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Block matrices

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Contents

Notations

X, Y, Z	Banach spaces on a field $\mathbb{K}$.
X^*	Dual space of X.
$\langle x, f \rangle$	The value $f(x)$ where $x \in X$ and $f \in X^*$.
$\oplus$	Direct sum.
I	The identity operator on X.
$\mathcal{L}(X, Y)$	Set of continuous operators from X to Y.
$\mathcal{K}(X, Y)$	Set of compact operators from X to Y.
$\mathrm{GL}(X)$	Set of invertible continuous operators on X.
$\Phi, \Phi_{\pm}$	Set of Fredholm and semi Fredholm operators.
$\mathcal{F}$	Set of Fredholm perturbations.
Φ_+, Φ_-	Set of upper, lower semi Fredholm operators.
$\mathcal{F}_+, \mathcal{F}_-$	Set of upper, lower semi Fredholm perturbations.
Φ_l, Φ_r	Set of left, right Fredholm operators.
$\mathcal{F}_l, \mathcal{F}_r$	Set of left, right Fredholm perturbations.
$\sigma_{e_1}, \sigma_{e_2}, \sigma_{e_3}, \sigma_{e_4}, \sigma_{e_5}$	Gustafson, Weidmann, Kato, Wolf, and Schechter essential spectrum.
$\sigma_{e_7}, \sigma_{e_8}$	Approximation point essential spectrum and defect essential spectrum.
$\ker(A)$	Kernel of A.
$R(A)$	Range of A.
$\alpha(A)$	Nullity of A.
$\beta(A)$	Defect of A.
$\rho(A)$	Resolvent set of A.
$\sigma(A)$	Spectrum of A.
n, m, l	Positive non null integers.
$n_1, \cdots, n_p$	Partition of n.
$m_1, \cdots, m_q$	Partition of m.
$l_1, \cdots, l_r$	Partition of l.
$A_{\mid V}$	Restriction of the operator A on the subspace V.

Introduction

In this work, we delve into the intricate study of block operator matrices, emphasising their algebraic and topological properties. Our exploration begins with a thorough examination of the foundational concepts of algebraic structures and topological spaces, setting the stage for analysis of block matrices. This investigation encompasses compactness, Fredholmness, and the essential spectrum of block operator matrices. These concepts are pivotal as they provide significant insights into the behaviour and characteristics of complex mathematical structures, which have far-reaching applications in various scientific and engineering disciplines.

This thesis is organised in three chapters. The first chapter discusses the basic properties of operators. It includes a study of compact operators, Fredholm operators, and their perturbations. These foundational concepts are crucial for understanding the more advanced topics discussed in the subsequent chapters.

In the second chapter we explore algebraic properties of block matrices, including the addition and multiplication, as well as the characteristics of square, triangular, and diagonal block matrices. This chapter also delves into the inverses and determinants of 2×2 block matrices, providing a comprehensive understanding of their algebraic structure. Furthermore, it investigates block operators, focusing on their compactness and Fredholmness, and examines Fredholm perturbations of block operators.

The last chapter is devoted to the essential spectra of operators, particularly block operators. It provides an analysis of the essential spectra, discussing various properties and implications. This chapter is crucial for understanding the spectral behaviour of block operator matrices, which has significant applications in the study of differential equations and other areas of functional analysis.

1 Preliminaries

The goal of this chapter consists in collecting some results which will be needed in the sequel. In particular, we recall some fundamental theorems on closed range operators, compact operators, Fredholm operators and Fredholm perturbations. For more details about these notions see [A. 15].

1.1 Basic properties

Definition 1.1.1 *Let E be a vectorial space on a valued field $(\mathbb{K}, |\cdot|)$, and let $||\cdot|| : E \to \mathbb{R}$ be a map. $||\cdot||$ is called norm on E if the following axioms are satisfied.*

1. $||x|| = 0 \Longrightarrow x = 0$ *for all* $x \in E$.
2. $||x + y|| \leqslant ||x|| + ||y||$ *for all* $x, y \in E$.
3. $||ax|| = |a| \cdot ||x||$ *for all* $a \in \mathbb{K}$, $x \in E$.

Lemma 1.1.2 *Let V, W be two subspaces of X. Then $V \times W$ is closed if and only if V and W are both closed.*

Proof Assume that V and W are closed. Let $(x_n, y_n)_{n\in\mathbb{N}} \in (V \times W)^{\mathbb{N}}$ be a sequence convergent to (x, y). Then $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are convergent sequences to x and y respectively. Since V and W are closed, then (x, y) belongs to $V \times W$. Hence $V \times W$ is closed.

Inversely, if $V \times W$ is closed, then let $(x_n) \in V^{\mathbb{N}}$ convergent to x. So $(x_n, 0) \in V^{\mathbb{N}}$ is convergent to $(x, 0)$. Since $V \times W$ is closed, then $(x, 0) \in V \times W$. Therefore $x \in \mathbb{V}$. Consequently, V is closed. The closure of W can be proved similarly. □

Definition 1.1.3 *Let $A \in \mathcal{L}(X, Y)$ and let W be a subspace of Y.*

$$A^{-1}(W) := \{x \in X \ / \ Ax \in W\}.$$

Definition 1.1.4 *Let $A \in \mathcal{L}(X, Y)$. A is called open operator if $A(\mathcal{O})$ is an open set for all open set $\mathcal{O}$ of X.*

Definition 1.1.5 *Let $A \in \mathcal{L}(X, Y)$. The null space of A, denoted by* $\ker(A)$, *is defined by*

$$\ker(A) := \{x \in X \ / \ Ax = 0\}.$$

The range (or image) of A, denoted by $R(A)$, is defined by

$$R(A) := \{y \in Y \ / \ \exists \ x \in X \ / \ Ax = y\}.$$

Definition 1.1.6 *The rank of an operator A is defined to be the dimension of its image.*

Theorem 1.1.7 *(Open Mapping Theorem) Let $A \in \mathcal{L}(X, Y)$ be a surjective operator. Then A is an open operator.*

Definition 1.1.8 *Let $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$. $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence if*

$$\lim_{n\to\infty} \operatorname{diam}(\{x_k \ / \ k \geqslant n\}) = 0,$$

where $\operatorname{diam}(\{x_k \ / \ k \geqslant n\})$ *designates the diameter of the set* $\{x_k \ / \ k \geqslant n\}$.

Definition 1.1.9 *Let $A \in \mathcal{L}(X, Y)$. A is bounded below if*

$$\inf_{||x||=1}(||Ax||) > 0.$$

This notion provides significant equivalences in the following theorem.

Theorem 1.1.10 *Let $A \in \mathcal{L}(X, Y)$. Then the following assertions are equivalent.*

1. *A is bounded below.*
2. *There exists $c > 0$ such that $||Ax|| \geqslant c \cdot ||x||$ for all $x \in E$.*
3. *A is injective and $R(A)$ is closed in Y.*
4. *$A^{-1} : R(A) \to X$ is continuous.*

Proof
(1) $\Longrightarrow$ (2) Let's assume that A is bounded below, then there exists $c > 0$ such that for all x verifying $||x|| = 1$, we have $||Ax|| \geqslant c$. Next, let $x \in X \setminus \{0\}$. Since $\frac{1}{||x||}x$ is normalised, then $||A\frac{1}{||x||}x|| \geqslant c$, hence for all vector x, we have $||Ax|| \geqslant c||x||$.

(2) $\Longrightarrow$ (3) Let's assume that there exists $c > 0$ such that for all $x \in E$, we have $||Ax|| \geqslant c \cdot ||x||$, then for a non null vector x, we have $||Ax|| > 0$, therefore $\ker(A) = \{0\}$. This means that A is injective. For the closure of $R(A)$, let's take a sequence $(Ax_n)_{n\in\mathbb{N}} \in R(A)^{\mathbb{N}}$ convergent in Y and $\epsilon > 0$. Since $(Ax_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, there exists a natural n such that for all $p, q \geqslant n$, we have $||Ax_p - Ax_q|| < c^{-1}\epsilon$, therefore, we have $||x_p - x_q|| < \epsilon$. Then $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. We deduce that $(x_n)_{n\in\mathbb{N}}$ is convergent in X, since X is a Banach space. Now as A is continuous, then $\lim_{n\to\infty} Ax_n = A(\lim_{n\to\infty} x_n)$. Consequently, $R(A)$ is closed.

(3) $\Longrightarrow$ (4) According to Theorem (1.1.7).

(4) $\Longrightarrow$ (1) Let's assume that $A^{-1} : R(A) \to X$ is continuous. Then there exists $c > 0$ such that for all $y \in R(A)$, we have $||A^{-1}y|| \leqslant c||y||$. Therefore for all $x \in E$, we have $||Ax|| \geqslant c^{-1}||x||$. So whenever $||x|| = 1$, we have $||Ax|| \geqslant c^{-1}$. Hence A is bounded below. □

Now, let us define of the adjoint operator. We denote by X^* the set of continuous linear maps from X to $\mathbb{K}$.

Definition 1.1.11 *Let $A \in \mathcal{L}(X, Y)$. Then there exists a unique operator $B \in \mathcal{L}(Y^*, X^*)$ such that*

$$\langle Ax, y\rangle = \langle x, By\rangle \text{ for all } x \in X \text{ and } y \in Y^*.$$

The operator B verifying the assertion of the theorem above is called adjoint of A.

Theorem 1.1.12 *[M. 02] (Hahn Banach Theorem) Let $x \in X$. Then there exists an element $y \in X^*$ such that $|\langle x, y\rangle| = ||y||^2$ and $|\langle x, y\rangle| = ||x||^2$.*

Remark 1.1 We deduce from Theorem (1.1.12) that if $\langle x, y\rangle = 0$ whenever $y \in X^*$, then $x = 0$, and inversely, if $\langle x, y\rangle = 0$ whenever $x \in X$, then $y = 0$.

Definition 1.1.13 *Let V be a subset of X. The orthogonal space of V, denoted $V^\perp$, is defined by*

$$V^\perp = \{y \in X^* \ / \ \forall \ x \in V, \ \langle x, y\rangle = 0\}.$$

Lemma 1.1.14 *Let $A \in \mathcal{L}(X,Y)$. Then the following relations hold.*

1. $\ker(A^*) = R(A)^\perp$.
2. $\ker(A)^\perp = R(A^*)$.

Proof
1) By definition of the kernel, we have

$$\ker(A^*) = \{f \in Y^* \ / \ A^* f = 0\}.$$

Applying the previous Theorem (1.1.12), we get

$$\ker(A^*) = \{f \in Y^* \ / \ \forall \ x \in X, \ \langle x, A^* f\rangle = 0\}.$$

Now, using the definition of the adjoint of A, we see that $\langle x, A^* f\rangle = \langle Ax, f\rangle$. Consequently,

$$\begin{aligned} \ker(A^*) &= \{f \in Y^* \ / \ \forall \ y \in R(X), \ \langle y, f\rangle = 0\} \\ &= R(A)^\perp. \end{aligned}$$

2) Can be proved similarly.

□

Theorem 1.1.15 *[Y. 02] Let M be a closed subspace of X. Then*

- *M^* is isomorphic to $X^*/M^\perp$.*
- *$(X/M)^*$ is isomorphic to $M^\perp$.*

Definition 1.1.16 *The nullity of an operator A, noted $\alpha(A)$, is defined by $\alpha(A) := \dim(\ker(A))$. The deficiency of an operator A, noted $\beta(A)$, is defined as $\beta(A) := \operatorname{codim}(R(A))$.*

The nullity and deficiency of A are either natural integers or infinite values.

Lemma 1.1.17 *Let $A \in \mathcal{L}(X,Y)$ such that $\alpha(A) < \infty$, and W be a finite dimensional subspace of X. Then $\dim(A^{-1}(W)) \leqslant \dim(W) + \dim(\ker(A))$.*

Proof It is well known that $A^{-1}(W)/\ker(A)$ is isomorphic to $A(A^{-1}W)$. Then

$$\dim(A^{-1}(W)) = \dim(A(A^{-1}(W))) + \dim(\ker(A)).$$

Since $A(A^{-1}(W)) \subset W$, we get $\dim(A^{-1}(W)) \leqslant \dim(W) + \dim(\ker(A))$. □

Lemma 1.1.18 *Let $A \in \mathcal{L}(X,Y)$ such that $\operatorname{codim}(R(A)) < \infty$, and let V be a subspace of X such that $\operatorname{codim}(V) < \infty$. Then $\operatorname{codim}(A(V)) \leqslant \operatorname{codim}(V) + \operatorname{codim}(R(A))$.*

Proof We know that $(X/A(V))/(X/R(A))$ is isomorphic to $R(A)/A(V)$. Vurthermore, we have the following isomorphismes

$$R(A) \simeq X/\ker(A) \quad , \quad A(V) \simeq V/(\ker(A) \cap V).$$

Then $\dim(R(A)/A(V)) \leqslant \dim(X/V)$. Which means that $\operatorname{codim}(A(V)) - \operatorname{codim}(R(A)) \leqslant \operatorname{codim}(V)$. Hence $\operatorname{codim}(A(V)) \leqslant \operatorname{codim}(V) + \operatorname{codim}(R(A))$. □

Lemma 1.1.19 *Let $A \in \mathcal{L}(X,Y)$ with a closed range. Then*

1. $\alpha(A^*) = \beta(A)$.

2. $\beta(A^*) = \alpha(A)$.

Proof 1) According to Lemma (1.1.14), we get $\dim(\ker(A^*)) = \dim(R(A)^{\perp})$. By using Theorem (1.1.15), we obtain $\dim(\ker(A^*)) = \dim((X/R(A))^*)$. Hence $\alpha(A^*) = \beta(A)$.

2) According to Lemma (1.1.14), we get $\dim(X^*/R(A^*)) = \dim(X^*/(\ker(A)^{\perp}))$. By using Theorem (1.1.15), we obtain $\dim(X^*/R(A^*)) = \dim(\ker(A)^*)$. Hence $\beta(A^*) = \alpha(A)$. □

Definition 1.1.20 *Let X, Y be two vectorial spaces. The direct sum of X and Y, noted $X \oplus Y$, is the space $X \times Y$ equipped with the following laws*

$$\begin{array}{ccc} (V \times W)^2 & \to & V \times W \\ ((v_1, w_1), (v_2, w_2)) & \mapsto & (v_1 + v_2, w_1 + w_2) \end{array}$$

and

$$\begin{array}{ccc} \mathbb{K} \times V \times W & \to & V \times W \\ (a, v_1, w_1) & \mapsto & (av_1, aw_1) \end{array}.$$

Definition 1.1.21 *Let V, W be two subspaces of a vectorial space X. W is called supplement of V if X is isomorphic to $V \oplus W$.*

Definition 1.1.22 *Let $A \in \mathcal{L}(X,Y)$. The range of A is closed if the limit of any convergent sequence of elements in $R(A)$ is in $R(A)$.*

Lemma 1.1.23 *Let $A \in \mathcal{L}(X,Y)$. Then the following assertions are equivalent.*

- *The set $R(A)$ is closed.*

- *There exists a closed subspace Z of Y such that $R(A) \cap Z = \{0\}$ and $R(A) \oplus Z$ be closed.*

Proof Let's assume that $R(A)$ is closed. We have $R(A) \cap \{0\} = \{0\}$ and $R(A) \oplus \{0\}$ is closed. Inversely, If there exists a closed subspace Z of Y such that $R(A) \cap Z = \{0\}$ and $R(A) \oplus Z$ be closed. Let $V = R(A) \oplus Z$ and

$$\begin{array}{cccc} J & : & R(A) & \to & V/Z \\ & & y & \mapsto & y + Z \end{array}.$$

It is easy to prove that J is an isomorphisme. Hence $R(A)$ is closed. □

Proposition 1.1.24 *Let $A \in \mathcal{L}(X,Y)$. If $\beta(A) < \infty$, then $R(A)$ is closed.*

Proof Let F be the supplement subspace of $R(A)$ in Y, since $\dim(F) < \infty$, then F is closed. According to Lemma (1.1.23), we deduce that $R(A)$ is closed. □

Theorem 1.1.25 *(Closed range theorem) Let $A \in \mathcal{L}(X,Y)$. Then $R(A^*)$ is closed if and only if $R(A)$ is closed.*

1.2 Compact operators

Definition 1.2.1 *An operator K is called compact if $(Kx_n)_{n\in\mathbb{N}}$ admits a convergent subsequence for all bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $X^{\mathbb{N}}$.*

Proposition 1.2.2 *[V. 17] Let $K \in \mathcal{L}(X)$. Then the following propositions are equivalent.*

- *The operator K is compact.*
- *For all $\epsilon > 0$, there exists a closed subspace M of X such that $\sup\limits_{\substack{x\in M\\ ||x||=1}}(||Kx||) < \epsilon$.*

This useful result is due to J. Schauder, see [N. 58].

Proposition 1.2.3 *An operator is compact if and only if its adjoint is also compact.*

Lemma 1.2.4 *The set of compact operators is topologically closed in $\mathcal{L}(X)$.*

Proof Let $(K_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}(X)^{\mathbb{N}}$ convergent to K in $\mathcal{L}(X)$. So for every sequence $(x_m)_{m\in\mathbb{N}}$ and an natural integer n, we can pick a convergent subsequence $\left(K_n x_{\sigma_n(m)}\right)_{m\in\mathbb{N}}$. For a positive integers n, p and q, we have

$$||Kx_{\phi_n(p)} - Kx_{\phi_n(q)}|| \leqslant ||Kx_{\phi_n(p)} - K_n x_{\phi_n(p)}|| + ||K_n x_{\phi_n(p)} - K_n x_{\phi_n(q)}|| + ||K_n x_{\phi_n(q)} - Kx_{\phi_n(q)}||.$$

This equality shows that by taking n large enough and $p, q \geqslant n$, we can minimise the value $||Kx_{\phi_n(p)} - Kx_{\phi_n(q)}||$ as much as desired. Consequently, $\left(Kx_{\phi_n(p)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence, then $(Kx_n)_{n\in\mathbb{N}}$ admits a convergent subsequence. This leads us to conclude that K is a compact operator. □

1.3 Fredholm operators

Now, we give some important results on the class of Fredholm operators.

Definition 1.3.1 *Let $A \in \mathcal{L}(X, Y)$. The operator A is called*

- *An upper semi Fredholm operator if it has closed range and finite nullity.*
- *A lower semi Fredholm operator if it has finite deficiency.*
- *A Fredholm operator if it is both upper and lower semi Fredholm operator.*
- *A semi Fredholm operator if it is upper or lower semi Fredholm operator.*
- *A left Fredholm operator if there exists a continuous operator A_0 such that $A_0A - I$ is compact.*
- *A right Fredholm operator if there exists a continuous operator A_0 such that $AA_0 - I$ is compact.*

Remark 1.2 It is possible to understand the concept of Fredholmness as an operator invertible except on a finite dimensional subspaces. As shown above, an upper semi Fredholm operator is "almost" injective, a lower semi Fredholm operator is "almost" surjective, A Fredholm operator is defined as the combination of these two properties.

The sets of upper, lower semi Fredholm operators, left, right Fredholm operators, semi Fredholm operators, and Fredholm operators are denoted respectively by $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_l(X,Y)$, $\Phi_r(X,Y)$, $\Phi_\pm(X,Y)$, and $\Phi(X,Y)$. If $Y = X$, they are respectively denoted by $\Phi_+(X)$, $\Phi_-(X)$, $\Phi_l(X)$, $\Phi_r(X)$, $\Phi_\pm(X)$, and $\Phi(X)$.

Remarks 1.3

1. $\Phi_+(X) \cap \Phi_-(X) = \Phi(X)$.
2. $\Phi_+(X) \cup \Phi_-(X) = \Phi_\pm(X)$.
3. The Fredholm operators set is obviously not a vectorial space, it is not even closed by addition. A simple counterexample can be given by taking a Fredholm operator A, so $-A$ is clearly a Fredholm operator. But their sum is the zero operator, which is not a Fredholm operator, since $X/R(\mathbf{0})$ is isomorphic to infinite dimensional space X, then it has an infinite deficiency.

Exemple 1.3.2 1. The identity operator I is a Fredholm operator. In fact, $\alpha(I) = 0$, and $\beta(A) = \dim(X/R(I))$ is zero.

2. The zero operator is neither upper nor lower semi Fredholm operator. Since $\alpha(\mathbf{0}) = \infty$ and $X/R(\mathbf{0})$ is isomorphic to infinite dimensional space X, then the nullity and deficiency of the null operator are both infinite.
3. Let $X = C^\infty([0,1],\mathbb{R})$ and
$$A \ : \ \begin{array}{ccc} X & \to & X \\ f & \mapsto & f' \end{array}.$$
The kernel of A is composed of the constant maps on some real number. Hence it is isomorphic to $\mathbb{R}$. For the range, since $AF = f$ for any map f where F is its primitive, then A is surjective. Consequently, we get $\alpha(A) = 1$ and $\beta(A) = 0$. Then A is a Fredholm operator.
4. Let $X = l^2(\mathbb{N},\mathbb{C})$ and A be the right shift operator on X defined as
$$A \ : \ \begin{array}{ccc} X & \to & X \\ (x_1, x_2, \cdots) & \mapsto & (0, x_1, x_2, \cdots) \end{array}.$$
Then A is obviously an injective map, its range is the set of the sequences in which the first component is zero, so $X/R(A)$ is isomorphic to $\mathbb{R}$. Consequently, we get $\alpha(A) = 0$ and $\beta(A) = 1$. Then A is a Fredholm operator.

Definition 1.3.3 *The index of a semi Fredholm operator A, denoted* $\mathrm{ind}(A)$*, is given by* $\mathrm{ind}(A) := \alpha(A) - \beta(A)$.

Remarks 1.4

1. If A is a Fredholm operator then its index is an integer number as defined above.
2. If $A \in \Phi_+(X) \setminus \Phi(X)$, then $\mathrm{ind}(A) = -\infty$.
3. If $A \in \Phi_-(X) \setminus \Phi(X)$, then $\mathrm{ind}(A) = \infty$.

Lemma 1.3.4 *Let $A \in \mathcal{L}(X,Y)$. The following propositions are equivalent.*

- *A is an upper semi Fredholm operator.*

- *There exists a closed subspace M of X such that* $\operatorname{codim}(M) < \infty$ *and* $\inf\limits_{\substack{x\in M\\ ||x||=1}}(||Ax||) > 0$.

Proof Let's assume that $A \in \Phi_+(X)$, and let M be the supplement subspace of $\ker(A)$. The operator $A : M \to R(A)$ is injective with a closed range. Then according to Proposition (1.1.10), $A_{|M}$ is bounded below. Therefore, $\inf\limits_{\substack{x\in M\\ ||x||=1}}(||Ax||) > 0$.

Inversely, if there exists a closed subspace M of X such that $\operatorname{codim}(M) < \infty$ and $\inf\limits_{\substack{x\in M\\ ||x||=1}}(||Ax||) > 0$, then $\ker(A) \cap M = \{0\}$. So $\dim(\ker(A)) < \infty$. Now, let F be the supplement subspace of M in X. We have $R(A) = A(F) + A(M)$ and A is continuous. Since $A_{|M}$ is bounded below, then $A(M)$ is closed. In the other part, $\dim(A(F)) < \infty$. Then $R(A)$ is closed. Hence $A \in \Phi_+(X)$. □

Here we establish some relations between Fredholmness of an operator and Fredholmness of its adjoint.

Proposition 1.3.5 *Let $A \in \mathcal{L}(X, Y)$. Then*

1. $A \in \Phi_+(X, Y)$ *if and only if* $A^* \in \Phi_-(Y^*, X^*)$.
2. $A \in \Phi_-(X, Y)$ *if and only if* $A^* \in \Phi_+(Y^*, X^*)$.
3. $A \in \Phi(X, Y)$ *if and only if* $A^* \in \Phi(Y^*, X^*)$.

Proof Let's assume that $A \in \Phi_+(X, Y)$, then $\alpha(A) < \infty$ and $R(A)$ is closed, which means that $\beta(A^*) < \infty$ according to Lemma (1.1.19). Hence $A^* \in \Phi_-(Y^*, X^*)$. Inversely, if $A^* \in \Phi_-(Y^*, X^*)$, then $\beta(A^*) < \infty$ and $R(A^*)$ is closed, by using Proposition (1.1.25) and Lemma (1.1.19) we conclude that $R(A)$ is closed and $\alpha(A) < \infty$.
Hence $A \in \Phi_+(X, Y)$. So the equivalence holds.

Let's assume that $A \in \Phi_-(X, Y)$, then $\beta(A) < \infty$, which means that $\alpha(A^*) < \infty$ and $R(A^*)$ is closed according to Lemma (1.1.19) and Proposition (1.1.25). Hence $A^* \in \Phi_+(Y^*, X^*)$. Inversely, if $A^* \in \Phi_+(Y^*, X^*)$, then $\alpha(A^*) < \infty$ and $R(A^*)$ is closed, by using Lemma (1.1.19) we conclude that $\beta(A) < \infty$. Hence $A \in \Phi_-(X, Y)$. So, the equivalence holds.

The third assertion follows from the first and the second assertion. □

Proposition 1.3.6 *Let $A \in \Phi_\pm(X, Y)$. Then* $\operatorname{ind}(A^*) = -\operatorname{ind}(A)$.

Proof By using the definition of index of the adjoint, we get $\operatorname{ind}(A^*) = \alpha(A^*) - \beta(A^*)$. According to Lemma (1.1.19), we deduce that $\operatorname{ind}(A^*) = \beta(A) - \alpha(A)$. Hence $\operatorname{ind}(A^*) = -\operatorname{ind}(A)$. □

Next, we give some formulas relating Fredholmness of two operators with the Fredholmness of their product.

Proposition 1.3.7 *Let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$. Then for $* = -, +, l, r$, If $A \in \Phi_*(X, Y)$ and $B \in \Phi_*(Y, Z)$, then $BA \in \Phi_*(X, Z)$.*

Proof Let's assume that A and B are lower Fredholm operators, Y_0, Z_0 be the supplements of $R(A)$ and $R(B)$ respectively. We get $Z = Z_0 + B(Y_0) + R(BA)$. By hypotheses, $\dim(Z_0) < \infty$ and $\dim(B(Y_0)) < \infty$, then $\operatorname{codim}(R(BA)) < \infty$. Hence $BA \in \Phi_-(X, Z)$. When A and B are upper Fredholm operators, the assertion can be proved by using

Proposition (1.3.5).

Let's assume that A and B are left Fredholm operators, then there exist two operators A_0, B_0 such that $A_0A - I$ and $B_0B - I$ are compacts, since

$$A_0B_0BA - I = A_0(B_0B - I)A + A_0A - I,$$

we conclude that BA is a left Fredholm operator. The same process can be followed to prove the assertion for right Fredholm operators. $\square$

Proposition 1.3.8 *Let $A \in \Phi(X, Y)$ and $B \in \Phi(Y, Z)$. Then $BA \in \Phi(X, Z)$. Moreover,*

$$\mathrm{ind}(BA) = \mathrm{ind}(A) + \mathrm{ind}(B).$$

Proof The first statement is obvious from Proposition (1.3.7).

Let $Y_0 = R(A) \cap \ker(B)$, and let Y_1, Y_2 be the supplements of Y_0 in $R(A)$ and $\ker(B)$ respectively. We have $Y_2 \cap R(A) \subset \ker(B) \cap R(A)$, so $Y_2 \cap R(A) = \{0\}$. Now, we choose a finite dimensional subspace Y_3 as the supplement of $R(A) \oplus Y_2$ in Y. Then we have

$$\begin{aligned} R(B) &= B(Y_1) \oplus B(Y_3) \\ &= B(Y_1 \oplus Y_0) \oplus B(Y_3) \\ &= B(R(A)) \oplus B(Y_3). \end{aligned}$$

So, $\mathrm{codim}(BA) = \mathrm{codim}(R(B)) + \dim(B(Y_3))$. Furthermore, $\dim(\ker(B)) = \dim(Y_0)$ and $\mathrm{codim}(R(A)) = \dim(Y_2) + \dim(Y_3)$. Let $\tilde{B} = B : \ker(BA) \to Y_0$. Since $\tilde{B}$ is injective and $\ker(\tilde{B}) = \ker(B)$. Hence $\dim(Y_0) = \dim(\ker(BA)) - \dim(\ker(A))$. Let us calculate the index of the operator BA, we have

$$\begin{aligned} \mathrm{ind}(BA) &= \alpha(BA) - \beta(BA) \\ &= \dim(Y_0) + \alpha(A) - \beta(B) - \dim(Y_3) \\ &= \alpha(B) - \dim(Y_2) + \alpha(A) - \beta(B) - \beta(A) + \dim(Y_2) \\ &= \mathrm{ind}(B) + \mathrm{ind}(A). \end{aligned}$$

$\square$

The converse of Proposition 1.3.7 is not true in general, we have the following theorem

Proposition 1.3.9 *Let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, Z)$.*

1. *If $BA \in \Phi_-(X, Z)$, then $B \in \Phi_-(Y, Z)$.*
2. *If $BA \in \Phi_+(X, Z)$, then $A \in \Phi_+(X, Y)$.*
3. *If $BA \in \Phi_l(X, Z)$, then $A \in \Phi_l(X, Y)$.*
4. *If $BA \in \Phi_r(X, Z)$, then $B \in \Phi_r(Y, Z)$.*
5. *If $BA \in \Phi(X, Z)$, then $A \in \Phi_+(X, Y) \cap \Phi_l(X, Y)$ and $B \in \Phi_-(Y, Z) \cap \Phi_r(Y, Z)$.*

Proof Let's assume that $BA \in \Phi_-(X, Z)$, then $R(BA) \subset R(B)$, which implies that $\beta(B) \leqslant \beta(BA)$, since $BA \in \Phi_-(X, Z)$ then $B \in \Phi_-(Y, Z)$.

Assume that $BA \in \Phi_+(X, Z)$, then $A^*B^* \in \Phi_-(Z^*, X^*)$, hence $A^* \in \Phi_-(Y^*, X^*)$. Consequently, $A \in \Phi_+(X, Y)$.

If $BA \in \Phi_l(X,Z)$, then there exists an operator B_0 such that $B_0(BA) - I$ is compact. Then $(B_0B)A - I$ is also compact, hence $A \in \Phi_l(X,Y)$.

If $BA \in \Phi_r(X,Z)$, then there exists an operator A_0 such that $(BA)A_0 - I$ is compact. Then $B(AA_0) - I$ is also compact, hence $B \in \Phi_r(Y,Z)$. □

Corollary 1.3.10 *Assume that AB and BA are both Fredholm operators. Then A and B are Fredholm operators.*

Proof By using assertion 5 of Proposition (1.3.9), we deduce that A and B are both upper and lower semi Fredholm operators. Hence they are both Fredholm operators. □

The classes of Fredholm operators and compact operators are disjoint.

Proposition 1.3.11 *In infinite dimension, a compact operator K cannot be a Fredholm operator.*

Proof Reasoning by absurd, assuming that K is a Fredholm operator, thus $\alpha(K) < \infty$. Let V and W be the supplements of $\ker(A)$ in X and $R(A)$ in Y respectively. Then V should have an infinite dimension. But since $K : V \to R(K)$ is a compact isomorphisme, so $\dim(V) < \infty$, which is a contradiction. Consequently, K is not a Fredholm operator. □

1.4 Fredholm perturbations

The purpose of this section is to describe the set of Fredholm perturbations.

Definition 1.4.1 *An operator $F \in \mathcal{L}(X,Y)$ is called*

- *A Fredholm perturbation if $A + F \in \Phi(X,Y)$ for all $A \in \Phi(X,Y)$.*
- *An upper semi Fredholm perturbation if $A + F \in \Phi_+(X,Y)$ for all $A \in \Phi_+(X,Y)$.*
- *A lower semi Fredholm perturbation if $A + F \in \Phi_-(X,Y)$ for all $A \in \Phi_-(X,Y)$.*
- *A left Fredholm perturbation if $A + F \in \Phi_l(X,Y)$ for all $A \in \Phi_l(X,Y)$.*
- *A right Fredholm perturbation if $A + F \in \Phi_r(X,Y)$ for all $A \in \Phi_r(X,Y)$.*

The sets of upper, lower semi Fredholm perturbations, left, right Fredholm perturbations, and Fredholm perturbations are respectively noted by $\mathcal{F}_+(X,Y)$, $\mathcal{F}_-(X,Y)$, $\mathcal{F}_l(X,Y)$, $\mathcal{F}_r(X,Y)$, and $\mathcal{F}(X,Y)$. If $Y = X$, they are respectively denoted by $\mathcal{F}_+(X)$, $\mathcal{F}_-(X)$, $\mathcal{F}_l(X)$, $\mathcal{F}_r(X)$ and $\mathcal{F}(X)$.

Proposition 1.4.2 *The sets $\mathcal{F}(X), \mathcal{F}_+(X), \mathcal{F}_-(X), \mathcal{F}_l(X)$ and $\mathcal{F}_r(X)$ are vectorial subspaces of $\mathcal{L}(X)$.*

Proof The null operator is clearly a Fredholm perturbation. Next, we choose F_1 and F_2 two Fredholm perturbations and λ a complex scalar. If $\lambda = 0$ then $\lambda F_1 + F_2$ is obviously a perturbation. In the other case, for every $A \in \mathcal{L}(X)$, we have $A + (F_1 + \lambda F_2) = \lambda(\lambda^{-1}A + F_1) + F_2$. Since F_1 and F_2 are two perturbations and λ is invertible, then $A + (F_1 + \lambda F_2)$ is a Fredholm operator. We see that $\lambda F_1 + F_2$ is a Fredholm perturbation. Hence $\mathcal{F}(X)$ is a vectorial subspace of $\mathcal{L}(X)$.

The others sets can be treated in the same way. □

Proposition 1.4.3 *[A. 09] Let $A \in \Phi_{\pm}(X)$ and $F \in \mathcal{F}(X)$. Then* $\text{ind}(A+F) = \text{ind}(A)$.

Fredholm operators are stable under compact perturbation, we have the following proposition.

Proposition 1.4.4 *The following statements are satisfied.*

1. $\mathcal{K}(X) \subset \mathcal{F}_l(X) \cap \mathcal{F}_r(X)$.
2. $\mathcal{K}(X) \subset \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$.
3. $\mathcal{K}(X) \subset \mathcal{F}(X)$.

Proof
1) Let $A \in \Phi_l$, then there exists an operator $A_0 \in \mathcal{L}(X)$ such that $A_0 A = I - K_0$ where K_0 is compact. Then $A_0(A+K) = I - (K_0 - A_0 K)$. Since $K_0 - A_0 K$ is compact, so $A_0(A+K) - I$ is compact. Therefore $A + K \in \Phi_l(X)$. Hence we deduce the inclusion $\mathcal{K}(X) \subset \mathcal{F}_l(X)$. In the same way, we prove the inclusion $\mathcal{K}(X) \subset \mathcal{F}_r(X)$.

2) Let $K \in \mathcal{K}(X)$ and $A \in \Phi_+(X)$. According to Proposition (1.3.4), it is possible to pick a closed subspace M_1 of X such that $\text{codim}(M_1) < \infty$ and $\inf\limits_{\substack{x \in M_1 \\ ||x||=1}}(||Ax||) > 0$. Let $c = \inf\limits_{\substack{x \in M_1 \\ ||x||=1}}(||Ax||)$. Since K is compact, by using Proposition (1.2.2) we conclude the existence of a closed subspace M_2 of X such that $\text{codim}(M_2) < \infty$ and $\sup\limits_{\substack{x \in M_1 \\ ||x||=1}}(||Kx||) < \frac{c}{2}$. Next, let $M = M_1 \cap M_2$. Since $\text{codim}(M_1) < \infty$ and $\text{codim}(M_2) < \infty$, then $\text{codim}(M) < \infty$. Moreover we have $\inf\limits_{\substack{x \in M \\ ||x||=1}}(||(A+K)x||) < \frac{c}{2}$. Applying the equivalence established in Proposition (1.3.4) on M and $A+K$, we deduce that $A + K \in \Phi_+(X)$. Hence $\mathcal{K}(X) \subset \mathcal{F}_+(X)$. Furthermore, the second part of the proof can be realised by using the adjoint operators. In other words, for $K \in \mathcal{K}(X)$ and $A \in \Phi_-(X)$, we have $A^* + K^* \in \Phi_+(X)$, then $A + K \in \Phi_-(X)$, which ends the proof.

The third inclusion is an obvious result from the first inclusion.

□

The following theorem establishes an important characterization of Fredholm operators.

Theorem 1.4.5 *(Atkinson Theorem) The following propositions are equivalent.*

1. *A is a Fredholm operator.*
2. *There exists a continuous operator A_0 such that $AA_0 - I$ and $A_0 A - I$ have finite ranks.*
3. *There exists a continuous operator A_0 such that $AA_0 - I$ and $A_0 A - I$ are compact.*

Proof
(1) $\Longrightarrow$ (2) Let V be the supplement of $\ker(A)$ in X. The operator $A : V \to R(A)$ is injective with a closed range. So it is bounded below according to Proposition (1.1.10). Let A_0 be its inverse, which is continuous. Now, for all $x \in V$ and $y \in R(A)$, we have

$(A_0A - I)x = 0$ and $(AA_0 - I)y = 0$, then both of $A_0A - I$ and $AA_0 - I$ have finite ranks.

(2) $\Longrightarrow$ (3) It is obvious since any finite rank operator is a compact operator.

(3) $\Longrightarrow$ (1) Let K_1, K_2 be two compact operators such that $A_0A = I - K_1$ and $AA_0 = I - K_2$, these two formulas due to Proposition (1.4.4) provides that A_0A and AA_0 are Fredholm operators. Next, according to Proposition (1.3.9), we get $A \in \Phi_+(X)$ and $A \in \Phi_-(X)$. Then A is a Fredholm operator.

□

Remarks 1.5

1. The operator A_0 in the previous Theorem is also a Fredholm operator for the same reason.

2. It is obvious from Theorem (1.4.5) that

$$\Phi_l(X) \cap \Phi_r(X) = \Phi(X).$$

Lemma 1.4.6 *The following assertions are equivalent.*

- *A can be written as $J + K$ where J is invertible and K is compact.*
- *A is Fredholm operator with zero index, i.e* $\text{ind}(A) = 0$.

Proof Suppose that A is Fredholm operator with zero index, P the surjective projection on $\ker(A)$, $(x_1, \cdots, x_n)$ be a base of $\ker(A)$, $y_1, \cdots, y_n \in X$ such that $X = R(A) \oplus \langle y_1, \cdots, y_n \rangle$, and let $U : \ker(A) \to X$ be the unique operator verifying for all i, $Ux_i = y_i$. It provides from the hypotheses that $\dim(R(UP)) < \infty$ and $A + UP$ is invertible. Then $A = A - UP + (-UP)$ where $A + UP$ is invertible and $-UP$ is compact.
For the proof of the inverse statement see [M. 02].

□

Proposition 1.4.7 *Let $A \in \Phi_\pm(X)$ and $K \in \mathcal{K}(X)$. Then* $\text{ind}(A + K) = \text{ind}(A)$.

Proof Let's assume first that $A \in \Phi(X)$, by using the Proposition (1.4.5), there exists $A_0 \in \mathcal{L}(X)$ such that $A_0A = I + K_0$ where K_0 is compact. Further, by using Lemma (1.4.6), we get $I + K_0$ is Fredholm operator with zero index, then $\text{ind}(A_0) = -\text{ind}(A)$. Next, we have $A_0(A + K) = I + (K_0 + A_0K)$. For the same reason as before we deduce that $\text{ind}(A + K) = -\text{ind}(A_0)$, so $\text{ind}(A + K) = \text{ind}(A)$.

Now, if $A \in \Phi_+(X) \setminus \Phi(X)$, then $A + K \in \Phi_+(X) \setminus \Phi(X)$, therefore $\text{ind}(A) = \infty$ and $\text{ind}(A + K) = \infty$. So the equality holds.

Next, $A \in \Phi_-(X) \setminus \Phi(X)$, then $A + K \in \Phi_-(X) \setminus \Phi(X)$, therefore $\text{ind}(A) = -\infty$ and $\text{ind}(A + K) = -\infty$. Hence $\text{ind}(A) = \text{ind}(A + K)$.

□

The goal of the next following lemmas is to prove that an upper or lower Fredholm perturbation is a perturbation.

Lemma 1.4.8 *Let $A \in \mathcal{L}(X, Y)$. The following assertions are equivalent.*

- *A is an upper semi Fredholm operator.*
- *For all compact operator K, $\alpha(A - K) < \infty$.*

Proof The proof of the first implication is immediate by using Proposition (1.4.7) and definition of upper semi Fredholm operator. Inversely, let's assume that A is not an upper semi Fredholm operator, so it is not bounded below, using an inductive process as detailed in [M. 02], we can construct a bi-orthogonal system $(x_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ such that for every n, we have $||x_n|| = 1$, $||Ax_n|| \leqslant 2^{1-2n}$, $||f_n|| \leqslant 2^{n-1}$, and $f_n(x_m) = \delta_{n,m}$. Next, for every n, we define the operator

$$K_n := \sum_{k=0}^{n} f_k(x_k)A_{x_k}.$$

then K_n is finite-dimensional operator. And for every n, m, we have

$$||K_nx - K_mx|| \leqslant \sum_{n>m} 2^{-n}||x||.$$

So (K_n) is a Cauchy sequence in $\mathcal{L}(X)$. We consider K the limit of the last sequence. Furthermore, for all element x_n, we have $Kx_n = Ax_n$, since these elements are linearly independent, we conclude that $\alpha(A - K) = \infty$. Hence the equivalence is established. ☐

Lemma 1.4.9 *Let $A \in \mathcal{L}(X,Y)$. The following assertions are equivalent.*

- *A is a lower semi Fredholm operator.*
- *For all compact operator K, $\beta(A - K) < \infty$.*

Proof The first implication is immediate by using Proposition (1.4.7) and definition of lower semi Fredholm operator. Inversely, for $K = 0$, we get $\beta(A) < \infty$. Then A is a lower semi Fredholm operator. Hence the equivalence is established. ☐

Lemma 1.4.10 *Let $A \in \Phi_+(X)$. Then there exists $\epsilon > 0$ such that for all operator B verifying $||B|| < \epsilon$, we have $A + B \in \Phi_+(X)$, $\alpha(A + B) \leqslant \alpha(A)$, and $\beta(A + B) \leqslant \beta(A)$.*

For a detailed proof, see [M. 02].

Lemma 1.4.11 *[M. 02] Let $A \in \Phi_-(X)$. Then there exists $\epsilon > 0$ such that for all operator B verifying $||B|| < \epsilon$, we have $A+B \in \Phi_-(X)$, $\alpha(A+B) \leqslant \alpha(A)$, and $\beta(A+B) \leqslant \beta(A)$.*

Proposition 1.4.12 *Let $F \in \mathcal{L}(X)$. Then the following propositions are equivalent.*

- *F is a Fredholm perturbation.*
- *$\alpha(A - F) < \infty$ for all $A \in \Phi(X)$.*
- *$\beta(A - F) < \infty$ for all $A \in \Phi(X)$.*

Proof Suppose that F is a Fredholm perturbation. This implies that $A+F$ is a Fredholm operator for all Fredholm operator A. Hence $\alpha(A - F) < \infty$. Inversely, let A be any Fredholm operator such that $\alpha(A - F) < \infty$, let $\lambda \in \mathbb{C}^*$ and K be any compact operator. Then the operator $\lambda^{-1}A - \lambda^{-1}K$ is a Fredholm operator by using compact perturbation and assertion 3 of Proposition (1.4.4). So $\alpha(A - K - \lambda F) < \infty$. Since K is arbitrary, then according to Lemma (1.4.8), we conclude that $A - \lambda F \in \Phi_+(X)$. By taking λ small enough and using Lemma (1.4.10), we get $\alpha(A - \lambda F) \leqslant \alpha(A)$ and $\beta(A - \lambda F) \leqslant \beta(A)$. This means that $A - \lambda F$ is a Fredholm operator. Consequently, F is a Fredholm perturbation.

Suppose that F is a Fredholm perturbation. This implies that $A+F$ is a Fredholm operator for all Fredholm operator A. Hence $\beta(A-F)<\infty$. Inversely, let A be any Fredholm operator such that $\beta(A-F)<\infty$, let $\lambda\in\mathbb{C}^*$ and K be any compact operator. Then the operator $\lambda^{-1}A-\lambda^{-1}K$ is a Fredholm operator by using compact perturbation and assertion 3 of Proposition (1.4.4). So $\beta(A-K-\lambda F)<\infty$. Since K is arbitrary, then according to Lemma (1.4.9), we conclude that $A-\lambda F\in\Phi_+(X)$. By taking λ small enough and using Lemma (1.4.11), we get $\alpha(A-\lambda F)\leqslant\alpha(A)$ and $\beta(A-\lambda F)\leqslant\beta(A)$. This means that $A-\lambda F$ is a Fredholm operator. Consequently, F is a Fredholm perturbation. □

Proposition 1.4.13 *The following inclusions hold.*

- $\mathcal{F}_+(X)\subset\mathcal{F}(X)$.
- $\mathcal{F}_-(X)\subset\mathcal{F}(X)$.

Proof
Let F be an upper semi Fredholm perturbation. Then $A+F$ is an upper semi Fredholm operator for all A an upper semi Fredholm operator. So $\alpha(A+F)<\infty$ for all Fredholm operator A. According to Proposition (1.4.12), we deduce that $F\in\mathcal{F}(X)$.

Let F be a lower semi Fredholm perturbation. Then $A+F$ is a lower semi Fredholm operator for all A a lower semi Fredholm operator. So $\alpha(A+F)<\infty$ for all Fredholm operator A. According to Proposition (1.4.12), we get $F\in\mathcal{F}(X)$.
□

Theorem 1.4.14 *[N. 01] The set of Fredholm perturbations is a bilateral ideal.*

We designate by $\mathrm{GL}(X)$ the algebra of invertible continuous operators on X. We have the following theorem.

Theorem 1.4.15 *Let $F\in\mathcal{F}(X)$ and $J_1,J_2\in\mathrm{GL}(X)$. Then $J_1FJ_2\in\mathcal{F}(X)$.*

Proof Let A be a Fredholm operator. After multiplying by invertible operators and adding Fredholm perturbations, we get $J_1(J_1^{-1}AJ_2^{-1}+F)J_2\in\Phi(X)$. Then $A+J_1FJ_2$ is a Fredholm operator by using Proposition 1.3.8. This means that J_1FJ_2 is a Fredholm perturbation. □

2 Block matrices

The goal of this section is to develop the notion of block matrices and their algebraic properties. By understanding the sum and product of block matrices, we explore specific classes such as triangular and square matrices.

In all the sequel, $M_{i,j}$ designate the entry at the i^{th} row and j^{th} column of matrix M. Which belongs to $\mathcal{M}_{n_i,n_j}(\mathbb{K})$ where $\mathbb{K}$ is a commutative field or to $\mathcal{L}(X_j, Y_i)$ if it is a continuous operator).

2.1 Algebraic properties

Definition 2.1.1 *A partitioned matrix, also called a block matrix, is a regular matrix that is divided into smaller rectangular sections. These sections are called blocks or sub-matrices.*

Here is an example of a matrix decomposition. We begin with the following large matrix

$$M = \begin{pmatrix} \pi & e & 0 & 1 & 99 & 2017 & 9901 & i & \gamma \\ e & \pi & 0 & 1 & 99 & 2017 & 9901 & i & \gamma \\ e & 0 & \pi & 1 & 99 & 2017 & 9901 & i & \gamma \\ e & 0 & 1 & \pi & 99 & 2017 & 9901 & i & \gamma \\ e & 0 & 1 & 99 & \pi & 2017 & 9901 & i & \gamma \\ e & 0 & 1 & 99 & 2017 & \pi & 9901 & i & \gamma \\ e & 0 & 1 & 99 & 2017 & 9901 & \pi & i & \gamma \\ e & 0 & 1 & 99 & 2017 & 9901 & i & \pi & \gamma \\ e & 0 & 1 & 99 & 2017 & 9901 & i & \gamma & \pi \end{pmatrix},$$

where γ denotes Euler constant.

This matrix can be partitioned into smaller submatrices, as follows

$$M_{1,1} = \begin{pmatrix} \pi & e & 0 & 1 \\ e & \pi & 0 & 1 \\ e & 0 & \pi & 1 \\ e & 0 & 1 & \pi \end{pmatrix}, \quad M_{1,2} = \begin{pmatrix} 99 & 2017 & 9901 \\ 99 & 2017 & 9901 \\ 99 & 2017 & 9901 \\ 99 & 2017 & 9901 \end{pmatrix}, \quad M_{1,3} = \begin{pmatrix} i & \gamma \\ i & \gamma \\ i & \gamma \\ i & \gamma \end{pmatrix}.$$

$$M_{2,1} = \begin{pmatrix} e & 0 & 1 & 99 \\ e & 0 & 1 & 99 \\ e & 0 & 1 & 99 \end{pmatrix}, \quad M_{2,2} = \begin{pmatrix} \pi & 2017 & 9901 \\ 2017 & \pi & 9901 \\ 2017 & 9901 & \pi \end{pmatrix}, \quad M_{2,3} = \begin{pmatrix} i & \gamma \\ i & \gamma \\ i & \gamma \end{pmatrix}.$$

$$M_{3,1} = \begin{pmatrix} e & 0 & 1 & 99 \\ e & 0 & 1 & 99 \end{pmatrix}, \quad M_{3,2} = \begin{pmatrix} 2017 & 9901 & i \\ 2017 & 9901 & i \end{pmatrix}, \quad M_{3,3} = \begin{pmatrix} \pi & \gamma \\ \gamma & \pi \end{pmatrix}.$$

According to this decomposition, the matrix M can be expressed in the following form

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{pmatrix}.$$

It is noteworthy that the matrix M can be decomposed into various block structures, and the maximum number of submatrices is equal to the total number in this matrix.

Here is another decomposition of the matrix M.

$$M_{1,1} = \begin{pmatrix} \pi & e & 0 \\ e & \pi & 0 \\ e & 0 & \pi \end{pmatrix} \quad , \quad M_{1,2} = \begin{pmatrix} 1 & 99 & 2017 \\ 1 & 99 & 2017 \\ 1 & 99 & 2017 \end{pmatrix} \quad , \quad M_{1,3} = \begin{pmatrix} 9901 & i & \gamma \\ 9901 & i & \gamma \\ 9901 & i & \gamma \end{pmatrix}.$$

$$M_{2,1} = \begin{pmatrix} e & 0 & 1 \\ e & 0 & 1 \\ e & 0 & 1 \end{pmatrix} \quad , \quad M_{2,2} = \begin{pmatrix} \pi & 99 & 2017 \\ 99 & \pi & 2017 \\ 99 & 2017 & \pi \end{pmatrix}.$$

$$M_{3,2} = \begin{pmatrix} 9901 & i & \gamma \\ 9901 & i & \gamma \\ 9901 & i & \gamma \end{pmatrix} \quad , \quad M_{3,3} = \begin{pmatrix} \pi & i & \gamma \\ i & \pi & \gamma \\ i & \gamma & \pi \end{pmatrix}.$$

In that case, M can be written as

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{1,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{pmatrix}.$$

The definition of block matrices shows that decompositions should be regular, i.e Matrices $M_{i,1}$ for $i = 1, \cdots, n$ should have the same number of columns. Same for $M_{i,2}, \cdots, M_{i,n}$. Matrices $M_{1,j}$ should have the same number of lines. Same for $M_{2,j}, \cdots, M_{n,j}$. The matrix in the following figure is not well decomposed.

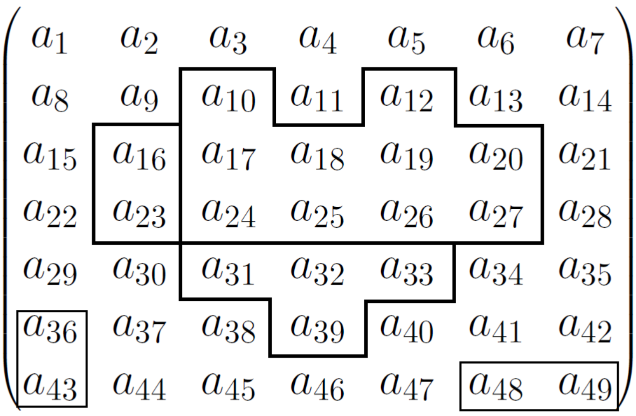

Figure 1: Wrong decomposition

2.1.1 Addition of two block matrices

In general, to make the sum of two matrices M and N, we know that they must have the same dimension. For blocks we must give identical partitions for M and N as above. We have the following proposition.

Proposition 2.1.2 *Let $M = (M_{i,j})_{i,j}$ and $N = (N_{i,j})_{i,j}$ be two block matrices. Then we have*

$$\begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & M_{n,2} & \cdots & M_{n,m} \end{pmatrix} + \begin{pmatrix} N_{1,1} & N_{1,2} & \cdots & N_{1,m} \\ N_{2,1} & N_{2,2} & \cdots & N_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ N_{n,1} & N_{n,2} & \cdots & N_{n,m} \end{pmatrix}$$
$$= \begin{pmatrix} M_{1,1}+N_{1,1} & M_{1,2}+N_{1,2} & \cdots & M_{1,m}+N_{1,m} \\ M_{2,1}+N_{2,1} & M_{2,2}+N_{2,2} & \cdots & M_{2,m}+N_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1}+N_{n,1} & M_{n,2}+N_{n,2} & \cdots & M_{n,m}+N_{n,m} \end{pmatrix}.$$

Exemple 2.1.3 Let

$$M = \begin{pmatrix} 2^{4095} & 2016 & 99 & 13 \\ 309 & 2017 & 101 & 17 \\ 6 & 1 & -1 & 0 \end{pmatrix} \quad , \quad N = \begin{pmatrix} 2^{4095} & 1 & 1 & 17 \\ -52 & -1 & -1 & 13 \\ 9901 & e^{i\pi} & 1 & 0 \end{pmatrix}.$$

The matrices M and N can be partitioned respectively as

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \quad , \quad N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix},$$

where

$$M_1 = \begin{pmatrix} 2^{4095} & 2016 & 99 \\ 309 & 2017 & 101 \end{pmatrix} \quad , \quad M_2 = \begin{pmatrix} 13 \\ 17 \end{pmatrix} \quad , \quad M_3 = \begin{pmatrix} 6 & 1 & -1 \end{pmatrix} \quad , \quad M_4 = \begin{pmatrix} 0 \end{pmatrix},$$

and

$$N_1 = \begin{pmatrix} 2^{4095} & 1 & 1 \\ -52 & -1 & -1 \end{pmatrix} \quad , \quad N_2 = \begin{pmatrix} 17 \\ 13 \end{pmatrix} \quad , \quad N_3 = \begin{pmatrix} 9901 & e^{i\pi} & 1 \end{pmatrix} \quad , \quad N_4 = \begin{pmatrix} 0 \end{pmatrix}.$$

The sum of M and N is given by

$$M + N = \begin{pmatrix} 2^{4096} & 2017 & 100 & 30 \\ 257 & 2016 & 100 & 30 \\ 9907 & 0 & 0 & 0 \end{pmatrix}.$$

We remark the following equalities

$$\begin{pmatrix} 2^{4095} & 2016 & 99 \\ 309 & 2017 & 101 \end{pmatrix} + \begin{pmatrix} 2^{4095} & 1 & 1 \\ -52 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2^{4096} & 2017 & 100 \\ 257 & 2016 & 100 \end{pmatrix}.$$

$$\begin{pmatrix} 6 & 1 & -1 \end{pmatrix} + \begin{pmatrix} 9901 & e^{i\pi} & 1 \end{pmatrix} = \begin{pmatrix} 9907 & 0 & 0 \end{pmatrix}.$$

$$\begin{pmatrix} 13 \\ 17 \end{pmatrix} + \begin{pmatrix} 17 \\ 13 \end{pmatrix} = \begin{pmatrix} 30 \\ 30 \end{pmatrix}.$$

$$\begin{pmatrix} 0 \end{pmatrix} + \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}.$$

Then $M + N$ can be written as

$$M + N = \begin{pmatrix} M_1 + N_1 & M_2 + N_2 \\ M_3 + N_3 & M_4 + N_4 \end{pmatrix}.$$

We remark blocks in every position of $M + N$ represents the sum of the corresponding block from $M + N$.

2.1.2 Multiplication of two block matrices

It is well known that the product of a matrix M with a matrix N, the length of M should equal to the width of N. For blocks, the partition of the length of M should be identical to the partition of the width of N, witch mean that for every i, j, k, the length of $M_{i,j}$ should be equal to the length of $N_{j,k}$.

Proposition 2.1.4 *Let $M = (M_{i,j})_{i,j}$ and $N = (N_{j,k})_{j,k}$ be two block matrices. Then we have*

$$MN = \left(\sum_{i=1}^{m} M_{i,j} N_{j,k} \right)_{i,j} .$$

Exemple 2.1.5 Let

$$M = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 11 & 13 & 17 & 19 \\ 23 & 29 & 31 & 37 \end{pmatrix} \quad , \quad N = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

The matrices M and N can be partitioned respectively as

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \quad , \quad N = \begin{pmatrix} N_1 & N_2 \\ N_3 & N_4 \end{pmatrix},$$

where

$$M_1 = \begin{pmatrix} 2 & 3 & 5 \\ 11 & 13 & 17 \end{pmatrix} \quad , \quad M_2 = \begin{pmatrix} 17 \\ 19 \end{pmatrix} \quad , \quad M_3 = \begin{pmatrix} 23 & 29 & 31 \end{pmatrix} \quad , \quad M_4 = \begin{pmatrix} 37 \end{pmatrix},$$

and

$$N_1 = \begin{pmatrix} 2 & 3 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} \quad , \quad N_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad , \quad N_3 = \begin{pmatrix} 0 & -1 \end{pmatrix} \quad , \quad N_4 = \begin{pmatrix} 0 \end{pmatrix}.$$

The product of M by N is given by

$$MN = \begin{pmatrix} -3 & -4 & 3 \\ -6 & -6 & 6 \\ -8 & -8 & 8 \end{pmatrix}.$$

We remark the following equalities

$$\begin{pmatrix} 2 & 3 & 5 \\ 11 & 13 & 17 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 7 \\ 9 \end{pmatrix} \begin{pmatrix} 0 & -1 \end{pmatrix} = \begin{pmatrix} -3 & -4 \\ -6 & -6 \end{pmatrix}.$$

$$\begin{pmatrix} 2 & 3 & 5 \\ 11 & 13 & 17 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 7 \\ 9 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}.$$

$$\begin{pmatrix} 23 & 29 & 31 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 37 \end{pmatrix} \begin{pmatrix} 0 & -1 \end{pmatrix} = \begin{pmatrix} -8 & -8 \end{pmatrix}.$$

$$\begin{pmatrix} 23 & 29 & 31 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 37 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 8 \end{pmatrix}.$$

Then MN can be written as

$$MN = \begin{pmatrix} M_1N_1 + M_2N_3 & M_1N_2 + M_2N_4 \\ M_3N_1 + M_4N_3 & M_3N_2 + M_4N_4 \end{pmatrix}.$$

Exemple 2.1.6 Here is an example of two partitions of the same matrix M that make it impossible to calculate the block product of M with itself

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \end{array}\right) \quad , \quad \left(\begin{array}{c|cc} 1 & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}\right).$$

2.1.3 Square block matrices

Definition 2.1.7 *A square block matrix is defined as a square matrix where the partitions of its dimensions are identical, in other words, each submatrix on the diagonal should be a square matrix.*

The matrix M defined in Example (2.1) with the first given partition is a good example of a square block matrix.

To make the product of a matrix M by itself, it is necessary and sufficient for M to be a square block matrix, for a such matrix, we have

$$M^2 = \left(\sum_{k=1}^{n} M_{i,j} M_{j,k} \right)_{i,j}.$$

Exemple 2.1.8 Let

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The matrix M can be partitioned as

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad M_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad M_3 = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad , \quad M_4 = \begin{pmatrix} 1 \end{pmatrix}.$$

The product of the matrix M by itself is given by

$$M^2 = \begin{pmatrix} 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}.$$

We remark the following equalities

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \begin{pmatrix}0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}2 & 0\end{pmatrix}.$$

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} + \begin{pmatrix}0\end{pmatrix}\begin{pmatrix}0\end{pmatrix} = \begin{pmatrix}2\end{pmatrix}.$$

Then M^2 can be written as

$$M^2 = \begin{pmatrix} M_1^2 + M_2M_3 & M_1M_2 + M_2M_4 \\ M_3M_1 + M_4M_3 & M_3M_2 + M_4^2 \end{pmatrix}.$$

2.1.4 Triangular and diagonal block matrices

In the following definition, M is not required to be square block matrix according to Definition (2.1.7), but it should only have the same number of blocks in length as in width.

Definition 2.1.9 *Let M be $n \times n$ block matrix. Then M is called*

- *An upper triangular block matrix if $M_{i,j} = 0$ for all $i > j$.*

$$\begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & \cdots & M_{1,n} \\ 0 & M_{2,2} & & & M_{2,n} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & M_{n,n} \end{pmatrix}.$$

- *A lower triangular block matrix if $M_{i,j} = 0$ for all $i < j$.*

$$\begin{pmatrix} M_{1,1} & 0 & \cdots & \cdots & 0 \\ M_{2,1} & M_{2,2} & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ M_{n,1} & \cdots & \cdots & M_{n,n-1} & M_{n,n} \end{pmatrix}.$$

- *A diagonal block matrix if all off diagonal matrices are zero matrices.*

$$\begin{pmatrix} M_{1,1} & 0 & \cdots & \cdots & 0 \\ 0 & M_{2,2} & & & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & M_{n,n} \end{pmatrix}.$$

Remark 2.1 A triangular block matrix is either an upper triangular block matrix or a lower triangular block matrix.

Exemple 2.1.10 1) Considering a partition of a matrix M into matrices $(M_{i,j})_{i,j}$ where each of them is filled with the number $\max(j-i+1,0)$, we obtain the following matrix

$$M=\left(\begin{array}{ccc|cc|ccc} 1&1&1&2&2&3&3&3\\ 1&1&1&2&2&3&3&3\\ 1&1&1&2&2&3&3&3\\ \hline 0&0&0&1&1&2&2&2\\ 0&0&0&1&1&2&2&2\\ 0&0&0&1&1&2&2&2\\ \hline 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1\\ 0&0&0&0&0&1&1&1 \end{array}\right).$$

This matrix can be partitioned into small matrices as follows,

$$M_{1,1}=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \quad , \quad M_{1,2}=\begin{pmatrix}2&2\\2&2\\2&2\end{pmatrix} \quad , \quad M_{1,3}=\begin{pmatrix}3&3&3\\3&3&3\\3&3&3\end{pmatrix}.$$

$$M_{2,1}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \quad , \quad M_{2,2}=\begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix} \quad , \quad M_{2,3}=\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix}.$$

$$M_{3,1}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} \quad , \quad M_{3,2}=\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} \quad , \quad M_{3,3}=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}.$$

Then we get

$$M=\begin{pmatrix}M_{1,1}&M_{1,2}&M_{1,3}\\M_{2,1}&M_{2,2}&M_{2,3}\\M_{3,1}&M_{3,2}&M_{3,3}\end{pmatrix}.$$

2) In a similar way, define every $M_{i,j}$ as the matrix filled by the number $\max(i-j+1,0)$, we get the lower triangular block matrix

$$M=\left(\begin{array}{ccc|cc|ccc} 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ 1&1&1&0&0&0&0&0\\ \hline 2&2&2&1&1&0&0&0\\ 2&2&2&1&1&0&0&0\\ 2&2&2&1&1&0&0&0\\ \hline 3&3&3&2&2&1&1&1\\ 3&3&3&2&2&1&1&1\\ 3&3&3&2&2&1&1&1 \end{array}\right).$$

This matrix can be partitioned into small matrices as follows,

$$M_{1,1}=\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \quad , \quad M_{1,2}=\begin{pmatrix}0&0\\0&0\\0&0\end{pmatrix} \quad , \quad M_{1,3}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

$$M_{2,1}=\begin{pmatrix}2&2&2\\2&2&2\\2&2&2\end{pmatrix} \quad , \quad M_{2,2}=\begin{pmatrix}1&1\\1&1\\1&1\end{pmatrix} \quad , \quad M_{2,3}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

$$M_{3,1} = \begin{pmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{pmatrix} \quad , \quad M_{3,2} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \\ 2 & 2 \end{pmatrix} \quad , \quad M_{3,3} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then we get

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} \\ M_{2,1} & M_{2,2} & M_{2,3} \\ M_{3,1} & M_{3,2} & M_{3,3} \end{pmatrix}.$$

3) Let

$$M = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 8 & 0 \\ 0 & 0 & 3 & 2 & 7 & 0 \\ 0 & 0 & 0 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}.$$

This matrix can be partitioned into small matrices as follows,

$$M_{1,1} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad , \quad M_{1,2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad M_{1,3} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$M_{2,1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad , \quad M_{2,2} = \begin{pmatrix} 1 & 0 & 8 \\ 3 & 2 & 7 \\ 0 & 4 & 6 \end{pmatrix} \quad , \quad M_{2,3} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$$M_{3,1} = \begin{pmatrix} 0 & 0 \end{pmatrix} \quad , \quad M_{3,2} = \begin{pmatrix} 0 & 0 & 0 \end{pmatrix} \quad , \quad M_{3,3} = \begin{pmatrix} 3 \end{pmatrix}.$$

We see that $M_{1,1}$, $M_{2,2}$ and $M_{3,3}$ are square matrices and all other matrices are null. Consequently, M is a diagonal block matrix.

$$M = \begin{pmatrix} M_{1,1} & 0 & 0 \\ 0 & M_{2,2} & 0 \\ 0 & 0 & M_{3,3} \end{pmatrix}.$$

2.2 Inverses and determinants of 2 by 2 block matrices

This subsection examines the determinants and invertibility of triangular block matrices, providing formulas for determinants and inverses of 2×2 matrices. These results are essential for understanding the broader implications of block matrix structures.

Particular matrices that will be used in this chapter :

$$M_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \quad M_U = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \quad , \quad M_L = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix},$$

where A, B, C, and D are particular matrices from $M_{i,j}$.

In all that follows in this chapter, these decompositions should taken in consideration

$$M_U = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \tag{1}$$

$$M_L = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \tag{2}$$

When D is invertible, then

$$M_2 = \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \tag{3}$$

It is important to note that $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$ are invertible, their inverses are respectively $\begin{pmatrix} I & -B \\ 0 & I \end{pmatrix}$, $\begin{pmatrix} I & 0 \\ -C & I \end{pmatrix}$, their determinants are equals to 1 as it will be shown below.

The last decomposition can be obtained by applying Gaussian reduction to the matrix M_2.

Proposition 2.2.1 *We have the following equalities*

$$\det \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \det(A) \quad , \quad \det \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} = \det(D) \quad , \quad \det \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} = 1 \quad , \quad \det \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} = 1.$$

Proof Let's prove the first statement, we put $A = (a_{i,j})_{i,j}$. Then

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ a_{n,1} & \cdots & a_{1,n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & & \vdots & \ddots & \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix},$$

we calculate the determinant from the bottom, we deduce that $\det \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \det(A)$. The other statements can be proved similarly. □

Proposition 2.2.2 *The following equalities hold.*

$$\det(M_U) = \det(A)\det(D) \quad , \quad \det(M_L) = \det(A)\det(D).$$

Proof

$$\begin{aligned} \det(M_U) &= \det \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \\ &= \det \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \det \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \det \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \\ &= \det(D) \cdot 1 \cdot \det(A) \\ &= \det(D)\det(A). \end{aligned}$$

In the same way, the $\det(M_L)$ can be calculated.

$$\begin{aligned}
\det(M_L) &= \det \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \\
&= \det \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \det \begin{pmatrix} I & 0 \\ C & I \end{pmatrix} \det \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \\
&= \det(A) \cdot 1 \cdot \det(D) \\
&= \det(A) \det(D).
\end{aligned}$$

□

For invertibility, we have the following results.

Proposition 2.2.3 *The following assertions are equivalent.*

- M_U *is invertible.*
- M_L *is invertible.*
- A *and* D *are invertible.*

Proof Since every matrix is invertible if and only if its determinant is different to zero, using the established equalities

$$\det(M_U) = \det(A) \det(D)$$

and

$$\det(M_L) = \det(A) \det(D)$$

provides the result directly. □

For general case, we have the following propositions.

Proposition 2.2.4 *Assume that* D *is invertible. Then* $\det(M_2) = \det(A - BD^{-1}C) \det(D)$.

Proof

$$\begin{aligned}
\det(M_2) &= \det \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \\
&= \det \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix} \det \begin{pmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{pmatrix} \det \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix} \\
&= \det(A - BD^{-1}C) \det(D).
\end{aligned}$$

□

Proposition 2.2.5 *The following assertions hold.*

- *The matrix* M_2 *is invertible if and only if* D *and* $A - BD^{-1}C$ *are invertible.*
- *If* D *and* $A - BD^{-1}C$ *are invertible, then*

$$M_2^{-1} = \begin{pmatrix} M_0^{-1} & -M_0^{-1}BD^{-1} \\ -D^{-1}CM_0^{-1} & D^{-1}CM_0^{-1}BD^{-1} + D^{-1} \end{pmatrix}$$

where $M_0 = A - BD^{-1}C$.

Proof The proof of the first statement is obvious according to Proposition (2.2.4). Next, the second statement can be proved by two methods :

1) The direct method is to propose a matrix which its product with M_2 gives the identity matrix.

2) A more illustrative method consists of using the decomposition of M_2 as follows

$$\begin{aligned} M_2^{-1} &= \begin{pmatrix} I & 0 \\ D^{-1}C & I \end{pmatrix}^{-1} \begin{pmatrix} M_0 & 0 \\ 0 & D \end{pmatrix}^{-1} \begin{pmatrix} I & BD^{-1} \\ 0 & I \end{pmatrix}^{-1} \\ &= \begin{pmatrix} I & 0 \\ -D^{-1}C & I \end{pmatrix} \begin{pmatrix} M_0^{-1} & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & -BD^{-1} \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} M_0^{-1} & -M_0^{-1}BD^{-1} \\ -D^{-1}CM_0^{-1} & D^{-1}CM_0^{-1}BD^{-1} + D^{-1} \end{pmatrix}. \end{aligned}$$

□

Remark 2.2 If A and $CA^{-1}B-D$ are invertible, then by using the following decomposition

$$M_2 = \begin{pmatrix} I & 0 \\ CA^{-1} & I \end{pmatrix} \begin{pmatrix} I & BM_1^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & M_1 \end{pmatrix},$$

we deduce that

$$M_2^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BM_1^{-1}CA^{-1} & -A^{-1}BM_1^{-1} \\ -M_1^{-1}CA^{-1} & M_1^{-1} \end{pmatrix},$$

where $M_1 = CA^{-1}B - D$.

In the following propositions, an extension of established results to an arbitrary dimension matrix will be presented.

Proposition 2.2.6 *Let M be a triangular block matrix. Then*

$$\det(M) = \prod_{i=1}^{n} \det(M_{i,i}).$$

Proof First, we treat the case of an upper triangular matrix. Let's consider the following decomposition

$$\left(\begin{array}{c|ccc} M_{1,1} & M_{1,2} & \cdots & M_{1,n} \\ \hline & M_{2,2} & \cdots & M_{2,n} \\ 0 & & \ddots & \\ & 0 & \cdots & M_{n,n} \end{array}\right) = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Assume that $\det(D) = \prod_{i=2}^{n} \det(M_{i,i})$, then by using Proposition (2.2.2), we get

$$\begin{aligned} \det(M) &= \det(A)\det(D) \\ &= \det(M_{1,1}) \prod_{i=2}^{n} \det(M_{i,i}) \\ &= \prod_{i=1}^{n} \det(M_{i,i}). \end{aligned}$$

Hence, by induction, we conclude the validity of the proposition for an arbitrary dimension matrix M. The proof of a lower triangular matrix is perfectly similar to the previous one.
□

For $n \times n$ block matrix, we have the following statements.

Proposition 2.2.7 *Let M be $n \times n$ block matrix. Then*

$$\det(M) = \prod_{k=1}^{n} \det(\alpha_{k,k}^{(n-k)}),$$

where

$$\alpha_{i,j}^{(0)} = M_{i,j} \quad , \quad \alpha_{i,j}^{(k)} = M_{i,j} - \sigma_{i,n-k+1}\widetilde{M_k}^{-1} m_{n-k+1,j} \text{ for } k \geqslant 1,$$

$$\sigma_{i,j} = (M_{i,j}, \cdots M_{i,n}),$$

$$m_{i,j} = \begin{pmatrix} M_{i,j} \\ \vdots \\ M_{n,j} \end{pmatrix},$$

$$\widetilde{M_k} = \begin{pmatrix} M_{n-k+1,n-k+1} & M_{n-k+1,n-k+2} & \cdots & M_{n-k+1,n} \\ M_{n-k+2,n-k+1} & M_{n-k+2,n-k+2} & \cdots & M_{n-k+2,n} \\ \vdots & \ddots & & \\ M_{n,n-k+1} & M_{n,n-k+2} & \cdots & M_{n,n} \end{pmatrix}.$$

Proof This proposition can be proved by inductive reasoning as follows, Assume that it is correct for any $(n-1) \times (n-1)$ block matrix, then let's consider the following decomposition

$$\left(\begin{array}{c|ccc} M_{1,1} & M_{1,2} & \cdots & M_{1,n} \\ \hline M_{1,2} & M_{2,2} & \cdots & M_{2,n} \\ \vdots & & \ddots & \\ M_{n,1} & M_{n,2} & \cdots & M_{n,n} \end{array}\right) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

According to Proposition (2.2.4), $\det(M) = \det(A - BD^{-1}C)\det(D)$. On the other hand, the following equalities hold

$$\begin{aligned} \alpha_{1,1}^{(n-1)} &= M_{1,1} - \sigma_{1,2}\widetilde{M_{n-1}}^{-1} m_{2,1} \\ &= M_{1,1} - (M_{1,2}, \cdots, M_{1,n}) \begin{pmatrix} M_{2,2} & \cdots & M_{2,n} \\ & \ddots & \\ M_{n,2} & \cdots & M_{n,n} \end{pmatrix}^{-1} \begin{pmatrix} M_{2,1} \\ \vdots \\ M_{n,1} \end{pmatrix} \\ &= A - BD^{-1}C. \end{aligned}$$

By hypothesis, we have

$$\det(D) = \prod_{k=2}^{n} \det(\alpha_{k,k}^{(n-k)}).$$

Therefore

$$\begin{aligned} \det(M) &= \det(A - BD^{-1}C)\det(D) \\ &= \det(\alpha_{1,1}^{(n-1)}) \cdot \prod_{k=2}^{n} \det(\alpha_{k,k}^{(n-k)}) \\ &= \prod_{k=1}^{n} \det(\alpha_{k,k}^{(n-k)}). \end{aligned}$$

□

Proposition 2.2.8 *Let M be a triangular block matrix. Then M is invertible if and only if the matrix $M_{i,i}$ is invertible for all i.*

Proof The previous result shows that $\det(M) \neq 0$ if and only if $\det(M_{i,i}) \neq 0$ for all i. Which ends the proof. □

2.3 Block operators

In this section, block matrices will be generalised to block operator matrices concept. Important results are introduced on compactness, Fredholmness and Fredholm perturbations of block operator.

Definition 2.3.1 *Let $X_1, \cdots, X_m, Y_1, \cdots, Y_n$ be Banach spaces. A block operator matrix M is defined a*

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & M_{1,m} \\ M_{2,1} & M_{2,2} & \cdots & M_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ M_{n,1} & M_{n,2} & \cdots & M_{n,m} \end{pmatrix},$$

where each $M_{i,j} : X_j \to Y_i$ is a continuous linear operator.

Lemma 2.3.2 *The following isomorphismes hold.*

- $\ker \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \simeq \ker(A).$
- $R \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \simeq R(A) \times Y.$

Proof By definition of the kernel, we have

$$\begin{aligned} \ker \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} &= \{(V, W) \in X \times Y \ / \ AV = 0 \text{ and } W = 0\} \\ &= \ker(A) \times \{0_Y\}. \end{aligned}$$

For the range, we have

$$\begin{aligned} R \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} &= \left\{ \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix} \ / \ (V, W) \in X \times Y \right\} \\ &= \left\{ \begin{pmatrix} AV \\ W \end{pmatrix} \ / \ (V, W) \in X \times Y \right\} \\ &= R(A) \times Y. \end{aligned}$$

□

2.3.1 Compactness of block operator matrices

Theorem 2.3.3 *If M_2 is a compact block operator matrix, then A and D are compact operators.*

Proof Let $(V_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$, since M is compact, then $\left(M_2 \cdot \begin{pmatrix} V_n \\ 0 \end{pmatrix} \right)_{n\in\mathbb{N}}$ admits a convergent subsequence, in other words, there exists a convergent subsequence $\left(AV_{\phi(n)}, CV_{\phi(n)} \right)_{n\in\mathbb{N}}$, so $\left(AV_{\phi(n)} \right)_{n\in\mathbb{N}}$ is a convergent subsequence of $(AV_n)_{n\in\mathbb{N}}$. Hence A is compact. Next, the compactness of D can be proved in the same way. □

2.3.2 Fredholmness of block operator matrices

Proposition 2.3.4 *The following equivalences holds*

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in \Phi_*(X \times Y) \text{ if and only if } A \in \Phi_*(X),$$

where $* = +, -, l, r$.

Proof The two isomorphismes concluded in Lemma (2.3.2) leads directly to the proposition for $* = +, -$.
For left Fredholm operator, let assume that $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in \Phi_l(X \times Y)$. Then by definition, there exists $\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \in \mathcal{L}(X \times Y)$ such that

$$\begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

is compact. Consequently, we get $\begin{pmatrix} A_1A - I & A_2 \\ A_3A & A_4 - I \end{pmatrix}$ is compact, therefore, according to Theorem (2.3.3), $A_1A - I$ is compact. Hence $A \in \Phi_l(X)$. Inversely, if $A \in \Phi_l(X)$, there exists $A_0 \in \mathcal{L}(X)$ such that $A_0A - I$ is compact. By using the following equality

$$\begin{pmatrix} A_0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} AA_0 - I & 0 \\ 0 & 0 \end{pmatrix},$$

we deduce that

$$\begin{pmatrix} A_0 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$$

is compact. Which means that $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in \Phi_l(X \times Y)$. Consequently, the equivalence is established.

The proof for Φ_r can be done similarly.

□

Remark 2.3 The above result still available for $\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$, and the proof is similar.

Proposition 2.3.5 *Let M be a triangular block operator matrix. Then $M \in \Phi_*(X^n)$ if $M_{i,i} \in \Phi_*(X)$ for all i, where $* = +, -, l, r$.*

Proof Let's prove this proposition for 2×2 upper triangular matrix M_U, which is noted by M_U above. We assume that A and D are in $\Phi_*(X)$, using Proposition (2.3.4), we deduce that $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ and $\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$ are in $\Phi_*(X \times Y)$. Now according to the decomposition of M_U given in equation (1), we conclude that $M_U \in \Phi_*(X \times Y)$. For an arbitrary dimension matrix, we consider the following decomposition

$$\left(\begin{array}{c|ccc} M_{1,1} & M_{1,2} & \cdots & M_{1,n} \\ \hline 0 & M_{2,2} & \cdots & M_{2,n} \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & M_{n,n} \end{array}\right) = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Suppose that the matrix $\begin{pmatrix} M_{2,2} & \cdots & M_{2,n} \\ & \ddots & \\ 0 & \cdots & M_{n,n} \end{pmatrix} \in \Phi_*(X^{n-1})$, by hypothesis we have $M_{i,i} \in \Phi_*(X)$ for all i, then D and A are in Φ_*. Next, according to the result already proved on M_U we conclude that $\left(\begin{array}{c|ccc} M_{1,1} & M_{1,2} & \cdots & M_{1,n} \\ \hline 0 & M_{2,2} & \cdots & M_{2,n} \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & M_{n,n} \end{array}\right)$ is in $\Phi_*(X^n)$. Consequently, it is possible by induction extend the result obtained form any upper triangular matrix of dimension n to an upper triangular matrix of dimension $n+1$. Hence the result is established for any arbitrary dimension upper triangular matrix.

For a lower triangular block operator matrix the process of the proof will be the same. The elements appearing in the decomposition of M_L are the matrices

$$\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \quad \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix},$$

and the invertible matrix

$$\begin{pmatrix} I & 0 \\ C & I \end{pmatrix},$$

so $M_L \in \Phi_*(X \times Y)$ whenever $A \in \Phi_*(X)$ and $D \in \Phi_*(Y)$. Next, we use the same reasoning by induction as before with the following decomposition

$$\left(\begin{array}{ccc|c} M_{1,1} & \cdots & 0 & 0 \\ & \ddots & & \vdots \\ M_{1,n-1} & \cdots & M_{n-1,n-1} & 0 \\ \hline M_{n,1} & \cdots & M_{n,n-1} & M_{n,n} \end{array}\right) = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}.$$

This ends the proof.

□

Remark 2.4 It is immediate to conclude that if $M_{i,i} \in \Phi(X)$ for all i, then $M \in \Phi(X^n)$.

Based on that, we have the following result.

Proposition 2.3.6 *Let M be a triangular block operator matrix such that $M_{i,i} \in \Phi_+(X)$ for all i. Then*

$$\alpha(M) \leqslant \sum_{i=1}^{n} \alpha(M_{i,i}).$$

Proof Let's prove this proposition for 2×2 upper triangular matrix M_U. We have

$$\ker(M_U) = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}^{-1} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}^{-1} \left(\ker \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}\right).$$

Let

$$V = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix}^{-1} \left(\ker \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}\right).$$

In this case, $\ker(M_U)$ can be expressed as

$$\ker(M_U) = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}^{-1} (V).$$

Since $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$ is bijective and $\ker \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}$ is isomorphic to $\ker(D)$, then $\dim(V) = \dim(\ker(D))$. By using Lemmas (1.1.17) and (2.3.2), we get

$$\alpha(M_U) \leqslant \alpha(A) + \alpha(D).$$

Consequently, the result can be extended to an arbitrary dimensional upper triangular matrix by inductive reasoning.

If M is a lower triangular matrix, then the proposition can be proved similarly.

□

Proposition 2.3.7 *Let M be a triangular block operator matrix such that $M_{i,i} \in \Phi_-(X)$ for all i. Then*

$$\beta(M) \leqslant \sum_{i=1}^{n} \beta(M_{i,i}).$$

Proof Let's prove this proposition for 2×2 upper triangular matrix M_U. We have

$$R(M_U) = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \left(R \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \right).$$

Let

$$W = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \left(R \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \right).$$

In that case $R(M_U)$ can be expressed as

$$R(M_U) = \begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} (W).$$

Since $\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$ is bijective and $R \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}$ is isomorphic to $R(A)$, then $\operatorname{codim}(W) = \operatorname{codim}(R(A))$. By using Lemmas (1.1.18) and (2.3.2), we get

$$\beta(M_U) \leqslant \beta(A) + \beta(D).$$

Consequently, the result can be extended to an arbitrary dimensional upper triangular matrix by inductive reasoning.

If M is a lower triangular matrix, then the proposition can be proved similarly.

□

Proposition 2.3.8 *Let M be a diagonal block operator matrix. Then*

1.
$$\alpha(M) = \sum_{i=1}^{n} \alpha(M_{i,i}).$$

2.
$$\beta(M) = \sum_{i=1}^{n} \beta(M_{i,i}).$$

3.
$$\operatorname{ind}(M) = \sum_{i=1}^{n} \operatorname{ind}(M_{i,i}),$$

whenever $M_{i,i} \in \Phi(X)$ for all i.

4. *M is an upper semi Fredholm operator if and only if $M_{i,i}$ is an upper semi Fredholm operator for all i.*

5. *M is a lower semi Fredholm operator if and only if $M_{i,i}$ is a lower semi Fredholm operator for all i.*

6. *M is a Fredholm operator if and only if $M_{i,i}$ is a Fredholm operator for all i.*

7. *M is invertible if and only if $M_{i,i}$ is invertible for all i.*

Proof
1) We have
$$\ker(M) = \{V \in X^n \ / \ MV = 0_{X^n}\}.$$
By decomposing M according to its blocks we get
$$\ker(M) = \left\{ \begin{pmatrix} V_1 \\ \vdots \\ V_n \end{pmatrix} \ / \ \begin{pmatrix} M_1 V_1 \\ \vdots \\ M_n V_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$
Therefore, a vector V is in the kernel of M if and only if $V_i \in \ker(M_{i,i})$ for all i. Then $\ker(M) = \prod_{i=1}^{n} \ker(M_{i,i})$. Hence
$$\alpha(M) = \sum_{i=1}^{n} \alpha(M_{i,i}).$$
2) In the same reasoning as before, we get $R(M) = \prod_{i=1}^{n} R(M_{i,i})$. Then
$$\beta(M) = \sum_{i=1}^{n} \beta(M_{i,i}).$$
3) It is obvious from assertion 1 and assertion 2.

4) It is well known that $\alpha(M_{i,i})$ are all finite if and only if their sum is finite. Then according to assertion 1, $\alpha(M)$ is finite if and only if $\alpha(M_{i,i})$ is finite for all i. Furthermore, $R(M)$ is closed if and only if $R(M_{i,i})$ is closed according to Lemma (1.1.2).

5) Can be checked in the same manner as 4.

6) Follows immediately from assertion 4 and assertion 5.

7) If $M_{i,i}$ is invertible for all i, then the construction of the inverse of M is obvious. Inversely, we treat the case of 2×2 block matrix. Assume that $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ is invertible, and let $\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}$ be its inverse. Then we have
$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \ , \ \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$
Hence
$$AB_1 = I \ , \ B_1 A = I \ , \ DB_4 = I \ , \ B_4 D = I.$$
Therefore, A and D are invertible. We prove the general result by induction.

□

Lemma 2.3.9 *Let M be a triangular block operator matrix such that $M_{i,i} \in \Phi(X)$ for all i. Then*

$$\operatorname{ind}(M) = \sum_{i=1}^{n} \operatorname{ind}(M_{i,i}).$$

Proof Using the same reasoning as in Proposition (2.2.6), let's deal with the matrix M_U, the isomorphismes established in Lemma (2.3.2) shows that

$$\operatorname{ind}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} = \operatorname{ind}(A)$$

and

$$\operatorname{ind}\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix} = \operatorname{ind}(D).$$

Next, according to the decomposition of M_U given in equation (1), we deduce that $\operatorname{ind}(M_U) = \operatorname{ind}(A) + \operatorname{ind}(D)$. Further, by using an inductive reasoning, we conclude that

$$\operatorname{ind}(M) = \sum_{i=1}^{n} \operatorname{ind}(M_{i,i}).$$

□

Proposition 2.3.10 *Let M be an upper triangular block operator matrix. Then we have the following assertions.*

1. *If $M \in \Phi_+(X^n)$, then $M_{1,1} \in \Phi_+(X)$.*
2. *If $M \in \Phi_l(X^n)$, then $M_{1,1} \in \Phi_l(X)$.*
3. *If $M \in \Phi_-(X^n)$, then $M_{n,n} \in \Phi_-(X)$.*
4. *If $M \in \Phi_r(X^n)$, then $M_{n,n} \in \Phi_r(X)$.*

Proof For a left Fredholm or an upper semi Fredholm operator, we present the matrix M as follows

$$\left(\begin{array}{c|ccc} M_{1,1} & M_{1,2} & \cdots & M_{1,n} \\ \hline 0 & M_{2,2} & \cdots & M_{2,n} \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & M_{n,n} \end{array}\right) = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

For $* = +, l$, if $M \in \Phi_*(X^n)$, then according to the decomposition of M_U given in equation (1), we get

$$\begin{pmatrix} I & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in \Phi_*(X^n).$$

This means that $\begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix} \in \Phi_*(X^n)$ by using Proposition (1.3.9). So according to Proposition (2.3.4), we deduce that $A \in \Phi_*(X^n)$, therefore $M_{1,1} \in \Phi_*(X)$.

For a right Fredholm or a lower semi Fredholm operators, we use the following decomposition

$$\left(\begin{array}{ccc|c} M_{1,1} & \cdots & M_{1,n-1} & M_{n,n} \\ \vdots & \ddots & & \vdots \\ 0 & \cdots & M_{n-1,n-1} & M_{n,n-1} \\ \hline 0 & \cdots & 0 & M_{n,n} \end{array}\right) = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$

Then due to the same reasoning as before, we deduce that $M_{n,n} \in \Phi_*(X)$.

□

For a lower triangular matrix, we have the following proposition.

Proposition 2.3.11 *Let M be a lower triangular block operator matrix. Then*

1. *If $M \in \Phi_-(X^n)$, then $M_{1,1} \in \Phi_-(X)$.*
2. *If $M \in \Phi_r(X^n)$, then $M_{1,1} \in \Phi_r(X)$.*
3. *If $M \in \Phi_+(X^n)$, then $M_{n,n} \in \Phi_+(X)$.*
4. *If $M \in \Phi_l(X^n)$, then $M_{n,n} \in \Phi_l(X)$.*

Proof For lower semi Fredholm and right Fredholm operators, we consider the following decomposition

$$\left(\begin{array}{c|ccc} M_{1,1} & 0 & \cdots & 0 \\ \hline M_{1,2} & M_{2,2} & \cdots & 0 \\ \vdots & & \ddots & \\ M_{1,2} & A_{n,2} & \cdots & M_{n,n} \end{array}\right) = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}.$$

For upper semi Fredholm and left Fredholm operators, this decomposition

$$\left(\begin{array}{ccc|c} M_{1,1} & \cdots & 0 & 0 \\ \vdots & \ddots & & \vdots \\ M_{n-1,1} & \cdots & M_{n-1,n-1} & 0 \\ \hline M_{n,1} & \cdots & M_{n,n-1} & M_{n,n} \end{array}\right) = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

will be used. In the same reasoning as in the proof of Proposition (2.3.10) on the matrix M_L, we get all the desired results. □

The following results are given in [N. 01].

Theorem 2.3.12 *Assume that B is compact on X. Then M_2 is Fredholm in X^2 if and only if*

1. $\beta(A) < \infty$.
2. $\alpha(D) < \infty$.
3. $\gamma_1 := \dim(\ker(A) \cap C^{-1}(R(D))) < \infty$
4. $\gamma_2 := \operatorname{codim}(R(D) + C(\ker(A))) < \infty$

Under conditions 1 - 4,

$$\operatorname{ind}(M_2) = \alpha(D) - \beta(A) + \gamma_1 - \gamma_2.$$

If, furthermore, $B = 0$, then

$$\alpha(M_2) = \alpha(D) + \gamma_1 \quad \text{and} \quad \beta(M_2) = \beta(A) + \gamma_2.$$

Corollary 2.3.13 *Assume that B or C is compact and A (or D) be Fredholm in X. Then M_2 is Fredholm if and only if D (or A, reps.) is Fredholm in X, and then*

$$\operatorname{ind}(M_2) = \operatorname{ind}(A) + \operatorname{ind}(D).$$

Corollary 2.3.14 *Assume that $B = 0$ and A (or D) be Fredholm in X. Then M_2 is Fredholm if and only if D (or A, reps.) is Fredholm in X, and then*

$$\alpha(M_2) = \alpha(A) + \alpha(D) - m \quad \text{and} \quad \beta(M_2) = \beta(A) + \beta(D) - m,$$

where $m = \dim(C(\ker(A))/(R(D) \cap C(\ker(A))))$.

2.3.3 Fredholm perturbation of block operators

Proposition 2.3.15 *Let $F \in \mathcal{L}(X^n)$. Then*

1. *$F \in \mathcal{F}_l(X^n)$ if and only if $F_{i,j} \in \mathcal{F}_l(X)$ for all i, j.*
2. *$F \in \mathcal{F}_r(X^n)$ if and only if $F_{i,j} \in \mathcal{F}_r(X)$ for all i, j.*
3. *$F \in \mathcal{F}(X^n)$ if and only if $F_{i,j} \in \mathcal{F}(X)$ for all i, j.*

Proof We treat the case of 2×2 block matrix.
1) Let $F = \begin{pmatrix} F_1 & F_2 \\ F_3 & F_4 \end{pmatrix}$ be a left Fredholm perturbation and $L = \begin{pmatrix} A & 0 \\ -F_3 & I \end{pmatrix}$ where $A \in \Phi_l(X)$. We have $L \in \Phi_l(X \times Y)$. So, $F + L = \begin{pmatrix} A + F_1 & F_2 \\ 0 & I + F_4 \end{pmatrix} \in \Phi_l(X \times Y)$. Then according to Proposition (2.3.11), we have $F_1 + A \in \Phi_l(X)$. Hence $F_1 \in \mathcal{F}_l(X)$. Next, to prove that any operator F_i for $i = 1, 2, 3$, it is possible according to Theorem (1.4.15), to permute the columns and the rows of F by creating a matrix of the form $J_1 F J_2$ where J_1 and J_2 are two invertible matrices, the goal of this process is to put the operator F_i in the first position in F. Consequently, we conclude that $F_i \in \mathcal{F}_l(X)$ for all i.

Inversely, let $F_1 \in \mathcal{F}_l(X)$, $L \in \Phi_l(X \times Y)$, and $N \in \mathcal{L}(X \times Y)$ such that $NL = I + K$. We put $\tilde{F_1} = \begin{pmatrix} F_1 & 0 \\ 0 & 0 \end{pmatrix}$, then we have

$$\begin{aligned} N(L + \tilde{F_1}) &= I + N\tilde{F_1} + K \\ &= \begin{pmatrix} N_1 F_1 + I & 0 \\ N_3 F_3 & I \end{pmatrix} + K \end{aligned}.$$

According to Theorem (1.4.14), $N_1 F_1$ is a left Fredholm perturbation. Then $N_1 + F_1 + I \in \Phi_l(X)$. Therefore $N(L + \tilde{F_1}) \in \Phi_l(X)$. So $L + \tilde{F_1} \in \Phi_l(X)$. Hence $\tilde{F_1} \in \mathcal{F}_l(X)$. Next, in the same reasoning as before, we can prove that if $F_{i,j} \in \mathcal{F}_l(X)$ for all i, j, then $\tilde{F_{i,j}} \in \Phi_l(X^n)$. Next, the sum of all matrices $\tilde{F_{i,j}}$ where $i, j = 1, \cdots, n$, we get $F \in \mathcal{F}_l(X)$.

2) The proof is similar to the previous one. Unless that the matrix should be an upper triangular matrix L of the form $\begin{pmatrix} A & -F_2 \\ 0 & I \end{pmatrix}$.

3) Is an immediate result from the first and second assertions.

Finally, by a simple induction reasoning, it will be easy to prove the results of an arbitrary dimension matrix F.

□

3 Essential spectra

This section focuses on the spectral analysis of linear operators and block operator matrices. We will explore some essential spectra of operators and block operator matrices, building on previous results to enhance our understanding of these spectral properties and their broad implications.

3.1 Essential spectra of an operator

In this subsection, various definitions of the essential spectra of continuous operators will be presented, with some fundamental results.

Definition 3.1.1 *Let $A \in \mathcal{L}(X)$. The spectrum of A, denoted $\sigma(A)$, is defined by*

$$\sigma(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \text{ is not invertible}\}.$$

The resolvent set of A, denoted by $\rho(A)$, is defined as the complement the spectrum of A in $\mathbb{C}$, i.e

$$\rho(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \text{ is invertible}\}.$$

The resolvent of any continuous operator is not empty.

Definition 3.1.2 *Let $A \in \mathcal{L}(X)$. The approximation point spectrum of A, denoted $\sigma_{\text{ap}}(A)$, is defined by*

$$\sigma_{\text{ap}}(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \text{ is not bounded below}\}.$$

The defect spectrum of A, denoted by $\sigma_\delta(A)$, is defined by

$$\sigma_\delta(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \text{ is not surjective}\}.$$

The concept of essential spectra was introduced and studied by several mathematicians, for example, H. Weyl, T. Kato, M. Schechter, F. E. Browder, and others. There are various and in general non-equivalent definitions of the essential spectrum of a bounded linear operator on a Banach space. In our work, we are interested by the following definitions. For $A \in \mathcal{L}(X)$.

- $\sigma_{e_1}(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \notin \Phi_+(X)\}$.
- $\sigma_{e_2}(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \notin \Phi_-(X)\}$.
- $\sigma_{e_3}(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \notin \Phi_\pm(X)\}$.
- $\sigma_{e_4}(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \notin \Phi(X)\}$.
- $\sigma_{e_5}(A) := \sigma_{e_4}(A) \cup \{\lambda \in \mathbb{C} \ / \ \text{ind}(\lambda - A) \neq 0\}$.
- $\sigma_{e_7}(A) := \sigma_{e_1}(A) \cup \{\lambda \in \mathbb{C} \ / \ \text{ind}(\lambda - A) > 0\}$.
- $\sigma_{e_8}(A) := \sigma_{e_2}(A) \cup \{\lambda \in \mathbb{C} \ / \ \text{ind}(\lambda - A) < 0\}$.
- $\sigma_{el}(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \notin \Phi_l(X)\}$.
- $\sigma_{er}(A) := \{\lambda \in \mathbb{C} \ / \ \lambda - A \notin \Phi_r(X)\}$.

The sets $\sigma_{e_1}(A)$, $\sigma_{e_2}(A)$, $\sigma_{e_3}(A)$, $\sigma_{e_4}(A)$, $\sigma_{e_5}(A)$, $\sigma_{el}(A)$, $\sigma_{er}(A)$, $\sigma_{e_7}(A)$, and $\sigma_{e_8}(A)$ are called respectively Gustafson, Weidmann, Kato, Wolf, Schechter, left, right essential spectrum, essential approximate point spectrum, and defect essential spectrum of A.

Remarks 3.1 1) We have these obvious equalities.

$$\sigma_{e_3}(A) = \sigma_{e_1}(A) \cap \sigma_{e_2}(A).$$

$$\sigma_{e_4}(A) = \sigma_{e_1}(A) \cup \sigma_{e_2}(A).$$

$$\sigma_{e_4}(A) = \sigma_{el}(A) \cup \sigma_{er}(A).$$

$$\sigma_{e_5}(A) = \sigma_{e_7}(A) \cup \sigma_{e_8}(A).$$

2) Clearly all essential spectra σ_{e_i} for $i = 1, \ldots, 5, 7, 8, l, r$ are invariant with respect to compact perturbations.

Proposition 3.1.3 *Let A be an invertible operator. Then*

$$\lambda \in \sigma_{e_i}(A) \text{ if and only if } \lambda^{-1} \in \sigma_{e_i}(A^{-1})$$

for $i = 1, \ldots, 5, 7, 8$.

Proof Suppose that $\lambda \in \sigma_{e_1}(A)$. By definition $\lambda - A$ is an upper semi Fredholm operator, then $-\lambda A(\lambda^{-1} - A^{-1})$ is an upper semi Fredholm operator, since $-\lambda A$ is invertible. According to the second assertion of Proposition 1.3.7, we see that $(\lambda^{-1} - A^{-1})$ is an upper semi Fredholm operator. So $\lambda^{-1} \in \sigma_{e_1}(A^{-1})$.

Suppose that $\lambda \in \sigma_{e_2}(A)$. By definition $\lambda - A$ is a lower semi Fredholm operator, then $-\lambda A(\lambda^{-1} - A^{-1})$ is a lower semi Fredholm operator, since $-\lambda A$ is invertible. If follows from the first assertion of Proposition 1.3.7 that $\lambda^{-1} - A^{-1}$ is a lower semi Fredholm operator. So $\lambda^{-1} \in \sigma_{e_1}(A^{-1})$.

For $i = 7, 8$, we have proven the equivalence for only the upper and lower semi Fredholmness of the operators (for $i = 1, 2$), it remains to verify the signs of the indices, since $-\lambda A$ is invertible, then $\text{ind}(\lambda - A) = \text{ind}(\lambda^{-1} - A^{-1})$, which means that they have the same sign.

The assertions for $i = 3, 4, 5$ can be proved directly by using the formulas of $\sigma_{e_3}(A)$, $\sigma_{e_4}(A)$ and $\sigma_{e_5}(A)$ noted in Remarks (3.1), and also the equivalences proved for $i = 1, 2, 7, 8$.

□

In the following proposition, we give an equivalent definitions of Schechter essential spectrum, essential approximate point spectrum and defect essential spectrum.

Proposition 3.1.4 *The following equalities hold.*

1. $\sigma_{e_5}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K)$.
2. $\sigma_{e_7}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma_{\text{ap}}(A + K)$.
3. $\sigma_{e_8}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma_{\delta}(A + K)$.

Proof
1) Let $\lambda \in \mathbb{C} \setminus \bigcap_{K\in\mathcal{K}(X)} \sigma(A+K)$, then there exists a compact operator K such that $\lambda-(A+K)$ is invertible. Then $\lambda-(A+K) \in \Phi(X)$ and $\operatorname{ind}(\lambda-(A+K)) = 0$. By using stability of Fredholm operators under compact perturbation $-K$, Propositions (1.4.7) and (1.4.4), we get $\lambda - A \in \Phi(X)$ and $\operatorname{ind}(\lambda - A) = 0$. Hence $\lambda \notin \sigma_{e_5}(A)$. This establish the first inclusion. Now we consider $\lambda \in \mathbb{C} \setminus \sigma_{e_5}(A)$, so $\lambda - A$ is a Fredholm operator with $\operatorname{ind}(\lambda - A) = 0$. By using Lemma (1.4.6), $\lambda - A$ can be written as $J + K$ where J is invertible and K is compact. since $\lambda-(A+K)$ is invertible, then $\lambda \notin \bigcap_{K\in\mathcal{K}(X)} \sigma(A+K)$.

2) Let $\lambda \in \mathbb{C} \setminus \bigcap_{K\in\mathcal{K}(X)} \sigma_{\mathrm{ap}}(A+K)$, then there exists a compact operator K such that $\lambda \notin \sigma_{\mathrm{ap}}(A+K)$. This means that $\lambda-(A+K)$ is bounded below. According to Proposition (1.1.10), $\lambda-(A+K)$ is injective with closed range. So $\lambda-(A+K) \in \Phi_+(X)$ and $\operatorname{ind}(\lambda-(A+K)) \leqslant 0$. By using stability of upper semi Fredholm operators under compact perturbation, we deduce that $\lambda - A \in \Phi_+(X)$ and $\operatorname{ind}(\lambda - A) \leqslant 0$, therefore $\lambda \notin \sigma_{e_7}(A)$. We conclude that $\sigma_{e_7}(A) \subset \bigcap_{K\in\mathcal{K}(X)} \sigma_{\mathrm{ap}}(A+K)$. For the proof of the converse inclusions we refer to [A. 09].
3) Let $\lambda \in \mathbb{C} \setminus \bigcap_{K\in\mathcal{K}(X)} \sigma_{\delta}(A+K)$, then there exists a compact operator K such that $\lambda \notin \sigma_{\mathrm{ap}}(A+K)$. This means that $\lambda-(A+K)$ is surjective, and consequently we have $\beta(\lambda-(A+K) = 0$. Hence $\lambda-(A+K) \in \Phi_-(X)$ and $\operatorname{ind}(\lambda-(A+K)) \geqslant 0$, then $\lambda - A \in \Phi_-(X)$ with $\operatorname{ind}(\lambda - A) \geqslant 0$. By using stability of lower semi Fredholm operators under compact perturbation, we get $\lambda \notin \sigma_{e_8}(A)$. We conclude that $\sigma_{e_8}(A) \subset \bigcap_{K\in\mathcal{K}(X)} \sigma_{\delta}(A+K)$.

For the proof of the converse inclusions we refer to [A. 09].

□

Proposition 3.1.5 *[A. 15] The following assertions hold.*

- *The set* $\mathbb{C} \setminus \sigma_{e_4}(A)$ *is open.*
- *The map* $\lambda \mapsto \operatorname{ind}(\lambda - A)$ *is constant on every component of* $\mathbb{C} \setminus \sigma_{e_4}(A)$.

Proposition 3.1.6 *Let* $A \in \mathcal{L}(X)$ *such that* $\mathbb{C} \setminus \sigma_{e_4}(A)$ *be connected. The following equalities hold.*

1. $\sigma_{e_5}(A) = \sigma_{e_4}(A)$.
2. $\sigma_{e_7}(A) = \sigma_{e_1}(A)$.
3. $\sigma_{e_8}(A) = \sigma_{e_2}(A)$.

Proof For he first statement, let $\lambda \in \mathbb{C} \setminus \sigma_{e_4}(A)$ and $\lambda_0 \in \rho(A)$, we have $\operatorname{ind}(\lambda_0 - A) = 0$. Since $\mathbb{C} \setminus \sigma_{e_4}(A)$ is connected, then according to Proposition (3.1.5) $\mu \mapsto \operatorname{ind}(\mu - A)$ is constant on $\mathbb{C} \setminus \sigma_{e_4}(A)$. We have $\lambda, \lambda_0 \in \mathbb{C} \setminus \sigma_{e_4}(A)$, then

$$\operatorname{ind}(\lambda - A) = \operatorname{ind}(\lambda_0 - A) = 0.$$

Hence $\lambda \notin \sigma_{e_5}(A)$. Consequently, $\sigma_{e_5}(A) = \sigma_{e_4}(A)$.

For the second statement, $\lambda \in \mathbb{C} \setminus \sigma_{e_1}(A)$, we distinguish two cases :

1) Let $\lambda \notin \sigma_{e_4}(A)$, then $\lambda \notin \sigma_{e_5}(A)$ according to the proof of the first statement, so $\text{ind}(\lambda - A) = 0$.

2) Let $\lambda \in \sigma_{e_4}(A)$, then $\lambda - A \in \Phi_+(X) \setminus \Phi(X)$, so $\text{ind}(\lambda - A) = -\infty$. Then in the two cases we have $\lambda \notin \sigma_{e_1}(A)$ and $\text{ind}(\lambda - A) \leqslant 0$, which is equivalent to $\lambda \notin \sigma_{e_7}(A)$. Further, the definition of $\sigma_{e_7}(A)$ shows that it contains $\sigma_{e_1}(A)$. Consequently, $\sigma_{e_7}(A) = \sigma_{e_1}(A)$.

The third assertion can be proved exactly as the second one, just by replacing σ_{e_1} , σ_{e_2}, Φ_+ respectively by $\sigma_{e_7}, \sigma_{e_8}$, Φ_-.

□

The following theorem shows the relation between the essential spectra of the sum of the two bounded linear operators and the essential spectra of each of these operators, where their products are Fredholm or semi Fredholm perturbations.

Theorem 3.1.7 *Let $A, B \in \mathcal{L}(X)$.*

1. *If $AB \in \mathcal{F}_+(X)$, then*

$$\sigma_{e_1}(A+B) \setminus \{0\} \subset (\sigma_{e_1}(A) \cup \sigma_{e_1}(B)) \setminus \{0\}.$$

Furthermore, if $BA \in \mathcal{F}_+(X)$, then

$$\sigma_{e_1}(A+B) \setminus \{0\} = (\sigma_{e_1}(A) \cup \sigma_{e_1}(B)) \setminus \{0\}.$$

2. *If $AB \in \mathcal{F}_-(X)$, then*

$$\sigma_{e_2}(A+B) \setminus \{0\} \subset (\sigma_{e_2}(A) \cup \sigma_{e_2}(B)) \setminus \{0\}.$$

Furthermore, if $BA \in \mathcal{F}_-(X)$, then

$$\sigma_{e_2}(A+B) \setminus \{0\} = (\sigma_{e_2}(A) \cup \sigma_{e_2}(B)) \setminus \{0\}.$$

3. *If $AB \in \mathcal{F}_l(X)$, then*

$$\sigma_{el}(A+B) \setminus \{0\} \subset (\sigma_{el}(A) \cup \sigma_{el}(B)) \setminus \{0\}.$$

Furthermore, if $BA \in \mathcal{F}_l(X)$, then

$$\sigma_{el}(A+B) \setminus \{0\} = (\sigma_{el}(A) \cup \sigma_{el}(B)) \setminus \{0\}.$$

4. *If $AB \in \mathcal{F}_r(X)$, then*

$$\sigma_{er}(A+B) \setminus \{0\} \subset (\sigma_{er}(A) \cup \sigma_{er}(B)) \setminus \{0\}.$$

Furthermore, if $BA \in \mathcal{F}_-(X)$, then

$$\sigma_{er}(A+B) \setminus \{0\} = (\sigma_{er}(A) \cup \sigma_{er}(B)) \setminus \{0\}.$$

5. *If $AB \in \mathcal{F}(X)$, then*

$$\sigma_{e_3}(A+B) \setminus \{0\} \subset (\sigma_{e_3}(A) \cup \sigma_{e_3}(B) \cup (\sigma_{e_1}(A) \cap \sigma_{e_2}(B)) \cup (\sigma_{e_2}(A) \cap \sigma_{e_1}(B))) \setminus \{0\}.$$

Furthermore, if $BA \in \mathcal{F}(X)$, then

$$\sigma_{e_3}(A+B) \setminus \{0\} = (\sigma_{e_3}(A) \cup \sigma_{e_3}(B) \cup (\sigma_{e_1}(A) \cap \sigma_{e_2}(B)) \cup (\sigma_{e_2}(A) \cap \sigma_{e_1}(B))) \setminus \{0\}.$$

6. If $AB \in \mathcal{F}(X)$, then

$$\sigma_{e_4}(A+B) \setminus \{0\} \subset (\sigma_{e_4}(A) \cup \sigma_{e_4}(B)) \setminus \{0\}.$$

Furthermore, if $BA \in \mathcal{F}(X)$, then

$$\sigma_{e_4}(A+B) \setminus \{0\} = (\sigma_{e_4}(A) \cup \sigma_{e_4}(B)) \setminus \{0\}.$$

7. If $AB \in \mathcal{F}_+(X)$, then

$$\sigma_{e_7}(A+B) \setminus \{0\} \subset (\sigma_{e_7}(A) \cup \sigma_{e_7}(B)) \setminus \{0\}.$$

Furthermore, if $AB, BA \in \mathcal{F}_+(X)$ and $\mathbb{C} \setminus \sigma_{e_4}(A)$ is connected, then

$$\sigma_{e_7}(A+B) \setminus \{0\} = (\sigma_{e_7}(A) \cup \sigma_{e_7}(B)) \setminus \{0\}.$$

8. If $AB \in \mathcal{F}_-(X)$, then

$$\sigma_{e_8}(A+B) \setminus \{0\} \subset (\sigma_{e_8}(A) \cup \sigma_{e_8}(B)) \setminus \{0\}.$$

Furthermore, if $BA \in \mathcal{F}_-(X)$ and $\mathbb{C} \setminus \sigma_{e_4}(A)$ is connected, then

$$\sigma_{e_8}(A+B) \setminus \{0\} = (\sigma_{e_8}(A) \cup \sigma_{e_8}(B)) \setminus \{0\}.$$

9. If $AB \in \mathcal{F}(X)$, then

$$\sigma_{e_5}(A+B) \setminus \{0\} \subset (\sigma_{e_5}(A) \cup \sigma_{e_5}(B)) \setminus \{0\}.$$

Furthermore, if $BA \in \mathcal{F}(X)$ and $\mathbb{C} \setminus \sigma_{e_4}(A)$ is connected, then

$$\sigma_{e_5}(A+B) \setminus \{0\} = (\sigma_{e_5}(A) \cup \sigma_{e_5}(B)) \setminus \{0\}.$$

Proof For the following proofs, we focus on the case where $\lambda \neq 0$. The case $\lambda = 0$ is considered trivial and omitted.

1) Let λ be an arbitrary complex number such that $\lambda \notin (\sigma_{e_1}(A) \cup \sigma_{e_1}(B)) \setminus \{0\}$. Then $\lambda \notin \sigma_{e_1}(A)$ and $\lambda \notin \sigma_{e_1}(B)$. Which means that $\lambda - A$ and $\lambda - B$ are both upper semi Fredholm operators. According to Proposition (1.3.7), the operator $(\lambda - A)(\lambda - B)$ is an upper semi Fredholm operator. By developing this product, using stability of upper semi Fredholm operators under upper semi Fredholm perturbation AB, we get $\lambda - (A+B)$ is an upper semi Fredholm operator. Consequently, $\lambda \notin \sigma_{e_1}(A+B) \setminus \{0\}$. Hence the inclusion is established.

For the converse inclusion, let $\lambda \notin \sigma_{e_1}(A+B) \setminus \{0\}$. By definition $\lambda - (A+B)$ is an upper semi Fredholm operator. By using again stability of upper semi Fredholm operators under upper semi Fredholm perturbations AB and BA, we conclude that $\lambda^2 - \lambda(A+B) + AB$ and $\lambda^2 - \lambda(A+B) + BA$ are upper semi Fredholm operators. Therefore $(\lambda - A)(\lambda - B)$ and $(\lambda - B)(\lambda - A)$ are both upper semi Fredholm operators, then by using Proposition (1.3.9), we conclude that $\lambda - A$ and $\lambda - B$ are upper semi Fredholm operators. This means that $\lambda \notin (\sigma_{e_1}(A) \cup \sigma_{e_1}(A)) \setminus \{0\}$. Hence we deduce the equality.

2) Let λ be an arbitrary complex number such that $\lambda \notin (\sigma_{e_2}(A) \cup \sigma_{e_2}(B)) \setminus \{0\}$. Then $\lambda \notin \sigma_{e_2}(A)$ and $\lambda \notin \sigma_{e_2}(B)$. Which means that $\lambda - A$ and $\lambda - B$ are both lower semi

Fredholm operators. According to Proposition (1.3.7), the operator $(\lambda - A)(\lambda - B)$ is a lower semi Fredholm operator. By developing this product, using stability of lower semi Fredholm operators under lower semi Fredholm perturbation AB, we get $\lambda - (A+B)$ is a lower semi Fredholm operator. Consequently, $\lambda \notin \sigma_{e_2}(A+B) \setminus \{0\}$. Hence the inclusion is established.

For the converse inclusion, let $\lambda \notin \sigma_{e_2}(A+B) \setminus \{0\}$, so by definition $\lambda - (A+B)$ is a lower semi Fredholm operator. By using again stability of lower semi Fredholm operators under lower semi Fredholm perturbations AB and BA, we conclude that $\lambda^2 - \lambda(A+B) + AB$ and $\lambda^2 - \lambda(A+B) + BA$ are lower semi Fredholm operators. Therefore $(\lambda - A)(\lambda - B)$ and $(\lambda - B)(\lambda - A)$ are both lower semi Fredholm operators, then by using Proposition (1.3.9), we conclude that $\lambda - A$ and $\lambda - B$ are lower semi Fredholm operators. This means that $\lambda \notin (\sigma_{e_2}(A) \cup \sigma_{e_2}(A)) \setminus \{0\}$. Hence we deduce the equality.

3) Let λ be an arbitrary complex number such that $\lambda \notin (\sigma_{el}(A) \cup \sigma_{el}(B)) \setminus \{0\}$. Then $\lambda \notin \sigma_{el}(A)$ and $\lambda \notin \sigma_{el}(B)$. Which means that $\lambda - A$ and $\lambda - B$ are both left Fredholm operators. According to Proposition (1.3.7), the operator $(\lambda - A)(\lambda - B)$ is a left Fredholm operator. By developing this product, using stability of left Fredholm operators under left Fredholm perturbation AB, we get $\lambda - (A+B)$ is a left Fredholm operator. Consequently, $\lambda \notin \sigma_{el}(A+B) \setminus \{0\}$. Hence the inclusion is established.

For the converse inclusion, let $\lambda \notin \sigma_{el}(A+B) \setminus \{0\}$, so by definition $\lambda - (A+B)$ is a left Fredholm operator. By using again stability of left Fredholm operators under left Fredholm perturbations AB and BA, we conclude that $\lambda^2 - \lambda(A+B) + AB$ and $\lambda^2 - \lambda(A+B) + BA$ are left Fredholm operators. Therefore $(\lambda - A)(\lambda - B)$ and $(\lambda - B)(\lambda - A)$ are both left Fredholm operators, then by using Proposition (1.3.9), we conclude that $\lambda - A$ and $\lambda - B$ are left Fredholm operators. This means that $\lambda \notin (\sigma_{el}(A) \cup \sigma_{el}(A)) \setminus \{0\}$. Hence we deduce the equality.

4) Let λ be an arbitrary complex number such that $\lambda \notin (\sigma_{er}(A) \cup \sigma_{er}(B)) \setminus \{0\}$. Then $\lambda \notin \sigma_{er}(A)$ and $\lambda \notin \sigma_{er}(B)$. Which means that $\lambda - A$ and $\lambda - B$ are both right Fredholm operators. According to Proposition (1.3.7), the operator $(\lambda - A)(\lambda - B)$ is a right Fredholm operator. By developing this product, using stability of right Fredholm operators under right Fredholm perturbation AB, we get $\lambda - (A+B)$ is a right Fredholm operator. Consequently, $\lambda \notin \sigma_{er}(A+B) \setminus \{0\}$. Hence the inclusion is established.

For the converse inclusion, let $\lambda \notin \sigma_{er}(A+B) \setminus \{0\}$, so by definition $\lambda - (A+B)$ is a right Fredholm operator. By using again stability of right Fredholm operators under right Fredholm perturbations AB and BA, we conclude that $\lambda^2 - \lambda(A+B) + AB$ and $\lambda^2 - \lambda(A+B) + BA$ are right Fredholm operators. Therefore $(\lambda - A)(\lambda - B)$ and $(\lambda - B)(\lambda - A)$ are both right Fredholm operators, then by using Proposition (1.3.9), we conclude that $\lambda - A$ and $\lambda - B$ are right Fredholm operators. This means that $\lambda \notin (\sigma_{er}(A) \cup \sigma_{er}(A)) \setminus \{0\}$. Hence we deduce the equality.

5) Consider the following inclusions

$$\begin{aligned}
\sigma_{e_3}(A+B) \setminus \{0\} &= \sigma_{e_1}(A+B) \cap \sigma_{e_2}(A+B) \setminus \{0\} \\
&\subset (\sigma_{e_1}(A) \cup \sigma_{e_1}(B)) \cap (\sigma_{e_2}(A) \cup \sigma_{e_2}(B)) \setminus \{0\} \\
&\subset (\sigma_{e_1}(A) \cap \sigma_{e_2}(A)) \cup (\sigma_{e_1}(B) \cap \sigma_{e_2}(B)) \cup (\sigma_{e_1}(A) \cap \sigma_{e_2}(B)) \\
&\cup (\sigma_{e_2}(A) \cap \sigma_{e_1}(B)) \setminus \{0\} \\
&\subset \sigma_{e_3}(A) \cup \sigma_{e_3}(B) \cup (\sigma_{e_1}(A) \cap \sigma_{e_2}(B)) \cup (\sigma_{e_2}(A) \cap \sigma_{e_1}(B)) \setminus \{0\}
\end{aligned}$$

The equality can be proved in the same way.

6) Let λ be an arbitrary complex number such that $\lambda \notin (\sigma_{e_4}(A) \cup \sigma_{e_4}(B)) \setminus \{0\}$. Then $\lambda \notin \sigma_{e_4}(A)$ and $\lambda \notin \sigma_{e_4}(B)$. Which means that $\lambda - A$ and $\lambda - B$ are both Fredholm operators. According to Proposition (1.3.7), the operator $(\lambda - A)(\lambda - B)$ is a Fredholm operator. By developing this product, using stability of Fredholm operators under Fredholm perturbation AB, we get $\lambda - (A + B)$ is a Fredholm operator. Consequently, $\lambda \notin \sigma_{e_4}(A + B) \setminus \{0\}$. Hence the inclusion is established.

For the converse inclusion, let $\lambda \notin \sigma_{e_4}(A + B) \setminus \{0\}$, so by definition $\lambda - (A + B)$ is a Fredholm operator. By using again stability of Fredholm operators under Fredholm perturbations AB and BA, we conclude that $\lambda^2 - \lambda(A + B) + AB$ and $\lambda^2 - \lambda(A + B) + BA$ are Fredholm operators. Therefore $(\lambda - A)(\lambda - B)$ and $(\lambda - B)(\lambda - A)$ are both Fredholm operators, then by using Proposition (1.3.9), we conclude that $\lambda - A$ and $\lambda - B$ are Fredholm operators. This means that $\lambda \notin (\sigma_{e_4}(A) \cup \sigma_{e_4}(A)) \setminus \{0\}$. Hence we deduce the equality.

7) Let $\lambda \notin (\sigma_{e_7}(A) \cup \sigma_{e_7}(B)) \setminus \{0\}$, since we have proved the inclusion of the first assertion then it remains to verify only the indexes, we have $\text{ind}(\lambda - A) \leqslant 0$ and $\text{ind}(\lambda - B) \leqslant 0$, then $\text{ind}((\lambda - A)(\lambda - B)) \leqslant 0$, following the same reasoning as in the first assertion, we get $\text{ind}(\lambda - (A + B)) \leqslant 0$. Hence $\lambda \notin \sigma_{e_7}(A + B) \setminus \{0\}$.

8) Let $\lambda \notin (\sigma_{e_8}(A) \cup \sigma_{e_8}(B)) \setminus \{0\}$, since we have proved the inclusion of the first assertion then it remains to verify only the indexes, we have $\text{ind}(\lambda - A) \geqslant 0$ and $\text{ind}(\lambda - B) \geqslant 0$, then $\text{ind}((\lambda - A)(\lambda - B)) \geqslant 0$, following the same reasoning as in the first assertion, we get $\text{ind}(\lambda - (A + B)) \geqslant 0$. Hence $\lambda \notin \sigma_{e_8}(A + B) \setminus \{0\}$.

The converse inclusions of assertion 7 and 8 are proved in [A. 15].

9) Let λ be an arbitrary complex number such that $\lambda \notin (\sigma_{e_5}(A) \cup \sigma_{e_5}(B)) \setminus \{0\}$, since we have proved the inclusion of the first assertion then it remains to verify only the indexes, we have $\text{ind}(\lambda - A) = 0$ and $\text{ind}(\lambda - B) = 0$, then $\text{ind}((\lambda - A)(\lambda - B)) = 0$, following the same reasoning as in the first assertion, we get $\text{ind}(\lambda - (A + B)) = 0$. Hence $\lambda \notin \sigma_{e_5}(A + B) \setminus \{0\}$.

For the converse inclusion, let $\lambda \notin \sigma_{e_5}(A + B) \setminus \{0\}$, by definition, $\lambda - (A + B)$ is a Fredholm operator with zero index, since AB and BA are Fredholm perturbations and λ is invertible, then following the same reasoning as in the first assertion, we get that $\lambda - A$ and $\lambda - B$ are both Fredholm operators and $\text{ind}(\lambda - A) + \text{ind}(\lambda - B) = 0$, since $\mathbb{C} - \sigma_{e_4}(A)$ is connected, then $\sigma_{e_5}(A) = \sigma_{e_4}(A)$ according to Proposition (3.1.6). Therefore $\text{ind}(\lambda - A) = 0$. Consequently, $\lambda - A$ and $\lambda - B$ are both Fredholm operators with zero indices. This means that $\lambda \notin (\sigma_{e_5}(A) \cup \sigma_{e_5}(A)) \setminus \{0\}$. Hence we deduce the equality.

□

3.2 Essential spectra of block operators

The following proposition is a direct consequence for recently viewed relations between Fredholmness of operators $M_{i,i}$ and M in Propositions (2.3.10) and (2.3.5).

Proposition 3.2.1 *Let M be a triangular block matrix operator. Then*

$$\sigma_{ek}(M) \subset \bigcup_{i=1}^{n} \sigma_{ek}(M_{i,i}) \text{ for } k = 1, \ldots, 5, 7, 8, l, r.$$

Proof For $k = 1, 2, 4, l, r$. Let $\lambda \in \mathbb{C} \setminus \bigcup_{i=1}^{n} \sigma_{ek}(M_{i,i})$, then $\lambda - M_{i,i} \in \Phi_*(X)$ for all i, where $*$ takes a value according to k. According to Proposition (2.3.5), we deduce that $\lambda - M \in \Phi_*(X^n)$. Hence $\lambda \notin \sigma_{ek}(M)$.

If $\lambda \notin \bigcup_{i=1}^{n} \sigma_{e_5}(M_{i,i})$, then all operators $\lambda - M_{i,i}$ are Fredholm operators with $\operatorname{ind}(\lambda - M_{i,i}) = 0$ for all i. According to Proposition (2.3.5) and Lemma (2.3.9), M is a Fredholm operator and $\operatorname{ind}(M) = \sum_{i=1}^{n} \operatorname{ind}(M_{i,i})$. Then M is a Fredholm operator with zero index. Hence $\lambda \notin \sigma_{e_5}(M)$.

If $\lambda \notin \bigcup_{i=1}^{n} \sigma_{e_7}(M_{i,i})$, then all operators $\lambda - M_{i,i}$ are upper semi Fredholm operators with $\operatorname{ind}(\lambda - M_{i,i}) \leqslant 0$ for all i. According to Proposition (2.3.5) and Lemma (2.3.9), M is an upper semi Fredholm operator and $\operatorname{ind}(M) = \sum_{i=1}^{n} \operatorname{ind}(M_{i,i})$. Then M is an upper semi Fredholm operator with less or equal to zero index. Hence $\lambda \notin \sigma_{e_7}(M)$.

If $\lambda \notin \bigcup_{i=1}^{n} \sigma_{e_8}(M_{i,i})$, then all operators $\lambda - M_{i,i}$ are lower semi Fredholm operators with $\operatorname{ind}(\lambda - M_{i,i}) \geqslant 0$ for all i. According to Proposition (2.3.5) and Lemma (2.3.9), M is a lower semi Fredholm operator and $\operatorname{ind}(M) = \sum_{i=1}^{n} \operatorname{ind}(M_{i,i})$. Then M is a lower semi Fredholm operator with greater or equal to zero index. Hence $\lambda \notin \sigma_{e_8}(M)$.

□

Proposition 3.2.2 *Let M be an upper triangular block operator matrix. The following inclusions hold.*

1. $\sigma_{e_1}(M_{1,1}) \subset \sigma_{e_1}(M)$.
2. $\sigma_{el}(M_{1,1}) \subset \sigma_{el}(M)$.
3. $\sigma_{e_2}(M_{n,n}) \subset \sigma_{e_2}(M)$.
4. $\sigma_{er}(M_{n,n}) \subset \sigma_{er}(M)$.
5. $\sigma_{e_1}(M_{1,1}) \cup \sigma_{el}(M_{1,1}) \cup \sigma_{e_2}(M_{n,n}) \cup \sigma_{er}(M_{n,n}) \subset \sigma_{e_4}(M)$.

Proof Suppose first that M is an upper triangular block operator matrix.

1) If $\lambda \in \mathbb{C} \setminus \sigma_{el}(M)$, then $\lambda - M \in \Phi_l(X^n)$. According to Proposition (2.3.10) we conclude that $\lambda - M_{1,1} \in \Phi_l(X)$, so $\lambda \notin \sigma_{el}(M_{1,1})$. Consequently, $\sigma_{el}(M_{1,1}) \subset \sigma_{el}(M)$.

2) If $\lambda \in \mathbb{C} \setminus \sigma_{e_1}(M)$, then $\lambda - M \in \Phi_+(X^n)$. According to Proposition (2.3.10) we conclude that $\lambda - M_{1,1} \in \Phi_+(X)$, so $\lambda \notin \sigma_{e_1}(M_{1,1})$. Consequently, $\sigma_{e_1}(M_{1,1}) \subset \sigma_{e_1}(M)$.

3) If $\lambda \in \mathbb{C} \setminus \sigma_{e_2}(M)$, then $\lambda - M \in \Phi_-(X^n)$. According to Proposition (2.3.10) we conclude that $\lambda - M_{n,n} \in \Phi_-(X)$, so $\lambda \notin \sigma_{e_2}(M_{n,n})$. Consequently, $\sigma_{e_2}(M_{n,n}) \subset \sigma_{e_2}(M)$.

4) If $\lambda \in \mathbb{C} \setminus \sigma_{er}(M)$, then $\lambda - M \in \Phi_r(X^n)$. According to Proposition (2.3.10) we conclude that $\lambda - M_{n,n} \in \Phi_r(X)$, so $\lambda \notin \sigma_{er}(M_{n,n})$. Consequently, $\sigma_{er}(M_{n,n}) \subset \sigma_{er}(M)$.

5) Can be directly established by taking the union of the sets in assertions 1 and 2 (or 3 and 4).

□

Proposition 3.2.3 *Let M be a block matrix operator such that $M = M_U + M_L$ where*

$$M_U = \begin{pmatrix} M_{1,1} & M_{1,2} & \cdots & \cdots & M_{1,n} \\ 0 & M_{2,2} & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & M_{n-1,n} \\ 0 & \cdots & \cdots & 0 & M_{n,n} \end{pmatrix}, \quad M_L = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ M_{1,2} & 0 & & & \vdots \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ M_{n,1} & \cdots & \cdots & M_{n,n-1} & 0 \end{pmatrix}.$$

Let $k = 1, \dots, 5, 7, 8, l, r$ and $*$ be its associated value i.e for $k = 1, 7$, $* = +$, for $k = 2, 8$, $* = -$, for $k = 4, 5$, there is no $*$, for $k = l, r$, $* = l, r$. Then

- *If $M_U M_L \in \mathcal{F}_*(X^n)$ then $\sigma_{ek}(M) \setminus \{0\} \subset \sigma_{ek}(M_U) \setminus \{0\}$.*
- *If furthermore $M_L M_U \in \mathcal{F}_*(X^n)$, then $\sigma_{ek}(M) \setminus \{0\} = \sigma_{ek}(M_U) \setminus \{0\}$.*

Proof Since $M = M_U + M_L$, by using Theorem (3.1.7), we get $\sigma_{ek}(M) \setminus \{0\} \subset (\sigma_{ek}(M_U) \cup \sigma_{ek}(M_L)) \setminus \{0\}$. Next, M_L is lower triangular matrix where all element on diagonal are nulls, then

$$\lambda - M_L \in \Phi_*(X) \Longleftrightarrow \lambda \neq 0.$$

for all λ. So $\sigma_{ek}(M_L) = \{0\}$. Hence $\sigma_{ek}(M) \setminus \{0\} \subset \sigma_{ek}(M_U) \setminus \{0\}$.

The second assertion can be proved in the same way and by using the equalities in Theorem (3.1.7). □

Theorem 3.2.4 *[S. 23] Let M be the block operator matrix defined above. Then the following inclusions hold.*

1. *$\sigma_{e_1}(M_{1,1}) \setminus \{0\} \subset \sigma_{e_1}(M) \setminus \{0\}$.*
2. *$\sigma_{el}(M_{1,1}) \setminus \{0\} \subset \sigma_{el}(M) \setminus \{0\}$.*
3. *$\sigma_{e_2}(M_{n,n}) \setminus \{0\} \subset \sigma_{e_2}(M) \setminus \{0\}$.*
4. *$\sigma_{er}(M_{n,n}) \setminus \{0\} \subset \sigma_{er}(M) \setminus \{0\}$.*
5. *$\sigma_{e_1}(M_{1,1}) \setminus \{0\} \cup \sigma_{el}(M_{1,1}) \setminus \{0\} \cup \sigma_{e_2}(M_{n,n}) \setminus \{0\} \cup \sigma_{er}(M_{n,n}) \setminus \{0\} \subset \sigma_{e_4}(M) \setminus \{0\}$.*
6. *$\sigma_{ek}(M) \setminus \{0\} \subset \bigcup_{i=1}^{n} \sigma_{ek}(M_{i,i}) \setminus \{0\}$ for $k = 1, \dots, 5, 7, 8, l, r$.*

Proof According to Proposition (3.2.3), we have $\sigma_{ek}(M) \setminus \{0\} = \sigma_{ek}(M_U) \setminus \{0\}$. Then by using the results of Propositions (3.2.1) and (3.2.2), we get all desired assertions. □

Conclusion

This work presents a thorough exploration of Fredholm block operator matrices, and the essential spectrum, establishing key theoretical results and demonstrating their practical relevance. The relationships and properties discussed provide a robust framework for further research and application in various scientific and engineering disciplines.

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