة أويلر - ماروياما، طريقة ميلشتين

Abstract

"Numerical Solutions of Stochastic Differential Equations" refers to approximating solutions of equations involving randomness. This approach uses analytical and probabilistic mathematics to find digital solutions to complex equations that change randomly over time.
Keywords : Numerical methods, Stochastic differential equations (SDEs), Monte Carlo simulations, Euler-Maruyama method, Milstein method, Weak convergence,

Résumé

"Les solutions numériques des équations différentielles stochastiques" font référence à l'approximation des solutions des équations impliquant de l'aléatoire. Cette approche utilise les mathématiques analytiques et probabilistes pour trouver des solutions numériques à des équations complexes qui changent de manière aléatoire avec le temps.

Mots cles : Méthodes numériques, Équations différentielles stochastiques (EDS), Simulations de Monte Carlo, Méthode d'Euler-Maruyama, Méthode de Milstein.

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Symbol or abbreviation	the description
SDE	Stochastic Differential Equations
NSE	Ordinary Differential Equations
E	Expectation
Var	Variance
RNG	Random Number Generator
EM	Euler-Maruyama Method
RK	Runge-Kutta Method
BM	Brownian Motion
DW	Differential Wiener
Ω	Open bounded by $\mathbb{R}^n$
B(x)	algebra of linear and bounded space operators X
$\mathbb{F}$	Body being either $\mathbb{C}$ either $\mathbb{R}$
$\mathbb{N}$	Natural numbers
$\sum$	Summation symbol

Reminder

In this thesis, the focus has been on studying and analyzing numerical solutions for stochastic differential equations. The emphasis has been on using numerical methods such as the Euler-Maruyama method and the Milstein method to obtain accurate numerical approximations of solutions, contributing to understanding phenomena influenced by random factors in nature, engineering, and economics.

The research presented includes practical applications and case studies exploring the effectiveness of these methods in solving real-world problems, with a focus on result accuracy, error analysis, and possible future enhancements.

This topic has been presented as an essential part of the research to enhance knowledge and understanding in applied science fields, aiding in the development of models and predictions under complex stochastic conditions.

Introduction

Stochastic differential equations (SDEs) driven by Brownian motions or Lévy processes are important tools in a wide range of applications, including biology, chemistry, mechanics, economics, physics and finance [2, 31, 33, 45, 58]. Those equations are interpreted in the framework of Itô calculus [2, 45] and examples are like, the geometric Brownian motion,

$$dX(t) = \mu X(t)dt + \sigma X(t)dW(t), X(0) = X_0$$

which plays a very important role in the Black-Sholes-Merton option pricing model,
This chapter provides the preliminaries for the whole dissertation. We give an overview of stochastic differential equations driven by Brownian motion or Lévy motion. We shall introduce the existence and uniqueness theorems of such equations. We shall also introduce the connection between stochastic differential equations driven by Brownian motion and partial differential equations, which is indispensable in the analysis of weak approximations of such stochastic differential equations. For a thorough introduction of the theory of stochastic differential equations, we refer to [2, 31, 39, 45].
the first chapter is concerned with the basic notion of SDE and we prove the properties of M.B we study the existence results of solutions of SDE of KOLMOGOROV
In chapter 02 ,we study the Strong Convergence of Euler-Maruyama Approximations of SDEs under Local Lipschitz Conditions and we state the Euler sleme
In addition we preuve the Euler Approximations with Superlinearly Growing Diffusion Coefficients
In chapter 03, Let us consider the following stochastic differential equation:

$$\begin{cases} dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW, t \in [0, T] \\ X(0) = x_0 \end{cases}$$

where $W(t)$ is a one-dimensional Wiener process starting at 0, $X(t)$ is a one- dimensional stochastic process and $\mu(t, x), \sigma(t, x)$ satisfy the following Lipschitz and linear growth condition

$$|\mu(t,x)-\mu(t,y)|+|\sigma(t,x)-\sigma(t,y)|<K(T)|x-y|,t\in[0,T]$$

$$|\mu(t,x)|+|\sigma(t,x)|\le K(T)(1+|x|),t\in[0,T]$$

such that the solution of (4.1) exists and is unique. Since we will use the second moment of the solution in our proof, we also assume that x_0 is independent of $(W(t),0\le t\le T)$ and $E[x_0^2]<\infty$such that

$$E[\sup_{0\le t\le T} X^2]<C(1+E[x_0^2])$$

where the constant C depends only on K and T.

We now give the Euler-Maruyama scheme. In this section, the time step is denoted by $\Delta=T/N$. For any integer i satisfying $0\le i\le N$, set $t_i=i\Delta$. We define at each node in $[0,T]:Y_0:=x_0$ and

$$Y_{i+1}:=Y_i+\mu(t_i,Y_i)\Delta+\sigma(t_i,Y_i)\Delta W_i,0\le i\le N-1$$

where $W_i=W(t_{i+1}-W(t_i)$. The continuous-time approximation is defined as:

$$Y(t):=Y_i+\mu(t_i,Y_i)(t-t_i)+\sigma(t_i,Y_i)(W(t)-W(t_i))$$

$$=Y_i+\int_{t_i}^{t}\mu(t_i,Y_i)ds+\int_{t_i}^{t}\sigma(t_i,Y_i)dW(s)for,t\in[t_i,t_{i+1}]$$

Let us also recall the definition of the weak convergence of a numerical scheme.

Let us also recall the definition of the weak convergence of a numerical scheme. We say that a time discrete approximation Y converges in the weak sense with order $\beta\in[0,\infty)$ if for any function g in a suitable function space there exists a finite constant C and a positive constant δ_0 such that

$$|E[g(X(T))]-E[g(Y_N)]|\le C\delta^{\beta}$$

for any time discretization with maximum step size $\delta\in(0,\delta_0)$

Before we continue, we first define some notations of function spaces. We denote by

$C_b^l([0,T]\times\mathbb{R})$ the space of l times continuously differentiable functions $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ for which all its partial derivatives up to order l are bounded uniformly in t (f may not be bounded).$C_b^l(\mathbb{R})$ is defined in a similar way. We also denote by $C_p^l([0,T]\times\mathbb{R})$ the space of l times continuously differentiable functions $f:[0,T]\times\mathbb{R}\to(\mathbb{R})$ for which all its partial derivatives up to order l have polynomial growth uniformly in t.

Maruyama scheme has weak convergence rate 1 [5, 33, 60]. For example, in [33], if both $\mu(t,x)$ and $\sigma(t,x)$ are homogeneous, it is required that $\mu(x),\sigma(x)^2$ and g(x) are all in the function space $C_p^4(\mathbb{R}^m)$, together with some other conditions. While in [5], although g is only required to be measurable and bounded (or has a polynomial growth),$\mu and \sigma$ are assumed to be homogeneous and to be C^∞ functions with bounded derivatives of any order. See also [25, 34] for other related results.

Due to the close relation between the weak approximation of the solution of (4.1) and the Kolmogorov backward partial differential equation, Malliavin calculus, which is powerful to deal with the derivatives of functions of random variables, can serve as an efficient tool to analyze the approximation error. For example, in [5, 25, 34, 35], techniques from Malliavin calculus, like integration by parts, are used very often to assist to get the expressions of the approximation errors

Chapter 1

Preliminaries on Stochastic Differential Equations

This chapter provides the preliminaries for the whole dissertation. We give an overview of stochastic differential equations driven by Brownian motion or Lévy motion. We shall introduce the existence and uniqueness theorems of such equations.We shall also introduce the connection between stochastic differential equations driven by Brownian motion and partial differential equations, which is indispensable in the analysis of weak approximations of such stochastic differential equations. For a thorough introduction of the theory of stochastic differential equations, we refer to [2, 31, 39, 45].

1.1 Existence and Uniqueness

Throughout this dissertation, $(\Omega, \mathcal{F}.\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}.P)$ will always denote a filtered probability space satisfying the usual hypothesis of right-continuity and completeness. We first consider the stochastic differential equations driven by Brownian motion:

$$\begin{cases} dX(t) = \mu(t.X(t)dt + \sigma(t.X(t))dW(t).t\in [0.T] \\ X(0) = x_0 \end{cases} \tag{1.1}$$

where $X(t) \in \mathbb{R}^n$ for all $t \in [0.t].W(t)$ is a $d-d$imensional Brownian motion (Wiener process) starting at 0, $\mu : [0.T] \times \mathbb{R}^m \longrightarrow \mathbb{R}^m$ and $\sigma : [0.T] \times \mathbb{R}^m \longrightarrow \mathbb{R}^{m\times d}$ also assume that $x_0$is $\mathcal{F}_0$ -measurable and independent of $(W(t), 0 \leq t \leq T)$ The boundedness condition on x_0 can be flexible. For now we only assume that $E[x_0^2]$.

The following Lipschitz and linear growth condition are standard in the theory of stochastic differential equations.

- (Lipschitz condition) For all $x, y \in \mathbb{R}^m$ and all $t \in [0.T]$.

$\|\mu(t.x) - \mu(t.y)\| + \|\sigma(t.x) - \sigma(t.y)\| \leq K(t)\|x - y\|$

- (Linear growth condition) For all $(t.x) \in [0.T] \times \mathbb{R}^m$

$$\|\mu(t.x)\| + \|\sigma(t.x)\| \leq K(t)(1 + \|x\|)$$

In (1.2) and (1.3), the constant K is positive and only depends on T. Sometimes, we also use the following Lipschitz and linear growth condition interchangeably.

- (Lipschitz condition) For all $x.y \in \mathbb{R}^m$ and all $t \in [0.T]$

$$||\mu(t.x) - \mu(t.y)||^2 \vee ||\sigma(t.x) - \sigma(t.y||^2 \leq K(T)||x - y||^2$$

- (Linear growth condition) For all $(t.x) \in [0.T] \times \mathbb{R}^m$

$$||\mu(t.x)||^2 \vee ||\sigma(t.x)||^2 \leq K(T)(1 + ||x||^2)$$

here , $a \vee b := \max(a.b)$ for any $a.b \in \mathbb{R}$

Theorem 1.1.1. *(existence and uniqueness, [31], Theorem 5.4). Suppose μ and σ satisfy the Lipschitz condition (1.2) and linear growth condition (1.3), and x_0 is independent of $.W(t), 0 \leq t \leq \infty$ with $E[||x_0||^2] \leq \infty$ then the equation (1.1) has a unique solution and satisfies*

$$E\Big[\sup_{0 \leq t \leq T} ||X(t)||^2\Big] < C(1 + E[||x_0||^2])$$

where C depends only on K and T.

Throughout this dissertation, we use C ¿ 0 to denote a generic constant which varies at different occurrences. If needed, the parameters on which C depends will also be specified in the parentheses after it.

Actually, the Lipschitz condition can be replaced by the local Lipschitz condition:

- *(Local Lipschitz condition) For every real number R ¿ 0 and T ¿ 0, there exists a positive constant K, depending on T and R, such that for all $t \in [0.T]$ and all $x.y \in \mathbb{R}^m$ with*

$||x||.||y|| \leq R$

$$||\mu(t.x) - \mu(t.y)|| + ||\sigma(t.x) - \sigma(t.y)|| \leq K(T.R)||x - y||$$

This condition is locally Lipschitz in x uniformly in t.

The existence and uniqueness theorem still holds under the local Lipschitz condition.

Theorem 1.1.2. *(existence and uniqueness, [31],) Suppose μ and σ satisfy the local Lipschitz condition (1.6) and linear growth condition (1.3), and x_0 is independent of $(W(t), 0 \leq t \leq T)$ with $E[||x_0||^2] < \infty$, then the equation (2.1) has a unique solution and satisfies*

$$E\left[\sup_{0\leq t\leq T} ||X(t)||^2\right] < C(1 + E[||x_0||^2])$$

where C depends only on K and T.

Having the local Lipschitz condition, many functions such as functions having continuous partial derivatives of first order with respect x_0 x on $[0.T] \times \mathbb{R}$ can serve as the drift and diffusion coefficients. But it still excludes some common functions like $-|x|^2 = x$ as the coefficients. The following theorem relaxes the linear growth condition.

Theorem 1.1.3. *(existence and uniqueness, [39]) Assume that the local*

Lipschitz condition (1.6) holds, but the linear growth condition (1.5) is replaced with the following monotone condition: there exists a positive constant C such that for all $(t.x) \in [0.T] \times \mathbb{R}$

$x^T\mu(t.x) + \frac{1}{2}||\sigma(t.x)||^2 \leq C(1 + ||x||^2)$

Then there exists a unique solution X(t) to equation (2.1) and satisfies

$$E\int_0^T ||X(t)||^2 dt < \infty$$

For example, consider the following SDE:

$dX(t) = [X(t) - X^3(t)dt + X^2(t)dW(t)], t \in [0.T]$ *// Although the coefficients are local Lipschitz continuous, they do not satisfy the linear growth condition. Nevertheless, the monotone condition is satisfied:*

$x(x - x^3) + \frac{1}{2}x^4 \leq x^2 \leq 1 + x^2$

Therefore by Theorem 1.3, it admits a unique solution.

We conclude this section by giving the L^p *-estimates of the solution of (1.1).*

Theorem 1.1.4. *Assume X(t) is the unique solution of the equation (1.1). Let* $p \geq 2$ *and* $x_0 \in L^p(\omega : \mathbb{R}^m)$*. Assume that there exists a constant*

$a > 0$ *such that for all* $(t.x) \in [0.T] \times \mathbb{R}^m$

$x^T\mu(t.x) + \frac{p-1}{2}||\sigma(t.x)||^2 \leq \alpha(1 + ||x||^2)$

them

$E[||X(t)||^p] \leq C := 2^{\frac{p-2}{2}}(1 + E[||x_0||^p e^{p\alpha t})$

for all $t \in [0.T]$

Note that the linear growth condition (2.3) is just a special case of (2.8). So the above L^p*-estimate is also true if the linear growth condition is fulfilled.*

Corollary 1.1.1. *Let* $p \geq 2$ *and* $x_0 \in (\omega; \mathbb{R}^m)$ *Assume that the linear growth condition (1.3) holds. Then inequality (1.9) holds with* $a = \sqrt{K} + K(p-1)/2$

1.2 Stochastic Differential Equations and Partial Differential Equations

There is a close relation between stochastic differential equations and partial differential equations (PDE). The Kolmogorov backward equation is one of the most important and *useful relations between the two. This PDE will play an important role in our analysis of weak approximations of the solutions of SDEs in Chapter 4.*

Theorem 1.2.1. *let $X(t)$ be the solution of the equation (1.1) with $m = 0$ Assume that the coefficients $\mu(t.x)$ and $\sigma(t.x)$ are locally Lipschitz and satisfy the linear growth condition. Assume in addition that they possess continuous partial derivatives with respect to x up to order two, and that they have at most polynomial growth. If $g(x)$ is twice continuously differentiable and satisfies together with its derivatives a polynomial growth condition, then the function*

$$f(t.x) = E[g(X(T))|X(T) = x] \text{ satisfies}$$

$$\begin{cases} \frac{\partial f}{\partial t}(t.x) + \mu(t.x) + \frac{1}{2}\sigma^2(t.x)\frac{\partial^2 f}{\partial x^2}(t.x) = 0, t \in [0.T].x \in \mathbb{R} \\ f(t.x) = g(x) \end{cases}$$

The multi-dimensional version can be found in, e.g. [45]. For convenience, we often use the following second order differential operator

$$L_t f(t.x) := \mu(t.x)\frac{\partial f}{\partial x}(t.x) + \frac{1}{2}\sigma^2(t.x)\frac{\partial^2 f}{\partial x^2}(t.x)$$

and write equation (1.10) as

$$\begin{cases} \frac{\partial f}{\partial t} + L_t f(t.x) = 0.t \in [0.T], x \in \mathbb{R} \\ f(T.x) = g(x) \end{cases}$$

To discuss the differential properties of the function f(t, x), we use a more general version of equation (1.1):

$$\begin{cases} dX(\theta) = \mu(\theta.X(\theta))d\theta + \sigma(\theta.X(\theta))dW(\theta).\theta \in [t.T] \\ X(t) = x \end{cases}$$

Note that here $X(\theta)$ is still $\mathbb{R}^m$ valued. We write the solution in the form $X^{t.x}(\theta)$ to represent the dependence of$X(\theta)$ on the initial data $(t.x)$ Therefore, $f(t.x)$ can be written as $f(t.x)[g(X^{t.x}(T))]$

The differentiability of $X^{t.x}(\theta)$ with respect to x depends on the smoothness of the coefficients μ and σ

Proposition 1.2.1. *([38], Theorem 2.3.3). Let k be a positive integer and* $0 < \alpha \leq 1$ *Suppose that coefficients* μ *and* σ *are* $C^{K.\alpha}$ *functions of* x *for some* α *and their derivatives up to k-th order are bounded. Then the solution* $X^{t.x}(\theta)$ *is a* $C^{K.\beta}$ *function of* x *for any* β *less than* α

With this theorem, we can discuss the differentiability of $f(t.x)$ to its definition $f(t.x) = E[X^{t.x}(T)]$. More details can be found in Chapter 3.

1.3 Numerical Solutions of Stochastic Differential Equations Driven by Brownian Motion

It is common that the equation (2.1) does not have a closed-form solution in many cases, where a numerical solution becomes a necessity. But unlike in the deterministic differential equations, there are two kinds of convergence of the numerical solutions of SDEs. The first kind of convergence is the strong convergence.

Definition 1.3.1. *Suppose* Y *is a discrete-time approximation of the solution* $X(t)$*of (1.1) with maximum step size* $\Delta > 0$ *We say that* Y *converges to* $X(t)$ *in the strong sense with order* $\gamma \in [0.\infty[$ *if there exists a finite constant* $C > 0$ *and a positive constant*$\Delta > 0$ *such that*

$$E[\|X(T) - Y(T)\|] \leq C\Delta^{\gamma}$$

for any time discretization with maximum step size $\in [0.\Delta_0]$
The other kind of convergence is the weak convergence.

Definition 1.3.2. *Suppose* Y *is a discrete-time approximation of the solution* $X(t)$ *of (1.1) with maximum step size* $\Delta > 0$ *We say that* Y *converges to* $X(t)$*in the weak sense with order* $\beta \in [0.\infty[$ *if for any function* $g : \mathbb{R}^m \rightarrow \mathbb{R}$ *in a suitable function space there exists a finite constant* $C > 0$ *and a positive constant* Δ_0 *such that*

$$|E[g(x(T))] - E[g(Y(T))]| \leq C\Delta^{\beta}$$

for any time discretization with maximum step size $\Delta \in [0.\Delta_0]$

The function space in the Definition 1.8 can be flexible. For example, it can be the space of all polynomial functions. It can also be the space $C_p^k(\mathbb{R}^m$ *in which all*

the functions are k-th continuously differentiable and all their partial derivatives up to order k have polynomial growth.

Complete reviews of the numerical solutions of SDEs driven by Brownian motion can be found in, e.g. [22, 33, 48].

In the following we first introduce the most commonly used numerical scheme to solve (2.1), the Euler-Maruyama (EM) scheme. Given a fixed integer $N > 0$, set the time step $\Delta t = T|N$. For any integer k satisfying $0 \leq k \leq N-1$,set $t_k = k\Delta t$ We define at each node in $[0.T] = Y_0 = x_0$ and

$$Y_{k+1} := Y_k + \mu(t_k.Y_k)\Delta t + \sigma(t_k.Y_k)\Delta W_k.0 \leq k \leq N-1$$

where $\Delta W_k = W(t_k+1) - W(t_k)$. Furthermore, we define the continuous-time approximation of the solution of (2.1) as:

$$\bar{Y}(t) := Y_k + \mu(t_k.Y_k)(t-t_k) + \sigma(t_k.Y_k)(W(t) - w(t_k))$$
$$= Y_k + \int_{t_k}^{t} (t_k.Y_k)ds + \int_{t_k}^{t} \sigma(t_k.Y_k)dW(s)$$

for $t \in [t_k.t_k+1]$

It is obvious that $\bar{Y}(t_k) = Y_k$ If we define the shift operator in the following way:

$$\eta(t) = t_k, t \in [t_k.t_{k+1}]$$

then scheme (2.16) can be written as

$$\bar{Y}(t) = Y_0 + \int_0^t \mu(\eta(t).\bar{Y}(\eta(t)))ds + \int_0^t \sigma(\eta.\bar{Y}(\eta(t)))dW(s).t \in [0.T]$$

Similar to getting Corollary 2.4.1, if $\mu(t.x)$ and $\sigma(t.x)$ satisfy the linear growth condition and $x_0 \in L^p(\omega), p \geq 2$, it then follows that,

$$\sup_{0\leq t\leq T} \|Y\bar{(}t)\| \in L^p(\omega)$$

There are also many other numerical schemes to solve (2.1), such as the Milstein scheme, the Runge-Kutta type scheme, the Itô-Taylor expansion scheme, etc.. We refer to [33] for a thorough treatment of the common numerical schemes we encounter in the field of numerical solutions of SDEs.

For the Euler-Maruyama approximate solutions, the strong convergence order is $\frac{1}{2}$ This is the following theorem.

Theorem 1.3.1. *([39], Theorem 2.7.3). Assume that the Lipschitz condition (2.2) and the linear growth condition (2.3) hold. Let $X(t)$ be the unique solution of equation (2.1), and $\bar{Y}(t)$ be its Euler-Maruyama approximate solution. Then*

$$E[\sup_{0\leq t\leq T} \|\bar{Y}(t) - X(t)\|^2] \leq \frac{C}{N}$$

if$\mu$and σ do not depend on the time variable t and satisfy the global Lipschitz condition, then for any $p > 1$, we also have

$$E[\sup_{0\leq t\leq T} \|X(t) - \bar{Y}(t)\|^p] \leq \frac{C(p.t}{N^{p|2}}$$

See e.g. Bouleau and Lepingle [8].

In fact, the above convergence is stronger than the strong convergence we defined in Definition 2.7. The error estimation is uniform with respect to the whole sample path rather than just the terminal value of it. This advantage can be very useful in many cases. For example, in the context of the Asian options, the payoff of the option at the expiration time depends on the average of the whole sample path

$$\frac{1}{T}\int_0^T S(t)dt$$

where $S(t)$ is the price of underlying stock.

We also remark that the assumed globally Lipschitz condition and the linear growth condition in Theorem 2.9 are very strong conditions and may fail to hold in many situations. For example, the one-dimensional stochastic Ginzburg-Landau equation takes the form

$$\begin{cases} dX(t) = (X(t) - X^3(t))dt + X(t)dW(t).t \in [0.1] \\ X(0) = 1 \end{cases}$$

The drift coefficient takes the form $\mu(x) = x - x^3$, which is clearly not globally Lipschitz continuous. But it is continuously differentiable and thus locally Lipschitz continuous. In fact, the family of SDEs with C^1 drift and diffusion coefficients consist of a very large part of SDEs we encounter in research. Hu [23] and Higham, Mao and Stuart [24] were the first to study the strong convergence problem of EM approximate solutions under the non-globally Lipschitz continuous conditions. After that, the study of numerical solutions of SDEs with local Lipschitz continuous coefficients has been a very active area. We will give a thorough introduction of such problems in Chapter 2.

As for the weak convergence of the Euler-Maruyama scheme, the convergence order is typically 1, but it can be under different conditions. For example, in Theorem 14.1.5 of [33], to achieve weak convergence order 1, it assumes $\mu(t.x)$ and $\sigma(t.x)$ being twice continuously differentiable and the test function $g(x)$ being fourth continuously differentiable, together with some other conditions (Hölder continuity, etc.). While in Theorem 14.5.1 of [33], it assumes a homogeneous equation and functions $\mu(x).\sigma(x)$ and $g(x)$ being in $C_p^4(\mathbb{R}^m)$ among some other conditions. Both of the two theorems require strongly smooth conditions on $\mu.\sigma$ and g measurable and bounded (or with polynomial growth), but $\mu(x)$ and $\sigma(x)$ are required to be c^∞ with bounded derivatives of any order. In Chapter 3, we will give some improvements of the conditions assumed on $\mu.\sigma$ and g

1.4 Stochastic Differential Equations Driven by Lévy Processes

Unlike Brownian motion, Lévy processes are stochastic processes allowing for jumps in their sample paths. In the following, we will give a short introduction to such processes. For an excellent and intuitive introduction to Lévy processes, we refer to [46]. We also refer to [56] for a thorough introduction to the infinitely divisible distributions and [2] for *stochastic calculus with respect to Lévy processes. For Lévy processes in finance, see e.g. [10, 49, 57, 58].*

In general, we say a càdlàg (right continuous with left limits) and adapted stochastic process $L = L(t), 0 \leq t \leq T$ *defined on a filtered probability space* $((\Omega), \mathcal{F}.\mathbb{F} = (\mathcal{F}_t)_t \leq 0.p)$ *is a Lévy process if the following conditions are satisfied:*

(**L1**) $L(0) = 0a.s.$;

(**L2**) *L has independent and stationary increments;*

(**L3**) *L is stochastically continuous, i.e. for all* $a > 0$ *and for all* $s \geq 0$

$$\lim_{t \to s} P(\|L(t) - L(s)\| > a = 0$$

By definition, Brownian motion is a special Lévy process. Other examples of Lévy processes are like Poisson process, compound Poisson process, $\alpha-$*stable process, etc. [2, 7, 10, 56].*

It is often convenient to use Poisson random measure to analyze the jumps of a Lévy process. Consider a set $A \in \mathcal{B}(\mathbb{R}^d|\{0\})$ *such that* $0 \notin \bar{A}$ *and let* $0 \leq t \leq T$ *Define the random measure of the jumps of the process L by*

$$N(\omega.t.A) = \sharp\{0 \leq s \leq t; \Delta L(s.\omega) \in A\} = \sum_{s \leq t} \mathbb{1}_A(\Delta L(s.\omega))$$

Therefore,$N(\omega.t.A)$ *counts the jumps of the process L of size in A up to time t. It can be verified that for fixed* $N(\omega.t.A)$ *is a Poisson process with intensity* $\nu(A) = E[N(\omega.1.A)]$ *Poisson random measure. The compensated Poisson measure is then defined as*

$$\bar{N} := N(t.A) - t\nu(A)$$

Definition 1.4.1. *The measure* ν *defined by*

$$\nu(A) = E[N(\omega.1.A)] = E\Big[\sum_{s \leq 1} \mathbb{1}_A(\Delta L(s.\omega))\Big]$$

is called the Lévy measure of the Lévy process L.

In general, the Lévy measure describes the expected number of jumps of a certain size in a time interval of length 1 and satisfies

$$\nu(\{0\}) = 0, and \int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\nu(dx) < \infty$$

It can be proved that if $\nu(\mathbb{R}^d) < \infty$then almost all paths of L have a finite number of jumps on every compact interval. In this case, the Lévy process has finite activity. If $\nu(\mathbb{R}^d) = \infty$ then almost all paths of L have an infinite number of jumps on everyvcompact interval. In this case, the Lévy process has infinite activity. See e.g. Theorem 21.3 in Sato [56] for the proof.

We can also define an integral with respect to the Poisson random measure N.Consider a set $A \in B(\mathbb{R}^d)/\{0\})$ $0 \notin \bar{A}$ and a function $f : \mathbb{R}^d \to \mathbb{R}^m$, Borel measurable and finite on A. We define the integral with respect to N as follows:

$$\int_A f(x)N(t.dx) = \sum_{s \le t} f(\Delta L(s)) 1_A(\Delta L(s)).$$

Note that the above integral is a $\mathbb{R}^m$-valued stochastic process. In the following, we use

$$\int_0^t \int_A f(x)N(ds.dx)$$

to denote this process. Similarly, for $f \in L^1(A)$, we define

$$\int_0^t \int_A f(x)N(d\bar{s}.dx) = \int_0^t \int_A f(x)N(ds.dx) - t \int_A f(x)\nu(dx)$$

With the help of Poisson random measure, we have the following decomposition of a Lévy process.

Theorem 1.4.1. *(Lévy-Itô Decomposition, [2]).*

Let L be a $\mathbb{R}^d$ -valued Lévy process, then there exists $b \in \mathbb{R}^d$, a Brownian motion WA(t) with covariance matrix A and an independent Poisson random measure N on $\mathbb{R}^+ \times$

$(\mathbb{R}^d - (\{0\}))$ *such that, for each* $t \geq 0$

$$L(t) = bt + W_A(t) + \int_0^t \int_{\|x\|<1} x\tilde{N}(ds.dx) + \int_0^t \int_{\|x\|\geq 1} xN(ds.dx)$$

Sometimes it is convenient to write

$$W_A(t) = (W_A^1(t), ..., W_A^d(t))$$

in the form

$$W_A^i(t) = \sum_{j=1}^{m} \sigma_j^i W^j(t)$$

where $W^1, ..., W^m$ *are standard one-dimensional Brownian motions and* σ *is a* $d \times m$ *real-valued matrix for which* $\sigma\sigma^T = A$ *If* L *is only a real-valued Lévy process,the term* $W_A(t)$ *can be replaced by* $\sigma W(t)$ *, where* $\sigma \geq 0$ *and*$W(t)$ *is a standard one-dimensional Brownian motion.*

It can be proved that $L(t) \in Lp(\Omega), \mathbf{p} \geq 1$ *if and only if* $\int_{\|x\|\geq 1} \|x\|^{\mathbf{p}} \nu(dx) \infty$ *In particular,* $L(t) \in L^1(\Omega)$ *if and only if* $\int_{\|x\|\geq 1} \|x\| \nu(dx) \infty$ *Therefore, if we assume*

$E[\|L(t)\|] < \infty$ *, we can rewrite (2.22) as*

$$L(t) = b_1 t + W_A(t) + \int_0^t \int_{\mathbb{R}^d} x\tilde{N}(ds.dx)$$

where $b_1 = b + \int_{\|x\|\geq 1} x\nu(dx)$ *In general, the simulation of a Lévy process is more complex than a Brownian motion. The simulation method varies from one kind of Lévy process to another. We refer to Cont and Tankov [10], Platen and Bruti-Liberati [49], Asmussen and Rosinski [3], Rosinski [52, 53] for the details.*

In view of (2.22), we consider the following stochastic differential equation driven by a stochastic process with jumps,

$$X(t) = X(0) + \int_0^t a(X(s-))ds + \int_0^t b(X(s-))dW(s)$$

$$+ \int_0^t \int_{\|y\|<1} f(X(s-).y)\tilde{N}(ds.dy) + \int_0^t \int_{\|y\|\geq 1} g(X(s-).y)N(ds.dy)$$

where $X(0)$ is $\mathcal{F}_0$ measurable,$X(t)$ is a $\mathbb{R}^m$ -valued stochastic process, W and N are independent of $\mathcal{F}_0, a; \mathbb{R}_m \to \mathbb{R}^m, b : \mathbb{R}^m \to \mathbb{R}^{m\times d}, f : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^m$ and $g : \mathbb{R}^m \times \mathbb{R}^d \to \mathbb{R}^m$ There exists a unique solution to the equation if the following conditions are satisfied [2, 50]:

- *Lipschitz condition: there exists a constant $C > 0$ such that for all $x_1, x_2 \in \mathbb{R}^m$.*

$$\|a(x_1) - a(x_2)\|^2 + \|B(x_1.x_1) - 2B(x_1.x_2) + B(x_2.x_2)\|$$

$$+ \int_{\|y\|<1} \|f(x_1.y) - f(x_2.y)\|^2 \nu(dy) \leq C\|x_1.x_2\|^2$$

- *Growth condition: there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^m$.*

$$\|a(x)\|^2 + \|B(x.x)\| + \int_{\|y\|<1} \|f(x.y)\|^2 \nu(dy \leq C(1 + \|x\|^2);$$

- *Big jump condition: g is jointly measurable and $x \longleftarrow g(x.y)$ is continuous for any $y \in [y : \|y\| \geq 1]$*

Here $B(x_1.x_2) = b(x_1)b(x_2)^T$ and we use the seminorm on the matrix B:

$$\|B\| = \sum_{i=1}^{m} |B_{ii}|$$

In view of (2.23), we sometimes consider the following SDE:

$$X(t) = X(0) + \int_0^t a(x(s-))ds + \int_0^t b(x(s-))dW(s) + \int_0^t \int_{\mathbb{R}^d} f(X(s-)), y)\tilde{N}(ds, dy).$$

It can be proved that there exists a unique solution to (2.25) if the following conditions are satisfied [21]:

$\mathbf{A-1}$. *There exists a constant C such that for any $x \in \mathbb{R}^m$*

$$\langle x, a(x)\rangle + \|\sigma(x)\|^2 + \int_{\mathbb{R}^d} \|f(x, y)\|^2 \nu(dy) \leq C(1 + \|x\|^2)$$

$\mathbf{A-2}$ *For every $R > 0$ there exists a constant $C(R)$ depending on R, such that for any $\|x_1, x_2\| \leq R$*

$$\langle x_1 - x_2, a(x_1) - a(x_2)\rangle + \|b(x_1) - b(x_2)\|^2 + \int_{\mathbb{R}^d} \|f(x_1, y) - f(x_2, y)\|^2 \nu(dy)$$

$$\leq C(R)\|x_1 - x_2\|^2$$

A − 3 *The function $a(x)$ is continuous in $x \in \mathbb{R}^m$ Condition A-1 is a monotone condition. Condition A-2 states that a satisfies the one-sided local Lipschitz condition and b and f satisfy the local Lipschitz condition. Furthermore, if $E[\|X(0)\|^p] < \infty$ for some $p \geq 2$ and if there exists a constant $C_1 > 0$ such that*

$$\int_{\mathbb{R}^d} \|f(x,y)\|^p \nu(dy) \leq C_1(1 + \|x\|^p)$$

for any $x \int \mathbb{R}^m$, then we have

$$E[\sup_{0\leq t\leq T} \|X(t)\|^p] < C$$

with $C := C(T.p, C_1, E[\|X(0)\|^p])$. See e.g. Lemma 1.2 in [11] for the proof.

Chapter 2

Strong Convergence of Numerical Approximations of SDEs Driven by Brownian Motion under Local Lipschitz Conditions

2.1 Strong Convergence of Euler-Maruyama Ap- proximations of SDEs under Local Lipschitz Conditions

We first consider the following stochastic differential equation with homogeneous coefficients:

$$\begin{cases} dX(t) = \mu(x(t))dt + \sigma(X(t))dW(t), t \in [0.T] \\ X(0) = x_0 \end{cases} \tag{2.1}$$

Like in the assumptions of (2.1), $X(t) \in \mathbb{R}^m$ for all $t \in [0,T], W(t)$ is a d-dimensional Brownian motion starting at 0, $\mu : \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : \mathbb{R}^m \to \mathbb{R}^{m\times d}, x_0$ is $\mathcal{F}-_0$ measurable and independent of $(W(t), 0 \leq t \leq T)$ But in this section, we assume that all the pth moments of $x_0, p > 0$ are finite.

In this case, the local Lipschitz condition is

• (Local Lipschitz Condition) For every real number $R > 0$, there exists a positive constant C, depending only on R, such that for all $\|x\|, \|y\| \leq R$

$$\|\mu(x) - \mu(y)\| + \|\sigma(x) - \sigma(y)\| \leq C(R)\|x - y\|$$

$$\|\mu(x) - \mu(y)\|^2 \vee \|\sigma(x) - \sigma(y)\|^2 \leq C(R)\|x - y\|^2$$

We still use the notations from section (2.3) to express the approximate solution of (3.1). Given the homogeneous coefficients, the discrete approximation in this section takes the form:

$$Y_{K+1} := Y_K + \mu(Y_K)\Delta t + \sigma(Y_K)\Delta W_K, 0 \leq K \leq N - 1$$

Furthermore, the continuous-time approximation of the solution of (3.1) is

$$\bar{Y}(t) := Y_K + \mu(Y)(t - t_K) + (Y_K)(W(t) - W(t_K))for, t \in [t_K, t_{K+1}]$$

Sometimes, it is more convenient to work with the equivalent definition

$$\bar{Y}(t) := Y_0 + \int_0^t \mu(Y(s))ds + \int_0^t \sigma(Y(s))dW(s)$$

where $Y(t)$ is defined by

$$Y(t) := Y_K, for, t \in t[t_K, t_{K+1}]$$

It is obvious that $\bar{Y}(t_K) = Y(t_K) = Y_K$. The following theorem is about the strong convergence of the Euler-Maruyama approximate solution of equation (3.1) under the local Lipschitz condition.

Theorem 2.1.1. *Suppose the coefficients μ and σ in equation (3.1) satisfy the local Lipschitz condition and for some $p > 2$ there is a constant A such that*

$$E\Big[\sup_{0\leq t\leq T} \|\bar{Y}(t)\|^p\Big] \vee E\Big[\sup_{0\leq t\leq T} \|X(t)\|^p\Big] \leq A$$

Then the Euler-Maruyama solution (3.5) satisfies

$$\lim_{\Delta t\to 0} E[\sup_{0\leq t\leq T} \|\bar{Y}(t) - X(t)\|^2] = 0$$

$$\tau_R := inf(t \geq 0 : \|Y\bar{(t)}\| \geq= R)$$

$$\rho_R := inf(t \geq 0 : \|X(t)\| \geq= R)$$

$$\theta_R := \tau_R \wedge \rho_R$$

and

$$e(t) := \bar{Y}(t) - X(t)$$

Recall the Young inequality: for $r_1 + q_1 = 1$

$$ab \leq \frac{\delta}{r} a^r + \frac{1}{q^{\delta q/r}} b^q, \forall a, b, \delta > 0$$

We thus have for any $\delta > 0$

$$E\Big[\sup_{0\leq t\leq T} \|e(t)\|^2\Big] = E[\sup_{0\leq t\leq T} \|e(t)\|^2 \mathbb{1}_{(\tau_R > T, \rho_R > T)}] + E\Big[\sup_{0\leq t\leq T} \|e(t)\|^2 \mathbb{1}_{(\tau_R > T, or, \rho_R > T)}\Big]$$

$$\leq E\Big[\sup_{0\leq t\leq T} \|e(t\wedge\theta_R)\|^2 \mathbb{1}_{(\theta_R)}\Big] + \frac{2\delta}{p} E\Big[\sup_{0\leq t\leq T} \|e(t)\|^p\Big]$$

$$+ \frac{1-2/p}{\delta^{2/(p-2)}} P(\tau_R \leq T or \rho_R \leq T)$$

Now

$$P(\tau_R \leq T) = E\Big[1_{(\tau_R\leq T)} \frac{\|\bar{Y}(\tau_R)\|^p}{R^p}\Big] \leq \frac{1}{R^p} E[\sup_{0\leq t\leq T} \|\bar{Y}(t)\|^p] \leq \frac{A}{R^p}$$

using (3.7). A similar result can be derived for ρ_R *such that*

$$P(\tau_R \leq T, or, \rho_R \leq T) \leq P(\tau_R \leq T) + P(\rho_R \leq T) \leq \frac{2A}{R^p}$$

Using these bounds along with

$$E\Big[\sup_{0\leq t\leq T} \|e(t)\|^p\Big] \leq 2^{p-1} E[\sup_{0\leq t\leq T} (\|\bar{Y}(t)\|^p + \|X((t)\|^p)] \leq 2^p A$$

in (3.9) gives

$$E\Big[\sup_{0\leq t\leq T} \|e(t)\|^2\Big] \leq E\Big[\sup_{0\leq t\leq T} (\|\bar{Y}(t\wedge\theta_R) - X(t\wedge\theta_R)\|^2)\Big]$$

$$+ \frac{2^{p+1}\delta A}{P} + \frac{(P-2)2A}{p\delta^{2/(P-2)} R^p}$$

Using

$$X(t\wedge\theta_R) := x_0 + \int_0^{t\wedge\theta_R} \mu(X(s))ds + \int_0^{t\wedge\theta_R} \sigma(X(s))dW(s)$$

(3.6) and Cauchy-Schwarz, we have

$$\|\bar{Y}(t \wedge \theta_R) - X(t\theta_R)\|^2$$

$$\| \int_0^{t\wedge\theta_R} \mu(Y(s)) - \mu(X(s))ds + \int_0^{t\wedge\theta_R} \sigma(Y(s)) - \sigma(X(s))dW(s)\|^2$$

$$\leq 2[T \int_0^{t\wedge\theta_R} \|\mu(Y(s)) - \mu(X(s))\|^2 ds + \| \int_0^{t\wedge\theta_R} \sigma(Y(s)) - \sigma(X(s))dW(s)\|^2]$$

From the local Lipschitz condition (3.3) and Doob's martingale inequality (A.9) we have for any $\tau \leq T$

$$E\Big[\sup_{0\leq t\leq\tau} \|\bar{Y}(t \wedge \theta_R) - X(t \wedge \theta_R)\|^2\Big]$$

$$\leq 2C(R)(T+4)E\int_0^{\tau\wedge\theta_R} \|Y(s) - X(s)\|^2 ds$$

$$\leq 4C(R)(T+4)E\int_0^{\tau\wedge\theta_R} (\|Y(s) - \bar{Y}(s)\|^2 + \|\bar{Y}(s) - X(s)\|^2) ds$$

$$\leq 4C(R)(T+4)(E\int_0^{\tau\wedge\theta_R} \|Y(s) - \bar{Y}(s)\| ds + E\int_0^{\tau} \|\bar{Y}(s-\theta_R) - X(s-\theta_R)\| ds)$$

$$\leq 4C(R)(T+4)(E\int_0^{\tau\wedge\theta_R} \|Y(s) - \bar{Y}(s)\|^2 ds$$

$$+ \int_0^{\tau} E[\sup_{0\leq r\leq s} \|\bar{Y}(r \wedge \theta_R) - X(r \wedge \theta_R)\|^2] ds)$$

To bound the first term in the parentheses on the right-hand side of (3.11), given $s \in [0, T \wedge \theta_R]$ *let* K_s *be the integer for which* $s \in [t_{K_s}, t_{K_s+1}]$ *. Then*

$$Y(s) - \bar{Y}(s) = Y_{K_s} - (Y_{K_s} + \int_{t_{K_s}}^{s} \mu(Y(s))ds + \int_{t_{K_s}}^{s} \sigma(Y(s))dW(s))$$

$$= -\mu(Y_{K_s})(s - t_{K_s}) - \sigma(Y_{K_s})(W(s) - W(t_{K_s}))$$

Therefore,

$$\|Y(s) - \bar{Y}(s)\|^2 \leq 2(\|\mu(Y_{K_s})\|^2 \Delta t^2 + \|\sigma(Y_{K_s})\|^2 \|W(s) - W(t_{k_s})\|^2)$$

By the local Lipschitz condition (3.3), for $\|x\| \leq R$*we have*

$$\|\mu(x)\|^2 \leq 2(\|\mu(x) - \mu(0)\|^2 + \|\mu(0)\|^2 \leq 2(C(R)\|x\|^2 + \|\mu(0)\|^2)$$

and, similarly,

$$\|\sigma(x)\|^2 \leq 2(C(R)\|x\|^2 + \|\sigma(0)\|^2$$

Hence, in (3.12),

$$\|Y(s) - \bar{Y}(s)\|^2 \leq 4(C(R)\|Y_{K_s}\|^2 + \|\mu(0)\|^2 \vee \|\sigma(0)\|^2)(\Delta t^2 + \|W(s) - W(t_{K_s})\|^2)$$

Using (3.7) and the Lyapunov inequality (A.8)

$E\int_0^{\tau\wedge\theta_R} \|Y(s) - \bar{Y}(s)\|^2 ds$

$\leq E\int_0^{\tau\wedge\theta_R} 4(C(R)\|Y_{K_s}\|^2 + \|\mu(0)\|^2 \vee \|\sigma(0)\|^2)((\Delta t)^2 + \|W(s) - W(t_{K_s})\|^2)ds$

$\leq \int_0^{\tau} 4E[(C(R)\|Y_{K_s}\|^2 + \|\mu(0)\|^2 \vee \|\sigma(0)\|^2)((\Delta t)^2 + \|W(s) - W(t_{K_s})\|^2)]ds$

$\leq \int_0^{\tau} 4(C(R)E[\|Y_{K_s}\|^2] + \|\mu(0)\|^2 \vee \|\sigma(0)\|^2)((\Delta t)^2 + \|W(s) - W(t_{K_s})\|^2)ds$

$\leq 4T(C(R)A^{2/p} + \|\mu(0)\|^2 \vee \|\sigma(0)\|^2\Delta t(\Delta t + d)$

In (3.11) we then have

$$E\Big[\sup_{0\leq t\leq\tau} \|\bar{Y}(t\wedge\theta_R)\|^2\Big]$$

$$\leq 16C(R)(T+4)T\Delta t(\Delta t + d)(C(R)A^{2/p} + \|\mu(0)\|^2 \vee \|\sigma(0)\|^2)$$

$$+4C(R)(T+4)\int_0^{\tau} E\sup_{0\leq r\leq s}[\|\bar{Y}(r\wedge\theta_R) - X(r\wedge\theta_R)\|^2]ds$$

Applying the Gronwall inequality (A.7) we obtain

$$E\sup_{0\leq t\leq T}[\|\bar{Y}(t\wedge\theta_R) - X(t\wedge\theta_R)\|^2] \leq C\Delta t(C(R)^2 + 1)e^{4C(R)(T+4)}$$

where C is a universal constant independent of Δt R and δ . Inserting this into (3.10) gives

$$E\Big[\sup_{0\leq t\leq T} \|e(t)\|^2\Big] \leq C\Delta t(C(R)^2 + 1)e^{4C(R)(T+4)} + \frac{2^{p+1}\delta A}{p} + \frac{(1-2/p)2A}{\delta^{2/(p-2)R^p}}$$

Given any $\varepsilon > 0$ we can choose δ so that $2^{p+1}\delta A/p < \varepsilon/3$, then choose R so that

$$\frac{(1-2/p)2A}{\delta^{2/(p-2)R^p}} < \varepsilon/3$$

and then choose Δt *sufficiently small for*

$$C\Delta t(C(R)^2+1)e^{4C(R)(T+4)} \le \varepsilon/3$$

so that in (3.13),

$$E\Big[\sup_{0\le t\le T} \|e(t)\|^2\Big] \le \varepsilon$$

as required.

This theorem establishes the strong convergence of Euler-Maruyama approximate solutions of (3.1). But the bounded condition it assumes for the pth moment of $X(t)$ *and* $\bar{Y}(t)$*is not satisfying. Although it may be possible to verify the bound of the pth moment of X(t), as many textbooks have done it, the bound of the pth moment of* $\bar{Y}(t)$ *is not satisfying. Although it may be possible to verify the bound of the pth moment of* $X(t)$ *, as many textbooks have done it, the bound of the pth moment of* $\bar{Y}(t)$ *is often very difficult to verify and sometimes may fail to hold. Besides, the convergence (3.8) is a very general one and does not involve the explicit convergence rate*

To remove the bound restriction on the pth moment of $\bar{Y}(t)$ *, the same author in [23] proposed a new numerical scheme called the split-step backward Euler (SSBE) method, which is defined by taking* $z_0 = x_0$ *and*

$$Z_K^* = Z_K + \Delta t \mu(Z_K^*)$$

$$Z_{K+1} = Z_K^* + \sigma(Z_K^*)\Delta W_K$$

They proved that the new SSBE method converges strongly without assuming any bound of the pth moment of the approximate solution. But more restrictions on the drift and diffusion coefficients are needed. This is the following theorem.

Theorem 2.1.2. . *Suppose the functions* μ *and* σ *in (3.1) are* C^1 *and there exist a* *constant $C > 0$ such that*

$$\langle x-y, \mu(x)-\mu(y)\rangle \le C\|x-y\|^2, \forall x, y \in \mathbb{R}^m$$

$$\|\sigma(x)-\sigma(y)\|^2 \le C\|x-y\|^2, \forall x, y \in \mathbb{R}^m$$

Consider the SSBE (3.14)-(3.15) applied to the SDE (3.1) under the above assumption. There exists a continuous-time extension $\bar{Z}(t)$ of the numerical solution (so that $\bar{Z}(t_K) = Z_K$) for which

$$\lim_{\Delta t \to 0} E\Big[\sup_{0\le t\le T} \|\bar{Z}(t) - X(t)\|^2\Big] = 0$$

Proof. See [23] for the details.

The condition this theorem assumes on the drift coefficient μ is called a one-sided Lipschitz condition. A good example is the following polynomial function

$$f(x) = -x^p + x, where.p \ge 3isanoddinteger.$$

It can be easily verified that it satisfies the condition (3.16). By taking $y = 0$, (3.16) and (3.17) also imply

$$\langle \mu(x), x\rangle \vee \|\sigma(x)\|^2 \le \alpha + \beta\|x\|^2, \forall x \in \mathbb{R}^m$$

where $\alpha := \frac{1}{2}\|\mu(0)\|^2 \vee 2\|\sigma(0)\|^2$ and $\beta := (C + \frac{1}{2}) \vee 2C$. Condition (3.18) is actually a monotone-type condition (see condition (2.7)). Note that μ and σ are also locally Lipschitz continuous $(\mu, \sigma are C^1)$. Therefore, by Theorem 2.3, under the assumptionsof Theorem (3.2), there exists a unique solution of (3.1).

To remove the bound restriction on the approximate solution and give an explicit convergence rate at the same time, [23] also proposed the Backward Euler scheme by

setting $U_0 = x_0$ and

$$U_{K+1} = U_K + \mu(U_{K+1})\Delta t + \sigma(U_K)\Delta W_K$$

Theorem 2.1.3. . *Suppose the conditions in Theorem (3.2) hold. Moreover, assume that there exist constants $C > 0$ and $q \in \mathbb{Z}^+$ such that all.*

$$\|\mu(x) - \mu(y)\|^2 \leq C(1 + \|x\|^q + \|y\|^q)\|x - y\|^2$$

Consider the backward Euler method (3.19) applied to SDE (3.1). There exists a continuous-time extension $\bar{U}(t)$ of the numerical solution (so that $\bar{U}(t_K) = U_K$) for which

$$E[\sup_{0\leq t\leq T} \|\bar{U}(t) - X(t)\|^2] = O(\Delta t)$$

Note that if μ satisfies condition (3.20), μ behaves polynomially or is superlinearly growing. Obviously, by the mean value theorem for derivatives, if the derivative of a function grows at most polynomially, this function must satisfy the condition (3.20). So sometimes we also define the polynomial growth of a function from the derivative perspective (see condition (3.23)).

This theorem provides a possibility of the computation of the numerical solutions of many SDEs which take nonlinear functions as the drift coefficients satisfying (3.16) and (3.20).

However, the backward Euler scheme (3.19) is an implicit method and its implementation requires too much computational effort. On the other hand, the explicit Euler-Maruyama scheme may not converge in the strong sense to the exact solution of an SDE with the one-sided Lipschitz continuous (inequality (3.16)) and superlinearly growing (inequality (3.20)) drift coefficients. Even worse, Theorem 1 in [27] also shows for such an SDE that the absolute moments of the explicit Euler approximations at a finite time point $T \in [0, \infty]$ diverge to infinity. To address this issue, Hutzenthaler, Jentzen and Kloeden [28] proposed a "tamed" version of

the explicit Euler scheme which is strongly convergent for SDEs with superlinearly growing drift coefficients. This is our next section.

2.2 A Tamed Euler Scheme

In this section, we still consider the SDE (2.1), but in a different form. Let $\sigma = (\sigma_1, \sigma_2, ..., \sigma_d)$ where the $\sigma_i^{,}s, 1 \leq i \leq d$, are the column vectors of the matrix σ Let $(W = (W_{(1)}, W^{(2)}, ..., W^{(d)})$ be the d-dimensional Brownian motion. The SDE is then expressed as

$$X(t) = x_0 + \int_0^t \mu(X(s))ds + \sum_{i=1}^{d} \int_0^t \sigma_i(X(s))dW^{(i)}(s)$$

for all $t \in [0.T]$

In this section, we still assume that σ is globally Lipschitz continuous. We also assume that the drift coefficient $\mu : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a continuously differentiable (i.e. C^1) and globally one-sided Lipschitz continuous function whose derivative grows at most polynomially. That is, there exists a positive real number C such that

$$\|\mu^{,}(x)\| \leq C(1 + \|x\|^C)$$

$$\|\sigma(x) - \sigma(y)\| \leq C\|x - y\|$$

$$\langle x - y.\mu(x) - \mu(y)\rangle \leq C\|x - y\|^2$$

Note that if μ satisfies (3.23), then by the mean value theorem for derivatives, it also satisfies (3.20). As we discussed in Theorem 3.3, the numerical scheme (3.19) converges strongly to the real solution of SDE (3.22) as long as (3.23), (3.24) and (3.25) are satisfied. However, in each time step of (3.19), the zero of a nonlinear equation has to be determined, which requires more computational effort. To solve this problem, in their seminal paper [27], Hutzenthaler, Jentzen and Kloeden proposed

a new explicit Euler scheme by defining $V_0 = x_0$and

$$V_{K+1} = V_K + \frac{\mu(V_K)\Delta t}{1 + \|\mu(V_K)\|\Delta t} + \sigma(V_K)\Delta W_K, 0 \leq K \leq N - 1$$

This scheme is called the tamed Euler scheme. Note that the drift term is "tamed" by the factor $1 + \|\mu(V_K)\|\Delta t$ and thus bounded by 1. And this prevents the large excursions generated by the drift term of the numerical scheme. Since the diffusion term σ is still required to be globally Lipschitz continuous, we can expect that the numerical scheme (3.26) behaves nicely and does not have a possibility of blowing up.

After a small transformation, the numerical scheme (3.26) becomes

$$V_{K+1} = V_K + \mu(V_K)\Delta t + \sigma(V_K)\Delta W_K - (\Delta t)^2 \frac{\mu(V_K)\|\mu(V_K)\|}{1 + \|\mu(V_K)\|\Delta t}$$

We can see that this is the Euler-Maruyama scheme added by a second-order term. As usual, we also define the continuous-time approximation of the tamed Euler scheme. It is natural to have

$$\bar{V}(t) := V_K + \frac{(t - t_K)\mu(V_K)}{1 + \|\mu(V_K)\|\Delta t} + \sigma(V_K)(W(t) - W(t_K))$$

for all $t \in [t_K, t_{K+1}], K = 0, 1, ..., N - 1$

Theorem 2.2.1. . *Suppose that the drift coefficient $\mu(x)$ is a continuously differentiable and globally one-sided Lipschitz continuous function whose derivative grows at most polynomially. Suppose also that the diffusion coefficient $\sigma(x)$ is globally Lipschitz continuous and $E[\|x_0\|^p] \leq \infty$ for all $C_p \in [0, \infty), p \in [1, \infty)$, of real numbers such that*

$$(E\Big[\sup_{t\in[0.T]} \|X(T) - \bar{V}(t)\|^p\Big])^{1/p} \leq C_p.N^{-1/2}$$

for all $N \in \mathbb{N}$ and all $p \in [1.\infty)$.

To prove this theorem, we need the following two lemmas.

Lemma 1. . *Let V_K be given by (3.26). Then we have that*

$$\sup_{N\in\mathbb{N}K\in(0,...,N)} E[\|V_K\|^p] < \infty$$

for all $\in [1, \infty)$

This lemma is very crucial in the proof of Theorem 2.4. See Lemma 3.1–3.9 for the proof.

Lemma 2. . *Let V_K be given by (3.26). Then we have that*

$$\sup_{N\in\mathbb{N}K\in(0,...,N)} E[\|\mu(V_K)\|^p] < \infty$$

$$\sup_{N\in\mathbb{N}K\in(0,...,N)} E[\|\sigma(V_K)\|^p] < \infty$$

for all $\in [1, \infty)$

Proof. First of all, by the polynomial growth property of $\mu(x)$ *, we have*

$$\|\mu(x) - \mu(0)\| \leq C(1 + \|x\|^{C})\|x\|$$

$$= C\|x\| + C\|x\|^{C+1}$$

$$\leq C(1 + \|x\|^{C+1}) + C(1 + \|x\|^{C+1})$$

$$\leq 2C(1 + \|x\|^{C+1}$$

Therefore, we have

$$\|\mu(x)\| \leq 2C + \|\mu(0)\|)(1 + \|x\|^{C+1}$$

Combining (3.33) and Lemma 3.4.1, we have

$$\sup_{N \in \mathbb{N} K \in (0,...,N)} \sup \|\mu(V_K)\|_{L^p(\Omega; R^m)}$$

$$\leq (2C + \|\mu(0)\|(1 + \sup_{N \in \mathbb{N} K \in (0,...,N)} \sup \|V_K\|^{(C+1)}_{L^{p(C+1)}(\Omega; R^m)}) < \infty$$

Additionally, the inequality $\|\sigma(x)\| \leq C\|x\| + \|\sigma(0)\|$ *for all* $x \in \mathbb{R}^m$ *and again Lemma 3.4.1 show that*

$$\sup_{N \in \mathbb{N} K \in (0,...,N)} \sup \|\sigma(V_K)\|_{L^p(\Omega; \mathbb{R}^m)}$$

$$\leq C(\sup_{N \in \mathbb{N} K \in (0,...,N)} \sup \|(V_K)\|_{L^p(\Omega; \mathbb{R}^m)}) + \|\sigma(0)\|$$

$$< \infty$$

for all $p \in [0, \infty)$

Next, we give the proof of Theorem 3.4.

Proof. We first define the shift operator

$$\eta(t) = t_i, if, t_i \leq t < t_{i+1}, i = 0, ..., N - 1$$

In this notation, equation (3.28) reads as

$$\bar{V}(s) = x_0 + \int_0^s \frac{\mu(\bar{V}(\eta(u)))}{1 + T/N\|\mu(\bar{V}(\eta(u)))\|} du + \int_0^s \sigma(\bar{V}(\eta(u)))dW(u)$$

for all $s \in [0,T]$ *P-a.s.. Our goal is then to estimate the quantity* $E\Big[\sup_{s\in[0,t]} \|X(s) - \bar{V}(s)\|^p\Big]$ *for all* $p \in [1,\infty)$. *Using (3.22) and (3.34), we get*

$$X(s) - \bar{V}(s) = \int_0^s (\mu(X(u)) - \frac{\mu(\bar{V}(\eta(u)))}{1 + T/N\|\mu(\bar{V}(\eta(u)))\|})du$$

$$+ \sum_{i=1}^d \int_0^s (\sigma_i(X(u)) - \sigma(\bar{V}(\eta(u))))dW^{(i)}(u)$$

for all $s \in [0,T]$ *P-a.s.. Itô's formula hence gives that*

$$\|X(s) - \bar{V}(s)\|^2 = 2\int_0^s \langle X(u) - \bar{V}(u), \mu(X(u)) - \mu(\bar{V}(u))\rangle du$$

$$+2\int_0^s \langle X(u) - \bar{V}(u), \mu(\bar{V}(u)) - \mu(\bar{V}(\eta(u))\rangle du$$

$$+\frac{2T}{N}\int_0^s \Big\langle X(u) - \bar{V}(u), \frac{\mu(\bar{V}(\eta(u)))\|\mu(\bar{V}(\eta(u)))\|}{1 + T/N\|\mu(\bar{V}(\eta(u)))\|}\Big\rangle du$$

$$+2\sum_{i=1}^d \int_0^s \langle X(u) - \bar{V}(u), \sigma_i(X(u)) - \sigma_i(\bar{V}(\eta(u)))\rangle dW^{(i)}(u)$$

$$+\sum_{i=1}^d \int_0^s \|\sigma_i(X(u)) - \sigma_i(\bar{V}(\eta(u)))\|^2 du$$

By the one-sided Lipschitz continuity of μ , *the global Lipschitz continuity of* σ *and Cauchy-Schwarz inequality, we have*

$$\|X(s) - \bar{V}(s)\|^2 \leq (2C + 2C^2 d)\int_0^s \|X(u) - \bar{V}(u)\|^2 du$$

$$+2\int_0^s \|X(u) - \bar{V}(u)\|\|\mu(\bar{V}(u)) - \mu(\bar{V}(\eta(u)))\|du$$

$$+\frac{2T}{N}\int_0^s \|X(u) - \bar{V}(u)\|\|\mu(\bar{V}(\eta(u)))\|^2 du$$

$$+2|\sum_{i=1}^d \int_0^s \langle X(u) - \bar{V}(u), \sigma_i(X(u)) - \sigma(\bar{V}(\eta(u)))\rangle dW^{(i)}(u)|$$

$$+2C^2d\int_0^s \|\bar{V}(u)-\bar{V}(\eta(u))\|du$$

for all $s\in[0,T]$ P-a.s.. Therefore,

$$\sup_{s\in[0,t]}\|X(s)-\bar{V}(s)\|^2$$

$$2(C+C^2d+1)\int_0^t \|X(s)-\bar{V}(s)\|^2ds+\int_0^T \|\mu(\bar{V}(s))-\mu(\bar{V}(\eta(s)))\|ds$$

$$\frac{T^2}{N^2}\int_0^T \|\mu(\bar{V}(\eta(s)))\|^4ds+2C^2d\int_0^T \|\bar{V}(s)-\bar{V}(\eta(s))\|^2ds$$

$$+2\sup_{s\in[0.t]}|\sum_{i=1}^d\int_0^s \langle X(u)-\bar{V}(u))-\sigma_i(\bar{V}(\eta(s)))\rangle dW^{(i)}(u)|$$

P-a.s.. for all $t\in[0,T]$. The Minkowski's inequality (A.5), Minkowski's integral inequality (A.6) and Burkholder-Davis-Gundy type inequality (A.11) yield that

$$\|\sup_{s\in[0,t]}\|X(s)-\bar{V}(s)\|^2_{L^{p/2}(\Omega;\mathbb{R})}$$

$$2(C+C^2d+1)\int_0^t \|X(s)-\bar{V}(s)\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

$$+\int_0^T \|\mu(\bar{V}(s))-\mu(\bar{V}(\eta(s)))\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

$$\frac{T^2}{N^2}\int_0^T \|\mu(\bar{V}(\eta(s)))\|^4_{L^{2p}(\Omega;\mathbb{R}^m)}ds$$

$$-2C^2d\int_0^T \|\bar{V}(s)-\bar{V}(\eta(s))\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

$$+p\Big(\sum_{i=1}^d\int_0^t \|\langle X(s)-\bar{V}(s),\sigma_i(X(s))-\sigma_i(\bar{V}(\eta(s)))\|^2_{L^{p/2}(\Omega;\mathbb{R}^m)}ds\Big)^{1/2}$$

for all $t\in[0,T]$ and all $p\in[4,\infty)$. Next the Cauchy-Schwarz inequality, the Hölder inequality and $ab\leq\frac{a^2}{2}+\frac{b^2}{2}$ imply that

$$p\Big(\sum_{i=1}^d\int_0^t \|\langle X(s)-\bar{V}(s),\sigma_i(X(s))-\sigma_i(\bar{V}(\eta(s)))\rangle\|^2_{L^{p/2}(\Omega;\mathbb{R})}ds\Big)^{1/2}$$

$$\leq p\Big(\sum_{i=1}^d\int_0^t \|X(s)-\bar{V}(s)\|^2_{L^p(\Omega;\mathbb{R}^m)}\|\sigma_i(X(s))-\sigma_i(\bar{V}(\eta(s)))\|^2_{L^p(\Omega;\mathbb{R}^m)}ds\Big)^{1/2}$$

$$\leq p\Big(\sup_{s\in[0,t]}\|X(s)-\bar{V}(s)\|_{L^p(\Omega;\mathbb{R}^m)}\Big)\Big(C^2d\int_0^t\|X(s)-\bar{V}(\eta(s))\|^2_{L^p(\Omega;\mathbb{R}^m)}ds\Big)^{1/2}$$

$$\leq \frac{1}{2}\sup_{s\in[0,t]}\|X(s)-\bar{V}(s)\|^2_{L^p(\Omega;\mathbb{R})}+\frac{p^2C^2d}{2}\int_0^t\|X(s)-\bar{V}(\eta))\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

$$\leq \frac{1}{2}\Big\|\sup_{s\in[0,t]}\|X(s)-\bar{V}(s)\|\Big\|^2_{L^p(\Omega;\mathbb{R})}+\frac{p^2C^2d}{2}\int_0^t\|X(s)-\bar{V}(\eta(s))\|^2_{L^p(\Omega;\mathbb{R})}ds$$

for all $t\in[0.T]$ *. Inserting inequality (3.36) into (3.35) and applying the estimate* $(a+b)^2\leq 2a^2+b^2$ *then yields that*

$$\Big\|\sup_{s\in[0.t]}\Big\|X(s)-\bar{V}(s)\|\|^2_{L^p(\Omega;\mathbb{R}^m)}$$

$$\|\sup_{s\in[0,t]}\|X(s)-\bar{V}(s)\|^2\|_{L^{p/2}(\Omega;\mathbb{R})}$$

$$\leq 2(C+C^2d+1+\frac{p^2C^2d}{2})\int_0^t\|X(s)-\bar{V}(s)\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

$$+\int_0^T\|\mu(\bar{V}(s))-\mu(\bar{V}(\eta(s)))\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

$$+\frac{T^2}{N^2}\int_0^T\|\mu(\bar{V}(\eta(s)))\|^4_{L^{2p}(\Omega;\mathbb{R}^m)}ds$$

$$+(2C^2d+p^2C^2d)\int_0^T\|\bar{V}(s)-\bar{V}(\eta(s))\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

$$\frac{1}{2}\|\sup_{s\in[0,t]}\|X(s)-\bar{V}(s)\|\|^2_{L^p(\Omega;\mathbb{R}^m)}$$

and therefore, we obtain that

$$\frac{1}{2}\|\sup_{s\in[0.t]}\|X(s)-\bar{V}(s)\|\|^2_{L^p(\Omega;\mathbb{R})}$$

$$\leq 2(C+C^2d+1+p^2C^2d)\int_0^t\|X(s)-\bar{V}(s)\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

$$+\int_0^T\|\mu(\bar{V}(s))-\mu(\bar{V}(\eta(s)))\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

$$+\frac{T^2}{N^2}\int_0^T\|\mu(\bar{V}(s)))\|^4_{L^{2/p}(\Omega;\mathbb{R}^m)}ds$$

$$+(2C^2d+p^2C^2d)\int_0^T\|\bar{V}(s)-\bar{V}(\eta(s)\|^2_{L^p(\Omega;\mathbb{R}^m)}ds$$

for all $t \in [0,T]$ and all $p \in [4,\infty)$. By Gronwall's lemma, we have

$$\| \sup_{t\in[0,T]} \|X(t) - \bar{V}(t)\| \|^2_{L^p(\Omega;\mathbb{R})}$$

$$\leq 2e^{4T(p^2C^2d+C+1)}(\int_0^T \|\mu(\bar{V}(s)) - \mu(\bar{V}(\eta(s)))\|^2_{L^p(\Omega;\mathbb{R}^m)} ds$$

$$+\frac{T^2}{N^2}\int_0^T \|\mu(\bar{V}(\eta(s)))\|^4_{L^{2p}(\Omega;\mathbb{R}^m)} ds$$

$$+2p^2C^2d\int_0^T \|\bar{V}(s) - \bar{V}(\eta(s))\|^2_{L^p(\Omega;\mathbb{R}^m)} ds$$

and hence, the inequality $\sqrt{a+b+c} \leq \sqrt{a}\sqrt{b}\sqrt{c}$ for all $a,b,c \in [0,\infty)$ gives that

$$\| \sup_{t\in[0,T]} \|X(t) - \bar{V}(t)\| \|^2_{L^p(\Omega;\mathbb{R})}$$

$$\leq \sqrt{2T}e^{2T(p^2C^2d+C+1)}(\sup_{t\in[0,T]} \|\mu(\bar{V}(t))) - \mu(\bar{V}(\eta(s)))\|_{L^p(\Omega;\mathbb{R}^m)}$$

$$\frac{T}{N}[\sup_{K\in(0,1,\ldots,N)} \|\mu(V_K)\|^2_{L^{2p}(\Omega;\mathbb{R}^m)}]$$

$$+pC\sqrt{2d}[\sup_{t\in[0,T]} \|\bar{V}(t) - \bar{V}(\eta(s))\|_{L^p(\Omega;\mathbb{R}^m)}]$$

for all $p \in [4,\infty)$. Additionally, the Burkholder-Davis-Dundy type inequality (A.11) shows that

$$\sup_{t\in[0,T]} \|\bar{V}(t) - \bar{V}(\eta(s))\|_{L^p(\Omega;\mathbb{R}^m)}$$

$$\frac{T}{N}(\sup_{t\in[0,T]} \| \frac{\mu(\bar{V}(\eta(s)))}{1+T/N\|\mu(\bar{V}(\eta(s)))} \|)$$

$$+ \sup_{t\in[0,T]} \| \int_0^t \sigma(\bar{V}(\eta(t)))dW(s)\|_{L^p(\Omega;\mathbb{R}^m)}$$

$$\leq \frac{T}{\sqrt{N}}(\sup_{K\in(0,1,\ldots,N)} \|\mu(V_K)\|_{L^p(\Omega;\mathbb{R}^m)})$$

$$+\frac{p\sqrt{Td}}{\sqrt{N}}(\sup_{i\in(1,2,\ldots,d)} \sup_{K\in(1,2,\ldots,N)} \|\sigma_i(V_K)\|_{L^p(\Omega;\mathbb{R}^m)}$$

for all $p \in [2, \infty)$ *. Lemma 3.4.2 hence implies that*

$$\sup_{N\in\mathbb{N}}(\sqrt{N}[\sup_{t\in[0,T]}\|\bar{V}(t)-\bar{V}(\eta(s))\|_{L^p(\Omega;\mathbb{R}^m)}]) < \infty$$

for all $p \in [1, \infty)$ *. In particular, we obtain that*

$$\sup_{N\in\mathbb{N}}\sup_{t\in[0,T]}\|\bar{V}(t)\|_{L^p(\Omega;\mathbb{R}^m)} < \infty$$

for all $p \in [1, \infty)$ *due to Lemma 3.4.1. Moreover, the estimate*

$$\|\mu(x)-\mu(y)\| \leq C(1+\|x\|^C+\|y\|^C)\|x-y\|, x, y \in \mathbb{R}^m$$

gives that

$$\sup_{t\in[0,T]}\|\mu(\bar{V}(t))-\mu(\bar{V}(\eta(s)))\|_{L^p(\Omega;\mathbb{R}^m)}$$

$$\leq C(1+2\sup_{t\in[0,T]}\|\bar{V}(t)\|^C_{L^{2pC}(\Omega;\mathbb{R}^m)})(\sup_{t\in[0.T]}\|\bar{V}(t)-\bar{V}(\eta(s))\|_{L^{2p}(\Omega;\mathbb{R}^m)})$$

for all $p \in [1, \infty)$ *. Inequalities (3.38) and (3.39) hence show that*

$$\sup_{N\in\mathbb{N}}(\sqrt{N}[\sup_{t\in[0,T]}\|\mu(\bar{V}(t))-\mu(\bar{V}(\eta(t)))\|_{L^p(\Omega;\mathbb{R}^m)}]) < \infty$$

for all $p \in [1, \infty)$ *. Combining (3.37), (3.38), (3.41) and Lemma 3.4.2 finally shows (3.29).*

2.3 Numerical Experiments

The example we use for our numerical experiment in this section is the 1- dimensional stochastic Ginzburg-Landau equation,

$$dX(t) = (X(t) - X^3(t()dt + X(t)dW(t), X(0) = 1$$

Here, $\mu(x) = x - x^3, \sigma(x) = x$ and $t \in [0, 1]$. Clearly,$\mu(x)$ satisfies a one-sided Lipschitz condition and grows superlinearly. We use 5 different time steps: $\Delta t = 2^{-12}, 2^{-11}, 2^{-10}, 2^{-9}, 2^{-8}$ and 1000 realizations for each discretisation. The following

figure is the loglog plot of the experimental error with respect to the 5 different time steps. We can see that the numerical scheme converges strongly with order $\frac{1}{2}$

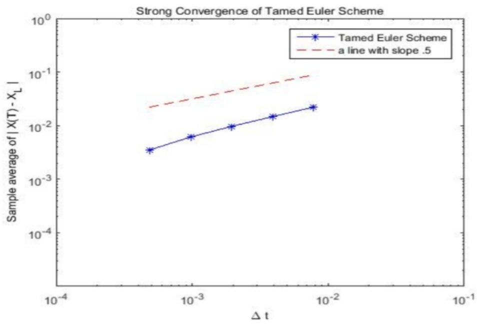

Figure 2.1: Log-log plot of the strong error from the tamed Euler approximation versus the time step Δt with the drift coefficients superlinearly growing.

2.4 Euler Approximations with Superlinearly Growing Diffusion Coefficients

The tamed Euler scheme we introduced in section 3.2 is a significant progress in the computation of numerical solutions of stochastic differential equations. It is an explicitb-numerical scheme and only assumes that the drift coefficient is one-sided Lipschitz and its derivative grows at most polynomially. However, it still requires the global Lipschitz continuity of the diffusion coefficient. In Sabanis [55], the author introduces a new explicit Euler-type numerical scheme to compute the numerical solutions of SDEs whose diffusion coefficients can be superlinearly growing. We introduce this new scheme in this section based on [55]. For the Milstein-type numerical scheme to address this issue, see e.g. [37].

In this section, we consider the following stochastic differential equation:

$$\begin{cases} dX(t) = \mu(t, X(t))dt + \sigma(t, X(t)dW(t), t \in [0, T] \\ X(t) = x_0 \end{cases}$$

where $X(t) \in \mathbb{R}^m$ for all $t \in [0, T], W(t)$is a d-dimensional Brownian motion starting at 0, $\mu : [0, T] \times \mathbb{R}^m \to \mathbb{R}^m$ and $\sigma : [0, T] \times \mathbb{R}^m \to \mathbb{R}^{m \times d}$. We also assume that x_0 is $\mathcal{F}_0$ -measurable, almost surely finite and independent of $(W(t), 0 \le t \le T)$. Let $p_0, p_1 \in [2, \infty)$ be positive constants. We consider the following conditions.

$\mathbf{A-1}.E[\|x_0\|^{p0} < \infty$

$\mathbf{A-2}$. $\mu(t,x)$ and $\sigma(t,x)$ are locally Lipschitz continuous in x for any $t \in [0,T]$ (see (2.6)).

$\mathbf{A-3}$. There exist positive constants l and L such that, for any $t \in [0,T]$

$$2\langle x-y, \mu(t,y)\rangle + (p_1-1)\|\sigma(t,y)\|^2 \leq L\|x-y\|^2$$

and

$$\|(t,x)-\mu(t,y)\| \leq L(1+\|x\|^l+\|y\|^l)\|x-y\|$$

for all $x, y \in \mathbb{R}^m$

$\mathbf{A-4}$. There exists a positive constant K such that,

$$2x^T\mu(t,x) + (p_0-1)\|\sigma(t,x)\|^2 \leq K(1+\|x\|^2)$$

for any $t \in [0,T] and x \in \mathbb{R}^d$

Note that, due to $A-2, \mu(t,x)$ and $\sigma(t,x)$ are locally bounded in x for any $t \in [0,T]$ That is, for every $R \geq 0$, there exists a positive constant N_R such that

$$\sup_{\|x\|\leq R} \|\mu(t,x)\| \leq N_R$$

$$\sup_{\|x\|\leq R} \|\sigma(t,x)\| \leq N_R$$

for any $t \in [0,T]$. We also observe that if $A-2, A-3, and, A-4$ hold, then

$$\|\mu(t,x)\| \leq \|\mu(t,x)-\mu(t.0)\| + \|\mu(t,0)\| \leq L(1+\|x\|^l)\|x\| + N_0 \leq C(1+\|x\|^{l+1})$$

for any $t \in [0,T] and, x \in \mathbb{R}^m, A-1, A-2, and A-4$, by Theorem 2.3, guarantee that there exists a unique solution of equation (3.42).

We now consider the numerical scheme. To be consistent with the two numerical schemes we are going to introduce in this section, we will use the following unified notation.

For every $N \geq 1$, the following numerical scheme is defined

$$dX_N(t) = \mu_N(t, X_N({}_{K_N}(t)))dt +_{\sigma_N} (t, X_N({}_{K_N}(t)))dW(t), \forall t \in [0, T]$$

with the same initial value x_0 as equation (3.42), where ${}_{\mu_N}(t,x)$ and ${}_{\sigma_N}(t,x)$ are $B(\mathbb{R}_+) \otimes B(\mathbb{R}^d)$ -measurable functions which take values in $\mathbb{R}^m and, \mathbb{R}^{m\times d}$ respectively and ${}_{K_N}(t) = [Nt]/N$. Note that the function ${}_{K_N}(t)$ jumps with size $1/N$, while the shift operator function $\eta(t)$ we defined in the last section jumps with size T/N . In other words, there are $N + 1$ nodes within the interval $[0, 1]$ in the numerical scheme (3.45), while there are $N + 1$ nodes within the interval $[0, T]$ in the numerical schemes

we introduced in the previous sections. We can see that, by defining the numerical scheme as in (3.45), we already have a continuous-time approximation of the solution of equation (3.42).

The following condition we assume is very important for our arguments.

B . There exists an $\alpha \in [0, 1/2]$ and a constant C such that, for every $N \geq 1$

$$\|\mu_N(t,x)\| \leq min(CN^\alpha, \|\mu(t,x)\|)\|\sigma_N(t,x)\| \leq min(CN^\alpha, \|\sigma(t,x)\|)$$

for any $t \in [0, T] and, x \in \mathbb{R}^m$
Let $\alpha \in [0, 1/2]$, we now define
Model1 :

$$\mu_N(t,x) := \frac{1}{1 + N^{-\alpha}\|\mu(t,x) + N^{-\alpha}\|\sigma(t,x)\|}\mu(t,x)$$

and

$$\sigma_N(t,x) := \frac{1}{1 + N^{-\alpha}\|\mu(t,x) + N^{-\alpha}\|\sigma(t,x)\|}\sigma(t,x)$$

for any $t \in [0, T], x \in \mathbb{R}^m, and, N \geq 1$

Model2 :

$$\mu_N(t,x) := \frac{1}{1+N^{-\alpha}\|x\|^{3l/2+2}}\mu(t,x)$$

and

$$\sigma_N(t,x) := \frac{1}{1+N^{-\alpha}\|x\|^{3l/2+2}}\sigma(t,x)$$

for any $t \in [0,T], x \in \mathbb{R}^m, and, N \geq 1$

It can be verified easily that both Model 1 and Model 2 satisfy the condition B for any $t \in [0,T], and, x \in \mathbb{R}^d$. Letp_0^* be the largest even number which is smaller than or equal to p_0 . In order to ease the notation, we say that the p-condition is satisfied if one of the following two cases hold true:

(**Model1**) The coefficients μ_N and σ_N are given by equations (3.47) and (3.48) with $\alpha = 1/2, p < p_1$ and either $p \leq \frac{p_9}{5l/2+3}$ if $l \in [0,2] \cap [0, \frac{p_0}{4} - 1]$ or $p \leq \frac{p_0^*}{2(l+1)}$ if $l \in [0, \frac{p_0^*}{4} - 1]$ and $m = d = 1$

(**Model2**) The coefficients μ_N and σ_N are given by equations (3.49) and (3.50) with $\alpha = 1/2, p < p_1, p \leq \frac{p_0}{5l/2+3}$ and $l \leq p_0 - 2$

We then can recover the optimal rate of strong convergence for Euler approximations.

Theorem 2.4.1. *([55], Theorem 2). Suppose A-1–A-4 and the p-condition hold, then the numerical scheme (3.45) converges to the true solution of SDE (3.42) in L^p-sense with order 1/2, i.e.*

$$\sup_{0\leq t\leq T} E[\|X(t) - X(t)\|^p] \leq CN^{-p/2}$$

where C is a constant independent of N.

The uniform L^p convergence for smaller values of p is given below.

Theorem 2.4.2. *([55], Theorem 3). Suppose A-1–A-4 and the p-condition hold, then the numerical scheme (3.45) converges to the true solution of SDE (3.42) in uniform L^p-sense with order 1/2, i.e.*

$$E \sup_{0\leq t\leq T} [\|X(t) - X(t)\|^q] \leq CN^{-q/2}$$

where C is a constant independent of N, for all $q < p$

To prove the above two theorems, we need the following five estimates.

Lemma 3. *([55], Lemma 1). Consider the numerical scheme (3.45) and let A-1–A-4 and B hold and*

$$\sup_{N\geq 1}\sup_{0\leq t\leq T} E[\|X_N(t)\|^q] < \infty$$

for some $q \geq 2$ *, then for any* $p \leq \frac{2}{l+2}q$ *and* $l \in [0, q-2]$

$$\sup_{0\leq t\leq T} E\|X_N(t) - X_N(K_N(t))\|^p \leq CN^{-p/2}$$

where C is a positive constant independent of N.

As in Section 3.1 and Section 3.2, the following three bound estimates of the numerical solution is crucial in our proof of strong convergence.

Lemma 4. *([55], Lemma 7). Consider the numerical scheme (3.45) with coeffi-cients given by (3.47) and (3.48) with* $\alpha = 1/2$ *. Suppose that A-1–A-4 with* $l \in (0, 2)$ *hold, then for some* $C := C(p, T, K, E[\|x_0\|^p])$

$$\sup_{N\geq 1}\sup_{0\leq t\leq T} E[\|X_N(t)\|^p] < C$$

for every $p \leq p_0$

Lemma 5. *([55], Lemma 5). Consider the numerical scheme (3.45) with coeffi-cients given by (3.49) and (3.50) with* $\alpha = 1/2$ *. Let also A-1–A-4 hold true. Then,for every* $p \leq p_0$

$$\sup_{N\geq 1}\sup_{0\leq t\leq T} E[\|X_N(t)\|^p] < C$$

where the constant $C := C(p, T, K, E[\|x_0\|^p])$

Lemma 6. *([55], Lemma 9). Consider the numerical scheme (3.45) with coef- ficients given by (3.47) and (3.48) with* $\alpha = 1/2$ *when* $m = d = 1$*. Suppose that A-1–A-4 hold, then for some* $C := C(p, T, K, E[\|x_0\|^p])$

$$\sup_{N\geq 1}\sup_{0\leq t\leq T} E[\|X_N(t)\|^p] < C$$

for every $p \leq p_0^*$

Lemma 7. *([55], Lemma 10). Consider the numerical scheme (3.45). Suppose A-1–A-4* *and the p-condition hold. Then,*

$$E[\int_0^T \|\mu(s, X_N(K_N(s))) - \mu_N(s, X_N(s)))\|^p ds] \leq CN^{-\alpha p}$$

and

$$E[\int_0^T \|\sigma(s, X_N(K_N(s))) - \sigma_N(s, X_N(s)))\|^p ds] \leq CN^{-\alpha p}$$

We refer to [55] for the proof of the above four lemmas. We now give the proof of the two main theorems of this section.

Proof of Theorem 3.5. We first consider, for every $N \geq 1$ and $t \in [0, T]$, the difference processes,

$$X_N(t) := X(t) - X_N(t)$$

$$\beta_N(t) := \mu(t, X(t)) - \mu_N(t, X_N(K_N(t)))$$

$$\alpha_N(t) := \sigma(t, X(t)) - \sigma_N(t, X_N(K_N(t)))$$

Therefore, by (3.42) and (3.45), we have

$$dX_N(t) = \beta_N(t)dt + \alpha_N(t)dW(t)$$

Note that $(|x|^p)^{‘} = ((x^2)^{p/2})^{‘} = px|x|^{p-2}$ and $(|x|^p)^{“} = p(p-1)|x|^{p-2}$

$$d\|X_N(t)\|^p = p\|X_N(t)\|^{p-2}(\langle X_N(t), \beta_N(t)\rangle + \frac{p-1}{2}\|\alpha_N(t)\|^2)dt$$

$$+\|X_N(t)\|^{p-2}\langle X_N(t), \alpha_N(t)dW(t)\rangle$$

Therefore,

$$\|X_N(t)\|^p \leq \frac{p}{2}\int_0^t \|X_N(s)\|^{p-2}(2\langle X_N(s), \beta_N(s)\rangle + (p-1)\|\alpha_N(s)\|^2)ds$$

$$+p\int_0^t \|X_N(s)\|^{p-2}\langle X_N(s), \alpha_N(s)dW(s)\rangle.$$

Note that in the above equality, for any $\epsilon > 0$,

$2\langle X_N(s), \beta_N(s)\rangle + (p-1)\|\alpha_N(s)\|^2$

$= \langle X(s) - X_N(s), \mu(s, X(s)) - \mu(s, X_N(s))\rangle$

$+2\langle X(s) - X_N(s), \mu(s, X_N(s)) - \mu(s, X_N(K_N(s)))\rangle$

$+2\langle X(s) - X_N(s), \mu(s, X_N(K_N(s))) - \mu_N(s, X_N(K_N(s)))\rangle$
$+(p-1)(\|\sigma(s, X(s)) - \sigma(s, X_N(s))\|^2$
$+\|\sigma(s, X_N(s)) - \sigma(s, X_N(K_N(s)))\|^2$
$+\|\sigma(s, X_N(K_N(s))) - \sigma_N(s, X_N(K_N(s)))\|^2$
$+2(\sqrt{\epsilon/2}\|\sigma(s, X(s)) - \sigma(s, X_N(s))\|)(\sqrt{2/\epsilon}\|\sigma(s, X_N(s)) - \sigma(s, X_N(K_N(s)))\|)$
$+2(\sqrt{\epsilon/2}\|\sigma(s, X(s)) - \sigma(s, X_N(s))\|)(\sqrt{2/\epsilon}\|\sigma(s, X_N(K_N(s))) - \sigma_N(s, X_N(K_N(s)))\|)$
$+2\|\sigma(s, X_N(s)) - \sigma(s, X_N(K_N(s)))\|\|\sigma(s, X_N(K_N(s))) - \sigma_N(s, X_N(K_N(s)))\|)$
$\leq 2\langle X(s) - X_N(s), \mu(s, X(s)) - \mu(s, X_N(s))\rangle$
$+2\langle X(s) - X_N(s), \mu(s, X_N(s)) - \mu(s, X_N(K_N(s)))\rangle$
$+2\langle X(s) - X_N(s), \mu(s, X_N(K_N(s))) - \mu_N(s, X_N(K_N(s)))\rangle$
$+(p-1)((1+\epsilon)\|\sigma(s, X(s)) - \sigma(s, X_N(s))\|^2$
$+(1+\frac{1}{\epsilon})\|\sigma(s, X_N(s)) - \sigma(s, X_N(K_N(s)))\|^2$
$+(1+\frac{1}{\epsilon})\|\sigma(s, X_N(K_N(s))) - \sigma_N(s, X_N(K_N(s)))\|^2)$ Due to $A-3$, we have that

$$(p_1 - 1)\|\sigma(t,x) - \sigma(t,y)\|^2 \leq L\|x-y\|^2 - 2\langle x-y, \mu(t,x) - \mu(t,y)\rangle$$

$$\leq C(1 + \|x\|^l + \|y\|^l)\|x-y\|^2.$$

If ϵ is small enough, one can get that $(1+\epsilon) \leq p_1 - 1$ based on the condition that $p < p_1$. By A-3, (3.63) and Cauchy-Schwarz inequality, we have

$$2\langle X_N(s), \beta_N(s)\rangle + (p-1)\|\alpha_N(s)\|^2$$

$$\leq C\|X_N(s)\|^2 + C(1 + \|X_N(s)\|^{2l} + \|X_N(K_N(s))\|^{2l})\|X_N(s) - X_N(K_N(s))\|^2$$

$$+\|\mu(s, X_N(K_N(s))) - \mu_N(s, X_N(K_N(s)))\|^2$$

$$+C\|\sigma(s, X_N(K_N(s))) - \sigma_N(s, X_N(K_N(s)))\|^2$$

Due to (3.46), Hölder's inequality, (3.44), Lemma 3.6.2, Lemma 3.6.3 and that $2p < p_0$ and $(l+2)p < p_0$(or $2p < p_0^*$ and $(l+2)p < p_0^*$ when we consider the Model 1 with $m = d = 1$) due to the p-condition, we have

$$E[\int_0^T \|X_N(s)\|^{2(p-2)}\|\alpha_N^T(s)\|^3 ds]$$

$$\leq 4E[\int_0^T \|X_N(s)\|^{2(p-1)}\|\sigma(s, X(s))\|^2 ds]$$

$$\leq 4\int_0^T (E[\|X_N(s)\|^2])^{(p-1)/p}(E[\|\sigma(s,X(s)\|^{2p}])^{1/p}ds$$

$$\leq C\int_0^T (E[\|X(s)\|^{2p}+\|X_N(s)\|^{2p}])^{(p-1)/p}(E[1+\|X(s)\|X(s)\|^{(l+2)p}]^{1/p}ds$$

$$\leq C$$

Therefore we get

$$E[\int_0^T \|X_N(t)\|^{p-2}\langle X_N(t),\alpha_N(t)dW(t)\rangle]=0$$

Note also that

$$\frac{1}{p/(p-2)}+\frac{1}{p/2}=1$$

Therefore, taking expectation on both sides of (3.61) and using (3.62), (3.64) and Young's inequality, we can get

$E[\|X_N(t)\|^p]$

$\leq CE[\int_0^t(\|X_N(s)\|^p+(1+\|X_N(s)\|^{2l}+\|X_N(K_N(s))\|^{2l})^{p/2}\|X_N(s)-X_N(K_N(s))\|^p$

$+\|\mu(s,X_N(K_N(s)))-\mu_N(s,X_N(K_N(s)))\|^p$

$+\|\sigma(s,X_N(K_N(s)))-\sigma_N(s,X_N(K_N(s)))\|^p ds]$

Moreover,

$$\epsilon(t):=E[\int_{0^t} C(1+\|X_N(s)\|^{lp}+\|X_N(K_N(s))\|^{lp})\|X_N(s)-X_N(K_N(s))\|^p ds]$$

$\leq C\int_0^t(\sqrt{E[\|X_N(s)-X_N(K_N(s))\|^{2p}]}ds$

due to Hölder's inequality, (3.56), (3.57) and the fact that $2lp<p_0$ (or $2lp<p_0^*$ when we consider the Model 1 with $m=d=1$) due to the p-condition. By (3.54), we have

$$\sup_{0\leq t\leq T}\epsilon(t)\leq CN^{-p/2}$$

By (3.66), (3.67), Lemma 3.6.1 (taking $\alpha=1/2$, we finally have

$$\sup_{0\leq t\leq T}E[\|X_N(t)\|^p]\leq CN^{-p/2}$$

To prove Theorem 3.6, we need another lemma.

Lemma 8. *([55], Lemma 11). Let $T\in[0,\infty)$ and let $f:=(f_t)_{t\in[0,T]}$ and $g:=(g_t)_{t\in[0,T]}$ be non-negative continuous $\mathcal{F}_t$ -adapted processes such that, for any constant $C>0$*

$$E[f_\tau\mathbb{1}_{(g0\leq C)}]\leq E[g_\tau\mathbb{1}_{(g0\leq C)}]$$

for any stopping time $\tau \leq T$. Then, for any stopping time $\tau \leq T$ and $\gamma \in (0,1)$

$$E[\sup_{t\leq\tau} f_t^{\gamma}] \leq \frac{2-\gamma}{1-\gamma} E[\sup_{t\leq\tau} g_t^{\gamma}]$$

Proof of Theorem 3.6. Let p satisfy the p-condition and let $X_N, \beta_N and \alpha_N$ be defined as in the proof of Theorem 3.5. Define $\phi(t) : [0,T] \to \mathbb{R}$ by

$$\phi(t) := exp(-(L+2)t)$$

Then, by Itô's formula,

$$d(\phi(t)\|X_N\|^2)^{p/2}$$

$$= \frac{p}{2}\phi(t)^{p/2}\|X_N(t)\|^{p-2}(2X_N(t)dX_N(t) + (p-1)\|\alpha_N(t)\|^2 d(t) - \frac{p(L+2)}{2}\phi(t)^{p/2}\|X_N(t)\|^p dt$$

$$= \frac{p}{2}\phi(t)^{p/2}\|X_N(t)\|^{p-2}(2X_N(t)\beta_N(s) + (p-1)\|\alpha_N(t)\|^2 d(t) - \frac{p(L+2)}{2}\phi(t)^{p/2}\|X_N(t)\|^p dt$$

$$+p\phi(t)^{p/2}\|X_N(t)\|^{p-2}X_N(s)\alpha_N(s)dW(t)$$

By (3.64), we have
$(\phi(t)\|X_N\|^2)^{p/2}$

$$\leq \int_0^t [\frac{p}{2}\phi(t)^{p/2}\|X_N(t)\|^{p-2}((L+2)\|X_N(t)\|^2 + \zeta_N(t)) - \frac{p(L+2)}{2}\phi(t)^{p/2}\|X_N(t)\|^p]dt$$

$$+ \int_0^t p\phi(t)^{p/2}\|X_N(t)\|^{p-2}X_N(s)\alpha_N(s)dW(t)$$

$$= \int_0^t (\frac{p}{2}\phi(t)^{p/2}\|X_N(t)\|^{p-2}\zeta_N(t))dt + \int_0^t p\phi(t)^{p/2}\|X_N(t)\|^{p-2}X_N(s)\alpha_N(s)dW(t)$$

where

$$\zeta_N(t) := C((1 + \|X_N(s)\|^{2l} + \|X_N(K_N(s))\|^{2l}\|X_N(s) - X_N(K_N(s))\|^2$$

$$+\|\mu(s, X_N(K_N(s))) - \mu_N(s, X_N(K_N(s)))\|^2$$

$$+\|\sigma(s, X_N(K_N(s))) - \sigma_N(s, X_N(K_N(s)))\|^2)$$

where $C > 0$ is independent of N. Since the equality (3.65) holds, it is immediately that

$$E\int_0^t p\phi(t)^{p/2}\|X_N(t)\|^{p-2}X_N(s)\alpha_N(s)dW(t) = 0$$

Note also that $\phi(t)^{p/2} \leq \phi(t)^{(p-2)/2}$ *. Then by (3.68), for any stopping time* $\tau \leq T$

$$E[(\phi(\tau)\|X_N(\tau)\|)^{p/2}] \leq \frac{p}{2}E[\int_0^{\tau}(\phi(t)\|X_N(t)\|^2)^{(p-2)/2}\zeta_N(t)dt]$$

Therefore, by Lemma 3.6.5,

$$E[\sup_{t\leq T}(\phi(t)\|X_N(t)\|^2)^{p\gamma/2}] \leq CE[(\int_0^T(\phi(t)\|^2)^{(p-2)/2}\zeta_N(t)dt)^{\gamma}]$$

for any $\gamma \in (0,1)$ *. Then, for* $p > 2$ *, by Young's inequality* ($\frac{1}{p/(p-2)} + \frac{1}{p} = 1$)

$$E[\sup_{t\leq T}(\phi(t)\|X_N(t)\|^2)^{p\gamma/2}] \leq \frac{1}{2}E[\sup_{t\leq T}(\phi(t)\|X_N(t)\|^2)^{p\gamma/2}] + CE[(\int_0^T\zeta(t)dt)^{p\gamma/2}]$$

It implies that

$$E[\sup_{t\leq T}(\phi(t)\|X_N(t)\|^2)^{p\gamma/2}] \leq CE[(\int_0^T\zeta_N(t)^{p/2}dt)^{\gamma}]$$

$$\leq C(E[\int_0^T\zeta(t)^{p/2}])^{\gamma}$$

where the last inequality is due to the concavity of the function x^{γ} *when* $\gamma \in (0,1)$ *By (3.68), it is very easy to see that the above inequality is also true if* $p = 2$ *. By the definition of* γ_N*, (3.67) and Lemma 3.6.5, we have*

$$E[\int_0^T\zeta_N(t)^{p/2}dt] \leq C(\epsilon(t) + E[\int_0^T\|\mu(s,X_N(K_N(s))) - \mu_N(s,X_N(K_N(s)))\|^p dt]$$

$$+E[\int\sigma(s,X_N(K_N(s))) - \sigma_N(s,X_N(K_N(s)))\|^p dt])$$

$$\leq CN^{-\alpha p}$$

Thus,

$$E[\sup_{t\leq R}(\phi(t)\|X_N(t)\|^2)^{p\gamma/2}] \leq CN^{-\alpha p\gamma}$$

which leads to

$$E[\sup_{t\leq T}\|X_N(t)\|^{p\gamma}] \leq exp((L+2)T)E[\sup_{t\leq T}(\phi(t)\|X_N(t)\|^2)^{p\gamma/2}]$$

$$\leq CN^{-\alpha p\gamma}$$

Since $\gamma \in (0,1)$ *, we are done*

2.5 Numerical Experiments

The example we use for our numerical experiment in this section is a 1-dimensional stochastic differential equation,

$$dX(t) = X(t)(1 - |X(t)|)dt + |X(t)|^{3/2}dW(t), X(0) = 1$$

Here, $\mu(x) = x(1 - |x|), \sigma(x) = |x|^{3/2}$ and $t \in [0, 1]$. Clearly, $\mu(x)$ and $\sigma(x)$ satisfy the monotone conditon and the polynomial growth condition with $l = 1$. We use Model 2 as our numerical scheme with $\alpha = 1/2$. We use 5 different time steps: $\Delta t = 2^{-12}, 2^{-11}, 2^{-10}, 2^{-9}, 2^{-8}$ and 1000 realizations for each discretisation. The following figure is the loglog plot of the experimental error with respect to the 5 different time steps. We can see that the numerical scheme converges strongly with order $\frac{1}{2}$

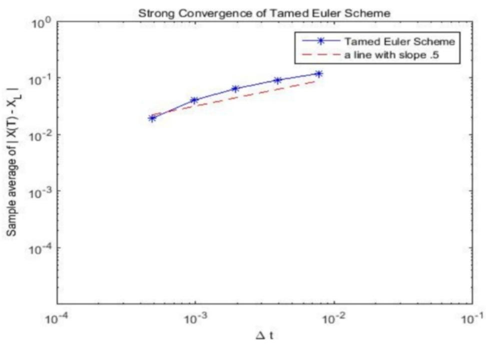

Figure 2.2: Log-log plot of the strong error from the numerical approximation versus the time step Δt with the drift and diffusion coefficients superlinearly growing

Chapter 3

Weak Convergence of Euler-Maruyama Approximation of SDEs Driven by Brownian Motion

3.1 Introduction

Let us consider the following stochastic differential equation:

$$\begin{cases} dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW, t \in [0, T] \\ X(0) = x_0 \end{cases}$$

where $W(t)$ is a one-dimensional Wiener process starting at 0, $X(t)$ is a one- dimensional stochastic process and $\mu(t,x), \sigma(t,x)$ satisfy the following Lipschitz and linear growth condition

$$|\mu(t,x) - \mu(t,y)| + |\sigma(t,x) - \sigma(t,y)| < K(T)|x - y|, t \in [0, T]$$

$$|\mu(t,x)| + |\sigma(t,x)| \leq K(T)(1 + |x|), t \in [0, T]$$

such that the solution of (4.1) exists and is unique. Since we will use the second moment of the solution in our proof, we also assume that x_0 is independent of $(W(t), 0 \leq t \leq T)$ and $E[x_0^2] < \infty$such that

$$E[\sup_{0 \leq t \leq T} X^2] < C(1 + E[x_0^2])$$

where the constant C depends only on K and T.

We now give the Euler-Maruyama scheme. In this section, the time step is denoted by $\Delta = T/N$. For any integer i satisfying $0 \leq i \leq N$, set $t_i = i\Delta$. We define at each node in $[0, T] : Y_0 := x_0$ and

$$Y_{i+1} := Y_i + \mu(t_i, Y_i)\Delta + \sigma(t_i, Y_i)\Delta Wi, 0 \leq i \leq N - 1$$

where $W_i = W(t_{i+1} - W(t_i)$. The continuous-time approximation is defined as:

$$Y(t) := Y_i + \mu(t_i, Y_i)(t - t_i) + \sigma(t_i, Y_i)(W(t) - W(t_i))$$

$$= Y_i + \int_{t_i}^{t} \mu(t_i, Y_i)ds + \int_{t_i}^{t} \sigma(t_i, Y_i)dW(s) for, t \in [t_i, t_{i+1}]$$

Let us also recall the definition of the weak convergence of a numerical scheme.

Let us also recall the definition of the weak convergence of a numerical scheme. We say that a time discrete approximation Y converges in the weak sense with order $\beta \in [0, \infty)$ if for any function g in a suitable function space there exists a finite constant C and a positive constant δ_0 such that

$$|E[g(X(T))] - E[g(Y_N)]| \leq C\delta^{\beta}$$

for any time discretization with maximum step size $\delta \in (0, \delta_0)$

Before we continue, we first define some notations of function spaces. We denote by $C_b^l([0, T] \times \mathbb{R})$ the space of l times continuously differentiable functions $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ for which all its partial derivatives up to order l are bounded uniformly in t (f may

not be bounded).$C_b^l(\mathbb{R}$ is defined in a similar way. We also denote by $C_p^l([0, T] \times \mathbb{R})$

the space of l times continuously differentiable functions $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ for which all its partial derivatives up to order l have polynomial growth uniformly in t.

Maruyama scheme has weak convergence rate 1 [5, 33, 60]. For example, in [33], if both $\mu(t, x)$ and $\sigma(t, x)$ are homogeneous, it is required that $\mu(x), \sigma(x)^2$ and g(x) are all in the function space $C_p^4(\mathbb{R}^m)$, together with some other conditions. While in [5], although g is only required to be measurable and bounded (or has a polynomial growth),$\mu and \sigma$

are assumed to be homogeneous and to be C^∞ functions with bounded derivatives of any order. See also [25, 34] for other related results.

Due to the close relation between the weak approximation of the solution of (4.1) and the Kolmogorov backward partial differential equation, Malliavin calculus, which is powerful to deal with the derivatives of functions of random variables, can serve as an efficient tool to analyze the approximation error. For example, in [5, 25, 34, 35], techniques from Malliavin calculus, like integration by parts, are used very often to assist to get the expressions of the approximation errors.

Another advantage of using Malliavin calculus to deal with the weak convergence problems is that it can also handle stochastic integrals with anticipating integrand. Therefore, the weak approximation problem of stochastic differential equations with terminal conditions can also be dealt with in the frame of Malliavin calculus, see e.g. [35]. In history, it had been believed for a long time that such equations with terminal conditions were not amenable to the analysis of approximation errors, due to the inability of Itô integral for anticipating integrands.

In this section, we do not assume the drift and diffusion terms are homogeneous or C^∞ functions. We only need $\mu(t,x) \in C_b^2([0,T] \times \mathbb{R}$,$\sigma(t,x) \in C_b^2([0,T] \times \mathbb{R}$ and $g(x) \in C_p^3(\mathbb{R})$. As we introduced in Section 2.2, if $\mu(t,x), \sigma(t,x)$ and $g(x)$ satisfy such conditions and the linear growth condition, then $f(t,x) := E[g(X(T))/X(t) = x]$ satisfies the following **Kolmogorovbackwardequation** :

$$\begin{cases} \frac{\partial f}{\partial t}(t,x) + L_t f(t,x) = 0, 0 \le t < T, x \in \mathbb{R} \\ f(T,x) = g(x) \end{cases}$$

where L_t is the second order differential operator defined by

$$L_t f(t,x) = \mu(t,x)\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}@\sigma^2(t,x)\frac{\partial^2 f}{\partial x^2}(t,x)$$

By the definition of f, we have

$$E[f(0,x_0)] = E[E[g(x(T)/X(0) = x_0)]] = E[g(X(T))]$$

By the boundary condition, we have

$$E[f(T, Y, Y(T))] = E[g(Y(T))]$$

The traditional technique in the proof of the weak convergence of the Euler scheme is to write

$$E[g(X(T))] - E[g(Y(T))]$$

$$= (E[f(T, Y(T))] - E[f(0, Y_0)](by(4.11)and(4.12))$$

$$= E \sum_{i=0}^{N-1} [f(\frac{(i+1)T}{N}, Y_{i+1} - f(\frac{iT}{N}, Y_i)]$$

and apply Taylor's formula on each difference of the above equality [59, 60]. In addition, equation (4.9) may also be used in the computations. In this section, we apply the techniques from Malliavin Calculus, such as the chain rule and integration by parts, in the computations, which results in less need of the smoothness of the drift, diffusion and test functions.

3.2 Preliminaries of Malliavin Calculus

There are many good monographs on Malliavin Calculus, see e.g. [4, 42, 43, 44]. In this section, we only introduce the materials that are necessary to our computations.

SupposeW $W(t)$ is a one dimensional Wiener process on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{R} = (\mathcal{F}_t)_{0 \le t \le T}, p)$. For $h(t) \in L^2([0, T])$, we denote by $W(h)$ the Itô stochastic integral $\int_0^T h(t)dW(t)$

Let S denote the set of all random variables of the form

$$f(W(W(h_1, ..., W(h_m))$$

where m is a positive integer, $f : \mathbb{R}^m \to \mathbb{R}$is a C^∞ function such that f and its partial derivatives have at most polynomial growth, and $h_i \in L^2([0, T]), i = 1, ..., m$ Before we continue, we point out a fact that the space S is dense in $L^p(\Omega)$ for every $p \geq 1$ [43].

Definition 3.2.1. *Let $F \in S$, the Malliavin derivative of F is a stochastic process defined* *by*

$$D_tF = \sum_{i=1}^{m} \frac{\partial f}{\partial x_i}(W(h_1), ..., (W(h_m))h_i(t)$$

We often write D_tF *as* DF *if there is no confusion. Specifically, if* $F = \int_0^T h(t)dW(t)$ *, then* $DF = D_tF = h(t)$

The operator $D : S \subset L^p(\Omega) \overset{L}{\to}^2 (\Omega, L^2([0,T]))$ *is closable for any* $p \in [1, \infty)$ *We denote by* $\mathbb{D}^{1,p}$ *the closure of* S *with respect to the norm*

$$\|F\|_{\mathbb{D}^{1,p}} = (E[|F|^p] + E[\|DF\|^p_{L^2}])^{1/2}$$

and

$$\mathbb{D}^{1,\infty} = \bigcap_{p \in \mathbb{N}} \mathbb{D}^{1,p}$$

It is immediate, using the definition of D*, that the product rule holds. That is, if* $F, G \in \mathbb{D}^{1,p}$ *, then* $FG \in \mathbb{D}^{1,p}$ *and* $D(F, G) = FDG + GDF$

Proposition 3.2.1. *(Chain rule, [4], Proposition 10). Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a continuously differentiable function with bounded derivative. Suppose* $F \in \mathbb{D}^{1,p}$ *for some* $p \geq 1$ *Then* $\phi(F) \in \mathbb{D}^{1,p}$ *and we have*

$$D\phi(F) = \phi^{'}(F)DF$$

If $F \in \mathbb{D}^{1\infty}$ *, then the conclusion is true for* ϕ *which is a continuously differentiable function with its derivative having a polynomial growth.*

The following proposition is very useful in our proof.

Proposition 3.2.2. *([44], Corollary 3.13). Let* $u = u(s), s \in [0, T]$ *, be an* $\mathcal{F}_t$ *-adapted stochastic process and assume that* $u(s) \in \mathbb{D}^{1,2}$ *for all s. Then*

1.$D_t(s), s \in [0, T], is \mathcal{F}_t$ *-adapted for all t;*

2.$D_tu(s) = 0, for, t > s.$

We now introduce the adjoint operator of D.

Definition 3.2.2. *We denote by* $Dom(\delta)$ *the subset of* $L^2(\Omega, L^2([0,T]))$ *composed of those elements* u *such that there exists a constant* $C > 0$ *satisfying*

$$|E[\langle DF, u\rangle_{L^2}]| \leq C\sqrt{E[F^2]} forall, F \in \mathbb{D}^{1,2}$$

Fix $u \in Dom(\delta)$. By (4.14), the linear operator $F \overset{(}{\to} E[\langle DF, u\rangle_{L^2}])$ is continuous from S, equipped with the $L^2(\Omega)$ norm, into $\mathbb{R}$. So we can extend it to a linear operator from $L^2(\Omega)$ into $\mathbb{R}$. By the Riesz representation theorem, there exists a unique element in $L^2(\Omega)$, noted $\delta(u)$, such that $E[\langle DF, u\rangle_{L^2}] = E[F\delta(u)]$. This is our next definition.

Definition 3.2.3. *If $u \in Dom(\delta)$ then $\delta(u)$ is the unique element of $L^2(\Omega)$ characterized by the following duality formula:*

$$E[F\delta(u)] = E[\langle DF, u\rangle_{L^2}]$$

for all $F \in \mathbb{R}^{1,2}$

Formula (4.15) is often called an **integrationbypartsformula**. *Usually, if u is a $\mathcal{F}_T$ -measurable stochastic process and is such that $E[\int_0^T u^2 dt] < \infty, \delta(u)$ is often written as $\int_0^T u\delta W(t)$ and we call it the* **Skorohodintegral**. *. We also point out that if u, in addition, is adapted to the filtration $\mathcal{F}_t$, the Skorohod integra $\int_0^T u\delta W(t)$ is nothing but the Itô integral $\int_0^T udW(t)$. Therefore, if $s_1 < s_2$ and u is a fixed $\mathcal{F}_{s1}$ measurable random variable , it is straightforward that $\int_{s1}^{s2} u\delta W(t) = \int_{s1}^{s2} udW(t) = u.(W(s_2) - W(s_1))$.*

The following proposition is useful in many situations.

Proposition 3.2.3. *([42], Proposition 2.5.4). Let $F \in \mathbb{D}^{1,2}$ and $u \in Dom(\delta)$ be such that the three expectations $E[F^2\|u\|_{L^2}^2, E[F^2\delta(u)^2]$ and $Fu \in Dom(\delta)$ and*

$$\delta(Fu) = F\delta(u) - \langle DF, u\rangle_{L^2}$$

For example, if $F \in \mathbb{D}^{1,2}$ and $u = 0$, then we have

$$\int_0^T F\delta W(t) = F\int_0^T 1\delta W(t) - \int_0^T D_tF, 1dt = F(W(T) - W(0)) - \int_0^T D_tFdt$$

We will use this trick very often in our proof to find $\int_0^T F\delta W(T)$.

3.3 Weak Convergence of the EM scheme using Malliavin Calculus

We now state the main theorem of this chapter, which assumes weaker conditions on the drift and diffusion coefficients. This new theorem is included in Rosiński and Wang [54].

Theorem 3.3.1. *([54], Theorem 3.1). Suppose the following conditions hold:*
1.$\mu(t,x) \in C_b^2([0,T] \times \mathbb{R}.\sigma^2(t,x) \in C_b^2([0,T] \times \mathbb{R}) and, g(x) \in C_p^3(\mathbb{R}$

2. *the linear growth condition for* $\mu(t,x) and \sigma(t,x) hold;$
3. *all the partial derivatives of* $\mu(t,x) and \sigma(t,x)$ *with respect to* x *up to order 2 are bounded by a constant* $M > 0$ *for any* t*;*
4. *there exists a positive number* L *such that* $|\sigma(t,x)| \geq L$ *for any* $(t,x) \in [0,T] \times \mathbb{R}$

Then the Euler-Maruyama scheme (4.5) has weak convergence order 1. That is, there exists a positive number C, which depends on M, T and L, such that

$$|E[g(X(T))] - E[g(Y(T))] \leq CN^{-1}$$

Before we prove this theorem, we first give two lemmas that are needed in ourproof.

Lemma 9. . *Suppose* $F, G \in \mathbb{D}^{1,2} and \int_0^T D_t G dt \neq 0$ *a.e.. Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a continuously differentiable function with bounded derivative, then*

$$E[F\phi^{'}(G)] = E[\phi(G)\delta(\frac{F}{\int_0^T D_t G dt})]$$

If $\phi(x)$ *is continuously differentiable with polynomial growth and* $G \in \mathbb{R}^{1\infty}$ *, the above conclusion is also true.*

Proof. By chain rule, we have

$$\int_0^T F D_t \phi(G) dt = \int_0^T F\phi^{'}(G) D_t G dt = F\phi^{'}(G) \int_0^T D_t G dt$$

Observing that $\int_0^T D_t G dt$ *is nonzero, by duality, we have*

$$E[F\phi^{'}(G)] = E[\frac{\int_0^T F D_t \phi(G) dt}{\int_0^T D_t G dt}]$$

$$= E[\langle D_t\phi(G), \frac{F}{\int_0^T D_tGdt}\rangle_{L2}]$$

$$E[\phi(G)\delta(\frac{F}{\int_0^T D_tGdt)}]$$

If

phi(x) is continuously differentiable with polynomial growth and $F \in \mathbb{D}^{1,\infty}$ *, by the same argument above and Proposition 4.2, we can easily get (4.19).*

See [5, 29] for a more general result with ϕ *defined on* $\mathbb{R}^m$

It is well known that, as long as $\mu(t,x) and \sigma(t,x)$ *satisfy the linear growth condition, one has [8], for any* $p \geq 1$,

$$\sup_{0\leq t\leq T} \|Y(t)\| \in L^p(\Omega)$$

Note also that

$$Y(s) = Y_i + \mu(t_i, Y_i)(s - t_i) + \sigma(t_i, Y_i)(W(s) - W(t_i))$$

By the chain rule (Proposition 4.2), one has the following:

Lemma 10. . *Suppose* $\mu(t,x), \sigma(t,x) and, g(x)$ *are assumed as in Theorem 4.7, then* $Y(s) \in D^{1,2}$ *and*

$$D_\tau Y(s) = \sigma(t_i, Y_i)\mathbb{1}_{t_i,s}(\tau), s \in [t_i, t_{i+1}], \tau \in [t_i, T]$$

Furthermore, $F(s) := f(s, Y(s)) \in \mathbb{D}^{1,2}$*and*

$$D_\tau F(s) = \frac{\partial f}{\partial x}(s, Y(s))D_\tau Y(s)$$

We now give the proof of Theorem 4.7.

Proof. First of all, by (4.10) and (4.13),

$$E[g(x(T))] - E[g(Y(T))]$$

$$= -E\sum_{i=0}^{N-1}[f(\tfrac{(i+1)T}{n}, Y_{i+1)} - f(\tfrac{iT}{n}, Y_i)]$$

$$= -E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}[\tfrac{\partial f}{\partial t}(s, Y(s)) + \mu(t_i, Y_i)\tfrac{\partial f}{\partial x}(s, Y(s)) + \tfrac{1}{2}\sigma^2(t_i, Y_i)\tfrac{\partial^2 f}{\partial x^2}(s, Y(s))]ds$$

$= -E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}[\mu(t_i,Y_i)\frac{\partial f}{\partial x}(s,Y(s))+\frac{1}{2}\sigma^2(t_i,Y_i)\frac{\partial^2 f}{\partial x^2}(s,Y(s))-L_t f(s,Y(s))]ds$

$= -E\sum_{i=0}^{N-1}\int_{t_i}^{ti+1}[(\mu(t_i,Y_i)-\mu(s,Y(s)))\frac{\partial f}{\partial x}(s,Y(s))$

$+\frac{1}{2}(\sigma^2(t_i,Y_i)-\frac{1}{2}\sigma^2(s,Y(s)))\frac{\partial^2 f}{\partial x^2}(s,Y(s))]ds$

$= E\sum_{i=0}^{N-1}\int_{t_i}^{ti+1}[(\mu(s,Y(s))-\mu(t_i,Y_i))\frac{\partial f}{\partial x}(s,Y(s))]ds$

$+E\sum_{i=0}^{N-1}\int_{t_i}^{ti+1}[\frac{1}{2}(\sigma^2(s,Y(s))-\sigma^2(t_i,Y_i))\frac{\partial^2 f}{\partial x^2}(s,Y(s))]ds$

$=: I_N + J_N$

In the following, we consider the individual differences in I_N and J_N

By (3.17), Lemma 3.7.1 and Lemma 3.7.2,

$$\int_{t_i}^{t_{i+1}} E[\mu(t_i,Y_i)\frac{\partial f}{\partial x}(s,Y(s))]ds$$

$$\int_{t_i}^{t_{i+1}} E[F(s)\delta(\frac{\mu(t_i,Y_i)\mathbb{1}_(t_i,t_{i+1})}{\sigma(t_i,Y_i)(s-t_i)})]ds$$

$$= \int_{t_i}^{t_{i+1}} E[F(s)\int_{t_i}^{t_{i+1}}\frac{\mu(t_i,Y_i)}{\sigma(t_i,Y_i)(s-t_i)}\delta W(\tau)]ds$$

$$\int_{t_i}^{t_{i+1}} E[F(s)\frac{\mu(t_i,Y_i)\Delta W_i}{\sigma(t_i,Y_i)(s-t_i)}]ds$$

Similarly,

$$\int_{t_i}^{t_{i+1}} E[\mu(s,Y(s))\frac{\partial f}{\partial x}(s,Y(s))]ds$$

$$= \int_{t_i}^{t_{i+1}} E[F(s)\delta(\frac{\mu(s,Y(s))\mathbb{1}_{(t_i},t_{i+1})}{\sigma(t_i,Y_i)(s-t_i)})]ds$$

$= \int_{t_i}^{t_{i+1}} E[F(s)\frac{\delta(\mu(s,Y(s))\mathbb{1}_{(t_i,t_{i+1})})}{\sigma(t_i,Y_i)(s-t_i)}]ds$
$= \int_{t_i}^{t_{i+1}} E[F(s)\frac{\mu(s,Y(s))\Delta W_i - \int_{t_i}^{t_{i+1}} D\mu(s,Y(s))d\tau}{\sigma(t_i,Y_i)(s-t_i)}]ds$
$= E\int_{t_i}^{t_{i+1}} F(s)\frac{\mu(s,Y(s))\Delta W_i - \int_{t_i}^{t_{i+1}} \frac{\partial\mu}{\partial x}(s,Y(s))D_\tau Y(s))D_\tau Y(s)d\tau}{\sigma(t_i,Y_i)(s-t_i)}ds$
$= E\int_{t_i}^{t_{i+1}} F(s)\frac{\mu(s,Y(s))\Delta W_i}{\sigma(t_i,Y_i)(s-t_i)} - F(s)\frac{\partial\mu}{\partial x(s,Y(s))ds}$
By (4.24) and (4.25), we have

$$I_N = E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}[(\mu(s,Y(s)) - \mu(t_i,Y_i))\frac{\partial f}{\partial x}(s,Y(s))]ds$$

$$= E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}} F(s)[\frac{(\mu(s,Y(s)) - \mu(t_i,Y_i))\Delta W_i}{\sigma(t_i,Y_i)(s-t_i)} - \frac{\partial\mu}{\partial x}(s,Y(s))]ds$$

similarly,for $F(t_i) = f(t_i,Y(t_i)) = f(t_i,Y_i)$ *,we have* $D_\tau F(t_i) = \frac{\partial f}{\partial x}(t_i,Y_i)D_\tau Y(t_i) = 0, \tau \in [t_i,t_{i+1}]$*.we have,by duality,*

$$0 = E\int_{t_i}^{t_{i+1}}\frac{\mu(s,Y(s)) - \mu(t_i,Y_i)}{\sigma(t_i,Y_i)(s-t_i)}D_\tau F(t_i,Y_i)d\tau$$

$$= E[F(t_i)\delta(\frac{\mu(s,Y(s)) - \mu(t_i,Y_i)}{\sigma(t_i,Y_i)(s-t_i)})]$$

$$= [F(t_i)(\frac{(\mu(s,Y(s)) - \mu(t_i,Y_i))\Delta W_i}{\sigma(t_i,Y_i)(s-t_i)} - \frac{\partial\mu}{\partial x}(s,Y(s)))]$$

Adding (4.27) to (4.26), and observing $\Delta W_i = W(s) - W(t_i) + W(t_{i+1}) - W(s)$*, we have*

$$I_N = E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}[(\mu(s,Y(s)) - \mu(t_i,Y_i))\frac{\partial f}{\partial x}(s,Y(s))]ds$$

$$= E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}(F(s) - F(t_i))[\frac{(\mu(s,Y(s)) - \mu(t_i,Y_i))\Delta W_i}{\sigma(t_i,Y_i)(s-t_i)} - \frac{\partial\mu}{\partial x}(s,Y(s))]ds$$

$$= E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}(F(s) - F(t_i))[\frac{(\mu(s,Y(s)) - \mu(t_i,Y_i))(W(s) - W(t_i))}{\sigma(t_i,Y_i)(s-t_i)}]ds$$

$$-\frac{\partial\mu}{\partial x}(s,Y(s))]ds$$

where in the last equality we used the fact that $W(t_{i+1}) - W(s)$ *is independent of* $(\mathcal{F}, 0 \leq t \leq s)$ *and* $E[W(t_{i+1}) - W(s)] = 0$

To reduce the notation burden in the following, we fix an interval $[t_i,t_{i+1}]$ *and denote* *by* $\Delta_s = s - t_i$ *and* $\Delta_{\mu\sigma} = \mu(t_i, Y_i)(s - t_i) + \sigma(t_i, Y_i)(W(s) - W(t_i)), s \in [t_i, t_{i+1}]$

In view of (4.20), it follows directly that, for any positive integer k,

$$E[|\mu(t_i, Y_i)|^K] \leq E[(1 + |Y_i|)^K] \leq C(K, T)$$

$$E[|\sigma(t_i, Y_i)|^K] \leq E[(1 + |Y_i|)^K] \leq C(K, T)$$

The following lemma gives the bound of the increment of $\Delta_{\mu\sigma} = \mu(t_i, Y_i)(s - t_i) + \sigma(t_i, Y_i)(W(s) - W(t_i))$

Lemma 11. *.suppose* $\mu(t, x)$ *and* $\sigma(t, x)$ *satisfy the linear growth condition, then there exists a positive constant* $C(T)$ *such that:*

$$E[|\Delta_{\mu\sigma}|] \leq C(T)\Delta_s^{1/2}$$

$$E[|_{\mu\sigma}|^2] \leq C(T)\Delta_s$$

Proof. This is straightforward by combining the linear growth condition, the bound of Y_i *and Cauchy-Schwarz.*

$$E[|\mu(t_i, Y_i)(s - t_i) + \sigma(t_i, Y_i)(W(s) - W(t_i))|]$$

$$\leq E[|\mu(t_i, Y_i)\Delta_s|] + E[|\sigma(t_i, Y_i)(W(s) - (W(t_i))|]$$

$$= \Delta_s E[|\mu(t_i, Y_i)|] + E[|W(s) - W(t_i)|]E[|\sigma(t_i, Y_i)|]$$

$$\leq \Delta_s E[1 + |Y_i|] + \Delta^{1/2} E[1 + |Y_i|]$$

$$= \Delta_s^{1/2}(\Delta_s^{1/2} E[1 + |Y_i|] + E[1 + |Y_i|])$$

$$\leq C(T)\Delta_s^{1/2}$$

The L^2 *bound of* $\Delta_{\mu\sigma}$ *can be proved in a similar way.*

In fact, applying the same argument as in Lemma 4.7.3, one obtains the following *estimate with no difficulty:*

$$E[|\Delta_{\mu\sigma}|^K] \leq C(K,T)\Delta_s^{K/2}$$

We now look at the terms in the last equality of (4.28). By Taylor expansion,

$$F(s) - F(t_i) = f(t_i + \Delta_s, Y_i + \Delta_{\mu\sigma}) - f(t_i, Y_i)$$

$$\Delta_s \frac{\partial f}{\partial t}(t_i, Y_i) + \Delta_{\mu\sigma}\frac{\partial f}{\partial x}(t_i, Y_i) + R_1$$

where R_1 is the Lagrange-type residual of the expansion and takes the form

$$R_1 = \frac{1}{2}\Delta_s^2 \frac{\partial^2 f}{\partial t^2}(t_i + c_1\Delta_s, Y_i + c_1\Delta_{\mu\sigma}) + \frac{1}{2}\Delta_{\mu\sigma}^2 \frac{\partial^2 f}{\partial x^2}(t_i + c_1\Delta_s, Y_i + c_1\Delta_{\mu\sigma})$$

$$+\Delta_s\Delta_{\mu\sigma}\frac{\partial^2 f}{\partial t \partial x}(t_i + c_1\Delta_s, Y_i + c_1\Delta_{\mu\sigma})$$

where $c_1 \in (0,1)$. Since $f \in c_p^3(\mathbb{R})$and the bound of $\Delta_{\mu\sigma}$ holds (Lemma 4.31), it is obvious that

$$E[|R_1|] \leq \sqrt{E[|R_1|^2]} \leq C(T,M)\Delta_s.$$

Similarly,

$$(\mu(s, Y(s)) - \mu(t_i, Y_i))(W(s) - W(t_i))$$
$$= (\Delta_s \frac{\partial \mu}{\partial t}(t_i, Y_i) + \Delta_{\mu\sigma}\frac{\partial \mu}{\partial x}(t_i, Y_i) + R_2).(W(s) - W(t_i)),$$

where R_2, similar to R_1, takes the form

$$R_2 = \frac{1}{2}\Delta_s^2 \frac{\partial^2 \mu}{\partial t^2}(t_i + c_2\Delta_s, Y_i + c_2\Delta_{\mu\sigma}) + \frac{1}{2}\Delta_{\mu\sigma}^2 \frac{\partial^2 \mu}{\partial x^2}(t_i + c_2\Delta_s, Y_i + c_2\Delta_s, Y_i + c_2\Delta_{\mu\sigma})$$

$$+\Delta_s\Delta_{\mu\sigma}\frac{\partial^2 \mu}{\partial t \partial x}(t_i + c_2\Delta_s, Y_i + c_2\Delta_{\mu\sigma})$$

where $c_2 \in (0,1)$, and has the following bound

$$E[|R_2|] \leq \sqrt{E[|R_2|^2]} \leq C(T,M)\Delta_s$$

By the same arguments of getting (4.35) and (4.38), we also have

$$E[|R_1R_2|^2] \le C(T,M)\Delta_s^4$$

$$E[|R_2(\Delta_s\frac{\partial f}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial f}{\partial x}(t_i,Y_i)|^2]/C(T,M)\Delta_s^3$$

$$E[|R_1(\Delta_s\frac{\partial \mu}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial \mu}{\partial x}(t_i,Y_i)|^2]/C(T,M)\Delta_s^3$$

Lastly,

$$\frac{\partial \mu}{\partial x}(s,Y(s)) = \frac{\partial \mu}{\partial x}(t_i,Y_i)+R_3$$

where R_3 takes the form

$$R_3 = \Delta_s\frac{\partial^2 \mu}{\partial t\partial x}(t_i+c_3\Delta_s, Y_i+c_3\Delta_{\mu\sigma})+\Delta_{\mu\sigma}\frac{\partial^2 \mu}{\partial x^2}(t_i+c_3\Delta_s, Y_i+c_3\Delta_{\mu\sigma})$$

where $c_3 \in (0,1)$. Similar to getting (3.35), R3 also satisfies

$$E[|R_3|] \le \sqrt{E[|R_3|^2]} \le C(T,M)\Delta_s^{1/2}$$

Similar to the arguments of getting (3.39) and (3.40), we can also have

$$E[|R_1R_3|] \le C(T,M)\Delta_s^{3/2}$$

$$E[|R_3(\Delta_s\frac{\partial f}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial f}{\partial x}(t_i,Y_i))|] \le C(T,M)\Delta_s$$

$$E[|R_1\frac{\partial \mu}{\partial x}(t_i,Y_i)|] \le C(T,M)\Delta_s$$

Combining (4.33) and (4.36), we get that

$$(F(s)-F(t_i))\frac{(\mu(s,Y(s))-\mu(t_i,Y_i))(W(s)-W(t_i))}{\sigma(t_i,Y_i)(s-t_i)}$$

$$\frac{W(s)-W(t_i)}{\sigma(t_i,Y_i)\Delta_s}[\Delta_s^2\frac{\partial f}{\partial t}(t_i,Y_i)\frac{\partial \mu}{\partial t}(t_i,Y_i)$$

$$+\Delta_s\Delta_{\mu\sigma}(\frac{\partial f}{\partial t}(t_i,Y_i)\frac{\partial \mu}{\partial x}(t_i,Y_i)+\frac{\partial f}{\partial x}(t_i,Y_i)\frac{\partial \mu}{\partial t}(t_i,Y_i))$$

$$+\Delta_{\mu\sigma}^2\frac{\partial f}{\partial x}(t_i,Y_i)\frac{\partial \mu}{\partial x}(t_i,Y_i)+R_2(\Delta_s\frac{\partial f}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial f}{\partial x}(t_i,Y_i))$$

$$R_1(\Delta_s\frac{\partial \mu}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial \mu}{\partial x}(t_i,Y_i))+R_1R_2].$$

Since $W(s)-W(t_i)$ is independent of $(\mathcal{F}_t, 0\le t\le t_i)$, the first term on the right hand side of (4.48) satisfies

$$E[\frac{W(s)-W(t_i)}{\sigma(t_i,Y_i)\Delta_s}\Delta_s^2\frac{\partial f}{\partial t}(t_i,Y_i)\frac{\partial \mu}{\partial t}(t_i,Y_i)]=0$$

Therefore, the second term on the right hand side of (4.48) satisfies

$$|E[\frac{W(s)-W(t_i)}{\sigma(t_i,Y_i)\Delta_s}\Delta_s\Delta_{\mu\sigma}(\frac{\partial f}{\partial t}(t_i,Y_i)\frac{\partial \mu}{\partial x}(t_i,Y_i)+\frac{\partial f}{\partial x}(t_i,Y_i)\frac{\partial \mu}{\partial t}(t_i,Y_i))]|$$

$$\le E[|\frac{W(s)-W(t_i)}{\sigma(t_i,Y_i)}\Delta_{\mu\sigma}(\frac{\partial f}{\partial t}(t_i,Y_i)\frac{\partial \mu}{\partial x}(t_i,Y_i)+\frac{\partial f}{\partial x}(t_i,Y_i)\frac{\partial \mu}{\partial t}(t_i,Y_i))]|$$

$$\le C(L,M)E[|W(s)-W(t_i)\Delta_{\mu\sigma}|]$$

$$\le C(L,M)\sqrt{E[|W(s)-W(t_i)^2]}\sqrt{E[\Delta_{\mu\sigma}|]}$$

$$\le C(L,M,T)\Delta_s$$

Similarly, observing $E[W(s)-W(t_i)]=0, E[(W(s)-W(t_i))^3]=0$, the third term in (3.48) satisfies

$$|E[\frac{W(s)-W(t_i)}{\sigma(t_i,Y_i)\Delta_s}\Delta_{\mu\sigma}^2\frac{\partial f}{\partial x}(t_i,Y_i)\frac{\partial \mu}{\partial x}(t_i,Y_i)]|$$

$$=|E[\frac{W(s)-W(t_i)}{\sigma(t_i,Y_i)\Delta_s}(\Delta_s^2\mu^2(t_i,Y_i)+2\Delta_s(W(s)-W(t_i))\mu(t_i,Y_i)\sigma(t_i,Y_i)$$

$$+(W(s)-W(t_i))^2\sigma^2(t_i,Y_i))\frac{\partial f}{\partial x}(t_i,Y_i)\frac{\partial \mu}{\partial x}(t_i,Y_i)]|$$

$$=|E[\frac{W(s)-W(t_i))^3\mu^2(t_i,Y_i)}{\sigma(t_i,Y_i)}]+2E[(W(s)-W(t_i))^2\mu(t_i,Y_i)]$$

$$+E[\frac{(W(s)-W(t_i))^3\sigma(t_i,Y_i)}{\Delta_s}\frac{\partial f}{\partial x}(t_i,Y_i)\frac{\partial \mu}{\partial x}(t_i,Y_i)]|$$

$$=|0+2E[(W(s)-W(t_i))^2]E[\mu(t_i,Y_i)]|$$

$$\le C(T)\Delta_s$$

By (3.40)), the fourth term on the right hand side of (3.48) satisfies

$$|E[\frac{W(s)-W(t_i)}{\sigma(t_i,Y_i)\Delta_s}R_2(\Delta_s\frac{\partial f}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial f}{\partial x}(t_i,Y_i))]|$$

$$\leq \frac{C(L)}{\Delta_s}E[|W(s)-W(t_i))R_2(\Delta_s\frac{\partial f}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial f}{\partial x}(t_i,Y_i))|]$$

$$\leq \frac{C(L)}{\Delta_s}\sqrt{E[W(s)-W(t_i))]}\sqrt{E[|R_2(\Delta_s\frac{\partial f}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial f}{\partial x}(t_i,Y_i))|^2]}$$

$$\leq \frac{C(L,M,T)}{\Delta_s}\Delta_s^{1/2}\Delta_s^{3/2}$$

$$\leq C(L,M,T)\Delta_s$$

Similarly, the fifth term on the right hand side of (4.48) satisfies

$$|E[\frac{W(s)-W(t_i)}{\sigma(t_i,Y_i)\Delta_s}R_1(\Delta_s\frac{\partial \mu}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial \mu}{\partial x}(t_i,Y_i)]|\leq C(L,M,T)\Delta_s$$

Finally, by (3.39), the last term on the right hand side of (3.48) satisfies

$$|E[\frac{W(s)-W(t_i)}{\sigma(t_i,Y_i)\Delta_s}R_1R_2]|\leq \frac{C(L)}{\Delta_s}E[|(W(s)-W(t_i))R_1R_2|]$$

$$\leq \frac{C(L)}{\Delta_s}\sqrt{E[(W_s-W_{t_i})^2]}\sqrt{E[|R_1R_2|^2]}$$

$$\leq \frac{C(L,M,T)}{\Delta_s}\Delta_s^{1/2}\Delta_s^{2}$$

$$= C(L,M,T)\Delta_s^{3/2}$$

Combining (3.48), (3.49), (3.50), (3.51), (3.52), (3.53) and (3.54), we have

$$|E[(F(s)-F(t_i))\frac{(\mu(s,Y(s))-\mu(t_i,Y_i))(W(s)-W(t_i))}{\sigma(t_i,Y_i)(s-t_i)}]|\leq C(L,M,T)\Delta_s$$

On the other hand, by (3.33) and (3.42), the other term in (3.28) is

$(F(s)-F(t_i))\frac{\partial \mu}{\partial x}(s,Y(s))$

$= (\Delta_s\frac{\partial f}{\partial t}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial f}{\partial x}(t_i,Y_i)+R_1)(\frac{\partial \mu}{\partial x}(t_i,Y_i)+R_3)$

$= \Delta_s\frac{\partial f}{\partial t}(t_i,Y_i)\frac{\partial \mu}{\partial x}(t_i,Y_i)+\Delta_{\mu\sigma}\frac{\partial f}{\partial x}(t_i,Y_i)\frac{\partial \mu}{\partial x}(t_i,Y_i)+R_1\frac{\partial \mu}{\partial x}(t_i,Y_i)$

$+R_3(\Delta_s \frac{\partial f}{\partial t}(t_i, Y_i) + \Delta_{\mu\sigma} \frac{\partial f}{\partial x}(t_i, Y_i)) + R_1 R_3$

Due to the assumption that $\mu(t,x) \in C_b^2(\mathbb{R})$ *and* $f \in C_p^3(\mathbb{R})$ *, the first term of the right hand side of (3.56) satisfies*

$$|E[\Delta_s \frac{\partial f}{\partial t}(t_i, Y_i) \frac{\partial \mu}{\partial x}(t_i, Y_i)]| \leq C(M)\Delta_s$$

Similarly, the second term of the right hand side of (3.56) satisfies

$|E[\Delta_{\mu\sigma} \frac{\partial f}{\partial x}(t_i, Y_i)]|$

$= |E[\Delta_s \mu(t_i, Y_i) \frac{\partial f}{\partial x}(t_i, Y_i) \frac{\partial \mu}{\partial x}(t_i, Y_i) + (W(s) - W(t_i))\sigma(t_i))\sigma(t_i, Y_i) \frac{\partial f}{\partial x}(t_i, Y_i) \frac{\partial \mu}{\partial x}(t_i, Y_i)]|$

$= |E[\Delta_s \mu(t_i, Y_i) \frac{\partial f}{\partial x}(t_i, Y_i) \frac{\partial \mu}{\partial x}(t_i, Y_i) + 0]|$

$\leq C(T, M)\Delta_s$

The estimates of the remaining terms on the right hand side of (3.56) are exactly (3.45), (3.46) and (3.47). Therefore, by (3.56), (3.57), (3.58), (3.45), (3.46) and (3.47), we have

$$|E[(F(s) - F(t_i)) \frac{\partial \mu}{\partial x}(s, Y(s))]| \leq C(T, M)\Delta_s$$

Finally, by (3.28), (3.55) and (3.59), we have

$|I_N| \leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} |E[(F(s) - F(t_i)) \frac{(\mu(s,Y(s)) - \mu(t_i,Y_i))(W_s - W_{t_i})}{\sigma(t_i,Y_i)(s - t_i)}]|$

$+|E[F(s) - F(t_i)) \frac{\partial \mu}{\partial x}(s, Y(s))]|ds$

$\leq \sum_{i=0}^{N-1} \int_{t_i}^{t_{i+1}} C(L, M, T)\Delta_s ds$

$\leq C(L, M, T) \sum_{i=0}^{N-1} \frac{1}{2}(t_{i+1} - t_i)^2$

$C(L, M, T) \sum_{i=0}^{N_1} \frac{1}{2}\Delta^2 \leq C(L, M, T)\Delta$

As for J_N *, let us compare* J_N *to* I_N *first.* I_N *takes the form*

$$I_N = E\sum_{i=0}^{N-1}\int_{t_i}^{t_{t+1}}[(\mu(s, Y(s)) - \mu(t_i, Y_i))\frac{\partial f}{\partial x}(s, Y(s))]ds$$

and J_N *takes the form*

$$J_N = E\sum_{i=0}^{N-1}\int_{t_i}^{t_{t+1}}[\frac{1}{2}(\sigma^2(s, Y(s)) - \sigma^2(t_i, Y_i))\frac{\partial^2 f}{\partial x^2}(s, Y(s))]ds$$

$$J_N = E\sum_{i=0}^{N-1}\int_{t_i}^{t_{t+1}}[\frac{1}{2}(\sigma^2(s, Y(s)) - \sigma^2(t_i, Y_i))\frac{\partial}{\partial x}\frac{\partial f}{\partial x}(s, Y(s))]ds$$

Therefore, if $\sigma^2(t, x)$ *satisfies the same conditions as* $\mu(t, x)$ *does and* $\frac{\partial f}{\partial x}(t, x)$ *satisfies the same conditions as* $f(t, x)$ *does, then* J_N *should have the similar estimates as* I_N *has, as shown in (3.60). Recall that our assumptions in Theorem 4.7 state that* $\mu(t, x) \in C_b^2([0, T] \times \mathbb{R}), \sigma^2(t, x) \in C_b^2([0, T] \times \mathbb{R})$ *and* $g(x) \in C_p^3(\mathbb{R})$*. Therefore, similar to (4.60),* J_N *also has the following estimate*

$$|J_N| = |E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}[\frac{1}{2}(\sigma^2(s, Y(s)) - \sigma^2(t_i, Y_i))\frac{\partial^2 f}{\partial x^2}(s, Y(s))]ds|$$

$$\leq C(L, M, T)\Delta.$$

Then, by (3.23), (3.60) and (3.63), it is straightforward that

$|E[g(X(T))] - E[g(Y(T))]|$

$= |E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}[\mu(s, Y(s)) - \mu(t_i, Y_i))\frac{\partial f}{\partial x}(s, Y(s))]ds$

$+E\sum_{i=0}^{N-1}\int_{t_i}^{t_{i+1}}[\frac{1}{2}\sigma^2(s, Y(s)) - \sigma^2(t_i, Y_i))\frac{\partial^2 f}{\partial x^2}(s, Y(s))]ds|$

$= |I_N + J_N| \leq |I_N| + |J_N|$

$\leq (L, T, M)\Delta.$

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