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UNIVERSITÉ AMAR TELIDJI LAGHOUAT
كلية العلوم
FACULTÉ DES SCIENCES
قسم الرياضيات
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Par :

HASSANE Doha Ihssane

Thème

Well posedness and stability result for a thermoelastic laminated Timoshenko beam with distributed delay term.

Devant le jury composé de:

SAF Salim	MCB	Université de Laghouat	Président
RAHMOUNE Abdel Aziz	MCA	Université de Laghouat	Examinateur
YAZID Fares	MCA	Université de Laghouat	Encadreur
DJERADI Fatima Siham	Doctorante	Université de Laghouat	Co-Encadreur

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Dedication

To my father and my mother, for their sacrifices, wisdom, and for always believing in me even when I doubted myself.

To my grandmothers Ma and Mayma and to my grandfather BAYA for their constant prayers for my success

To my brother Younes, my sisters Wissal and Mafaz .

To my uncle the professor CHETTIH Mohamed and my aunts CHETTIH Reguia, CHETTIH Saliha, CHETTIH Souad and my aunt BENDAOUED Zohra .

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And to everyone who has been a part of this incredible journey, your support has been invaluable.

Thank you all for being the pillars of my success. This work is as much yours as it is mine.

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Abstract

In this memory , we considered a linear thermoelastic laminated Timoshenko beam with distributed delay, where the heat conduction is given by Cattaneo's law. we established the global existence and the uniqueness of the solution by using the semi-group theory. Finally, we studied the asymptotic behavior of solution by using the multiplier method, and we proved the exponential an the polynomial stabilities of the system for the cases of equal and non-equal speeds of wave propagation.

Keys words:*Timoshenko; stability; existence and .uniqueness; distributed delay term*

تلخيص

في هذه المذكرة، اعتبرنا نظاما خطيا للحزمة المصفحة لتيموشينكو مع تأخير موزع.
أولا، برهنا وجود وحدانية الحل باستخدام نظرية هيل-يوشيدا.
أخيرا، قمنا بدراسة السلوك المقارب للحل باستخدام طريقة الطاقة، و برهنا الإستقرار الأسي و متعدد الحدود وفقا لافتراضات مناسبة حول ردود الفعل المتأخرة في حالات تساوي و عدم تساوي سرعة انتشار الموجة.

الكلمات المفتاحية : حد التأخير الموزع، الإستقرار الأسي، متعدد الحدود، الوجود، الوحدانية ،تيموشينكو.

Résumé

dane ce mémoire, nou considérons un systéme lineaire thermoélastic de poutres lamellaires Timochenko avec un terme de retard distribué . nous établissons l'existence globale et l'unicité de la solution en utilisoant la theorie de semi groupe. En fin, nous étudions les comportement asymptotiques de solution en utilisant la méthode des multiplicateurs et nous montrons la stabilité exponentielle et polynomiale de systéme.

les mots clés:*Terme de retard distribué,stabilité,exponentielle,plynomial,existence,unicité,Timoshenko.*

Contents

GENERAL INTRODUCTION

The study of thermoelastic laminated Timoshenko beams with distributed delay is an advanced area of structural analysis that integrates beam theory, composite material science, thermoelasticity, and time-dependent effects. This approach provides a detailed and accurate understanding of the beam's behavior under complex loading conditions, making it crucial for the design and analysis of modern engineering structures.

1.Thimoshinko Beam Theory

Timoshenko beam theory, named after Stephen Timoshenko, is an extension of the classical Euler-Bernoulli beam theory. It accounts for both bending and shear deformations, making it more accurate for analyzing beams, especially those with a relatively short length or those made from materials where shear effects are non-negligible. The Timoshenko beam theory incorporates:

- **Shear Deformation:** Unlike Euler-Bernoulli beams, which assume plane sections remain plane and perpendicular to the neutral axis, Timoshenko beams allow for shear deformation, providing a more realistic representation of the beam's behavior under load.
- **Rotary Inertia:** It includes the effects of rotary inertia, which becomes significant in dynamic analysis.

2.Thermoelasticity

Thermoelasticity refers to the study of the interaction between thermal and elastic effects in materials. For a thermoelastic beam, temperature changes can induce thermal stresses and strains, affecting the beam's overall mechanical behavior. Key considerations include:

- **Thermal Expansion:** Different layers in a laminated beam may expand or contract differently with temperature changes, leading to additional stresses.

- **Temperature Distribution:** Temperature gradients across the beam's cross-section can cause bending and warping.

3.Distributed Delay

Distributed delay refers to a type of time-dependent behavior where the response of the system is influenced by its past states over a continuous range of time. This concept is crucial in many real-world applications where the effects of past states are not instantaneous but spread over time. In the context of a thermoelastic laminated Timoshenko beam, distributed delay can be related to:

- **Material Relaxation:** The gradual adjustment of material stresses and strains over time.
- **Heat Transfer Dynamics:** The time-dependent process of temperature distribution within the beam.
- **Viscoelastic Effects:** Time-dependent deformation behavior of materials, particularly relevant in composite materials.

4.Thermoelastic Laminated Timoshenko Beam Analysis

The analysis of a thermoelastic laminated Timoshenko beam involves combining the principles of Timoshenko beam theory, laminated composite materials, and thermoelasticity. This complex interaction is analyzed to predict the beam's behavior under various loading and environmental conditions, including:

- **Aerospace Engineering:** For components like wings and fuselage sections where weight savings and performance are critical.
- **Civil Engineering:** In structures like bridges and buildings that may be subjected to varying thermal environments.
- **Mechanical Engineering:** For designing lightweight, high-performance components in machinery and vehicles.

5.Applications

Thermoelastic laminated Timoshenko beams are widely used in:

- **Aerospace Engineering:** For components like wings and fuselage sections where weight savings and performance are critical.
- **Civil Engineering:** In structures like bridges and buildings that may be subjected to varying thermal environments.
- **Mechanical Engineering:** For designing lightweight, high-performance components in machinery and vehicles.

In this work, we shall examine how we can apply the distributed delay term laminate beam. The latter was introduced for the first time by Hansen and Spies.[6, 7]. Also, this is due to the relevance of the research topic to the applicability of these materials in the industry. Whereas, they presented a mathematical model of two-layer beams with structural damping due to the interslip obtained by:

$$\begin{cases} \rho\omega_{tt} + G\left(\psi - \omega_x\right)_x = 0, \\ \\ I_\rho\left(3s_{tt} - \psi_{tt}\right) - G\left(\psi - \omega_x\right) - D\left(3s_{xx} - \psi_{xx}\right) = 0, \\ \\ I_\rho s_{tt} - G\left(\psi - \omega_x\right) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t - Ds_{xx} = 0, \end{cases} \tag{0.1}$$

There are some results related to laminated beam equations that study global existence and stability of the relevant system. In addition, by adding appropriate damping effects such as frictional damping (boundary), viscoelastic, or internal damping. However, if we add linear damping terms to two of the three equations, System (0.1) is exponentially stable under the assumption of equal wave speeds $(\rho/I_\rho) = (G/D)$. But if we add these conditions to all equations, the system decays exponentially without assuming equal wave speeds [8, 11].

As for thermoelastic laminated Timoshenko beam, there are few results including the work of Liu et al. [12] and Apalara [13]. In a previous study [12], the authors considered the following laminated beams with past history:

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x + \theta_x = 0, \\ \\ I_\rho(3\omega - \psi)_{tt} - D(3\omega - \psi)_{xx} + \int_0^t g(s)(3\omega - \psi)_{xx}(t-s)ds - G\left(\psi - \varphi_x\right) - \theta = 0, \\ \\ I_\rho\omega_{tt} - D\omega_{xx} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma\omega + \frac{4}{3}\beta\omega_t = 0, \\ \\ k\theta_t - \tau\theta_{xx} + \varphi_{tx} + (3\omega - \psi)_t = 0 \end{cases}$$

They established the global well-posedness of solutions to the system and the stability of the system. If $\beta \neq 0$, they proved exponential and polynomial stabilities depending on the behavior of the kernel function g. If $\beta = 0$, they established exponential stability in the case of the equal wave speeds assumption, and exponential stability does not exist in the case of nonequal wave speeds assumption. Apalara [13] considered a laminated beam with second sound in the following system:

$$\begin{cases} \rho\omega_{tt} + G\left(\psi - \omega_x\right)_x = 0, \\ I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G\left(\psi - \omega_x\right) + \delta\theta_x = 0, \\ I_\rho s_{tt} - Ds_{xx} + G\left(\psi - \omega_x\right) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t = 0, \\ \rho_3\theta_t + q_x + \delta(3s-\psi)_{tx} = 0, \\ \tau q_t + \alpha q + \theta_x = 0. \end{cases} \tag{0.2}$$

They established the global well-posedness and proved exponential and polynomial stabilities depending on the parameter χ. Feng [14] considered a Timoshenko-Coleman-Gurtin system and studied the long-time dynamics of the system. In the study by Aouadi et al.[15], the authors considered two classes of nonuniform thermoelastic Timoshenko systems and established global well-posedness and stability results. We can mention two new results: one on laminated beams with thermal damping [16] and another on a coupled system of hyperbolic equations with a heat equation of second sound [17]. Recently, Feng considered the following system:

$$\begin{cases} \rho\omega_{tt} + G\left(\psi - \omega_x\right)_x + \delta\theta_x = 0, \\ I_\rho(3s-\psi)_{tt} - D(3s-\psi)_{xx} - G\left(\psi - \omega_x\right) = 0, \\ I_\rho s_{tt} - Ds_{xx} + G\left(\psi - \omega_x\right) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t = 0, \\ \rho_3\theta_t + q_x + \delta\omega_{tx} = 0, \\ \tau q_t + \alpha q + \theta_x = 0. \end{cases} \tag{0.3}$$

They established the global well-posedness and stability of systems. Introducing a distributed delay term makes our problem different from those considered so far in the literature. The importance of this term appears in many works, and this is due to the fact

that many phenomena depend on their past. Also, its influence on the asymptotic behavior of the solution for different types of problems such as the Timoshenko system [13], transmission problem [19], wave equation [20], and thermoelastic system [21].
In complement to Feng's work [18] and previous works, we are working to prove the well-posedness and stability results with distributed delay for the cases of equal and nonequal speeds of wave propagation under appropriate assumptions. We prove these results using the energy method. We use c throughout this paper to denote a generic positive constant.

Chapter 1

Preliminaries

In this chapter, we recall some basic knowledge in functional analysis, most of which will be used in the subsequent chapter. The reader can easily find the details in the related literature, see, e.g.,[1, 2, 3, 4, 5].

1.1 Functional Spaces

We denote by $\mathbb{R}^n$ the Euclid space, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain,$C^k(\Omega)$ is the k^{th} differentiable continuous function space in Ω, $C^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space in Ω, $C_c^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space with compact support in Ω .

Definition 1. *Let X be a vector space over the filed $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then a semi-norm on X is a function $\|.\| : X \longrightarrow \mathbb{R}$, such that :*
a) $\|x\| \geqslant 0$ for all $x \in X$,
b) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$,
c) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$.
A norm on X is a semi-norm which also satisfies :
d) $\|x\| = 0 \Rightarrow x = 0$. A vector space X together with a norm $\|.\|$ is called a normed linear space, a normed vector space or simply, a normed space.

Definition 2. *(Convergent and Cauchy sequences)*
Let X be a normed space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.
a) $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

i.e. if

$$\forall \varepsilon > 0; \exists N > 0, \forall n \geq N, \|x_n - x\| < \varepsilon.$$

b) $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence if*

$$\forall \varepsilon > 0; \exists N > 0, \forall m, n \geq N, \|x_m - x_n\| < \varepsilon.$$

Normed spaces in which every Cauchy sequence is convergent are called complete normed spaces. In general a normed space is not complete.

Definition 3. *(Banach Spaces)*
A normed spaces is called a Banach space if it is complete i.e. if any Cauchy sequence inside the space converges to a point of the space. Its dual space X' is the linear space of all continuous linear functional $f : X \to \mathbb{R}$.

Proposition 1. *X' equipped with the norm $\|.\|_{X'}$ defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}$$

is also a Banach space.

Remark 1. *From X' we construct the bidual or second dual $X'' = (X')'$. Furthermore, with each $u \in X$ we can define $\varphi(u) \in X''$ by $\varphi(u)(f) = f(u)$, $f \in X'$, this satisfies clearly $\|\varphi(x)\| \leq \|u\|$. Moreover, for each $u \in X$ there is an $f \in X'$ with $f(u) = \|u\|$ and $\|f\| = 1$, so it follows that $\|\varphi(x)\| = \|u\|$.*

Definition 4. *Since φ is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X onto a closed subspace of X'', we denote this by

$$X \hookrightarrow X''.$$

Definition 5. *If φ (in the above definition) is onto X'' we say X is reflexive, $X \approxeq X''$.*

1.1.1 The weak and weak star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\begin{aligned} \varphi_f :& X \to \mathbb{R} \\ & x \mapsto \varphi_f \end{aligned}$$

When f cover X', we obtain a family $(\varphi_f)_{f\in X'}$ of appmications to X in $\mathbb{R}$.

Definition 6. *The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X for which every $(\varphi_f)_{f\in X'}$ is continuous.*

We will define the topology on X', the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_x :& X' \to \mathbb{R} \\ & f \mapsto \varphi_x(f) = \langle f, x\rangle_{X',X} \end{aligned}$$

Definition 7. *The weak star topology on X' is the weakest topology on X' for which every $(\varphi_x)_{x\in X'}$ is continuous.*

Remark 2. *Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest than the topology $\sigma(X', X'')$, and this later is weaker than the strong topology.*

Definition 8. *A sequence (x_n) in X is weakly convergent to x if and only if*

$$\lim_{n\to\infty} f(x_n) = f(x)$$

for evey $f \in X'$, and this is denoted by $x_n \rightharpoonup x$.

Remark 3.

1. *If the weak limit exists, it is unique.*
2. *If $x_n \to x \in X$ (strongly), then $x_n \rightharpoonup x$ (weakly).*
3. *If $\dim X < \infty$, then the weak convergent implies the strong convergent.*

1.1.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analyses, known as Hilbert spaces. Then, we must give some important result on these spaces here.

Definition 9. *A Hilbert space $\mathcal{H}$ is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let $\mathcal{H}$ complete.*

Theorem 1. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in the Hilbert space $\mathcal{H}$, then it possess a subsequence which converges in the weak topology of $\mathcal{H}$.*

Theorem 2. *In the Hilbert space, all sequence which converges in the weak topology is bounded.*

Theorem 3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence which converges to x, in the weak topology and $(y_n)_{n\in\mathbb{N}}$ is an other sequence which converges weakly to y, then*

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

Proposition 2. *Let X and Y be two Hilbert spaces, let $(x_n)_{n\in\mathbb{N}} \in X$ be a sequence which converges weakly to $x \in X$, let $A \in \mathcal{L}(X, Y)$. Then, the sequence $(A(x_n))_{n\in\mathbb{N}}$ converges to $A(x)$ in the weak topology of Y.*

Theorem 4. *(The Lax-Milgram Theorem)*
Let X be a Hilbert space and let $a : X \times X \to \mathbb{R}$ be a bilinear functional. Assume that there exist two constants $C < \infty, \alpha > 0$ such that:
i) $|a(u, v)| \le C\|u\|.\|v\|$ *for all* $(u, v) \in X \times X$ *(continuity);*
ii) $a(u, u) \ge \alpha\|u\|^2$ *for all* $u \in X$ *(coerciveness).*
Then, for every $f \in X^*$ (the dual space of X), there exists a unique $u \in X$ such that $a(u, v) = \langle f, v\rangle$ for all $v \in X$.

1.1.3 The $L^p(\Omega)$ spaces

Definition 10. *Let $1 \le p \le \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \ \textit{is measurable and} \int_\Omega |f(x)|^p dx < \infty \right\}$$

Notation 1 : For $p \in \mathbb{R}$ and $1 \le p < \infty$, denote by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}$$

. If $p = \infty$, we have

$$L^p(\Omega) = \{ f : \Omega \to \mathbb{R} : f \ \textit{is measurable and there exists } C \textit{ such that}, |f(x)| \le C \textit{ in } \Omega \}$$

. **Notation 2 :** Let $1 \le p \le \infty$, we denote by q the conjugate of p i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1.$

Theorem 5. *It is well known that $L^p(\Omega)$ supplied with the norm $\|.\|_p$ is a Banach space, for all $1 \le p \le \infty$.*

Remark 4. *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

is a Hilbert space .

Theorem 6. *For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space.*

1.1.4 The Sobolev space $W^{m,p}(\Omega)$

Definition 11. *i) Let $m \in \mathbb{N}$ and $p \in [0, \infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega), such\ \ that\ \ \partial^\alpha f \in L^p(\Omega) for\ \ all\ \ \alpha \in \mathbb{N}^m\}$$

such that $|\alpha| = \sum_{j=1}^n \alpha_j \leq m$ where, $\partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} .... \partial_n^{\alpha_n}$.
ii) if $f \in W^{m,p}(\Omega)$, we define its norm to be

$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|<k} \int_\Omega |D^\alpha f|^p dx\right)^{\frac{1}{p}} & ;(1 \leq p < \infty), \\ \sum_{|\alpha|<k} ess \sup |D^\alpha f| & ;(p = \infty). \end{cases}$$

Definition 12. *We denote by*

$$W_0^{m,p}(\Omega)$$

the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.

Remark 5. *i) if $p = 2$ we usually write*

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Supplied with the norm

$$\|f\|_{H^m} = \left(\sum_{|\alpha|\leq m} (\|\partial^\alpha f\|_{L^2})^2\right)^{\frac{1}{2}}$$

The letter H is used, since - as we will see - $H^m(\Omega)$ is a Hilbert space.
with usual scalar product

$$\langle u, v\rangle = \sum_{|\alpha|\leq m} \int_\Omega \partial^\alpha u \partial^\alpha v \ dx.$$

Note that $H^0(\Omega) = L^2(\Omega)$.

Theorem 7. *1. $H^m(\Omega)$ supplied with inner product $\langle ., .\rangle_{H^m(\Omega)}$ is a Hilbert space.*
2. If $m \geq m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$.

Theorem 8. *Assume that Ω is an open domain in $\mathbb{R}^n, n \geq 1$, with smooth boundary Γ. Then,*
i) if $1 \leq p \leq n$, we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, p^*]$, where $p^* = \dfrac{np}{n-p}$.
ii) if $p = n$ we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, \infty)$.
iii) if $p > n$ we have $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, where $\alpha = \dfrac{p-n}{p}$.

1.1.5 The $L^p(0, T, X)$ space

Definition 13. *Let X be a Banach space, denote by $L^p(0, T, X)$ the space of measurable functions*

$$f :]0, T[\to X$$
$$t \mapsto f(t)$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \qquad 1 \leq p < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t \in]0,T[} ess \|f(t)\|_X$$

Theorem 9. *$L^p(0, T, X)$ equipped with the norm $\|.\|_{L^p(0,T,X)}$ is a Banach space .*

Proposition 3. *Let X be a reflexive Banach space, X' it's dual, and $1 \leq p < \infty$, $1 \leq q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0, T, X)$ is identify algebraically and topologically with $L^q(0, T, X')$.*

1.2 Some useful inequalities

In this section, we shall recall some inequalities which will be used in the subsequent chapters.

1.2.1 Young inequalities

Theorem 10. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad , a, b > 0.$$

Theorem 11. *(Young inequality with ε)*
Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$ab \leq \varepsilon \frac{a^p}{p} + \frac{1}{\varepsilon^{\frac{q}{p}}} \frac{b^q}{q} \quad , a, b > 0.$$

The Young inequality has several variants in the following.

Corollary 1. *Let* $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. *Then*
i) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.
ii) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p\varepsilon^{\frac{1}{q}}} + \frac{b\varepsilon^{\frac{1}{p}}}{q} \qquad ,\forall \varepsilon > 0.$

iii) $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \qquad ,0 < \alpha < 1.$

1.2.2 The Hölder inequalities

Theorem 12. *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,

then, if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, *we have*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} . \|g\|_{L^q(\Omega)}.$$

Theorem 13. *(Generalized Hölder inequality)*
Let $1 \leq p_1, ..., p_m \leq \infty$, $\frac{1}{p_1} + ... + \frac{1}{p_m} = 1$, *then, if* $f_k \in L^{p_k}(\Omega)$ *for* $k = 1, ..., m$ *we have*

$$\int_{\Omega} |f_1 .... f_m| dx \leq \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\Omega)}.$$

Remark 6. *We have the corresponding weighted Hölder inequality of the integral form. Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, $\omega(x) > 0$ *on* Ω. *Then*

$$\int_{\Omega} |fg| \omega(x) dx \leq \left(\int_{\Omega} |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \left(\int_{\Omega} |g(x)|^q \omega(x) dx \right)^{\frac{1}{q}}.$$

1.2.3 The Minkowski inequality

Theorem 14. *Assume* $1 \leq p \leq \infty$, $f, g \in L^p(\Omega)$, *then*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

If $0 \neq p < 1$, *then*

$$\|f+g\|_{L^p(\Omega)} \geq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

In the applications, the integral from of the Minkowski inequality is used frequently.

1.2.4 The Poincaré inequality

In this subsection, we shall recall the Poincaré inequality in different forms.

Theorem 15. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $f \in H_0^1(\Omega)$. *Then there is a positive constant* C *such that*

$$\|f\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}, \qquad \forall f \in H_0^1(\Omega).$$

Theorem 16. *Let* Ω *be a bounded domain of* C^1 *in* $\mathbb{R}^n$. *There is a positive constant* C *, such that for any* $f \in H^1(\Omega)$.

$$\|f - \tilde{f}\|_{L^2(\Omega)} \leq C\|\nabla f\|_{L^2(\Omega)}$$

Where $\tilde{f} = \frac{1}{|\Omega|}\int_\Omega f(x)dx$ *is the integral average of* f *over* Ω, *and* $|\Omega|$ *is the volume of* Ω.

Theorem 17. *Under assumption of Theorem* (16) *for any* $f \in H^1(\Omega)$, *we have*

$$\|f\|_{L^2(\Omega)} \leq C\left(\|\nabla f\|_{L^2(\Omega)} + \left|\int_\Omega f dx\right|\right).$$

1.3 Basic theory of semi-groups

In this section, we recall some basic knowledge in semigroups,most of which will be used in the subsequent chapters.

1.3.1 C_0-Semi-groups of Linear Operators

Definition 14. *(Semi-groups)*
Let X *be a Banach space, the one-parameter family* $S(t),\ 0 \leq t < \infty$ *from* X *to* X *is called a Semigroups if*
(i) $S(0) = I$ *(* I *is the identity operator on* X *).*
(ii) $S(t+s) = S(t).S(s)$ *for every* $t, s \geq 0$ *(the Semigroup property).*

Definition 15. *The linear operator* $\mathcal{A}$ *defined by*

$$D(\mathcal{A}) = \left\{x \in X : \lim_{t\to 0^+} \frac{(S(t)x - x)}{t} \quad exists\right\}$$

and

$$\mathcal{A}x = \lim_{t\to 0^+} \frac{(S(t)x - x)}{t} = \frac{d(S(t)x)}{dt}|_{t=0} \quad for \quad all \quad x \in D(\mathcal{A})$$

is called the infinitesimal generator of the Semigroup $S(t)$, $D(\mathcal{A})$ is called the domain of $\mathcal{A}$.

Definition 16. *(C_0-Semigroups).*
A Semigroup $S(t)$, $0 \le t < \infty$, from X to X is called a strong continuous Semigroup of bounded linear operators if

$$\lim_{t\to 0^+} S(t)x = x, \quad for \quad all \quad x \in X,$$

or

$$\lim_{t\to 0^+} \|S(t)x - x\| = 0, \quad for \quad all \quad x \in X.$$

i.e $S(t)$ C_0- Semigroup.

Definition 17. *A semigroup $S(t)$, $0 \le t < \infty$ is called a semigroup of contraction if there exists a constant $\alpha > 0$ $(0 < \alpha < 1)$ such that for all $t > 0$,*

$$\|S(t)x - S(t)y\| \le \alpha\|x - y\|, \quad for \quad all \quad x, y \in X.$$

1.3.2 Hille-Yosida Theorem

Definition 18. *An unbounded linear operator $\mathcal{A} : D(\mathcal{A}) \subset H \to H$[1] is said to be monotone[2], if it satisfies*

$$\langle Av, v\rangle \ge 0 \qquad \forall v \in D(\mathcal{A}).$$

It is called maximal monotone if, in addition; $R(I + \mathcal{A}) = H$, i.e

$$\forall f \in H, \quad \exists u \in D(\mathcal{A}) \quad such \quad that \quad u + \mathcal{A}u = f.$$

Proposition 4. *Let $\mathcal{A}$ be a maximal monotone operator. Then*
1. $D(\mathcal{A})$ is dense in H.
2. $\mathcal{A}$ is closed operator.
3. For every $\lambda > 0, (I + \lambda\mathcal{A})$ is bijective from $D(\mathcal{A})$ onto H, $(I + \lambda\mathcal{A})^{-1}$ is a bounded operator, and $\|(I + \lambda\mathcal{A})^{-1}\|_{\mathcal{L}(H)} \le 1$.

Theorem 18. *(Hille-Yosida)*
Let $\mathcal{A}$ be a maximal monotone operator. Then, given any $u_0 \in D(\mathcal{A})$ there exists a unique

[1] H denotes a Hilbert space.
[2] Some authors say that A is accretive or $-\mathcal{A}$ is dissipative.

function

$$u \in C^1([0,+\infty);H) \cap C([0,+\infty);D(\mathcal{A}))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 & \textit{on } [0,+\infty) \\ u(0) = u_0. \end{cases}$$

Chapter 2

WELL POSEDNESS

In this work, we are concerned with the following system:

$$\begin{cases} \rho\omega_{tt} + G\left(\psi - \omega_x\right)_x + \delta\theta_x = 0, \\ \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G\left(\psi - \omega_x\right) = 0, \\ \\ I_\rho s_{tt} - Ds_{xx} + G\left(\psi - \omega_x\right) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t + \frac{4}{3}\int_{\tau_1}^{\tau_2} \left|\mu_2(\varrho)\right| s_t(x, t-\varrho)d\rho = 0, \\ \\ \rho_3\theta_t + q_x + \delta\omega_{tx} = 0, \\ \\ \tau q_t + \alpha q + \theta_x = 0. \end{cases} \tag{2.1}$$

where

$$(x, \rho, t) \in (0,1) \times (\tau_1, \tau_2) \times (0, \infty),$$

with the Neumann-Dirichlet boundary conditions

$$\begin{gathered} \omega_x(0,t) = \omega_x(1,t) = \psi(0,t) = \psi(1,t) = 0, t \geq 0, \\ s(1,t) = s(0,t) = \theta(0,t) = \theta(1,t) = 0, t \geq 0, \end{gathered} \tag{2.2}$$

and the initial data

$$\begin{aligned} &\omega(x,0) = \omega_0(x), \omega_t(x,0) = \omega_1(x), \psi(x,0) = \psi_0(x), \\ &\psi_t(x,0) = \psi_1(x), s(x,0) = s_0(x), s_t(x,0) = s_1(x), \\ &\theta(x,0) = \theta_0(x), q(x,0) = q_0(x), \end{aligned} \tag{2.3}$$

where

$$(x,t) \in (0,1) \times (0, \infty)$$

Here, $\rho, G, I_\rho, D, \gamma, \beta, \delta, \rho_3, \alpha$ and τ are positive constants. τ_1, τ_2 are tow real numbers with $0 \leq \tau_1 \leq \tau_2, \mu_2$ is an L^∞ function satisfying: (H1) $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \, d\varrho < \beta. \tag{2.4}$$

In this section, we give the existence and uniqueness result of the system (2.5) , (2.3) using the semigroup theory. To achieve our goal, as in a previous study, [20] taking the following new variable:

$$y(x, \rho, \varrho, t) = s_t(x, t - \varrho\rho),$$

then, we obtain

$$\begin{cases} \rho y_t(x, \rho, \rho, t) + y_\rho(x, \rho, \rho, t) = 0, \\ y(x, 0, \varrho, t) = s_t(x, t), \end{cases}$$

Consequently, the problem is equivalent to

$$\begin{cases} \rho\omega_{tt} + G\left(\psi - \omega_x\right)_x + \delta\theta_x = 0, \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G\left(\psi - \omega_x\right) = 0, \\ I_\rho s_{tt} - Ds_{xx} + G\left(\psi - \omega_x\right) + \frac{4}{3}\gamma s + \frac{4}{3}\beta s_t + \frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, y(x, 1, \varrho, t)d\varrho = 0, \\ \rho_3\theta_t + q_x + \delta\omega_{tx} = 0, \\ \tau q_t + \alpha q + \theta_x = 0, \\ \rho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \end{cases} \tag{2.5}$$

where:

$$(x, \rho, \varrho, t) \in (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty),$$

with the following boundary and initial conditions:

$$\begin{aligned} \omega_x(0, t) = \omega_x(1, t) = \psi(0, t) = \psi(1, t) = 0, t \geq 0, \\ \theta(0, t) = \theta(1, t) = s(1, t) = s(0, t) = 0, t \geq 0, \\ \omega(x, 0) = \omega_0(x), \omega_t(x, 0) = \omega_1(x), \psi(x, 0) = \psi_0(x), \\ \psi_t(x, 0) = \psi_1(x), s(x, 0) = s_0(x), s_t(x, 0) = s_1(x), \\ q(x, 0) = q_0(x), \theta(x, 0) = \theta_0(x), \\ y(x, \rho, \varrho, 0) = f_0(x, \rho\varrho)/x \in (0.1), \rho \in (0.1), s \in (0, \tau_2)\,. \end{aligned}$$

Meanwhile, from (2.5) we have:

$$\rho\omega_{tt} + G(\psi - \omega_x) + \delta\theta_x = 0,$$

Hence,

$$\rho \int_0^1 \omega_{tt}\, dx + G \int_0^1 (\psi - \omega_x)\, dx + \delta \int_0^1 \theta_x\, dx = 0,$$

that is

$$\rho \int_0^1 \omega_{tt}\, dx + G(\psi - \omega_x)\Big|_0^1 + \delta\theta\Big|_0^1 = 0,$$

By (2.3) it follows that $(\psi - \omega_x)\Big|_0^1 = \theta\Big|_0^1 = 0$, and therefore :

$$\int_0^1 \int_0^1 \omega_{tt}\, dx = 0.$$

Let us now solve :

$$\frac{d^2}{dt^2} \int_0^1 \omega(x,t)\, dx = 0. \tag{2.6}$$

We have:

$$\begin{aligned} \frac{d^2}{dt^2} \int_0^1 \omega(x,t)\, dx = 0 &\Rightarrow \frac{d}{dt} \int_0^1 \omega(x,t)\, dx = a \\ &\Rightarrow \int_0^1 \omega(x,t)\, dx = at + b, \quad (a,b) \in \mathbb{R}. \end{aligned}$$

Now, we fix (a,b) using the initial data to finally obtain:

$$a = \int_0^1 \omega_1(x)\, dx, \quad b = \int_0^1 \omega_0(x)\, dx.$$

Hence,

$$\int_0^1 \omega(x,t)\, dx = t \int_0^1 \omega_1(x)\, dx + \int_0^1 \omega_0(x)\, dx,$$

Consequently, if we let

$$\bar{\omega}(x,t) = \omega(x,t) - t \int_0^1 \omega_1(x)\, dx - \int_0^1 \omega_0(x)\, dx, \tag{2.7}$$

we get

$$\int_0^1 \bar{\omega}(x,t)\,dx = 0 \quad \forall t \geq 0$$

Using the same method from the fifth equation we have :

$$\delta \int_0^1 q_t\,dx + \alpha \int_0^1 q\,dx + \theta\Big|_0^1 = 0, \tag{2.8}$$

and by the initial condition, we have $\theta\Big|_0^1 = 0$. Therefore :

$$\delta \frac{d}{dt} \int_0^1 q\,dx + \alpha \int_0^1 q\,dx = 0,$$

To solve it we proceed as follows:

$$\delta \frac{d}{dt} \int_0^1 q\,dx + \alpha \int_0^1 q\,dx = 0 \Rightarrow \delta \frac{d}{dt} \int_0^1 q\,dx = -\alpha \int_0^1 q\,dx$$

$$\Rightarrow \frac{\frac{d}{dt}\int_0^1 q\,dx}{\int_0^1 q\,dx} = -\frac{\alpha}{\delta}$$

$$\Rightarrow \ln\left[\int_0^1 q\,dx\right] = -\frac{\alpha}{\delta}t + c$$

$$\Rightarrow \int_0^1 q\,dx = Ke^{-\frac{\alpha}{\delta}t}.$$

Using the initial data we get :

$$x = \int_0^1 q_0(x)\,dx.$$

Therefore, if we let

$$\bar{q}(x,t) = q(x,t) - e^{-\frac{\alpha}{\delta}t} \int_0^1 q_0(x)dx, \tag{2.9}$$

we get :

$$\int_0^1 \bar{q}(x,t)dx = 0 \quad \forall t \geq 0.$$

Therefore, the use of Poincare's inequality for $\bar{\omega}, \bar{q}$ is justified. In addition, simple substitution shows that $(\bar{\omega}, \psi, s, \theta, \bar{q})$ satisfies System (2.5). Henceforth, we work with $\bar{\omega}, \bar{q}$ instead of ω, q but write ω, q for simplicity of notation.

At this step, in order to define the energy functional of the previous system, we multiply the equations of system ((2.5)) by ω_t, $(3s-\psi)_t$, s_t, θ, q, and $y|\mu_2(\rho)|$ respectively and integrating over $(0,1)$, we find:

$$\rho\int_0^1 \omega_{tt}\omega_t\,dx + G\int_0^1 \omega_t(\psi-\omega_x)\,dx + \delta\int_0^1 \theta_x\omega_t\,dx = 0,$$

$$I_\rho\int_0^1 (3s-\psi)_{tt}(3s-\psi)_t\,dx - D\int_0^1 (3s-\psi)_{xx}(3s-\psi)_t\,dx - G\int_0^1 (3s-\psi_x)(3s-\psi)_t\,dx = 0,$$

$$I_\rho\int_0^1 s_{tt}s_t\,dx - D\int_0^1 s_{tt}s_t\,dx + G\int_0^1 s_t(\psi-\omega_x)\,dx + \frac{4}{3}\gamma\int_0^1 s_t s\,dx + \frac{4}{3}\beta\int_0^1 s_t^2\,dx,$$

$$+\frac{4}{3}\int_0^1\int_{\theta_1}^{\theta_2} s_t|\mu_2(\varrho)|y(x,1,\varrho,t)\,dx = 0,$$

$$\rho_3\int_0^1 \theta_t\theta\,dx + \int_0^1 q_x\theta\,dx + \delta\int_0^1 \theta\omega_{tx}\,dx = 0,$$

and

$$\theta\int_0^1 q_t q\,dx + \alpha\int_0^1 q^2\,dx + \int_0^1 q\theta_x\,dx = 0,$$

Summing up and integrating by parts, we get:

$$\frac{1}{2}\frac{d}{dt}\left\{\int_0^1 \omega_t^2 + I_\rho(3s-\psi)_t^2 + 3I_\rho s_t^2 + 4\gamma s^2 + 3Ds_x^2 + D(3s-\psi)^2 + G(\psi-\omega_x)^2 + \rho_3\theta^2 + \theta q^2\,dx\right\}$$
$$= -4\beta\int_0^1 s_t^2\,dx - \alpha\int_0^1 q^2\,dx - 4\int_0^1 s_t^2\int_{\theta_1}^{\theta_2} |\mu(\varrho)|y(x,1,h,t)\,d\varrho\,dx, \tag{2.10}$$

then, the energy functional is given by

$$E(t) = \frac{1}{2}\int_0^1 \left\{\rho\omega_t^2 + I_\rho(3s-\psi)_t^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2 + D(3s-\psi)_x^2\right.$$
$$\left. + D\left(\psi-\omega_x\right)^2 + \rho_3\theta^2 + \tau q^2 + 4\int_0^1\int_{\tau_1}^{\tau_2} \rho\left|\mu_2(\rho)\right| y^2(x,\rho,\rho,t)d\rho d\rho\right\} dx.$$

For the existence and uniqueness of the solution we first, we introduce the vector function

$$U = \left(\omega, \omega_t, 3s-\psi, (3s-\psi)_t, s, s_t, \theta, q, y\right)^T,$$

and the two new dependent variables $v = \omega_t, u = \psi_t, \varphi = s_t$, then system (2.5) can be written as follows:

$$\begin{cases} U_t = \mathcal{A}U \\ \\ U(0) = U_0 = (\omega_0, \omega_1, 3s_0 - \psi_0, 3s_1 - \psi_1, s_0, s_1, \theta_0, q_0, f_0)^T \end{cases} \tag{2.11}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} :\to \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \\ -\frac{1}{\rho}\left[G\left(\psi_x - \omega_x\right)_x + \delta\theta_x\right], \\ \\ 3\varphi - u \\ \\ \frac{1}{I_\rho}\left[D(3s-\psi)_{xx} + G\left(\psi_x - \omega_x\right)\right], \\ \\ \varphi \\ \\ \frac{1}{I_\rho}\left[Ds_{xx} - G\left(\psi_x - \omega_x\right) - \frac{4}{3}\gamma s - \frac{4}{3}\beta\varphi - \frac{4}{3} - \int_{\tau_1}^{\tau_2}\left|\mu_2(\varrho)\right| y(x,1,\varrho,t)d\rho\right] \\ \\ -\frac{1}{\rho_3}\left[q_x + \delta v_x\right] \\ \\ -\frac{1}{\tau}\left[q + \theta_x\right] \\ \\ -\frac{1}{\rho}y_\rho \end{pmatrix}. \tag{2.12}$$

and $\mathcal{H}$ is the energy space given by

$$\begin{aligned} \mathcal{H} = & H_*^1 \times L_*^2(0,1) \times H_0^1 \times L^2(0,1) \times H_0^1 \times L^2(0,1) \times L^2(0,1) \times L_*^2(0,1) \\ & \times L^2\left((0,1)\times(0,1)\times(\tau_1,\tau_2)\right), \end{aligned}$$

where

$$\begin{aligned} L_*^2(0,1) &= \left\{\varphi \in L^2(0,1) / \int_0^1 \varphi(x)dx = 0\right\}, \\ H_*^1(0,1) &= H^1(0,1) \cap L_*^2(0,1), \\ H_*^2(0,1) &= \left\{\varphi \in H^2(0,1) / \varphi_x(1) = \varphi_x(0) = 0\right\}, \end{aligned}$$

For any

$$U = (\omega, v, 3s - \psi, 3\varphi - u, s, \varphi, \theta, q, y)^T \in \mathcal{H},$$
$$\hat{U} = (\widehat{\omega}, \widehat{v}, 3\widehat{s} - \widehat{\psi}, 3\hat{\varphi} - \widehat{u}, \widehat{s}, \hat{\varphi}, \widehat{\theta}, \widehat{q}, \widehat{y})^T \in \mathcal{H},$$

we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned}
< U, \widehat{U} >_{\mathcal{H}} = & \rho \int_0^1 v\hat{v}dx + I_\rho \int_0^1 (3\varphi - u)(3\hat{\varphi} - \widehat{u})dx + 3I_\rho \int_0^1 \varphi\hat{\varphi}dx \\
& + G \int_0^1 (\psi - \omega_x) \left(\widehat{\psi} - \widehat{\omega}_x\right) dx + \rho_3 \int_0^1 \theta\hat{\theta}dx + \tau \int_0^1 q\widehat{q}dx \\
& + D \int_0^1 (3s - \psi)_x (3\hat{s} - \widehat{\psi})_x dx + 4\gamma \int_0^1 \hat{s}\hat{s}dx + 3D \int_0^1 s_x \widehat{S}_x dx \\
& + 4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho \left|\mu_2(\varrho)\right| y\hat{y}d\rho d\rho dx.
\end{aligned} \tag{2.13}$$

The domain of $\mathcal{A}$ is given by:

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} U \in \mathcal{H}/\omega \in H_*^2 \cap H_*^1, 3s - \psi, s \in H^2 \cap H_*^1, \\ \\ v, q \in H_*^1, 3\varphi - u, \varphi, \theta \in H_0^1(0,1), \\ \\ y, y_\rho \in L^2\left((0,1) \times (0,1) \times (\tau_1, \tau_2)\right), y(x, 0, \varrho, t) = \varphi \end{array} \right\}$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we can give the following existence result:

Theorem 2.0.1. *Let $U_0 \in \mathcal{H}$ and assume that* (2.4) *holds. Then, there exists a unique solution $U \in \mathcal{C}(\mathbb{R}_+, \mathcal{H})$ of Problem* (2.10)*. Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in \mathcal{C}\left(\mathbb{R}_+, \mathcal{D}(\mathcal{A})\right) \cap \mathcal{C}^1\left(\mathbb{R}_+, \mathcal{H}\right).$$

Proof. First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U_0 \in \mathcal{D}(\mathcal{A})$ and by using (2.12), we have:

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle &= \rho \int_0^1 \frac{-}{\rho}\left[G(\psi-\omega_x)_x + \delta\theta_x\right] v\, dx \\
&+ I_\rho \int_0^1 \frac{1}{I_\rho}\left[D(3s-\psi)_{xx} + G(\psi_x-\omega_x)\right](3\varphi-u)\, dx \\
&+ \frac{3}{I_\rho}\int_0^1 \left[Ds_{xx} - G(\psi-\omega_x) - \frac{4}{3}\gamma s - \frac{4}{3}\beta\varphi - \frac{4}{3}\int_{\theta_1}^{\theta_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho\right]\varphi\, dx \\
&+ G\int_0^1 (u - v_x)(\psi-\omega_x)\, dx \\
&+ \rho_3 \int_0^1 \frac{-1}{\rho_3}(q_x + \delta v_x)\theta\, dx + \theta\int_0^1 \frac{-1}{\theta}(\alpha q + \theta_x) q\, dx \\
&+ D\int_0^1 (3\varphi-u)_x(3s-\psi)_x\, dx + 4\gamma\int_0^1 \varphi s\, dx + 3D\int_0^1 \varphi_x s_x\, dx \\
&+ 4\int_0^1\int_0^1\int_{\theta_1}^{\theta_2} \varrho|\mu_2(\varrho)|\frac{-1}{\varrho} y_\rho y\, d\varrho\, d\rho\, dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle &= -G\int_0^1 (\psi-\omega_x)_x v\, dx - \delta\int_0^1 s_{xx}\varphi\, dx - 3G\int_0^1 (\psi-\omega_x)\varphi\, dx \\
&- G\int_0^1 (\varphi-\omega_x) v_x\, dx + G\int_0^1 u(\psi-\omega_x)\, dx - 4\gamma\int_0^1 s\varphi\, dx \\
&- 4\beta\int_0^1 \varphi^2\, dx - c\int_0^1\int_{\theta_1}^{\theta_2} |\mu_2(\varrho)|\varphi y(x,1,\varrho,t)\, d\varrho\, dx - \int_0^1 q_x\theta\, dx \\
&- \delta\int_0^1 v_x\theta\, dx - \alpha\int_0^1 q^2\, dx - \int_0^1 \theta_x\, dx + D\int_0^1 (3\varphi-u)_x(3s-\psi)_x\, dx \\
&+ 4\gamma\int_0^1 \varphi s\, dx + 3D\int_0^1 \varphi_x s_x\, dx - 4\int_0^1\int_0^1\int_{\theta_1}^{\theta_2} |\mu_2(\varrho)| y_\rho y\, d\varrho\, d\rho\, dx,
\end{aligned}$$

Now using the integration by parts we find

$$
\begin{aligned}
\langle \mathcal{A}U, U\rangle = & G\int_0^1 (\psi-\omega_x)v_x + \delta\int_0^1 \theta v_x\, dx - D\int_0^1 (3s-\psi)_x(3\varphi-u)_x\, dx\\
& - 3D\int_0^1 s_x\varphi_x\, dx - 4\beta\int_0^1 \varphi^2\, dx - 4\int_0^1\int_{\theta_1}^{\theta_2} |\mu_2(\varrho)|\varphi y(x,1,\varrho,t)\, d\varrho\, dx\\
& + \int_0^1 q\theta_x\, dx - \delta\int_0^1 v_x\theta\, dx - \alpha\int_0^1 q^2\, dx - \int_0^1 \theta_x q\, dx\\
& + D\int_0^1 (3\varphi-u)_x(3s-\psi)_x\, dx + 3D\int_0^1 \varphi_x s_x\, dx\\
& - 4\int_0^1\int_0^1\int_{\theta_1}^{\theta_2} |\mu_2(\varrho)| y_\rho y\, d\varrho\, d\rho\, dx.
\end{aligned}
$$

As a result we write

$$
\begin{aligned}
< \mathcal{A}U, U >_{\mathcal{H}} = & -4\beta\int_0^1 \varphi^2 dx - 4\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, \varphi y(x,1,\varrho,t) d\rho dx\\
& - 4\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, y_\rho y d\rho d\rho dx - \alpha\int_0^1 q_x^2 dx.
\end{aligned}
\tag{2.14}
$$

For the third term of the right-hand side of (2.13), we have

$$
\begin{aligned}
-\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, y_\rho y d\rho d\rho dx = & -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 |\mu_2(\varrho)|\, \frac{d}{d\rho} y^2 d\rho d\rho dx\\
= & -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\, y^2(x,1,\varrho,t) d\rho dx\\
& + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, y^2(x,0,\rho,t) d\rho dx.
\end{aligned}
\tag{2.15}
$$

By using Young's inequality, we get

$$
\begin{aligned}
-\int_0^1\int_{\tau_1}^{\tau_2} \rho\, |\mu_2(\varrho)|\, \varphi y(x,1,\varrho,t) d\rho dx \leq & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, d\varrho\right)\int_0^1 \varphi^2 dx\\
& + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, y^2(x,1,\varrho,t) d\rho dx,
\end{aligned}
\tag{2.16}
$$

Substituting (2.14) (2.15) into (2.13), using the fact that $y(x,0,\varrho,t) = \varphi(x,t)$ and (2.4), we obtained

$$
\begin{aligned}
< \mathcal{A}U, U >_{\mathcal{H}} \leq & -4\left(\beta - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, d\varrho\right)\int_0^1 \varphi^2 dx - \alpha\int_0^1 q^2 dx,\\
\leq & \ 0.
\end{aligned}
\tag{2.17}
$$

Hence, the operator $\mathcal{A}$ is dissipative. Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $(Id - \mathcal{A})$ is subjective. Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}$, we prove that there exists a unique $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9) \in \mathcal{D}(\mathcal{A})$ such that

$$(Id - \mathcal{A})V = F. \tag{2.18}$$

That is,

$$\begin{cases} v_1 - v_2 = f_1, \\ \rho v_2 - G v_{1xx} - G v_{3x} + \delta v_{7x} = \rho f_2, \\ v_3 - v_4 = f_3, \\ I_\rho v_4 - D v_{3xx} - G v_5 + G v_3 + G v_{1x} = I_\rho f_4, \\ v_5 - v_6 = f_5, \\ \left(I_\rho + \frac{4}{3}\beta\right) v_6 - D v_{5xx} - G v_3 - G v_{1x} + \left(3G + \frac{4}{3}\gamma\right) v_5 \\ +\frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\, y(x,1,\rho,t) d\rho = I_\rho f_6, \\ \rho_3 v_7 + v_{8x} + \delta v_{2x} = \rho_3 f_7, \\ (\tau + \alpha) v_8 + v_{7x} = \tau f_8, \\ \varrho y(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = \varrho f_9. \end{cases} \tag{2.19}$$

We note that the last equation in (2.18), we have

$$\varrho y + y_\rho = \varrho f_9,$$

first, we solve the homogeneous equation, as follows

$$\varrho y + y_\rho = 0 \Rightarrow y_\rho = -\varrho y \Rightarrow y = K e^{-\varrho\rho},$$

then, using the variation of constants, we see that

$$\varrho K e^{-\varrho\rho} - \varrho K e^{-\varrho\rho} + K' e^{-\varrho\rho} = \varrho f_9,$$

that is

$$K'(x,\rho) = \varrho f_9 e^{\varrho\rho} \Rightarrow K(x,\rho) = \varrho \int_0^\rho f_9 e^{\varrho\sigma} d\sigma + c_1,\ c_1 \in \mathbb{R}.$$

Hence, using the boundary conditions

$$y(x,\rho,\varrho,t) = e^{-\rho\rho}\varphi + \varrho e^{\rho\rho} \int_0^\rho e^{\rho\sigma} f_9(x,\sigma,\rho,t) d\sigma, \tag{2.20}$$

then,

$$y(x,1,\varrho,t) = e^{-\rho}\varphi + \varrho e^{\rho}\int_0^1 e^{\rho\sigma} f_9(x,\sigma,\varrho,t)d\sigma, \tag{2.21}$$

and we infer from $(2.18)_8$ that

$$v_{7x} = \tau f_8 - (\tau+\alpha)v_8 \Rightarrow v_7 = \tau\int_0^x f(\sigma)d\sigma - (\tau+\alpha)\int_0^x v_8(\sigma)d\sigma. \tag{2.22}$$

we have

$$v_2 = v_1 - f_1, \quad v_4 = v_3 - f_3, \quad v_6 = v_5 - f_5. \tag{2.23}$$

Inserting (2.20) (2.21) and (2.22) in (2.18),(2.18) (2.18) and (2.18), we get:

$$\begin{cases} \rho(v_1 - f_1) - Gv_{1xx} - Gv_{3x} + \delta(\tau f_8 - (\tau+\alpha)v_8) = \rho f_8, \\ \\ I_\rho(v_3 - f_3) - Dv_{3xx} - Gv_5 + Gv_3 + Gv_{1x} = I_\rho f_4, \\ \\ (I_\rho + \frac{3}{4}\beta)(v_5 - f_5) - Dv_{5xx} - Gv_3 - Gv_{1x} + (3G + \frac{3}{4}\gamma)v_5 + \\ \\ \frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|(e^{-\varrho}\varphi + \varrho e^{\varrho}\int_{\tau_1}^{\tau_2} e^{\varrho\sigma} f_9(x,\sigma,\varrho,\tau)d\sigma) = I_\rho f_6, \\ \\ \rho_3(\tau\int_0^x f_8(\sigma)d\sigma - (\tau+\alpha)\int_0^x v_8 d\sigma + v_{8x} + \delta(v_{1x} - f_{1x})) = \rho_3 f_7. \end{cases}$$

by simplifications, we get:

$$\begin{cases} \rho v_1 - Gv_{1xx} - Gv_{3x} + \delta(\tau f_8 - (\tau+\alpha)v_8) = \rho\left(f_1 + f_2\right) - \tau f_8, \\ \\ I_\rho - Dv_{3xx} - Gv_5 + Gv_3 + Gv_{1x} = I_\rho\left(f_3 + f_4\right), \\ \\ \left(I_\rho + 3G + \frac{4}{3}\beta + \frac{4}{3}\gamma + \frac{4}{3}\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\, e^{-\rho}d\rho\right) v_5 - Dv_{5xx} - Gv_3 - Gv_{1x} = \\ I_\rho\left(f_5 + f_6\right) + \frac{4}{3}\left(\beta - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, e^{-\rho}d\varrho\right) f_5, \\ \\ -\rho_3(\tau+\alpha)\int_0^x v_8(\sigma)d\sigma + v_{8x} + \delta v_{1x} = \rho_3 f_7 - \rho_3\tau\int_0^x f_8(\sigma)d\sigma + \delta f_{1x}. \end{cases}$$

finally ,we get:

$$\begin{cases} \rho v_1 - G v_{1xx} - G v_{3x} + 3G v_{5x} - \delta(\tau+\alpha) v_8 = h_1, \\ \\ I_\rho v_3 - D v_{3xx} - 3G v_5 + 3G v_{1x} = h_2, \\ \\ \mu_3 v_5 - D v_{5xx} - G v_3 - G v_{1x} = h_3, \\ \\ -\rho_3(\tau+\alpha) \int_0^x v_8(\sigma) d\sigma + v_{8x} + \delta v_{1x} = h_4. \end{cases} \tag{2.24}$$

where:

$$\begin{cases} \mu_3 = I_\rho + 3G + \frac{4}{3}\beta + \frac{4}{3}\gamma + \frac{4}{3} \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| e^{-\rho} d\rho, \\ \\ h_1 = \rho (f_1 + f_2) - \tau f_8, \\ \\ h_2 = I_\rho (f_3 + f_4), \\ \\ h_3 = I_\rho (f_5 + f_6) + \frac{4}{3} \left(\beta - \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| e^{-\rho} d\varrho \right) f_5 \\ \\ \quad -\frac{4}{3} \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| e^{\rho} \int_0^1 e^{\varrho\sigma} f_9(x,\sigma,\varrho,t) d\sigma d\rho, \\ \\ h_4 = \rho_3 f_7 - \rho_3 \tau \int_0^x f_8(\sigma) d\sigma + \delta f_{1x}. \end{cases} \tag{2.25}$$

We multiply (2.23) by $\widehat{v}_1, \widehat{v}_3, \widehat{v}_5$, and $(\tau+\alpha) \int_0^x \widehat{v}_8(\sigma) d\sigma$, respectively, and integrate their sum over $(0,1)$ to get

$$\begin{cases} \rho \int_0^1 \widehat{v}_1 v_1 dx - G \int_0^1 \widehat{v}_1 v_{1xx} dx - G \int_0^1 \widehat{v}_1 v_{3x} dx + 3G \int_0^1 \widehat{v}_1 v_{5x} dx - \delta(\tau+\alpha) \int_0^1 \widehat{v}_1 v_8 dx = \int_0^1 \widehat{v}_1 h_1 dx, \\ \\ I_\rho \int_0^1 \widehat{v}_3 v_3 - D \int_0^1 \widehat{v}_3 v_{3xx} dx - 3G \int_0^1 \widehat{v}_3 v_5 dx + 3G \int_0^1 \widehat{v}_3 v_{1x} dx = \int_0^1 \widehat{v}_3 h_2 dx, \\ \\ \mu_3 \int_0^1 \widehat{v}_5 v_5 dx - D \int_0^1 \widehat{v}_5 v_{5xx} dx - G \int_0^1 \widehat{v}_5 v_3 dx - G \int_0^1 \widehat{v}_5 v_{1x} dx = \int_0^1 \widehat{v}_5 h_3 dx, \\ \\ -\rho_3(\tau+\alpha)^2 \int_0^1 \left(\int_0^x v_8(\sigma) d\sigma \right) \left(\int_0^x \widehat{v}_8(\sigma) d\sigma \right) dx + (\tau+\alpha) \int_0^1 \left(\int_0^x v_8(\sigma) d\sigma \right) v_{8x} dx \\ \\ +\delta(\tau+\alpha) \int_0^1 \left(\int_0^x v_8(\sigma) d\sigma \right) v_{1x} dx = (\tau+\alpha) \int_0^1 \left(\int_0^x v_8(\sigma) d\sigma \right) h_4 dx, \end{cases} \tag{2.26}$$

then, the following variational formulation:

$$B\left((v_1, v_3, v_5, v_8), (\hat{v}_1, \widehat{v}_3, \hat{v}_5, \hat{v}_8) \right) = \Gamma \left(\widehat{v}_1, \widehat{v}_3, \hat{v}_5, \hat{v}_8 \right), \tag{2.27}$$

where

$$B : \left(H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times L_*^2(0,1)\right)^2 \to \mathbb{R},$$

is the bilinear form defined by

$$\begin{aligned} B\left((v_1,v_3,v_5,v_8),(\widehat{v}_1,\hat{v}_3,\widehat{v}_5,\hat{v}_8)\right) =&\mu_4 \int_0^1 v_5\widehat{v}_5 dx + \rho \int_0^1 v_1\widehat{v}_1 dx \\ &+ I_\rho \int_0^1 v_3\widehat{v}_3 dx + (\tau+\alpha)\int_0^1 v_8\hat{v}_8 dx \\ &+ D\int_0^1 v_{3x}\hat{v}_{3x} dx + 3D\int_0^1 v_{5x}\widehat{v}_{5x} dx \\ &+ G\int_0^1 \left(-v_{1x} - v_3 + 3v_5\right)\left(-\widehat{v}_{1x} - \widehat{v}_3 + 3\widehat{v}_5\right) dx \\ &+ \rho_3(\tau+\alpha)^2 \int_0^1 \left(\int_0^x v_8(\sigma)d\sigma\right)\left(\int_0^x \hat{v}_8(\sigma)d\sigma\right) dx, \end{aligned} \tag{2.28}$$

where $\mu_4 = 3I_\rho + 4\beta + 4\gamma + 4\int_{\tau_1}^{\tau_2} |\mu_2(\rho)|\, e^{-\rho} d\rho$ and

$$\Gamma : \left(H_*^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times L_*^2(0,1)\right) \to \mathbb{R},$$

is the linear functional given by

$$\begin{aligned} \Gamma\left(\widehat{v}_1,\widehat{v}_3,\widehat{v}_5,\widehat{v}_8\right) =& \int_0^1 h_1\widehat{v}_1 dx + \int_0^1 h_2\widehat{v}_3 dx + \int_0^1 h_3\widehat{v}_5 dx \\ &+ \int_0^1 h_4\left(-(\tau+\alpha)\int_0^x \widehat{v}_8(\sigma)d\sigma\right) dx. \end{aligned} \tag{2.29}$$

Now, for $V = H_*^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L) \times L_*^2(0,L)$, equipped with the norm

$$\begin{aligned} \|(v_1,v_3,v_5,v_8)\|_V^2 =& \|(-v_{1x} - v_3 + 3v_5)\|_2^2 + \|v_1\|_2^2 \\ &+ \|v_8\|_2^2 + \|v_{3x}\|_2^2 + \|v_{5x}\|_2^2, \end{aligned}$$

then, we have

$$
\begin{aligned}
B((\varphi,\psi,\theta,P),(\varphi,\psi,\theta,P)) =& \mu_4\int_0^1 v_5^2dx+\rho\int_0^1 v_1^2dx \\
&+I_\rho\int_0^1 v_3^2dx+(\tau+\alpha)\int_0^1 v_8^2dx \\
&+D\int_0^1 v_{3x}^2dx+3D\int_0^1 v_{5x}^2dx \\
&+G\int_0^1 \left(-v_{1x}-v_3+3v_5\right)^2dx \\
&+\rho_3(\tau+\alpha)^2\int_0^1\left(\int_0^x v_8(\sigma)d\sigma\right)^2dx \\
\geq& \rho\int_0^1 v_1^2dx+(\tau+\alpha)\int_0^1 v_8^2dx \\
&+D\int_0^1 v_{3x}^2dx+3D\int_0^1 v_{5x}^2dx \\
&+G\int_0^1 \left(-v_{1x}-v_3+3v_5\right)^2dx,
\end{aligned}
\tag{2.30}
$$

thus, for some $M_0 > 0$,

$$B\left(\left(v_1, v_3, v_5, v_8\right), \left(v_1, v_3, v_5, v_8\right)\right) \geq M_0 \left\|\left(v_1, v_3, v_5, v_8\right)\right\|_V^2, \tag{2.31}$$

Thus B is coercive. Consequently, using Lax-Milgram theorem, we conclude that (2.5) has a unique solution:

$$\begin{aligned} &v_1 \in H_*^1(0,1), \\ &v_3, v_5 \in H_0^1(0,1), \\ &v_8 \in L_*^2(0,1). \end{aligned} \tag{2.32}$$

Substituting v_1, v_3, v_5, and v_8 into (2.20), (2.21), and (2.22), respectively, we have

$$\begin{aligned} &v_2 \in H_*^1(0,1), \\ &v_4, v_6 \in H_0^1(0,1), \\ &v_7 \in H_*^1(0,1), \\ &y, y_\rho \in L^2\left((0,1) \times (0,1) \times (\tau_1, \tau_2)\right). \end{aligned} \tag{2.33}$$

Let $\widehat{v}_1 \in H_0^1(0,1)$ and denote

$$\widehat{v}_1 = \widehat{v}_1(x) - \int_0^1 \widehat{v}_1(\xi) d\xi, \tag{2.34}$$

which gives us $\widehat{v}_1 \in H_*^1(0,1)$. Now we replace $(\widehat{v}_1, \widehat{v}_3, \widehat{v}_5, \widehat{v}_8)$ by $(\widehat{v}_1, 0, 0, 0)$ in (2.25) to obtain

$$G \int_0^1 \left(-v_{1x} - v_3 + 3v_5\right)\left(-\widehat{v}_{1x}\right) dx + \rho \int_0^1 v_1 \widehat{v}_1 dx = \int_0^1 h_1 \widehat{v}_1 dx,$$

we get

$$\begin{aligned} G \int_0^1 v_{1xx} \widehat{v}_{1x} dx =& \rho \int_0^1 v_1 \hat{v}_1 dx - G \int_0^1 v_{3x} \hat{v}_{1x} dx \\ &+ G \int_0^1 v_{5x} \hat{v}_{1x} dx - \int_0^1 h_1 \widehat{v}_1 dx, \quad \forall \widehat{v}_1 \in H_0^1(0,1), \end{aligned} \tag{2.35}$$

which yields

$$G v_{1xx} = \rho v_1 - G v_{3x} + G v_{5x} - h_1 \in L^2(0,1).$$

Thus

$$v_1 \in H^2(0,1). \tag{2.36}$$

Moreover, (39) also holds for any $\Phi \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$\begin{aligned} &Gv_{1x}(1)\Phi(1) - Gv_{1x}(0)\Phi(0) - G\int_0^1 v_{1xx}\Phi dx + \rho\int_0^1 v_1\Phi dx - G\int_0^1 v_{3x}\Phi dx \\ &+ G\int_0^1 v_{5x}\Phi dx - \int_0^1 h_1\Phi dx = 0. \end{aligned} \tag{2.37}$$

Then, we get for any $\Phi \in C^1([0,1])$

$$Gv_{1x}(1)\Phi(1) - Gv_{1x}(0)\Phi(0) = 0. \tag{2.38}$$

Because Φ is arbitrary, we get that $v_{1x}(0) = v_{1x}(1) = 0$. Hence, $v_1 \in H^2_*(0,1)$. Using similar arguments as above, we can obtain

$$\begin{aligned} &v_3, v_5 \in H^2(0,1) \cap H^1_0(0,1), \\ &v_7 \in H^1_0(0,1), \\ &v_8 \in H^1_*(0,1). \end{aligned} \tag{2.39}$$

Finally, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in \mathcal{D}(\mathcal{A})$ such that (2.17) is satisfied. Consequently, we conclude that $\mathcal{A}$ is a maximal dissipative operator. Hence, by Lumer-Philips theorem, [12, 22], we have the well posedness result. This completes the proof. □

Chapter 3

EXPONENTIAL DECAY

In this section, we state and prove our stability result. We need the following lemmas.

Lemma 3.0.1. *The energy functional E, defined by*

$$\begin{aligned} E(t) =& \frac{1}{2}\int_0^1 \left\{\rho\omega_t^2 + I_\rho(3s-\psi)_t^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2 + D(3s-\psi)_x^2 \right. \\ & \left. + D\left(\psi-\omega_x\right)^2 + \rho_3\theta^2 + \tau q^2 + 4\int_0^1\int_{\tau_1}^{\tau_2} \rho\left|\mu_2(\rho)\right| y^2(x,\rho,\rho,t)d\rho d\rho \right\} dx, \end{aligned} \tag{3.1}$$

satisfies:

$$E'(t) \leq -\alpha \int_0^1 q^2 dx - 4\eta_0 \int_0^1 s_t^2 dx \leq 0, \tag{3.2}$$

where: $\eta_0 = \beta - \int_{\tau_1}^{\tau_2} \left|\mu_2(\rho)\right| d\rho > 0$,

Proof. Multiplying the equations of (2.5) by ω_t, $(3s-\psi)_t$, s_t, θ, and Q respectively and integrating over $(0,1)$, we find:

$$\begin{cases} \rho\int_0^1 \omega_{tt}\omega_t\,dx + G\int_0^1 \omega_t(\psi-\omega_x)\,dx + \delta\int_0^1 \theta_x\omega_t\,dx = 0, \\ \\ I_\rho\int_0^1 (3s-\psi)_{tt}(3s-\psi)_t\,dx - D\int_0^1 (3s-\psi)_{xx}(3s-\psi)_t\,dx - G\int_0^1 (3s-\psi_x)(3s-\psi)_t\,dx = 0, \\ \\ I_\rho\int_0^1 s_{tt}s_t\,dx - D\int_0^1 s_{tt}s_t\,dx + G\int_0^1 s_t(\psi-\omega_x)\,dx + \frac{4}{3}\gamma\int_0^1 s_ts\,dx + \frac{4}{3}\beta\int_0^1 s_t^2\,dx \\ \\ +\frac{4}{3}\int_0^1\int_{\theta_1}^{\theta_2} s_t|\mu_2(\varrho)|y(x,1,\varrho,t)\,dx = 0, \\ \\ \rho_3\int_0^1 \theta_t\theta\,dx + \int_0^1 q_x\theta\,dx + \delta\int_0^1 \theta\omega_{tx}\,dx = 0, \\ \\ \theta\int_0^1 q_tq\,dx + \alpha\int_0^1 q^2\,dx + \int_0^1 q\theta_x\,dx = 0. \end{cases}$$

Summing up and integrating by parts, we get:

$$\frac{1}{2}\frac{d}{dt}\left\{\int_0^1 \omega_t^2 + I_\rho(3s-\psi)_t^2 + 3I_\rho s_t^2 + 4\gamma s^2 + 3Ds_x^2 + D(3s-\psi)^2 + G(\psi-\omega_x)^2 + \rho_3\theta^2 + \theta q^2\,dx\right\}$$
$$= -4\beta\int_0^1 s_t^2\,dx - \alpha\int_0^1 q^2\,dx - 4\int_0^1 s_t^2\int_{\theta_1}^{\theta_2} |\mu(\varrho)| y(x,1,h,t)\,d\varrho\,dx \tag{3.3}$$

Using Young's inequality, we arrive at

$$\int_0^1 S_t \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, y(x,1,\varrho,t)d\rho dx \leq \left(\frac{1}{2}\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,d\varrho\right)\int_0^1 s_t^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,y^2(x,1,\varrho,t)d\rho dx. \tag{3.4}$$

Now, multiplying the last equation in (2.5) by $y\,|\mu_2(\rho)|$ and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}
&\frac{d}{dt}\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\rho\,|\mu_2(\rho)|\,y^2(x,\rho,\varrho,t)d\rho d\rho dx,\\
&= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\,yy_\rho(x,\rho,\varrho,t)d\rho d\rho dx,\\
&= -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|\,\frac{d}{d\rho}y^2(x,\rho,\varrho,t)d\rho d\rho dx,\\
&= \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\left(y^2(x,0,\varrho,t) - y^2(x,1,\varrho,t)\right)d\rho dx,\\
&= \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|\,d\rho\right)\int_0^1 s_t^2 dx,\\
&-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\rho)|\,y^2(x,1,\varrho,t)d\rho dx.
\end{aligned}\tag{3.5}$$

From (3.1) (3.3), (3.4), and(3.5) , we get (3.2).

$$E'(t) \leq -\alpha\int_0^1 q^2 dx - 4\left(\beta - \int_{\tau_1}^{\tau_2}|\mu_2(s)|\right)\int_0^1 s_t^2 dx \tag{3.6}$$

then, by (2.4), there exists a positive constant η_0 such that

$$E'(t) \leq -4\eta_0\int_0^1 s_t^2 dx - \alpha\int_0^1 q^2 dx \leq 0, \tag{3.7}$$

then, we obtain E is a nonincreasing function. □

Lemma 3.0.2. *The functional*

$$F_1(t) := I_\rho \int_0^1 (3s-\psi)(3s-\psi)_t dx - \rho \int_0^1 \omega_t \left(\int_0^x (3s-\psi)(y)dy \right) dx$$

satisfies

$$\begin{aligned} F_1'(t) \leq & -\frac{D}{2} \int_0^1 (3s-\psi)_x^2 dx + \varepsilon_1 \int_0^1 \omega_t^2 dx \\ & + c\left(1+\frac{1}{\varepsilon_1}\right) \int_0^1 (3s-\psi)_t^2 dx + c\int_0^1 \theta^2 dx \end{aligned} \tag{3.8}$$

Proof. The derivative with respect to t is:

$$\begin{aligned} F_1'(t) = & I_\rho \int_0^1 (3s-\psi)_t^2 + I_\rho \int_0^1 (3s-\psi)(3s-\psi)_{tt} dx \\ & -\rho \int_0^1 \omega_{tt} \left(\int_0^x (3s-\psi)(y)\, dy \right) - \rho \int_0^1 \omega_t \left(\int_0^x (3s_t-\psi_t)(y)\, dy \right) dx \end{aligned} \tag{3.9}$$

By integrationg$(2.5)_2$ we can wee that:

$$I = D\int_0^1 (3s-\psi)_{xx}(3s-\psi)dx - G\int_0^1 (\psi-\omega_x)(3s-\psi)dx,$$

using integration by part we get:

$$I = D\left[(3s-\psi)(3s-\psi)\big|_0^1 - \int_0^1 (3s-\psi)_x^2 \right] + G\int_0^1 (\psi-\omega_x)(3s-\psi)dx$$

using the initial data we get and simplifying we get

$$I = -D\int_0^1 (3s-\psi)_x^2 dx + G\int_0^1 (\psi-\omega_x)(3s-\psi)dx$$

now, from (2.1) we have:

$$\rho\omega_{tt} + G(\psi-\omega_x)_x + \delta\theta_x = 0$$

by integrating and multiplying by $\int_0^1 (3s-\psi)(y)dy$ and we get:

$$\rho\int_0^1 \omega_{tt} \int_0^1 (3s-\psi)(y)dydx + G\int_0^1 (\psi-\omega_x) \int_0^1 (3s-\psi)(y)dydx_x + \delta\int_0^1 \theta_x \int_0^1 (3s-\psi)(y)dydx = 0,$$

now, we have:

$$
\begin{aligned}
F_1'(t) =& I_\rho \int_0^1 (3s-\psi)_t^2 dx - D\int_0^1 (3s-\psi)_x^2 dx + G\int_0^1 (\psi-\omega_x)(3s-\psi)dx \\
&+ G\int_0^1 (\psi-\omega_x)\int_0^1 (3s-\psi)(y)dydx + \delta\int_0^1 \theta_x \int_0^1 (3s-\psi)(y)dydx \\
&- \rho\int_0^1 \omega_{tt}\int_0^1 (3s-\psi)(y)dydx
\end{aligned}
$$

after simplificaations and using integration by part we get:

$$
\begin{aligned}
F_1'(t) =& I_\rho \int_0^1 (3s-\psi)_t^2 dx - D\int_0^1 (3s-\psi)_x^2 dx \\
&+ \delta\int_0^1 \theta_x \left(\int_0^x (3s-\psi)(y)dy\right) dx \\
&- \rho\int_0^1 \omega_t \left(\int_0^x (3s-\psi)_t(y)dy\right) dx
\end{aligned}
\tag{3.10}
$$

Using Young's, Cauchy-Schwarz, and poincare's inequalities, we obtain (3.9). □

Lemma 3.0.3. *The functional*

$$
F_2(t) := \rho\int_0^1 (\psi-\omega_x)\left(\int_0^x \omega_t(y)dy\right) dx \tag{3.11}
$$

satisfies

$$
\begin{aligned}
F_2'(t) \leq& -\frac{G}{2}\int_0^1 (\psi-\omega_x)^2 dx + \varepsilon_2\int_0^1 (3s-\psi)_t^2 dx \\
&+ c\left(1+\frac{1}{\varepsilon_2}\right)\int_0^1 \omega_t^2 dx + c\int_0^1 \theta^2 dx + c\int_0^1 s_t^2 dx
\end{aligned}
\tag{3.12}
$$

Proof. Differentiating $F_2(t)$ with respect to t and using integration by parts, we have:

$$
\begin{aligned}
F_2'(t) =& \rho\int_0^1 (\psi-\omega_x)_t \int_0^x \omega_t(y)\,dy\,dx \\
&+ \rho\int_0^1 (\psi-\omega_x)\int_0^x \omega_{tt}(y)\,dy\,dx
\end{aligned}
$$

Now using the first equation of the system (2.1), we obtain:

$$
\begin{aligned}
&\rho\int_0^1 (\psi-\omega_x)\int_0^x \omega_{tt}(y)\,dy\,dx \\
&= -\int_0^1 G(\psi-\omega_x)\int_0^x \rho\int_0^1 G(\psi-\omega_y) - \gamma\int_0^x (\psi-\omega_x)\int_0^x \theta_y\,dy\,dx
\end{aligned}
$$

Direct computation:

$$\begin{aligned}
&\rho \int_0^1 (\psi - \omega_x) \int_0^x \omega_{tt}(y)\, dy\, dx \\
&= -G \int_0^1 (\psi - \omega_x) \left[(\psi - \omega_x)\Big|_0^1 \right] dx \\
&\quad - \gamma \int_0^1 \theta(\psi - \omega_x)\Big|_0^1 dx \\
&= -G \int_0^1 (\psi - \omega_x)^2\, dx - \gamma \int_0^1 \theta(\psi - \omega_x)\, dx
\end{aligned}$$

Therefore:

$$\begin{aligned}
F_t'(2) = \rho &\int_0^1 \psi_t \int_0^x \omega_t(y)\, dy\, dx \\
&- \rho \int_0^1 \omega_{xt} \int_0^x \omega_t(y)\, dy\, dx \\
&- G \int_0^1 (\psi - \omega_x)^2\, dx - \gamma \int_0^1 \theta(\psi - \omega_x)\, dx
\end{aligned}$$

Using integration by parts, we conclude:

$$F_2'(t) = \rho \int_0^1 \psi_t \int_0^1 \omega_t(y)\, dy\, dx + \rho \int_0^1 \omega_t^2\, dx - G \int_0^1 (\psi - \omega_x)^2\, dx - \gamma \int_0^1 \theta(\psi - \omega_x)\, dx$$

Using the fact that $\psi_t = 3s_t - (3s - \psi)_t$, we find:

$$\begin{aligned}
F_2'(t) = 3\rho \int_0^1 s_t \int_0^x \omega_t(y)\, dy\, dx - \rho \int_0^1 (3s - \psi)_t \int_0^x \omega_t(y)\, dy\, dx \\
+ \rho \int_0^1 \omega_t^2\, dx - G \int_0^1 (\psi - \omega_x)^2\, dx - \gamma \int_0^1 (\psi - \omega_x)\theta\, dx.
\end{aligned} \tag{3.13}$$

By Young's Poincaré and Cauchy-Schwartz inequalities, we get:

$$
\begin{aligned}
&3\rho \int_0^1 s_t \int_0^x \omega_t(y)\,dy\,dx \\
&\quad \le c\int_0^1 s_t^2\,dx + c\int_0^1 \left(\int_0^x \omega_t(y)\,dy\right)^2 dx \\
&\quad \le c\int_0^1 s_t^2\,dx + c\int_0^1 \omega_t^2\,dx \\
&-\rho \int_0^1 (3s-\psi)_t \int_0^x \omega_t(y)\,dy\,dx \\
&\quad \le \epsilon_2 \int_0^1 (3s-\psi)_t^2\,dx + \frac{1}{\epsilon_2}\int_0^1 \left(\int_0^x \omega_t(y)\,dy\right)^2 dx \\
&\quad \le \epsilon_2 \int_0^1 (3s-\psi)_t^2\,dx + \frac{1}{\epsilon_2}\int_0^1 \int_0^x \omega_t^2\,dy\,dx \\
&-\delta \int_0^1 (\psi-\omega_x)\theta\,dx \\
&\quad \le \frac{G}{2}\int_0^1 (\psi-\omega_x)^2\,dx + c\int_0^1 \theta^2\,dx
\end{aligned}
$$

Combining the above inequalities, we obtain (3.12) Using the fact that $\psi_t = 3s_t - (3s-\psi)_t$, Young's and Cauchy-Schwarz inequalities, we obtain (3.12)

□

Lemma 3.0.4. *The functional*

$$
F_3(t) := \tau\rho_3 \int_0^1 \theta \left(\int_0^x q(y)dy\right) dx \tag{3.14}
$$

satisfies

$$
\begin{aligned}
F_3'(t) \le &-\frac{\rho_3}{2}\int_0^1 \theta^2 dx + \varepsilon_3 \int_0^1 \omega_t^2 dx \\
&+ c\left(1+\frac{1}{\varepsilon_3}\right)\int_0^1 q^2 dx
\end{aligned} \tag{3.15}
$$

Proof. Differentiating $F_3(t)$ with respect to t, we find:

$$
F_3'(t) = \rho_3 \int_0^1 \theta_t \int_0^x q(y)\,dy\,dx + \rho_3 \int_0^1 \theta \int_0^x q_t(y)\,dy\,dx \tag{3.16}
$$

Using equations 4 and 5 of system (2.1), we get:

$$\rho_3 \int_0^1 \theta_t \int_0^x q(y)\,dy\,dx = -\rho_3 \int_0^1 q_x \int_0^x q(y)\,dy\,dx - \rho_3\delta \int_0^1 \omega_{tx} \int_0^x q(y)\,dy\,dx$$
$$\rho_3 \int_0^1 \theta \int_0^x q_t(y)\,dy\,dx = -\alpha\rho_3 \int_0^1 \theta \int_0^x q(y)\,dy\,dx - \rho_3 \int_0^1 \theta \int_0^x \theta_y(y)$$

Integrating by parts and using the initial conditions, it follows that:

$$\begin{aligned} \rho_3 \int_0^1 \theta_t \int_0^x q(y)\,dy\,dx &= \rho_3 \int_0^1 q^2\,dx + \rho_3\gamma \int_0^1 q\omega_t\,dx \\ \rho_3 \int_0^1 \theta \int_0^x q_t(y)\,dy\,dx &= -\alpha\rho_3 \int_0^1 \theta \int_0^x q(y)\,dy\,dx - \rho_3 \int_0^1 \theta^2\,dx \end{aligned}$$

Replacing in $F_3'(t)$, we obtain:

$$F_3'(t) = \rho_3 \int_0^1 q^2\,dx + \rho_3\delta \int_0^1 \omega_t q\,dx - \rho_3 \int_0^1 \theta^2\,dx - \alpha\rho_3 \int_0^1 \theta \int_0^x q(y)\,dy\,dx$$

Now, we use Young's Poincare and Cauchy-Schwartz inequalities

$$\theta\delta \int_0^1 \int_0^x \theta q(y)\,dy\,dx \le \epsilon_3 \int_0^1 (\omega^2 - t)\,dx + \frac{1}{\epsilon_3} \int_0^1 \left(\int_0^x q(y)\,dy \right)^2 dx$$

and

$$\begin{aligned} -\rho\alpha \int_0^1 \theta \int_0^x q(y)\,dy\,dx &\le \frac{\rho_3}{2} \int_0^1 \theta^2\,dx + c \int_0^1 \left(\int_0^x q(y)\,dy \right)^2 dx \\ &\le \frac{\rho_3}{2} \int_0^1 \theta^2\,dx + c \int_0^1 q^2\,dx \end{aligned}$$

Hence, estimate (3.15) follows by combining $F_3'(t)$ and the above inequalities. □

Lemma 3.0.5. *The functional*

$$F_4(t) := -\rho\rho_3 \int_0^1 \theta \left(\int_0^x \omega_t(y) dy \right) dx \tag{3.17}$$

satisfies

$$\begin{aligned} F_4'(t) \le &- \frac{\rho\delta}{2} \int_0^1 \omega_t^2 dx + \varepsilon_4 \int_0^1 (\psi - \omega_x)^2\,dx \\ &+ c\left(1 + \frac{1}{\varepsilon_4}\right) \int_0^1 \theta^2 dx + c \int_0^1 q^2 dx \end{aligned} \tag{3.18}$$

Proof. Taking the derivative of F_4 with respect to t, we have:

$$F_4'(t) = -\rho\rho_3 \int_0^1 \theta_t \int_0^\alpha \omega_t(y)\, dy\, dx \\ - \rho\rho_3 \int_0^1 \theta \int_0^\alpha \omega_{tt}(y)\, dy\, dx$$

Using equations (0.3) of system (2.1), we find:

$$- \rho\rho_3 \int_0^1 \theta_t \int_0^x \omega_{tt}(y)\, dy\, dx = \rho \int_0^1 q_x \int_0^x \omega_t(y)\, dy\, dx \\ + \rho\delta\alpha \int_0^1 \omega_{tx} \int_0^x \omega_t(y)\, dy\, dx$$

and

$$-\rho\rho_3 \int_0^1 \theta \int_0^x \omega_{tt}(y)\, dy\, dx = G\rho_3 \int_0^1 \theta \int_0^x (\psi - \omega_y)_y\, dy\, dx - \rho$$

Hence, using integration by parts, we see that

$$F_4'(t) = -\rho \int_0^1 q\omega_t\, dx - \rho\delta \int_0^1 \omega_t^2\, dx + \rho_3 G \int_0^1 \theta(\psi - \omega_x)\, dx + \delta\rho_3 \int_0^1 \theta^2\, dx. \tag{3.19}$$

Finally, applying Young's Poincaré and Cauchy-Schwarz inequalities, we find:

$$-\rho \int_0^1 q\omega_t\, dx \leq \rho\delta \int_0^1 \omega_t^2\, dx + c \int_0^1 q^2\, dx \\ \rho_3 G \int_0^1 \theta(\psi - \omega_x)\, dx \leq \epsilon_4 \int_0^1 (\psi - \omega_x)^2\, dx + \frac{1}{\epsilon_4} \int_0^1 \theta^2\, dx$$

With these inequalities, estimate (3.18) follows directly. □

Lemma 3.0.6. *The functional*

$$\begin{aligned} F_5(t) :=& \tau\delta G I_\rho \int_0^1 (3s - \psi)_t\, (\psi - \omega_x)\, dx - \tau\delta D\rho \int_0^1 \omega_t (3s - \psi)_x dx \\ &+ \rho_3 \tau\, (D\rho - G I_\rho) \int_0^1 \theta (3s - \psi)_t dx \\ &- \tau\, (D\rho - G I_\rho) \int_0^1 q (3s - \psi)_x dx \end{aligned} \tag{3.20}$$

satisfies

$$\begin{aligned} F_5'(t) \leq & -\frac{\tau\delta GI_\rho}{2}\int_0^1 (3s-\psi)_t^2 dx + \varepsilon_5 \int_0^1 (3s-\psi)_x^2 dx \\ & + \varepsilon_6 \int_0^1 \theta^2 dx + c\left(1+\frac{1}{\varepsilon_6}\right)\int_0^1 (\psi-\omega_x)^2\,dx + \frac{c}{\varepsilon_5}\int_0^1 q^2 dx \\ & + c\int_0^1 s_t^2 dx + \chi \int_0^1 \theta_x (3s-\psi)_x dx \end{aligned} \tag{3.21}$$

where

$$\chi = \tau\delta^2 D - (D\rho - GI_\rho)\left(\frac{\tau\rho_3 D}{I_\rho} - 1\right) \tag{3.22}$$

Proof. Differentiating F_5 with respect to t, we find:

$$\begin{aligned} F_5'(t) = & \theta\delta GI_\rho \int_0^1 (3s-\psi)_{tt}(\psi-\omega_x)\,dx + \theta\delta GI_\rho \int_0^1 (3s-\psi)_t(\psi-\omega_x)\,dx \\ & - \theta\delta D\rho \int_0^1 \omega_{tt}(3s-\psi)_x\,dx - \theta\delta D\rho\int_0^1 \omega_t(3s-\psi)_{xt}\,dx \\ & + \rho_3\tau(D\rho - GI_\rho)\int_0^1 \theta_t(3s-\psi)_t\,dx \\ & + \rho_3\tau(D\rho - GI_\rho)\int_0^1 \theta(3s-\psi)_{tt}\,dx \\ & - \tau(D\rho - GI_\rho)\int_0^1 q_t(3s-\psi)_x\,dx - \tau(D\rho - GI_\rho)\int_0^1 q(3s-\psi)_{xt}\,dx \end{aligned}$$

Using equation (1.2.4.5) of system (2.1) along with integration by parts, we find:

$$\begin{aligned} F_5'(t) = & \rho G^2\delta \int_0^1 (\psi-\omega_x)^2\,dx + \theta\delta GI_\rho\int_0^1 (3s-\psi)_t\psi_t\,dx \\ & + \delta^2\theta\int_0^1 \theta_x(3s-\psi)_x\,dx + \frac{\theta\rho_3 D}{I_\rho}(D\rho - GI_\rho)\int_0^1 \theta_x(3s-\psi)_x\,dx \\ & + \frac{\theta\rho_3 D}{I_\rho}(D\rho - GI_\rho)\int_0^1 \theta(\psi-\omega_x)\,dx + \alpha(D\rho - GI_\rho)\int_0^1 q(3s-\psi)_x\,dx \\ & + (D\rho - GI_\rho)\int_0^1 \theta_x(3s-\psi)_x\,dx \end{aligned} \tag{3.23}$$

Using Young's inequality, we get:

$$\begin{aligned} 3\theta\delta GI_\rho \int_0^1 s_t(3s-\psi)_t\,dx \quad & \leq \frac{\theta\delta GI_\rho}{2}\int_0^1 (3s-\psi)_t^2\,dx + c\int_0^1 s_t^2\,dx \\ \frac{\theta\rho_3}{I_\rho}(D\rho - GI_\rho)\int_0^1 \theta(\psi-\omega_x)\,dx \quad & \leq \epsilon_6\int_0^1 \theta^2\,dx + \frac{1}{\epsilon_6}\int_0^1 (\psi-\omega_x)^2\,dx \end{aligned}$$

Exploiting these inequalities, we establish the proof. □

Lemma 3.0.7. *The functional*

$$F_6(t) := 3I_\rho \int_0^1 ss_t dx + 2\beta \int_0^1 s^2 dx \tag{3.24}$$

satisfies

$$F'(t) \leq -3D \int_0^t s^2\,dx - 2y \int_0^t s\,dx + c\int_0^t (y-w_2)\,dx + c\int_0^1 s^2\,dx + c\int_{x_1}^1 |u_2(t)|^2(x_1, e_1, \theta)\,d\theta\,dx \tag{3.25}$$

Proof. Using the third equation of system (2.1), we get:

$$\begin{aligned} &3I_\rho \int_0^1 ss_{tt}\,dx - 3D\int_0^1 s_{xx}s\,dx + 3G\int_0^1 (\psi - \omega_x)s\,dx \\ &+ 4\delta \int_0^1 s^2\,dx + 4\beta \int_0^1 ss_t\,dx + 4\int_0^1 s\int_{\tau_1}^{\tau_2} |\mu_2(\rho)| y(x,1,\varrho,t)\,d\varrho\,dx = 0 \end{aligned}$$

Integrating by parts, we have:

$$\begin{aligned} F_6'(t) = &-3D\int_0^1 s_x^2\,dx - 3G\int_0^1 s(\psi - \omega_x)\,dx + 3I_\rho \int_0^1 s_t^2\,dx \\ &- 4\delta \int_0^1 s^2\,dx - 4\int_0^1 s\int_{\tau_1}^{\tau_2} |\mu_2(\rho)| y(x,1,\varrho,t)\,d\rho\,dx \end{aligned} \tag{3.26}$$

Using Young's Poincaré and Cauchy-Schwarz inequalities, we obtain:

$$\begin{aligned} -3\int_0^1 s(\psi - \omega_x)\,dx &\leq \gamma \int_0^1 s^2\,dx + c\int_0^1 (\psi - \omega_x)^2\,dx \\ -4\int_0^1 s\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)\,d\varrho\,dx &\leq \gamma \int_0^1 s^2\,dx + c\int_0^1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)\,d\varrho\right)^2 dx \\ &\leq \gamma \int_0^1 s^2\,dx + c\int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|^2 y^2(x,1,\varrho,t)\,d\varrho\,dx \end{aligned}$$

which concludes the proof. □

Lemma 3.0.8. *The functional*

$$F_7(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho e^{-\rho\rho} \left|\mu_2(\rho)\right| y^2(x,\rho,\rho,t) d\rho d\rho dx$$

satisfies,

$$\begin{aligned} F_7'(t) \leq & -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho \left|\mu_2(\rho)\right| y^2(x,\rho,\rho,t) d\rho d\rho dx + \beta \int_0^1 s_t^2 dx \\ & -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} \left|\mu_2(\rho)\right| y^2(x,1,\rho,t) d\rho dx \end{aligned} \tag{3.27}$$

where η_1 is a positive constant.

Proof. Using the third equation of system (2.1), we get:

$$\begin{aligned} 3I_\rho \int_0^1 s s_{tt}\, dx - 3D \int_0^1 s_{xx} s\, dx + 3G \int_0^1 (\psi - \omega_x) s\, dx \\ + 4\delta \int_0^1 s^2\, dx + 4\beta \int_0^1 s s_t\, dx + 4 \int_0^1 s \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| y(x,1,\varrho,t)\, d\varrho\, dx = 0 \end{aligned}$$

Integrating by parts, we have:

$$\begin{aligned} F_6'(t) = & -3D \int_0^1 s_x^2\, dx - 3G \int_0^1 s(\psi - \omega_x)\, dx + 3I_\rho \int_0^1 s_t^2\, dx \\ & -4\delta \int_0^1 s^2\, dx - 4 \int_0^1 s \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| y(x,1,\varrho,t)\, d\rho\, dx \end{aligned}$$

Using Young's Poincaré and Cauchy-Schwarz inequalities, we obtain:

$$-3\int_0^1 s(\psi - \omega_x)\, dx \leq \gamma \int_0^1 s^2\, dx + c \int_0^1 (\psi - \omega_x)^2\, dx$$

$$\begin{aligned} -4 \int_0^1 s \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)\, d\varrho\, dx & \leq \gamma \int_0^1 s^2\, dx + c \int_0^1 \left(\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)\, d\varrho \right)^2 dx \\ & \leq \gamma \int_0^1 s^2\, dx + c \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|^2 y^2(x,1,\varrho,t)\, d\varrho\, dx \end{aligned}$$

which concludes the proof. □

3.1 Exponential stability

In this subsection, we study the exponential stability of systems, and we consider the case $\chi = 0$.

Theorem 3.1.1. *Assume* (2.4)*, there exist positive constants λ_1 and λ_2 such that the energy functional given by* (3.1) *satisfies*

$$E(t) \leq \lambda_2 e^{-\lambda_1 t}, \forall t \geq 0 \tag{3.28}$$

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) := NE(t) + F_1(t) + \sum_{i=2}^{i=7} N_i F_i(t) \tag{3.29}$$

where N and $N_i, i = 2 \dots 7$, are positive constants to be selected later. By differentiating (3.29) and using (3.2), (3.9),(3.12),(3.15),(3.18),(3.21),(3.25), and (3.27), we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \left[\frac{D}{2} - \varepsilon_5 N_5\right] \int_0^1 (3s-\psi)_x^2 dx \\
& - [4\eta_0 N - cN_2 - cN_3 - cN_6 - \beta N_7] \int_0^1 s_t^2 dx \\
& - \left[\frac{\tau G \delta I_\rho}{2} N_5 - c\left(1 + \frac{1}{\varepsilon_1}\right) - \varepsilon_2 N_2\right] \int_0^1 (3s-\psi)_t^2 dx \\
& - \left[\frac{G}{2} N_2 - \varepsilon_4 N_4 - c\left(1 + \frac{1}{\varepsilon_6}\right) N_5 - cN_6\right] \int_0^1 (\psi - \omega_x)^2 dx \\
& - \left[\frac{\rho\delta}{2} N_4 - \varepsilon_1 - c\left(1 + \frac{1}{\varepsilon_2}\right) N_2 - \varepsilon_3 N_3\right] \int_0^1 \omega_t^2 dx \\
& - \left[\frac{\rho_3}{2} N_3 - \varepsilon_6 N_5 - c\left(1 + \frac{1}{\varepsilon_4}\right) N_4 - cN_2 - c\right] \int_0^1 \theta^2 dx \\
& - \left[\alpha N - cN_4 - c\left(1 + \frac{1}{\varepsilon_3}\right) N_3 - \frac{c}{\varepsilon_5} N_5\right] \int_0^1 q^2 dx \\
& - [2\gamma N_6] \int_0^1 s^2 dx - [3DN_6] \int_0^1 s_x^2 dx \\
& - [\eta_1 N_7 - cN_6] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\rho)| \, y^2(x,1,\rho,t) d\rho dx \\
& - [N_7 \eta_1] \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho \, |\mu_2(\varrho)| \, y^2(x,\rho,\varrho,t) d\rho d\rho dx
\end{aligned} \tag{3.30}$$

By setting

$$\varepsilon_1 = 1, \varepsilon_2 = \frac{\tau G \delta I_\rho N_5}{4N_2}, \varepsilon_3 = \frac{1}{N_3}, \varepsilon_4 = \frac{GN_2}{4N_4}, \varepsilon_5 = \frac{D}{4N_5}, \varepsilon_6 = \frac{1}{N_5}$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & - \left[\frac{D}{4}\right] \int_0^1 (3s-\psi)_x^2 dx \\
& - [4\eta_0 N - cN_2 - cN_3 - cN_6 - \beta N_7] \int_0^1 s_t^2 dx \\
& - \left[\frac{\tau G\delta I_\rho}{4} N_5 - 2c\right] \int_0^1 (3s-\psi)_t^2 dx \\
& - \left[\frac{G}{4} N_2 - c\left(1+N_5\right) N_5 - cN_6\right] \int_0^1 \left(\psi - \omega_x\right)^2 dx \\
& - \left[\frac{\rho\delta}{2} N_4 - c\left(1 + \frac{N_2}{N_5}\right) N_2 - 2\right] \int_0^1 \omega_t^2 dx \\
& - \left[\frac{\rho_3}{2} N_3 - c\left(1 + \frac{N_4}{N_2}\right) N_4 - cN_2 - c - 1\right] \int_0^1 \theta^2 dx \\
& - \left[\alpha N - cN_4 - c\left(1+N_3\right) N_3 - cN_5^2\right] \int_0^1 q^2 dx \\
& - [2\gamma N_6] \int_0^1 s^2 dx - [3DN_6] \int_0^1 s_x^2 dx \\
& - [\eta_1 N_7 - cN_6] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\, y^2(x,1,\varrho,t) d\varrho dx \\
& - [N_7 \eta_1] \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho \,|\mu_2(\varrho)|\, y^2(x,\rho,\varrho,t) d\varrho d\rho dx,
\end{aligned} \tag{3.31}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. We fixed N_6, and we choose N_5, N_7 large enough such that

$$\alpha_1 = \frac{\tau G\delta I_\rho}{4} N_5 - 2c > 0$$
$$\alpha_2 = \eta_1 N_7 - cN_6 > 0$$

then, we choose N_2 large enough such that

$$\alpha_3 = \frac{G}{4} N_2 - c\left(1+N_5\right) N_5 - cN_6 > 0$$

then, we choose N_4 large enough such that

$$\alpha_4 = \frac{\rho\delta}{2} N_4 - c\left(1 + \frac{N_2}{N_5}\right) N_2 - 2 > 0$$

then, we choose N_3 large enough such that

$$\alpha_5 = \frac{\rho_3}{2} N_3 - c\left(1 + \frac{N_4}{N_2}\right) N_4 - cN_2 - c - 1 > 0$$

thus, we arrive at

$$
\begin{aligned}
\mathcal{L}'(t) \leq & -\frac{D}{4}\int_0^1 (3s-\psi)_x^2 dx - \alpha_3 \int_0^1 (\psi-\omega_x)^2 dx - [N\eta_0 - c]\int_0^1 s_t^2 dx \\
& - \alpha_2 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho \left|\mu_2(\varrho)\right| y^2(x,1,\varrho,t) d\rho dx - \alpha_1 \int_0^1 (3s-\psi)_t^2 dx \\
& - \alpha_5 \int_0^1 \theta^2 dx - \alpha_7 \int_0^1 s_x^2 dx - [\alpha N - c]\int_0^1 q^2 dx - \alpha_4 \int_0^1 \omega_t^2 dx \\
& - \alpha_6 \int_0^1 s^2 dx - \alpha_8 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho \left|\mu_2(\varrho)\right| y^2(x,\rho,\varrho,t) d\rho d\rho dx
\end{aligned}
\tag{3.32}
$$

where $\alpha_6 = 2\gamma N_6, \alpha_7 = 2DN_6, \alpha_8 = \eta_1 N_7$. On the other hand, if we let

$$
\mathfrak{T}(t) = F_1(t) + \sum_{i=2}^{i=7} N_i F_i(t)
$$

then,

$$
\begin{aligned}
|\mathfrak{I}(t)| \leq & I_\rho \int_0^1 |(3s-\psi)(3s-\psi)_t| \, dx + \rho \int_0^1 \left|\omega_t \left(\int_0^x (3s-\psi)(y)dy\right)\right| dx \\
& + N_2\rho \int_0^1 \left|(\psi-\omega_x)\left(\int_0^x \omega_t(y)dy\right)\right| dx + N_3\tau\rho_3 \int_0^1 \left|\theta\left(\int_0^x q(y)dy\right)\right| dx \\
& + N_4\rho\rho_3 \int_0^1 \left|\theta\left(\int_0^x \omega_t(y)dy\right)\right| dx + N_5\tau \left|(D\rho - GI_\rho)\right| \int_0^1 |q(3s-\psi)_x| \, dx \\
& + N_5\tau\delta GI_\rho \int_0^1 \left|(3s-\psi)_t(\psi-\omega_x)\right| dx + N_5\tau\delta D\rho \int_0^1 |\omega_t(3s-\psi)_x| \, dx \\
& + N_5\rho_3\tau \left|(D\rho - GI_\rho)\right| \int_0^1 |\theta(3s-\psi)_t| \, dx + 3I_\rho N_6 \int_0^1 |ss_t| \, dx \\
& + 2\beta \int_0^1 s^2 dx + N_7 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho e^{-\rho\rho} \left|\mu_2(\rho)\right| y^2(x,\rho,\rho,t) d\rho d\rho dx.
\end{aligned}
$$

Exploiting Young's, Cauchy-Schwarz, and Poincaré inequalities, we get

$$
\begin{aligned}
|\mathfrak{I}(t)| \leq & c \int_0^1 \left(\omega_t^2 + (3s-\psi)_t^2 + (3s-\psi)_x^2 + (\psi-\omega_x)^2 + s^2 + s_x^2 + s_t^2\right) dx \\
& + c\int_0^1 \left(\theta^2 + q^2\right) dx + c\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho \left|\mu_2(\rho)\right| y^2(x,\rho,\rho,t) d\rho d\rho dx.
\end{aligned}
$$

then,

$$
|\mathfrak{T}(t)| \leq cE(t)
$$

Consequently, we obtain

$$|\mathfrak{T}(t)| = |\mathcal{L}(t) - NE(t)| \leq cE(t)$$

that is,

$$(N-c)E(t) \leq \mathcal{L}(t) \leq (N+c)E(t) \tag{3.33}$$

Now, by choosing N large enough such that

$$N-c>0, \alpha N-c>0, \eta_0 N-c>0$$

and exploiting (3.1), estimates (3.32), and (3.33), respectively, give

$$\mathcal{L}'(t) \leq -k_2 E(t) \tag{3.34}$$

for some $k_2 > 0$, and

$$c_1 E(t) \leq \mathcal{L}(t) \leq c_2 E(t), \forall t \geq 0 \tag{3.35}$$

for some $c_1, c_2 > 0$, we have

$$\mathcal{L}(t) \sim E(t)$$

A combination with (3.34) gives

$$\mathcal{L}'(t) \leq -\lambda_1 \mathcal{L}(t) \tag{3.36}$$

where $\lambda_1 = \frac{k_2}{c_2}$. Finally, a simple integration of 3.48, we obtain (3.28). Then the proof is complete. □

3.2 Polynomial stability

In this subsection, we study the polynomial stability of systems, and we consider the case $\chi \neq 0$.

Theorem 3.2.1. *Assume* (2.4)*, there exist positive constant* C_1 *such that the energy functional given by* (3.1) *satisfies*

$$E(t) \leq \frac{C_1}{t}, \forall t > 0 \tag{3.37}$$

Proof. First, we introduce second-order energy functional $E_2(t)$ by

$$E_2(t) = E_1(\omega_t, \psi_t, s_t, \theta_t, q_t) = E(\omega_t, \psi_t, s_t, \theta_t, q_t)$$

satisfies

$$
\begin{aligned}
E_2'(t) &\leq -\alpha \int_0^1 q_t^2 dx - 4\eta_0 \int_0^1 s_{tt}^2 dx \\
&\leq -\alpha \int_0^1 q_t^2 dx
\end{aligned}
\tag{3.38}
$$

And thanks to $(2.1)_5$ and Young's inequality, the last term of $F_5'(t)$ gives

$$
\begin{aligned}
\chi \int_0^1 \theta_x (3s-\psi)_x dx &= -\chi\tau \int_0^1 q_t (3s-\psi)_x dx - \chi \int_0^1 q(3s-\psi)_x dx \\
&\leq \frac{c}{\varepsilon_7} \int_0^1 q_t^2 dx + \frac{c}{\varepsilon_7} \int_0^1 q^2 dx + 2\varepsilon_7 \int_0^1 (3s-\psi)_x^2 dx
\end{aligned}
$$

We define a Lyapunov functional

$$
\mathcal{G}(t) := N\left(E(t) + E_1(t)\right) + F_1(t) + \sum_{i=2}^{i=7} N_i F_i(t) \tag{3.39}
$$

where N and $N_i, i = 2 \dots 7$, are positive constants to be selected later..By differentiating (3.39)and using (3.2), (3.9) (3.12),(3.15),(3.18),(3.21),(3.25), (3.27), and (3.38), we have:

$$\begin{aligned}
\mathcal{G}'(t) \leq & -\left[\frac{D}{2} - (\varepsilon_5 + 2\varepsilon_7) N_5\right] \int_0^1 (3s-\psi)_x^2 dx \\
& - [4\eta_0 N - cN_2 - cN_3 - cN_6 - \beta N_7] \int_0^1 s_t^2 dx \\
& - \left[\frac{\tau G \delta I_\rho}{2} N_5 - c\left(1 + \frac{1}{\varepsilon_1}\right) - \varepsilon_2 N_2\right] \int_0^1 (3s-\psi)_t^2 dx \\
& - \left[\frac{G}{2} N_2 - \varepsilon_4 N_4 - c\left(1 + \frac{1}{\varepsilon_6}\right) N_5 - cN_6\right] \int_0^1 (\psi - \omega_x)^2 dx \\
& - \left[\frac{\rho\delta}{2} N_4 - \varepsilon_1 - c\left(1 + \frac{1}{\varepsilon_2}\right) N_2 - \varepsilon_3 N_3\right] \int_0^1 \omega_t^2 dx \\
& - \left[\frac{\rho_3}{2} N_3 - \varepsilon_6 N_5 - c\left(1 + \frac{1}{\varepsilon_4}\right) N_4 - cN_2 - c\right] \int_0^1 \theta^2 dx \\
& - \left[\alpha N - cN_4 - c\left(1 + \frac{1}{\varepsilon_3}\right) N_3 - \left(\frac{c}{\varepsilon_5} + \frac{c}{\varepsilon_7}\right) N_5\right] \int_0^1 q^2 dx \\
& - [2\gamma N_6] \int_0^1 s^2 dx - [3DN_6] \int_0^1 s_x^2 dx \\
& - [\eta_1 N_7 - cN_6] \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \, y^2(x,1,\varrho,t) d\rho dx \\
& - [N_7 \eta_1] \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho \, |\mu_2(\varrho)| \, y^2(x,\rho,\varrho,t) d\rho d\rho dx \\
& - \left[N\alpha - \frac{c}{\varepsilon_7} N_5\right] \int_0^1 q_t^2 dx.
\end{aligned} \tag{3.40}$$

By setting

$$\varepsilon_1 = 1, \varepsilon_2 = \frac{\tau G \delta I_\rho N_5}{4N_2}, \varepsilon_3 = \frac{1}{N_3}, \varepsilon_4 = \frac{GN_2}{4N_4}, \varepsilon_5 = \frac{D}{8N_5}, \varepsilon_6 = \frac{1}{N_5}, \varepsilon_7 = \frac{D}{16N_5}$$

we obtain

$$
\begin{aligned}
\mathcal{G}'(t) \leq & -\left[\frac{D}{4}\right]\int_0^1 (3s-\psi)_x^2 dx \\
& -\left[4\eta_0 N - cN_2 - cN_3 - cN_6 - \beta N_7\right]\int_0^1 s_t^2 dx \\
& -\left[\frac{\tau G\delta I_\rho}{4}N_5 - 2c\right]\int_0^1 (3s-\psi)_t^2 dx \\
& -\left[\frac{G}{4}N_2 - c\left(1+N_5\right)N_5 - cN_6\right]\int_0^1 \left(\psi-\omega_x\right)^2 dx \qquad (3.41)\\
& -\left[\frac{\rho\delta}{2}N_4 - c\left(1+\frac{N_2}{N_5}\right)N_2 - 2\right]\int_0^1 \omega_t^2 dx \\
& -\left[\frac{\rho_3}{2}N_3 - c\left(1+\frac{N_4}{N_2}\right)N_4 - cN_2 - c - 1\right]\int_0^1 \theta^2 dx \\
& -\left[\alpha N - cN_4 - c\left(1+N_3\right)N_3 - cN_5^2\right]\int_0^1 q^2 dx \\
& -\left[2\gamma N_6\right]\int_0^1 s^2 dx - \left[3DN_6\right]\int_0^1 s_x^2 dx \\
& -\left[\eta_1 N_7 - cN_6\right]\int_0^1\int_{\tau_1}^{\tau_2} \left|\mu_2(\rho)\right| y^2(x,1,\varrho,t)\,d\rho dx \\
& -\left[N_7\eta_1\right]\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \rho\left|\mu_2(\varrho)\right| y^2(x,\rho,\varrho,t)\,d\rho d\rho dx \\
& -\left[N\alpha - cN_5^2\right]\int_0^1 q_t^2 dx
\end{aligned}
$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. We fixed N_6, and we choose N_5, N_7 large enough such that

$$
\begin{aligned}
\alpha_1 &= \frac{\tau G\delta I_\rho}{4}N_5 - 2c > 0 \\
\alpha_2 &= \eta_1 N_7 - cN_6 > 0
\end{aligned}
$$

then, we choose N_2 large enough such that

$$\alpha_3 = \frac{G}{4}N_2 - c\left(1+N_5\right)N_5 - cN_6 > 0$$

then, we choose N_4 large enough such that

$$\alpha_4 = \frac{\rho\delta}{2}N_4 - c\left(1+\frac{N_2}{N_5}\right)N_2 - 2 > 0$$

then, we choose N_3 large enough such that

$$\alpha_5 = \frac{\rho_3}{2} N_3 - c\left(1 + \frac{N_4}{N_2}\right) N_4 - cN_2 - c - 1 > 0$$

thus, we arrive at

$$\begin{aligned}
\mathcal{G}'(t) \leq & - \frac{D}{4}\int_0^1 (3s-\psi)_x^2 dx - \alpha_3 \int_0^1 (\psi - \omega_x)^2 dx - [N\eta_0 - c]\int_0^1 s_t^2 dx \\
& - \alpha_2 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho \left|\mu_2(\varrho)\right| y^2(x,1,\varrho,t) d\rho dx - \alpha_1 \int_0^1 (3s-\psi)_t^2 dx \\
& - \alpha_5 \int_0^1 \theta^2 dx - \alpha_7 \int_0^1 s_x^2 dx - [\alpha N - c]\int_0^1 q^2 dx - \alpha_4 \int_0^1 \omega_t^2 dx \\
& - \alpha_6 \int_0^1 s^2 dx - \alpha_8 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho \left|\mu_2(\varrho)\right| y^2(x,\rho,\varrho,t) d\rho d\rho dx \\
& - [N\alpha - c]\int_0^1 q_t^2 dx.
\end{aligned} \tag{3.42}$$

where $\alpha_6 = 2\gamma N_6, \alpha_7 = 2DN_6, \alpha_8 = \eta_1 N_7$. On the other hand, if we let

$$\mathcal{K}(t) = F_1(t) + \sum_{i=2}^{i=7} N_i F_i(t)$$

$$\begin{aligned}
|\mathcal{K}(t)| \leq & I_\rho \int_0^1 \left|(3s-\psi)(3s-\psi)_t\right| dx + \rho \int_0^1 \left|\omega_t \left(\int_0^x (3s-\psi)(y) dy\right)\right| dx \\
& + N_2 \rho \int_0^1 \left|(\psi - \omega_x)\left(\int_0^x \omega_t(y) dy\right)\right| dx + N_3 \tau \rho_3 \int_0^1 \left|\theta \left(\int_0^x q(y) dy\right)\right| dx \\
& + N_4 \rho \rho_3 \int_0^1 \left|\theta \left(\int_0^x \omega_t(y) dy\right)\right| dx + N_5 \tau \left|(D\rho - GI_\rho)\right| \int_0^1 \left|q(3s-\psi)_x\right| dx \\
& + N_5 \tau \delta G I_\rho \int_0^1 \left|(3s-\psi)_t (\psi - \omega_x)\right| dx + N_5 \tau \delta D \rho \int_0^1 \left|\omega_t (3s-\psi)_x\right| dx \\
& + N_5 \rho_3 \tau \left|(D\rho - GI_\rho)\right| \int_0^1 \left|\theta (3s-\psi)_t\right| dx + 3 I_\rho N_6 \int_0^1 \left|s s_t\right| dx \\
& + 2\beta \int_0^1 s^2 dx + N_7 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\rho\rho} \left|\mu_2(\varrho)\right| y^2(x,\rho,\varrho,t) d\rho d\rho dx.
\end{aligned}$$

Exploiting Young's, Cauchy-Schwarz, and Poincaré inequalities, we get

$$\begin{aligned}
|\mathcal{K}(t)| \leq & c\int_0^1 \left(\omega_t^2 + (3s-\psi)_t^2 + (3s-\psi)_x^2 + (\psi - \omega_x)^2 + s^2 + s_x^2 + s_t^2\right) dx \\
& + c\int_0^1 \left(\theta^2 + q^2\right) dx + c\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \rho \left|\mu_2(\varrho)\right| y^2(x,\rho,\rho,t) d\rho d\rho dx
\end{aligned}$$

then,

$$|\mathcal{K}(t)| \leq cE_1(t)$$

Consequently, we obtain

$$|\mathcal{K}(t)| = |\mathcal{G}(t) - N\left(E_1(t) + E_2(t)\right)| \leq cE(t) = cE_1(t)$$

that is,

$$(N-c)E_1(t) + NE_2(t)(t) \leq \mathcal{L}(t) \leq (N+c)E_1(t) + NE_2(t) \tag{3.43}$$

Now, by choosing N large enough such that

$$N - c > 0, N\alpha - c > 0, N\eta_0 - c > 0$$

and exploiting (3.1), estimates (3.43) and (3.42), respectively, give

$$m_1\left(E_1(t) + E_2(t)\right) \leq \mathcal{G}(t) \leq m_2\left(E_1(t) + E_2(t)\right), \forall t \geq 0 \tag{3.44}$$

for some $m_1, m_2 > 0$.
we have

$$\mathcal{G}(t) \sim \left(E_1(t) + E_2(t)\right)$$

and we have

$$\mathcal{G}'(t) \leq -d_1 E_1(t) \tag{3.45}$$

for some $d_1 > 0$.
Integrating (3.45), we get

$$\begin{aligned}\int_0^t E_1(y)dy &\leq \frac{1}{d_1}(\mathcal{G}(0) - \mathcal{G}(t)) \leq \frac{1}{d_1}\mathcal{G}(0) \\ &\leq \frac{m_2}{d_1}\left(E_1(0) + E_2(0)\right)\end{aligned} \tag{3.46}$$

using the fact that

$$(tE_1(t))' = E_1(t) + tE_1'(t) \leq E_1(t) \tag{3.47}$$

we get that

$$tE_1(t) \leq \frac{m_2}{d_1}\left(E_1(0) + E_2(0)\right) \tag{3.48}$$

which gives us

$$E_1(t) \leq \frac{C_1}{t} \tag{3.49}$$

where $C_1 = \frac{m_2}{d_1}\left(E_1(0) + E_2(0)\right)$, we obtain (3.49). Then the proof is complete. $\square$

3.3 CONCLUSION

This work studies the linear thermoelastic laminated Timoshenko beam with distributed delay, where the heat conduction is given by Cattaneoâs law, where the well posedness of the system is established. Moreover, the exponential and polynomial stabilities of the system for both cases, equal and nonequal speeds of wave propagation are proven. In the next work, we will apply the distributed delay in our studied problems in previous studies [22, 23, 24].

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