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# Exponential Stability in Thermoelastic Timoshenko and Bresse Systems

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A Thesis Presented for the Degree of

**DOCTOR in Mathematics**
In The Field of Applied Mathematics
By
Zineb Khalili
Title

# Exponential Stability in Thermoelastic Timoshenko and Bresse Systems

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AMAR TELIDJI UNIVERSITY, LAGHOUAT
Faculty of Sciences
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# Exponential Stability in Thermoelastic Timoshenko and Bresse Systems

by

ZINEB KHALILI

Under the Supervision of

**Dr. Djamel Ouchenane**

This thesis is submitted in order to obtain the degree of **Doctor LMD**

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# Publications

1. **Z. Khalili**, D. Ouchenane. A stability result for a Timoshenko system with infinite history and distributed delay term. *Kragujevac J. Math.*, 2020.

2. **Z. Khalili**, D. Ouchenane and A. Choucha. Well-posedness and Stability result of a nonlinear damping Porous-elastic system in thermoelasticity of second sound with infinite memory and distributed delay terms. *J. DCDIS., Series A: Mathematical Analysis.,* July 8, 2020

3. D. Ouchenane, **Z. Khalili**, F. Yazid. Global nonexistence of solution for coupled nonlinear Klein-Gorden with degenerate damping and source terms. accepted in Journal of Studia Mathematica., May 2020

4. F. Yazid, D. Ouchenane, **Z. Khalili**. A stability result for thermoelastic Timoshenko system of second sound with distribued delay term. J. MESA 11(N 1 2020):163-175., February 2020

5. D. Ouchenane, **Z. Khalili**, F. Yazid, M. Abdalla, B. Belkacem Cherifand, I. Mekawy. A New Result of Stability for Thermoelastic-Bresse System of Second Sound Related with Forcing, Delay, and Past History Terms. Journal of Function Spaces., (2021) Volume 2021, Article ID 9962569

6. **Z. Khalili**, A. El-hamidi, D. Ouchenane. Existence and stability results for laminated beam with thermodiffusion effects and localized nonlinear damping. Submitted to Journal of Mathematical Methods in the Applied Sciences.

7. **Z. Khalili**, D. Ouchenane. Exponential stability for a Timoshenko thermoelastic system with second sound and a time-varying delay term in the internal feedback. Submitted to Journal of Asymptotic analysis.

8. **Z. Khalili**, D. Ouchenane, A. El-hamidi. Stability result for Laminated beam with Thermo-Visco-Elastic effect and localizes nonlinear damping. Submitted to Journal of Nonlinear Analysis: Real World Applications.

9. **Z. Khalili**, D. Ouchenane. General decay of a nonlinear damping-Porous thermoelastic system of second sound with infinite memory. Has been submitted to Journal of Applicable Analysis.

10. **Z. Khalili**, D. Ouchenane, A. El-hamidi. Existence and stability rate of a thermodiffusion Laminated beam with viscoelastic damping. Submitted to Journal of Open Mathematics.

# Participation's

1. Workshops on Pure and Applied Mathematics in M'sila, WPAM'2018, December 2018
2. Third International conference in Operator theory EDP and Applications in El Oued, CITO2019, April 2019
3. International seminar on mathematics and computer science in Oran, February 2021
4. National Conference on applied mathematics and Mathematics Modeling in El-Oued, A3M'2021, March 2021
5. First National Conference of Pure and Applied Mathematics in Tebessa, CNMPA2021, April 2021
6. The International E-Conference on Mathematical Advances and Applications, ICOMAA-2021, held on May 26-29, 2021, Istanbul, Turkey.
7. The First Conference on Mathematics and Applications of Mathematics, CMAM-2021, 30 Jun-1 Jul 2021 Jijel (Algeria)
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# *Abstract*

In this thesis we considered some elastic, thermoelastic and viscoelastic systems with the presence of different mechanisms of dissipation damping and for more general forms of nonlinearities addressed from a different angle, and under assumptions on initial data and boundary conditions, conditions on damping and source terms. In our study, we focused on studying the stability of the solution and the instability of the solution (the explosion of the solution at a finite instant in time), and it started with study the existence and uniqueness of the solution we have demonstrated the existence and uniqueness of the solution by using the Semi-group method if the problems studied are linear, and if the problems studied are not linear we applied the Feado-Glarken method

**Keywords**: Timoshenko systems, Bresse systems, Porous systems, Thermoelasticity, Second sound, Distributed delay term, Infinite memory, Source term, Positive initial energy, Degenerate damping, Nonlinear viscoelastic wave equations, Global nonexistence, Semi-groupe method, Feado-Galarkin method, Concavity method, Energy method, Lyapounov functional, Exponential decay.

# Résumé

Dans cette thèse, nous avons considéré certains systèmes élastiques, thermoélastiques et viscoélastiques avec la présence de différents mécanismes d'amortissement de dissipation et pour des formes plus générales de non-linéarités abordées sous un angle différent, et sous des hypothèses sur les données initiales et les conditions aux limites, les conditions d'amortissement et les termes sources. Dans notre étude, nous nous sommes concentrés sur l'étude de la stabilité de la solution et de l'instabilité de la solution (l'explosion de la solution à un instant fini dans le temps), et cela a commencé d'abord par étudier l'existence et l'unicité de la solution que nous avons démontré l'existence et l'unicité de la solution en utilisant la méthode du Semi-groupe si les problèmes étudiés sont linéaires, et si les problèmes étudiés n'est pas linéaires on appliquée la méthode de Feado-Glacken.



**Mots clés** : Systèmes de Timochenko, Systèmes de Bresse, Systèmes de poreux, Thermoélasticité, Terme de retard distribué, Deuxième son, Mémoire infinie, Terme source, Énergie initiale positive, Amortissement dégénéré, Équations d'ondes viscoélastiques non linéaires, Non-existence globale, Méthode de concavité, Méthode de Semi-groupe, Méthode de Feado-Galarkin, Méthode de l'énergie, Lyapounov fonctionnel, Décroissance exponentielle.

# الملخص

في هذه الأطروحة، درسنا بعض الأنظمة المرنة والمطاطية الحرارية واللزجة مع وجود آليات تخميد وتبديد مختلفة وللحصول على أشكال أكثر عمومية من اللاخطية المقاربة من زاوية مختلفة، وتحت افتراضات على شروط المعطيات الأولية والحدودية وكذلك شروط التخميد والمصدر، ركزنا في دراستنا على دراسة استقرار الحل وعدم استقرار الحل (انفجار الحل في لحظة زمنية منتهية)، وهذا بدءا اولا بدراسة وجود و وحدانية الحل، نشير هنا اننا برهنا وجود و وحدانية الحل باستعمال طريقة سومي ڨروب Semi-groupe لان المشاكل المدروسة خطية، واذا كانت المشاكل المدروسة غير خطية فيتم تطبيق عليها طريقة فايدوغالركين Faedo-Galerkin.

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# Introduction

The aim of this thesis is to investigate the stability of some elastic, thermoelastic and viscoelastic evolution problems, such as the Timoshenko systems, Bresse systems, Porous systems and the systems of viscoelastic wave equations.

## The thermoelasticity theory:

The classical model of thermoelasticity is based on Fourier's law, i.e. the heat flux is proportional to the gradient of temperature. Over the past two decades, there have been a lot of work on local existence, global existence, well-posedeness, and asymptotic behavior of solutions to some initial-boundary value problems as well as to Cauchy problems in both one-dimensional and multi-dimensional thermoelasticity.

**Thermoelasticity of type I, II and III:**

In [18, 19, 20] Green and Naghdi introduced the equation of heat conduction :

$$K^*\nabla^2\theta + K\nabla^2\dot{\theta} = \rho C_v\ddot{\theta} + vT_0\ddot{u}_{i,i}, \tag{1}$$

where $u$ is the displacement field, $\theta$ is the temperature above reference temperature $T_0$, $C_v$ is the specific heat of the solid at constant volume, $K^*$ is the thermal conductivity rate and $K$ is the thermal conductivity of the medium. Notice that if we put $K^* = 0$ in equation (1) i.e. the thermal conductivity rate is absent, then the equation is acknowledged by the heat conduction equation for **GN-I** theory of thermoelasticity and if we substitute $K = 0$ in equation (1) i.e. the thermal conductivity rate is absent, we obtain the heat conduction equation of **GN-II** theory of thermoelasticity.

**Thermoelasticity with second sound**

One theory suggests that we should replace Fourier' s law

$$q + K\theta_x = 0,$$

by the so-called Cattaneo' s law

$$\tau q_t + q + k\theta_x = 0.$$

Result concerning existence, blow up, and asymptotic behavior of smooth, as well as weak solutions in thermoelasticity with second sound have been established over the past two decades. See [60, 65] and references therein.

# The Bresse systems

The study of Bresse systems started in 1859 in the work of Bresse [4] in which he gave the following system consists of three wave equations

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_1 w_{tt} = N_x - lQ + F_3, \end{cases} \tag{2}$$

where

$$N = k_0\left(w_x - l\varphi\right), \quad Q = k\left(\varphi_x + lw + \psi\right), \quad M = b\psi_x, \tag{3}$$

we use $N, Q$ and $M$ to denote the axial force, the shear force and the bending moment. By $w, \varphi$ and $\psi$ we are denoting the longitudinal, vertical and shear angle displacements. Here $\rho_1 = \rho A, \rho_2 = \rho I, b = EI, k_0 = EA, k = k'GA$ and $l = R^{-1}$. To material properties, we use $\rho$ for density, $E$ for the modulus of elasticity, $G$ for the shear modulus, $K$ for the shear factor, $A$ for the cross-sectional area, $I$ for the second moment of area of the cross-section and $R$ for the radius of curvature and we assume that all this quantities are positives. Also by $F_i$ we are denote external forces.

We will present some works, which studied the stability of the dissipatif Bresse system. The paper [1] was concerned with asymptotic stability of a Bresse system with two frictional dissipations

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + lw + \psi\right)_x - k_0 l\left(w_x - l\varphi\right) = -\gamma_1\varphi_t, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + lw + \psi\right) = -\gamma_2\psi_t, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + lw + \psi\right) = 0. \end{cases} \tag{4}$$

Under the condition of equal speeds of wave propagation, the authors proved that the system is exponentially stable. Otherwise, they show that the system is not exponentially stable. Then, they proved that the solution decays polynomially to zero with optimal decay rate, depending on the regularity of initial data.

In [14], the authors considered two Cauchy problems related to the Bresse model with two dissipative mechanisms corresponding to the heat conduction coupled to the system. The first of them is the Bresse system with thermoelasticity of Type I:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x - \psi - l\omega)_x - k_0 l(\omega_x - l\varphi) + l\gamma\theta_1 = 0, & \text{in } \mathbb{R}\times(0,\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} - k(\varphi_x - \psi - l\omega) + \gamma\theta_{2x} = 0, & \text{in } \mathbb{R}\times(0,\infty) \\ \rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x - kl(\varphi_x - \psi - l\omega) + \gamma\theta_{1x} = 0, & \text{in } \mathbb{R}\times(0,\infty) \\ \theta_{1t} - k_1\theta_{1xx} + m_1(\omega_x - l\varphi)_t = 0, & \text{in } \mathbb{R}\times(0,\infty) \\ \theta_{2t} - k_2\theta_{2xx} + m_2\psi_{xt} = 0, & \text{in } \mathbb{R}\times(0,\infty), \end{cases} \tag{5}$$

with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t, \theta_1, \theta_2)(x,0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_{10}, \theta_{20})(x).$$

The second one, is the Bresse system with thermoelasticity of Type III:

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x - \psi - l\omega)_x - k_0 l(\omega_x - l\varphi) + l\gamma\theta_{1t} = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} - k(\varphi_x - \psi - l\omega) + \gamma\theta_{2xt} = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ \rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x - kl(\varphi_x - \psi - l\omega) + \gamma\theta_{1xt} = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ \theta_{1tt} - k_1\theta_{1xx} - \alpha_1\theta_{1xxt} + m_1(\omega_x - l\varphi)_t = 0 & \text{in } \mathbb{R}\times(0,\infty), \\ \theta_{2tt} - k_2\theta_{2xx} - \alpha_2\theta_{2xxt} + m_2\psi_{xt} = 0 & \text{in } \mathbb{R}\times(0,\infty), \end{cases} \tag{6}$$

with the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t, \theta_1, \theta_2, \theta_{1t}, \theta_{2t})(x,0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_{10}, \theta_{20}, \theta_{11}, \theta_{21})(x),$$

where $\alpha_1$, $\alpha_2$, $\rho_1, \rho_2, \gamma, b, k, k_0, k_1, k_2, l$ $m_1$ and $m_2$ are positive constants. The authors proved that the decay rate of the solutions are very slow in the whole line, where they show that the solutions decay with the rate of $(1+t)^{-1/8}$ in the $L^2$-norm, whenever the initial data belongs to $L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ for a suitable $s$. The main tool used to prove this results is the energy method in the Fourier space.

- is easy to see that for $l = 0$ where the longitudinal displacement $w$ is not considered, the system (2) leads to the Timoshenko system.

## The Timoshenko systems

The study of Timoshenko systems started in 1921 in the work of Timoshenko [66] in which he gave the following system of coupled hyperbolic equations

$$\begin{cases} \rho\varphi_{tt} = (K(\varphi_x - \psi))_x, & \text{in } (0,L)\times(0,+\infty) \\ I_\rho\psi_{tt} = (EI\psi_x)_x + K(\varphi_x - \psi), & \text{in } (0,L)\times(0,+\infty), \end{cases} \tag{7}$$

where $t$ denotes the time variable, $x$ is the space variable along the beam of length $L$, in its equilibrium configuration, $\varphi$ is the transverse displacement of the beam and $\psi$ is the rotation angle of the filament of the beam. The coefficients $\rho, I_\rho, E, I$ and $K$ are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia a cross a section and the shear modulus. In the aim to find the minimal dissipation such that the solution of the coupled system (7) decays uniformly to zero, as time goes to infinity, several authors introduced different types of dissipative mechanisms to stabilize system (7). For example, two frictional linear dampings $\varphi_t, \psi_t$ acting on the first and the second equations respectively have been used in [61]. Also many authors proved that the presence of only one damping (frictional damping $\varphi_t$, localized frictional damping $\alpha(x)\varphi_t$, of memory type $\int_0^t g(t-\tau)\psi_{xx}(\tau)d\tau$ ) acting in the domain on a some part of it, suffices to stabilize the system. Frictional damping with an indefined sign has also considered lately in [49]. Kim and Renardy [24] considered (7) together with two boundary controls of the form

$$\begin{cases} K\psi(L,t) - K\varphi_x(L,t) = \alpha\varphi_t(L,t), & \forall t \geq 0 \\ EI\psi_x(L,t) = -\beta\varphi_t(L,t), & \forall t \geq 0, \end{cases}$$

and used the multiplier techniques to establish an exponential decay result for the total energy of (7). They also provided numerical estimates to the eigenvalues of the operator associated with system (7).

Concerning the Timoshenko system with viscoelastic damping, Ammar-Khodja et *al.* [25] considered a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + \displaystyle\int_0^t g(t-s)\psi_{xx}(s)ds + K(\varphi_x + \psi) = 0, \end{cases} \tag{8}$$

in $(0,L)\times(0,+\infty)$, together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ and $g$ decays uniformly. Precisely, they proved an exponential decay if $g$ decays in an exponential rate and polynomially if $g$ decays in a polynomial rate.

For the Timoshenko systems in classical thermoelasticity, Rivera and Racke [47] considered

$$\begin{cases} \rho_1\varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in } (0, L) \times (0, +\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in } (0, L) \times (0, +\infty) \\ \rho_3\theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0, & \text{in } (0, L) \times (0, +\infty). \end{cases} \tag{9}$$

Under appropriate conditions of $\sigma, \rho_i, b, k, \gamma$, they proved several exponential decay results for the linearized system and a non exponential stability result for the case of different wave speeds.

In the other hand, for the Timoshenko systems in thermoelasticity of type III, we have the recent papers of Messaoudi and Said-Houari [43, 44] in which the authors proved several stability results. More precisely, in [43] they investigated the asymptotic behavior of the problem

$$\begin{cases} \rho_1\varphi_{tt} - K(\varphi_x + \psi)_x = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \beta\theta_x = 0, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0, \end{cases} \tag{10}$$

in $(0, +\infty) \times (0, 1)$ and proved an exponential decay result similar to the one in [47]. We recall that the heat conduction in (10) is given by Green and Naghdi's theory. The same problem (10) with an additional damping of history type of the form

$$\int_0^\infty g(s)\psi_{xx}(x,\ t-s)\,ds, \tag{11}$$

acting in the second equation has been analyzed in [44]. The authors of [44] proved an exponential and polynomial stability results for the equal and nonequal wave-speed propagation under conditions on the relaxation function $g$ weaker than those in [40] and [48]. For the general stability of (10) with (11), see [16].

Concerning the Timoshenko system with delay, Said-Houari & Rahali [62] considered the following Timoshenko system with infinite history and a delay term in the internal feedback

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x + \psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \int_0^\infty g(s)\psi_{xx}(x,t-s)\,ds \\ +K(\varphi_x + \psi)(x,t) + \mu_1\psi_t(x,t) + \mu_2\psi_t(x,t-\tau) = 0. \end{cases} \tag{12}$$

They established the well-posedness of problem (12) and the exponential stability of solution. In the absence of the viscoelastic damping ($g \equiv 0$), problem (12) has been studied recently by Said-Houari & Laskri [63]. Under some assumption, they proved the well-posedness and established for $\mu_1 > \mu_2$ an exponential decay result for the case of equal-speed wave propagation, i.e.

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}. \tag{13}$$

Subsequently, the work in [63] has been extended to the case of time-varying delay of the form $\psi_t(x, t - \tau(t))$ by Kirane, Said-Houari & Anwar [25]. First, by using the variable norm technique of Kato and under some restriction on the parameters $\mu_1, \mu_2$ and on the delay function $\tau(t)$, the system has been shown to be well-posed. Second, under relationship between the weight of the delay term in the feedback, the weight of the term without delay and the wave speeds, an exponential decay result of the total energy has been proved.

Recently, Nicaise et al. in [52], introduced the case of time-varying delay in the wave equation and proved the exponential stability under the condition

$$\mu_2 < \sqrt{1-d}\mu_1, \tag{14}$$

where the constant $d$ satisfies

$$\tau'(t) \le d < 1, \quad \text{for all } t > 0, \tag{15}$$

and

$$\tau \in W^{2,\infty}([0,T]), \quad \text{for all } T > 0. \tag{16}$$

## The Porous systems

The study of Porous systems started in 1972, where the one-dimensional Porous-elastic model is given by

$$\begin{aligned} \rho_0 u_{tt} &= \mu u_{xx} + \beta\varphi_x, \text{ in } (0,l)\times(0,L) \\ \rho_0 k\varphi_{tt} &= \alpha\varphi_{tt} - \beta u_x - \tau\varphi_t - \xi\varphi, \text{ in } (0,l)\times(0,L), \end{aligned}$$

and it has been studied by many authors. The first contribution in this direction was obtained by [59]. To be more precise, which was developed in [15], the authors showed that the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to the usual elastic effects, the materials with voids possess a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This concept was introduced in the pioneered work in [54], when the authors have advanced nonlinear theory of elastic materials with voids (See [8], [9]). In the other hand, the basic evolution equations for one-dimensional theories of porous materials with memory effect are given by

$$\rho u_{tt} = T_x, \qquad J\phi_{tt} = H_x + G, \tag{17}$$

where $T$ is the stress tensor, $H$ is the equilibrated stress vector and $G$ is the equilibrated body force. The variables $u$ and $\phi$ are the displacement of the solid elastic material and the volume fraction, respectively. The constitutive equations are

$$T = \mu u_x + b\phi, \qquad H = \delta\phi_x - \int_0^t g\,(t-s)\,\phi_x\,(s)\,ds, \qquad G = -bu_x - \xi\phi. \tag{18}$$

In [2] substituting (18) into (17) is concerned

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & \text{in } (0,1)\times(0,\infty) \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \int_0^t g\,(t-s)\,\phi_{xx}\,(x,s)\,ds = 0, & \text{in } (0,1)\times(0,\infty)\,. \end{cases} \tag{19}$$

A porous-elastic system with memory term and Neumann-Dirichlet boundary conditions where $g$ is the relaxation function it has been proved a general decay result, for more detail (see [2]). Apalara established the stability result of solutions to the system for the case of equal speeds of wave propagation. Feng et al [12], developed the work for the case of nonequal speeds of wave propagation . In [30] the authors considered the following system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + a\phi + \displaystyle\int_0^\infty g(s)\phi_{xx}(t-s)ds + \alpha(t)f(\phi_t) = 0, \end{cases} \tag{20}$$

they proved the global well posedness and stability results of a nonlinear damping porous-elastic system with past history for the case of equal speeds of wave propagation. And in [31], The authors have developed the work for the case of nonequal speeds of wave propagation.

Choucha et al [7] considered the following system with memoery and distributed delay terms

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \displaystyle\int_0^t g(s)\phi_{xx}(t-s)ds \\ +\mu_1\phi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\phi_t(x, t-\varrho)d\varrho = 0, \end{cases} \tag{21}$$

they proved the exponential stability results for the case of equal speeds of wave propagation under assumptions suitables.

## The system of damped wave equation

Originally the interaction between the source term and the damping term in the wave equation is given by :

$$u_{tt} - \Delta u + a\left|u_t\right|^{m-2} u_t = b\left|u\right|^{p-2} u, \qquad \text{in } \Omega \times (0,T), \tag{22}$$

where $\Omega$ is a bounded domain of $\mathbb{R}^N$, $N \geq 1$ with a smooth boundary $\partial\Omega$, has an exciting history. It has been shown that the existence and the asymptotic behavior of solutions depend on a crucial way on the parameters $m$, $p$ and on the nature of the initial data. More precisely, it is well known that in the absence of the source term $\left|u\right|^{p-2} u$ then a uniform estimate of the form

$$\|u_t(t)\|_2 + \|\nabla u(t)\|_2 \leq C, \tag{23}$$

holds for any initial data $(u_0, u_1) = (u(0), u_t(0))$ in the energy space $H_0^1(\Omega) \times L^2(\Omega)$, where $C$ is a positive constant independent of $t$. The estimate (23) shows that any local solution $u$ of problem (22) can be continued in time as long as (23) is verified. This result has been proved by several authors. See for example [21, 27].

On the other hand in the absence of the damping term $\left|u_t\right|^{m-2} u_t$, the solution of (22) ceases to exist and there exists a finite value $T^*$ such that

$$\lim_{t\to T^*} \|u(t)\|_p = +\infty, \tag{24}$$

the reader is refereed to Ball [3] and Kalantarov & Ladyzhenskaya [23] for more details. When both terms are present in equation (22), the situation is more delicate. This case has been considered by Levine in [35, 36], where he investigated problem (22) in the linear damping case ($m = 2$) and showed that any local solution $u$ of (22) cannot be continued in $(0,\infty) \times \Omega$ whenever the initial data are large enough (negative initial energy). The main tool used in [35] and [36] is the "concavity method". This method has been a widely applicable tool to prove the blow up of solutions in finite time of some evolution equations. The basic idea of this method is to construct a positive functional $\theta(t)$ depending on certain norms of the solution and show that for some $\gamma > 0$, the function $\theta^{-\gamma}(t)$ is a positive concave function of $t$. Thus there exists $T^*$ such that $\lim_{t\to T^*} \theta^{-\gamma}(t) = 0$. Since then, the concavity method became a powerful and simple tool to prove blow up in finite time for other related problems. Unfortunately, this method is limited to the case of a linear damping. Georgiev and Todorova [17] extended Levine's result to the nonlinear damping case ($m > 2$). In their work, the authors considered the problem (22) and introduced a method different from the one known as the concavity method. They showed that the solutions with negative energy continue to exist globally 'in time' if the damping term dominates the source term (i.e.$m \geq p$) and blow up in finite time in the other case (i.e.$p > m$) if the initial energy is sufficiently negative. Their method is based on the construction of an auxiliary function $L$ which is a perturbation of the total energy of the system and satisfies the differential inequality

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t). \tag{25}$$

In $[0,\infty)$, where $\nu > 0$. Inequality (25) leads to a blow up of the solutions in finite time $t \geq L(0)^{-\nu}\xi^{-1}\nu^{-1}$, provided that $L(0) > 0$. However the blow up result in [17] was not optimal in terms of the initial data causing the finite time blow up of solutions. Thus several improvement have been made to the result in [17] (see for example [33, 34, 39, 67]. In particular, Vitillaro in [67] combined the arguments in [17] and [34] to extend the result in [17] to situations where the damping is nonlinear and the solution has positive initial energy.

In [69], Yang.Z, studied the problem

$$\begin{gathered} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta-2}\nabla u_t\right) \\ +a\,|u_t|^{m-2}u_t = b|u|^{p-2}u, \end{gathered} \tag{26}$$

in $(0,T)\times\Omega$ with initial conditions and boundary condition of Dirichlet type. He showed that the solutions is blow up in finite time $T^*$ under the condition $p > \max\{\alpha, m\}$, $\alpha > \beta$, and the initial energy is sufficiently negative (see condition $(ii)$ in [69][Theorem 2.1]). In fact this condition made it clear that there exists a certain relation between the blow-up time and $|\Omega|$([69][Remark 2]).

Messaoudi and Said-Houari [42] improved the result in [69] and showed that the blow up of solutions of problem (26) takes place for negative initial data only regardless of the size of $\Omega$.

### The main results of this thesis

This thesis contains five chapters.

**Chapter 1** : In this chapter, we consider a nonlinear Thermoelastic Bresse systems of second sound with delay term in the internal feedback and infinity history acting on the shear angle displacement also with forcing term in the second equation. Under suitable assumptions on the data, we prove the well-posedness of our problem. Furthermore, an exponential stability result will be shown without the usual equal wave speeds and an additional assumption on the coefficients. To achieve our goal, we make use the semigroup and energy method. This work has been published in [56]

**Chapter 2** : We show in this chapter the general stability of solution for one-dimensional Timoshenko systems with infinite history and distributed delay term regardless also of the speeds of wave propagation. We prove our result by using the energy method combined with some properties of convex functions. This work has been recently published in [28]

**Chapter 3** : The goal of this chapter is to investigate the exponential stability of the Timoshenko systems in thermoelasticity of second sound with a time-varying delay term in the internal feedback. The well-posedness of the problem is assured by using the variable norm technique of Kato. Furthermore the stability of the system is shown by applying the energy method. This work submited to the journal of Asymptotic analysis.

**Chapter 4** : In this chapter we consider a one-dimensional Thermoelastic Porous systems with nonlinear damping, infinite memory and distributed delay terms, where the heat conduction is given by Cattaneo's law. We establish the well posedness of the system. And we prove the stability results of the system for the cases of equal and nonequal speeds of wave propagation. This work has been recently published in [29]

**Chapter 5** : In the last chapter we will substantiation that the positive intial-energy of solution for coupled nonlinear Klein-Gordon equations with degenerate damping and source terms. We prove, with positive initial energy, the global nonexistence of solution by concavity method. This work has been recently accepted in the journal of Studia Mathematica.

# Chapter 1

## A New result of stability for thermoelastic-Bresse system of second sound related with forcing, delay, and past history terms

### 1.1 Introduction

In this chapter, we treated (with Ouchenane et al.[56]) the following problem:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + lw + \psi\right)_x - k_0 l\left(w_x - l\varphi\right) + \mu_1\varphi_t\left(x,t\right) \\ +\mu_2\varphi_t\left(x,t-\tau\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + lw + \psi\right) + \int_0^\infty g\left(s\right)\psi_{xx}\left(x,t-s\right)ds \\ +\gamma\theta_x + f\left(\psi\right) = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + kl\left(\varphi_x + lw + \psi\right) = 0, \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \kappa\theta_x = 0, \end{cases} \tag{1.1}$$

where $(x, t) \in (0, 1) \times (0, \infty)$, with initial-boundary conditions

$$\begin{aligned} \varphi(0,t) = \varphi_x(1,t) &= \psi_x(0,t) = \psi(1,t) \\ &= w_x(0,t) = w(1,t) \\ &= \theta(0,t) = q(1,t) = 0, t \geq 0, \end{aligned} \tag{1.2}$$

and

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), x \in (0,1) \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), x \in (0,1) \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), x \in (0,1) \\ \theta(x,0) = \theta_0(x), q(x,0) = q_0(x), x \in (0,1) \\ \varphi_t(x,t-\tau) = f_0(x,t-\tau), x \in (0,1), \end{cases} \tag{1.3}$$

with $\tau > 0$ is a time delay, $\mu_1$ and $\mu_2$ are positive real numbers. The function $\theta$ is the temperature difference, $q$ is the heat flux, $\rho_1, \rho_2, \rho_3, k, l, k_0, b, \gamma, \kappa, \alpha, \beta$ are positive constants, and the relaxation function $g$ satisfies the following assumptions:

(H1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a $C^1$ function satisfying

$$g(0) > 0, \qquad b - \int_0^\infty g(s)\,ds = b - g_0 = L > 0.$$

(H2) There exists a positive constant $\zeta$ such that

$$g'(t) \leq -\zeta g(t), \qquad \forall t \geq 0.$$

(H3) $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing $C^0$-function such that there exist positive constants $\nu_1, \nu_2, \epsilon$ and a strictly increasing function $G \in C^1([0,\infty))$ with $G(0) = 0$ and $G$ is a linear or strictly convex $C^2$–function on $(0, \epsilon.]$ such that

$$\begin{cases} s^2 + f^2(s) \leq G^{-1}(sf(s)), & \forall |s| \leq \epsilon \\ \nu_1 |s| \leq |f(s)| \leq \nu_2 |s|, & \forall |s| \geq \epsilon, \end{cases}$$

which implies that $sf(s) > 0$ for all $s \neq 0$. $f$ also satisfies the following property:

$$\left| f\left(\psi^2\right) - f\left(\psi^1\right) \right| \leq k_0 \left( \left|\psi^1\right|^\theta - \left|\psi^2\right|^\theta \right) \left|\psi^1 - \psi^2\right| \quad \text{for all } \psi^1, \psi^2 \in \mathbb{R}, \tag{1.4}$$

where $k_0 > 0$ and $\theta > 0$. We prove the well-posedness and establish a stability results depending on the following parameters

$$\tilde{\eta} = \left(1 - \frac{\alpha k \rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\gamma^2 \alpha}{b} \qquad \text{and} \qquad k = k_0. \tag{1.5}$$

It is well known that, in the single wave equation, if $\mu_2 = 0$, that is, in the absence of a delay, the energy of system exponentially decays (see [5, 6, 26, 50]). On the contrary, if $\mu_1 = 0$, that is, there exists only the delay part in the interior, the system becomes unstable (see [11]). It is shown that a small delay in a boundary control can turn such a well-behaved hyperbolic system into a wild one and therefore, delay becomes a source of instability. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [51, 53, 68]).

## 1.2 Preliminaries and well-posedness

Firstly, we assume the following hypotheses:

$$|\mu_2| < \mu_1.$$

Using semi-group theory, we will prove the systems (1.1)-(1.3) are well posed by introduce the following new variable [53]

$$z(x,\rho,t) = \varphi_t\,(x, t - \tau\rho)\,, \qquad x \in (0,1)\,, \rho \in (0,1)\,, t > 0. \tag{1.6}$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \qquad \text{in } (0,1)\times(0,1)\times(0,\infty)\,. \tag{1.7}$$

We then set the auxiliary variable [10]

$$\eta^t\,(x,s) = \psi\,(x,t) - \psi\,(x,t-s)\,, \qquad s \geq 0.$$

For this reason, we find the following equation

$$\eta^t_t\,(x,s) + \eta^t_s\,(x,s) = \psi_t\,(x,t)\,.$$

Therefore, problem (1.1) takes the form

$$\begin{cases} \rho_1\varphi_{tt} - k\,(\varphi_x + lw + \psi)_x - lk_0\,(w_x - l\varphi) \\ +\mu_1\varphi_t\,(x,t) + \mu_2 z\,(x,1,t) = 0 \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \\ \rho_2\psi_{tt} - L\psi_{xx} + k\,(\varphi_x + lw + \psi) + \int_0^\infty g\,(s)\,\eta^t_{xx}\,(x,s)\,ds \\ +\gamma\theta_x + f\,(\psi\,(x,t)) = 0 \\ \rho_1 w_{tt} - k_0\,(w_x - l\varphi)_x + lk\,(\varphi_x + lw + \psi) = 0 \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0 \\ \alpha q_t + \beta q + \theta_x = 0 \\ \eta^t_t\,(x,s) + \eta^t_s\,(x,s) = \psi_t\,(x,t)\,. \end{cases} \tag{1.8}$$

With the boundary conditions:

$$\begin{aligned} \varphi(0,t) &= \varphi_x(1,t) = \psi_x(0,t) = \psi(1,t) = w_x(0,t) = w(1,t) \\ &= \theta(0,t) = q(1,t) = 0, \quad t \geq 0, \end{aligned} \tag{1.9}$$

and the initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \ x \in (0,1) \\ \psi_t(x,0) = \psi_1(x), w(x,0) = w_0(x), \\ w_t(x,0) = w_1(x), x \in (0,1) \\ \theta(x,0) = \theta_0(x), q(x,0) = q_0(x), x \in (0,1) \\ \varphi_t(x,-t) = f_0(x,t) \text{ in } (0,1) \times (0,\tau) \\ z(x,1,t) = f(x,t-\tau) \text{ in } (0,1) \times (0,\tau) \\ \eta^t(x,0) = 0, \qquad \forall t \geq 0 \\ \eta^t(0,s) = \eta^t(1,s) = 0 \qquad \forall s \geq 0 \\ \eta^0(x,s) = \eta_0(s) = 0 \qquad \forall s \geq 0. \end{cases} \tag{1.10}$$

Let $\xi$ be positive constants such that:

$$\tau |\mu_2| < \xi < \tau (2\mu_1 - |\mu_2|), \tag{1.11}$$

where, $\tau$ is a real number with $0 < \tau$ and $\mu_1, \mu_2$ are a positive constants, and the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, f, \theta_0, q_0, \eta_0)$ belong to a suitable space (see below).
If we set

$$U = (\varphi, \varphi_t, z, \psi, \psi_t, w, w_t, \theta, q, \eta^t)^T,$$

then

$$U' = (\varphi_t, \varphi_{tt}, z_t, \psi_t, \psi_{tt}, w_t, w_{tt}, \theta_t, q_t, \eta_t^t)^T.$$

Therefore, problem (1.8)-(1.10) can be written as

$$\begin{cases} U'(t) = AU(t) + F \\ U(0) = (\varphi_0, \varphi_1, f_1(.,\tau), \psi_0, \psi_1, w_0, w_1, \theta_0, q_0, \eta_0), \end{cases} \tag{1.12}$$

where the operator $A$ is defined by

$$A\begin{pmatrix}\varphi\\ u\\ z\\ \psi\\ v\\ w\\ \varpi\\ \theta\\ q\\ \phi\end{pmatrix}=\begin{pmatrix}u\\ \frac{k}{\rho_1}(\varphi_x+lw+\psi)_x+\frac{k_0 l}{\rho_1}(w_x-l\varphi)-\frac{\mu_1}{\rho_1}u-\frac{\mu_2}{\rho_1}z(.,1)\\ -\left(\frac{1}{\tau}\right)z_\rho\\ v\\ \frac{L}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+lw+\psi)+\frac{1}{\rho_2}\int_0^\infty g(s)\,\phi_{xx}(s)\,ds-\frac{\gamma}{\rho_2}\theta_x\\ \varpi\\ \frac{k_0}{\rho_1}(w_x-l\varphi)_x-\frac{kl}{\rho_1}(\varphi_x+lw+\psi)\\ -\frac{1}{\rho_3}q_x-\frac{\gamma}{\rho_3}v_x\\ -\frac{\beta}{\alpha}q-\frac{k}{\alpha}\theta_x\\ -\phi_s+v\end{pmatrix}, \tag{1.13}$$

$$F=\begin{pmatrix}0\\0\\0\\0\\-\frac{1}{\rho_2}f(\psi)\\0\\0\\0\\0\\0\end{pmatrix}.$$

We consider the following spaces

$$H_*^1(0,1)=\left\{h\in H^1(0,1):h(0)=0\right\},$$

$$\widetilde{H}_*^1(0,1)=\left\{h\in H^1(0,1):h(1)=0\right\},$$

$$H_*^2(0,1)=H^2(0,1)\cap H_*^1(0,1),$$

$$\widetilde{H}_*^2(0,1)=H^2(0,1)\cap\widetilde{H}_*^1(0,1),$$

and

$$\begin{aligned}\mathcal{H} \;=\;& H_*^1(0,1)\times L^2(0,1)\times L^2\left((0,1),H_0^1(0,1)\right)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)\times\widetilde{H}_*^1(0,1)\\ &\times L^2(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)\times L_g^2\left(\mathbb{R}^+,H_0^1(0,1)\right),\end{aligned}$$

where $L_g^2\left(\mathbb{R}^+, H_0^1(0,1)\right)$ denotes the Hilbert space of $H_0^1-$valued functions on $\mathbb{R}^+$, endowed with the inner product

$$(V_1, V_2)_{L_g^2(\mathbb{R}^+, H_0^1(\Omega))} = \int_0^1 \int_0^1 g(s) V_{1x}(s) V_{2x}(s)\, ds dx.$$

We will show under the assumption (1.11) that $A$ generates a $C_0$ semigroup on $\mathcal{H}$.
Now, we consider the vectors

$$U = (\varphi, u, z, \psi, v, w, \varpi, \theta, q, \phi)^T, \overline{U} = \left(\overline{\varphi}, \overline{u}, \overline{z}, \overline{\psi}, \overline{v}, \overline{w}, \overline{\varpi}, \overline{\theta}, \overline{q}, \bar{\phi}\right)^T,$$

and we define the inner product

$$\begin{aligned}\left\langle U, \overline{U}\right\rangle_{\mathcal{H}} &= k\int_0^1 (\varphi_x + \psi + lw)\left(\overline{\varphi}_x + \overline{\psi} + l\overline{w}\right) dx + \rho_2 \int_0^1 v\overline{v}dx + \rho_1 \int_0^1 \varpi\overline{\varpi}dx \\ &+ k_0 \int_0^1 (w_x - l\varphi)(\overline{w}_x - l\overline{\varphi})\, dx + l\int_0^1 \psi_x \overline{\psi_x} dx \\ &+ \rho_1 \int_0^1 u\overline{u}dx + \xi \int_0^1 \int_0^1 z\overline{z}d\rho dx + \rho_3 \int_0^1 \theta\overline{\theta}dx \\ &+ \alpha \int_0^1 q\overline{q}dx + \int_0^1 \int_0^\infty g(s)\, \phi_x(s)\, \bar{\phi}_x(s)\, dxds. \end{aligned} \tag{1.14}$$

$\mathcal{H}$ is a Hilbert space for $l$ small enough since, in this case, the above inner product is equivalent to the natural inner product defined on $\mathcal{H}$.
The domain of $A$ is given by

$$D(A) = \left\{ \begin{array}{c} U \in \mathcal{H}/\varphi \in H_*^2(0,1); \psi, w \in \widetilde{H}_*^2(0,1), u, \theta \in H_*^1(0,1); \\ v, \varpi, q \in \widetilde{H}_*^1(0,1), u = z(.,0), z_\rho \in L^2\left((0,1); L^2(0,1)\right) \\ , \varphi_x(1) = 0, w_x(0) = \psi_x(0) = 0, \\ \phi_s \in L_g^2\left(\mathbb{R}^+, H_0^1(0,1)\right), \quad \phi(x,0) = 0, \end{array} \right\}. \tag{1.15}$$

Important properties of the above metrics are stated in the following lemmas. Although most of these results are followed straightforwardly from the known results, they are crucial for what follows. So for the convenience of the reader, we give their proofs here.

**Lemma 1.1.** *The operator A is dissipative and satisfies, for any $U \in D(A)$,*

$$\begin{aligned}\langle AU, U\rangle_{\mathcal{H}} &= -\beta\int_0^1 q^2 dx + \Big(-\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau}\Big)\int_0^1 u^2 dx \\ &\quad + \Big(\frac{\mu_2}{2} - \frac{\xi}{2\tau}\Big)\int_0^1 z^2(x,1)\,dx \\ &\quad + \frac{1}{2}\int_0^1\int_0^\infty g'(s)\,|\phi_x(x,s)|^2\,ds dx \\ &\le 0. \end{aligned} \tag{1.16}$$

*Proof.* For any $U \in D(A)$, using the inner product,

$$\langle AU, U\rangle_{\mathcal{H}} = \left\langle \begin{pmatrix} u \\ \frac{k}{\rho_1}(\varphi_x + lw + \psi)_x + \frac{k_0 l}{\rho_1}(w_x - l\varphi) - \frac{\mu_1}{\rho_1}u - \frac{\mu_2}{\rho_1}z(.,1) \\ -\left(\frac{1}{\tau}\right) z_\rho \\ v \\ \frac{L}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + lw + \psi) + \frac{1}{\rho_2}\int_0^\infty g(s)\,\phi_{xx}(s)\,ds - \frac{\gamma}{\rho_2}\theta_x \\ -\varpi \\ \frac{k_0}{\rho_1}(w_x - l\varphi)_x - \frac{kl}{\rho_1}(\varphi_x + lw + \psi) \\ -\frac{1}{\rho_3}q_x - \frac{\gamma}{\rho_3}v_x \\ -\frac{\beta}{\alpha}q - \frac{1}{\alpha}\theta_x \\ -\phi_s + v \end{pmatrix}, \begin{pmatrix} \varphi \\ u \\ z \\ \psi \\ v \\ w \\ \varpi \\ \theta \\ q \\ \phi \end{pmatrix} \right\rangle_{\mathcal{H}}.$$

Then

$$\begin{aligned}\langle AU, U\rangle_{\mathcal{H}} &= k\int_0^1 (u_x + v + l\varpi)(\varphi_x + lw + \psi)\,dx + k_0\int_0^1 (\varpi_x - lu)(w_x - l\varphi)\,dx \\ &\quad + k\int_0^1 (\varphi_x + lw + \psi)\,u dx + k_0 l\int_0^1 (w_x - l\varphi)\,u dx \\ &\quad - \mu_1\int_0^1 u^2 dx - \mu_2\int_0^1 z(x,1)\,u dx + L\int_0^1 \psi_{xx} v dx \\ &\quad - k\int_0^1 (\varphi_x + lw + \psi)\,v dx - \gamma\int_0^1 \theta_x v dx \\ &\quad + k_0\int_0^1 (w_x - l\varphi)\,\varpi dx - kl\int_0^1 (\varphi_x + lw + \psi)\,\varpi dx \\ &\quad + L\int_0^1 v_x\psi_x dx + \int_0^1\int_0^\infty g(s)\,\phi_x(s)\,(-\phi_s + v)\,dx ds \\ &\quad - \int_0^1 q_x\theta dx - \gamma\int_0^1 u_x\theta dx - \beta\int_0^1 q^2 dx \\ &\quad - \int_0^1 \theta_x q dx - \xi\int_0^1\int_0^1 z z_\rho d\rho dx. \end{aligned} \tag{1.17}$$

By the fact that

$$
\begin{aligned}
& -\beta \int_0^1 q^2 dx - \mu_1 \int_0^1 u^2 dx - \mu_2 \int_0^1 z(x,1)\, u dx + \int_0^1 \int_0^\infty g(s)\, \phi_x(s)\, (-\phi_s + v)\, dx ds \\
& -\frac{\xi}{\tau} \int_0^1 \int_0^1 z(x,\rho)\, z_\rho(x,\rho)\, d\rho dx \\
= & -\beta \int_0^1 q^2 dx - \mu_1 \int_0^1 u^2 dx - \mu_2 \int_0^1 z(x,1)\, u dx + \int_0^1 \int_0^\infty g(s)\, \phi_x(s)\, (-\phi_s + v)\, dx ds \\
& -\frac{\xi}{2\tau} \int_0^1 \int_0^1 \frac{\partial}{\partial \rho} z^2(x,\rho)\, d\rho dx \\
= & -\beta \int_0^1 q^2 dx - \mu_1 \int_0^1 u^2 dx - \mu_2 \int_0^1 z(x,1)\, u dx + \int_0^1 \int_0^\infty g(s)\, \phi_x(s)\, (-\phi_s + v)\, dx ds \\
& -\frac{\xi}{2\tau} \int_0^1 \left\{ z^2(x,1) - z^2(x,0) \right\} dx \\
= & -\beta \int_0^1 q^2 dx - \mu_1 \int_0^1 u^2 dx - \mu_2 \int_0^1 z(x,1)\, u dx + \int_0^1 \int_0^\infty g(s)\, \phi_x(s)\, (-\phi_s + v)\, dx ds \\
& -\frac{\xi}{2\tau} \int_0^1 z^2(x,1)\, dx + \frac{\xi}{2\tau} \int_0^1 u^2 dx,
\end{aligned}
$$

and using Young's inequality, we find

$$
\begin{aligned}
\langle AU, U \rangle_H \leq & -\beta \int_0^1 q^2 dx + \left( -\mu_1 + \frac{\mu_2}{2} + \frac{\xi}{2\tau} \right) \int_0^1 u^2 dx + \left( \frac{\mu_2}{2} - \frac{\xi}{2\tau} \right) \int_0^1 z^2(x,1)\, dx \\
& + \frac{1}{2} \int_0^1 \int_0^\infty g'(s)\, |\phi_x(x,s)|^2\, ds dx.
\end{aligned}
$$

Keeping in mind condition (1.11), the desired result yields. □

**Lemma 1.2.** *The operator $I - A$ is surjective.*

*Proof.* We need to show that for all $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7 f_8, f_9, f_{10})^T \in \mathcal{H}$, there exists $U \in D(A)$ such that

$$
U - AU = \mathcal{F}, \tag{1.18}
$$

that is

$$\begin{cases} -u+\varphi = f_1 \in H_*^1(0,1) \\ -k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + \rho_1 u + \mu_1 u + \mu_2 z(.,1) = \rho_1 f_2 \in L^2(0,1) \\ z + \tau^{-1} z_\rho = \tau f_3 \in L^2\left((0,1), H^1(0,1)\right) \\ -v + \psi = f_4 \in \widetilde{H}_*^1(0,1) \\ -L\psi_{xx} + k(\varphi_x + lw + \psi) + \rho_2 v - \int_0^\infty g(s)\phi_{xx}(s)\,ds + \gamma\theta_x = \rho_2 f_5 \in L^2(0,1) \\ -\varpi + w = f_6 \in \widetilde{H}_*^1(0,1) \\ -k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \rho_1 \varpi = \rho_1 f_7 \in L^2(0,1) \\ q_x + \gamma v_x + \rho_3 \theta = \rho_3 f_8 \in L^2(0,1) \\ (\beta + \alpha) q + \theta_x = \alpha f_9 \in L^2(0,1) \\ \phi + \phi_s - v = f_{10} \in L^2(0,1). \end{cases} \tag{1.19}$$

From (1.19), we define

$$\theta = \frac{\alpha}{k}\int_0^x f_9(y)\,dy - \frac{\alpha}{k}(\beta+\alpha)\int_0^x q(y)\,dy, \tag{1.20}$$

so $\theta(0,t) = 0$.

Inserting $u = \varphi - f_1, v = \psi - f_4, \varpi = w - f_6$ and (1.19) into (1.20) , we get

$$\begin{cases} -k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + \rho_1 \varphi + \mu_1 u + \mu_2 z(.,1) = h_1 \in L^2(0.1) \\ -L\psi_{xx} + k(\varphi_x + lw + \psi) + \rho_2 \psi - \int_0^\infty g(s)\phi_{xx}(s)\,ds - \gamma(\beta+\alpha) q = h_2 \in L^2(0.1) \\ -k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \rho_1 w = h_3 \in L^2(0.1) \\ q_x + (\beta+\alpha)\int_0^x q(y)\,dy - \gamma\psi_x = h_4 \in L^2(0.1) \\ z + \tau^{-1} z_\rho = h_5 \in L^2(0.1) \\ \phi + \phi_s - v = h_6 \in L^2(0.1), \end{cases} \tag{1.21}$$

where

$$\begin{cases} h_1 = \rho_1(f_1 + f_2) \\ h_2 = \rho_2(f_4 + f_5) - \frac{\alpha}{k}\gamma f_9 \\ h_3 = \rho_1(f_6 + f_7) \\ h_4 = \gamma f_{4x} + \rho_3\left(f_8 - \frac{\alpha}{k}\int_0^x f_9(y)\,dy\right) \\ h_5 = z + \tau^{-1} z_\rho \\ h_6 = \phi + \phi_s - v. \end{cases} \tag{1.22}$$

Furthermore, by (1.19) we can find as $z(x,0) = u(x)$ for $x \in (0,1)$. Following the same approach as in ([51]), we obtain, by using equation for $z$ in (1.19)

$$z(x,\rho) = u(x)e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho f_3(x,s)e^{\tau\rho s}ds.$$

From (1.19), we obtain

$$z(x,\rho) = \varphi(x)e^{-\tau\rho} - f_1 e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^{\rho} f_3(x,s)e^{\tau\rho s}ds,$$

then

$$z(x,1) = \varphi(x)e^{-\tau} + z_0(x),$$

such that

$$z_0(x) = -f_1 e^{-\tau} + \tau e^{-\tau}\int_0^{\rho} f_3(x,s)e^{\tau s}ds.$$

We note that the last equation in (1.21) with $\phi(x,0) = 0$ has a unique solution

$$\begin{aligned}\phi(x,s) &= \left(\int_0^x e^y (f_{10}(x,y) + v(x)\,dy)\,e^{-s}\right)\\ &= \left(\int_0^x e^y (f_{10}(x,y) + \psi(x) - f_4(x)\,dy)\,e^{-s}\right).\end{aligned} \tag{1.23}$$

To solve (1.22) we consider

$$a\left((\varphi,\psi,w,q),(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q})\right) = L\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right), \tag{1.24}$$

where

$$a : \left[H_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times L^2(0,1)\right]^2 \longrightarrow \mathbb{R},$$

is the bilinear form given by

$$\begin{aligned}a\left((\varphi,\psi,w,q),(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q})\right) &= k\int_0^1 (\varphi_x + lw + \psi)\left(\widetilde{\varphi}_x + l\widetilde{w} + \widetilde{\psi}\right)dx\\ &\quad +(\beta+\alpha)\int_0^1 q\widetilde{q}dx + b\int_0^1 \psi_x\widetilde{\psi}_x dx + \rho_2\int_0^1 \psi\widetilde{\psi}dx\\ &\quad -\gamma(\beta+\alpha)\int_0^1 q\widetilde{\psi}dx + \rho_1\int_0^1 \psi\widetilde{\psi}dx + \gamma(\beta+\alpha)\int_0^1 \psi\widetilde{q}dx\\ &\quad +\rho_1\int_0^1 w\widetilde{w}dx + k_0\int_0^1 (w_x - l\varphi)(\widetilde{w}_x - l\widetilde{\varphi})\,dx\\ &\quad +\int_0^1 \varphi\widetilde{\varphi}(\mu_1 + \mu_2 e^{-\tau})\,dx\\ &\quad +\rho_3(\beta+\alpha)\int_0^1\left(\int_0^x q(y)\,dy\int_0^x \widetilde{q}(y)\,dy\right)dx,\end{aligned} \tag{1.25}$$

and

$$L : \left[H_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times \widetilde{H}_*^1(0,1)\times L^2(0,1)\right] \longrightarrow \mathbb{R},$$

is the linear form defined by

$$\begin{aligned} L\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right) &= \int_0^1 h_1\widetilde{\varphi}dx+\int_0^1 h_2\widetilde{\psi}dx+\int_0^1 h_3\widetilde{w}dx \\ &+(\alpha+\beta)\int_0^1 h_4\int_0^x \widetilde{q}(y)\,dydx \\ &+\int_0^1 (\mu_1 f_1\mu_2 z_0)\,\widetilde{\varphi}dx. \end{aligned} \tag{1.26}$$

It is easy to verify that $a$ is continuous and coercive, and $L$ is continuous. So applying the Lax-Milgram theorem, we deduce that for all $\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right)\in H_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)$ problem (1.24) admits a unique solution $(\varphi,\psi,w,q)\in H_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times\widetilde{H}_*^1(0,1)\times L^2(0,1)$. Since $D(A)$ is dense in $\mathcal{H}$ Consequently, using Lemmas 2.1 and 2.2, we conclude that $A$ is a maximal monotone operator. Hence, by Hille-Yosida theorem (see [37] and [58]) we have the following well-posedness result such that (1.12) is satisfied. □

**Lemma 1.3.** *The operator $F$ defined in (1.13) is locally Lipschitz in $\mathcal{H}$*

*Proof.* Let $U=(\varphi,u,z,\psi,v,w,\varpi,\theta,q,\phi)^T$, $\overline{U}=\left(\overline{\varphi},\overline{u},\overline{z},\overline{\psi},\overline{v},\overline{w},\overline{\varpi},\overline{\theta},\overline{q},\overline{\phi}\right)^T$, then we have

$$\left\|F(U)-F\left(\overline{U}\right)\right\|_{\mathcal{H}}\le\left\|f(\psi)-f\left(\overline{\psi}\right)\right\|_{L^2}.$$

By using (1.4), Holder's and Poincaré's inequalities, we can obtain

$$\begin{aligned} \left\|f(\psi)-f\left(\overline{\psi}\right)\right\|_{L^2} &\le \left(\|\psi\|_{2\theta}^{\theta}+\left\|\overline{\psi}\right\|_{2\theta}^{\theta}\right)\left\|\psi-\overline{\psi}\right\| \\ &\le c_1\left\|\psi-\overline{\psi}\right\|, \end{aligned}$$

wich gives us

$$\left\|F(U)-F\left(\overline{U}\right)\right\|_{\mathcal{H}}\le c_1\left\|\psi-\overline{\psi}\right\|_{\mathcal{H}}.$$

Then the operator $F$ is locally Lipschitz in $\mathcal{H}$, the proof is hence complete. □

**Theorem 1.4.** *Let $U_0\in\mathcal{H}$, then there exists a unique weak solution $U\in C(\mathbb{R}^+,\mathcal{H})$ of problem (1.1)-(1.3). Moreover, if $U_0\in D(A)$, then $U\in C(\mathbb{R}^+,D(A))\cap C^1(\mathbb{R}^+,\mathcal{H})$.*

## 1.3 Exponential stability

In this section, we state and prove our stability result for the energy of the solution of system (1.1)-(1.3), by using the multiplier technique. So we define the energy of our system by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+b\psi_x^2+\rho_3\theta^2+\alpha q^2\right. \\ &\quad \left.+k(\varphi_x+\psi+lw)^2+k_0(w_x-l\varphi)^2\right]dx+\frac{\xi}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx \\ &\quad +\frac{1}{2}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx+\int_0^1 \hat{f}(\psi(t))dx. \end{aligned} \tag{1.27}$$

The proof of the stability for our system is based on the following lemmas

**Lemma 1.5.** *Let $(\varphi,\psi,w,\theta,q,z,\eta^t)$ be the solution of (1.8)-(1.10). Then the energy functional, defined by (1.27) satisfies*

$$\begin{aligned} E'(t) &\le -\beta\int_0^1 q^2dx-C\int_0^1\psi_t^2dx-\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\|\varphi_t\|_2^2 \\ &\quad -\left(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)\|z(x,1,t)\|_2^2 \\ &\quad +\frac{1}{2}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx, \end{aligned} \tag{1.28}$$

*such that $C>0$*

*Proof.* Multiplying $(1.1)_1$, $(1.1)_2$, $(1.1)_3$, $(1.1)_4$, and $(1.1)_5$ by $\varphi_t$, $\psi_t$, $w_t$, $\theta$, and $q$, respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions, and adding the results, we obtain (1.28).

With, the fact

$$\frac{d}{dt}\hat{f}(\psi)=f(\psi)\psi,$$

give us (1.28). □

**Lemma 1.6.** *Let $(\varphi,\psi,w,\theta,q,z,\eta^t)$ be the solution of (1.8)-(1.10). Then the functional*

$$F_1(t):=\alpha\rho_3\int_0^1\theta\int_0^x q(y)\,dydx, \tag{1.29}$$

*satisfies, for any $\varepsilon_1 > 0$, the estimate*

$$F_1'(t) \leq -\frac{\rho_3}{2}\int_0^1 \theta^2 dx + \varepsilon_1 \int_0^1 \psi_t^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 q^2 dx. \tag{1.30}$$

*Proof.* Taking the derivative of $F_1$, using the fourth and fifth equations in (1.1) and performing integration by parts, we get

$$\begin{aligned} F_1'(t) &= -\rho_3 k \int_0^1 \theta^2 dx - \alpha k \int_0^1 q^2 dx - \alpha\gamma \int_0^1 \psi_{tx} \int_0^1 q(y)\,dydx \\ &\quad -\beta\rho_3 \int_0^1 \theta \int_0^x q(y)\,dydx. \end{aligned} \tag{1.31}$$

Using Cauchy–Schwarz and Young's inequalities with $\varepsilon_1 > 0$, we get (1.30). □

**Lemma 1.7.** *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (1.8)-(1.10). Then the functional*

$$F_2(t) := -\frac{\rho_2\rho_3}{\gamma}\int_0^1 \theta \int_0^x \psi_t(y)\,dydx, \tag{1.32}$$

*satisfies, for any $\varepsilon_1, \varepsilon_2, \delta_1 > 0$, the estimate*

$$\begin{aligned} F_2'(t) &\leq -\frac{\rho_2}{\gamma}\int_0^1 \psi_t^2 dx + \varepsilon_2 \int_0^1 (\varphi_x + \psi + lw)^2 dx \\ &\quad + \left(\varepsilon_3 + \frac{\rho_3}{\gamma}\left(\frac{\varepsilon_2}{b^2\lambda_2} + \frac{b^2}{2\varepsilon_2\lambda_2}\right)\right)\int_0^1 \psi_x^2 dx \\ &\quad + c\left(1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)\int_0^1 \theta^2 dx + c\int_0^1 q^2 dx \\ &\quad + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \\ &\quad + (\delta_1 + C_1)\int_0^1 \theta_x^2 dx. \end{aligned} \tag{1.33}$$

*Proof.* Differentiation of $F_2$, using equations in (1.1) and integration by parts, we obtain

$$\begin{aligned} F_2'(t) &= -\rho_2 \int_0^1 \psi_t^2 dx - \frac{\rho_2 k}{\gamma}\int_0^1 q\psi_t dx + \rho_3 \int_0^1 \theta^2 dx - \frac{b\rho_3}{\gamma}\int_0^1 \theta\psi_x dx \\ &\quad + \frac{k\rho_3}{\gamma}\int_0^1 (\varphi_x + \psi + lw)\int_0^x \theta(y)\,dydx \\ &\quad + \frac{\rho_3}{\gamma}\int_0^1\int_0^\infty g(s)\eta_x^t(x,s)\,ds\int_0^x \theta_x(y)\,dydx \\ &\quad + \frac{\rho_3}{\gamma}\int_0^1 \theta\int_0^x f(\psi)\,dydx. \end{aligned} \tag{1.34}$$

Estimate (1.33) follows by using Cauchy–Schwarz, Young's and Poincaré's inequalities that,

$$\int_0^1 |f(\psi)\,\theta|\,dx \le \int_0^1 |\psi|^\theta\,|\psi|\,|\theta|\,dx \le \|\psi\|_{2(\theta+1)}^\theta\,\|\psi\|_{2(\theta+1)}\,\|\theta\| \le C_1\int_0^1 \theta^2 dx.$$

□

**Lemma 1.8.** *Let* $(\varphi,\psi,w,\theta,q,z,\eta^t)$ *be the solution of (1.8)-(1.10). Then the functional*

$$F_3(t) := -\rho_1\int_0^1 (\varphi\varphi_t + ww_t)\,dx, \tag{1.35}$$

*satisfies the estimate*

$$\begin{aligned} F_3'(t) \le\; & -\left(\rho_1 - \frac{1}{4\varepsilon_4}\right)\int_0^1 \varphi_t^2 dx + c\int_0^1 \psi_x^2 dx \\ & + k_0\int_0^1 (w_x - l\varphi)^2\,dx + c\int_0^1 (\varphi_x+\psi+lw)^2\,dx \\ & - \rho_1\int_0^1 w_t^2 dx + (\varepsilon_5\mu_2 + \mu_1\varepsilon_4)\int_0^1 \varphi^2 dx \\ & + \frac{\mu_2}{4\varepsilon_5}\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{1.36}$$

*Proof.* Direct computations, using (1.1)-(1.3), give

$$\begin{aligned} F_3'(t) =\; & -\rho_1\int_0^1 \varphi_t^2 dx + k\int_0^1 (\varphi_x+\psi+lw)^2\,dx \\ & - k\int_0^1 (\varphi_x+\psi+lw)\,\psi dx - \rho_1\int_0^1 w_t^2 dx \\ & + k_0\int_0^1 (w_x - l\varphi)^2\,dx + \mu_1\int_0^1 \varphi\varphi_t dx + \mu_2\int_0^1 \varphi z(x,1,t)\,dx. \end{aligned} \tag{1.37}$$

Using Young's and Poincaré's inequalities, estimate (1.36) is established. □

**Lemma 1.9.** *Let* $(\varphi,\psi,w,\theta,q,z,\eta^t)$ *be the solution of (1.8)-(1.10). Then the functional*

$$F_4(t) := \rho_2\int_0^1 \psi\psi_t dx, \tag{1.38}$$

*satisfies for any* $\delta_2 > 0$ *the estimate*

$$\begin{aligned} F_4'(t) \leq & \left(\frac{b}{2}+\delta_2+C_2\right)\int_0^1 \psi_x^2 dx+\rho_2\int_0^1 \psi_t^2 dx \\ & +\frac{k^2}{b}\int_0^1 (\varphi_x+\psi+lw)^2 dx + c\int_0^1 \theta^2 dx \\ & +\frac{g_0}{4\delta_2}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned} \tag{1.39}$$

*Proof.* Taking the derivative of $F_4$ and using the second equation in (1.1), it follows that

$$\begin{aligned} F_4'(t) = & -b\int_0^1 \psi_x^2 dx+\rho_2\int_0^1 \psi_t^2 dx+\gamma\int_0^1 \psi_x\theta dx - k\int_0^1 (\varphi_x+\psi+lw)\,dx \\ & +\int_0^1 \psi_x(x)\int_0^\infty g(s)\,\eta_x^t(x,s)\,dsdx. - \int_0^1 \psi f(\psi)\,dx, \end{aligned} \tag{1.40}$$

where

$$\int_0^1 |f(\psi)\,\psi|\,dx \leq \int_0^1 |\psi|^\theta |\psi|\,|\psi|\,dx \leq \|\psi\|_{2(\theta+1)}^\theta \|\psi\|_{2(\theta+1)} \|\psi\| \leq C_2\int_0^1 \psi_x^2 dx.$$

Young's and Poincaré's inequalities for (1.40) yields (1.39). □

**Lemma 1.10.** *Let* $(\varphi,\psi,w,\theta,q,z,\eta^t)$ *be the solution of (1.8)-(1.10). Then the functional*

$$F_5(t) := -\rho_1\int_0^1 \varphi_t(w_x-l\varphi)\,dx - \rho_1\int_0^1 w_t(\varphi_x+\psi+lw)\,dx, \tag{1.41}$$

*satisfies the estimate*

$$\begin{aligned} F_5'(t) \leq & -\left(lk_0-\frac{\mu_1}{4\varepsilon_6}-\frac{\mu_2}{4\varepsilon_7}\right)\int_0^1 (w_x-l\varphi)^2 dx - \frac{l\rho_1}{2}\int_0^1 w_t^2 dx \\ & +(l\rho_1+\varepsilon_6\mu_1)\int_0^1 \varphi_t^2 dx + c\int_0^1 \psi_t^2 dx \\ & +lk\int_0^1 (\varphi_x+\psi+lw)^2 dx + \varepsilon_7\mu_2\int_0^1 z^2(x,1,t)\,dx. \end{aligned} \tag{1.42}$$

*Proof.* Differentiation of $F_5$, using $(1.1)_1$ and $(1.1)_3$ , we arrive at

$$\begin{aligned} F_5'(t) = & -lk_0\int_0^1 (w_x-l\varphi)^2 dx - l\rho_1\int_0^1 w_t^2 dx + l\rho_1\int_0^1 \varphi_t^2 dx \\ & +lk\int_0^1 (\varphi_x+\psi+lw)^2 dx - \rho_1\int_0^1 \psi_t w_t dx \\ & +\mu_1\int_0^1 \varphi_t(w_x-l\varphi)\,dx + \mu_2\int_0^1 z(x,1,t)(w_x-l\varphi)\,dx. \end{aligned} \tag{1.43}$$

Young's inequality for (1.43) yields (1.42) . □

**Lemma 1.11.** *Let* $(\varphi, \psi, w, \theta, q, z, \eta^t)$ *be the solution of (1.8)-(1.10) and let* $k = k_0$. *Then the functional*

$$F_6(t) := -\rho_1 \int_0^1 (w_x - l\varphi) \int_0^x w_t(y)\, dydx - \rho_1 \int_0^1 \varphi_t \int_0^x (\varphi_x + \psi + lw)\, dydx, \tag{1.44}$$

*satisfies the estimate*

$$\begin{aligned} F_6'(t) \quad &\leq \quad -\frac{\rho_1}{2} \int_0^1 \varphi_t^2 dx - k_0 \int_0^1 (w_x - l\varphi)^2 dx + \rho_1 \int_0^1 w_t^2 dx \\ &\quad +k \int_0^1 (\varphi_x + \psi + lw)^2 dx + \frac{\rho_1}{2} \int_0^1 \psi_t^2 dx. \end{aligned} \tag{1.45}$$

*Proof.* A simple differentiation of $F_6$, using the first and third equations in (1.1), leads to

$$\begin{aligned} F_6'(t) \quad &= \quad -\rho_1 \int_0^1 \varphi_t^2 dx - k_0 \int_0^1 (w_x - l\varphi)^2 dx + \rho_1 \int_0^1 w_t^2 dx \\ &\quad -\rho_1 \int_0^1 \varphi_t \int_0^x \psi_t(y)\, dy + k \int_0^1 (\varphi_x + \psi + lw)^2 dx \\ &\quad +l(k - k_0) \int_0^1 (w_x - l\varphi) \int_0^x (\varphi_x + \psi + lw)\, dydx. \end{aligned} \tag{1.46}$$

Where we have used integration by parts and the boundary conditions in (1.3). Young's and Cauchy-Schwarz inequalities, with the fact that $k = k_0$, give (1.45). □

**Lemma 1.12.** *Let* $(\varphi, \psi, w, \theta, q, z, \eta^t)$ *be the solution of (1.8)-(1.10) and let (1.5) holds. Then the functional*

$$\begin{aligned} F_7(t) \quad : \quad &= \rho_2 \int_0^1 \psi_t (\varphi_x + \psi + lw)\, dx + \frac{b\rho_1}{k} \int_0^1 \varphi_t \psi_x dx \\ &+ \frac{b\rho_3}{\gamma} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 \theta \varphi_t dx - \frac{b}{\gamma} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \int_0^1 q(\varphi_x + \psi + lw)\, dx \\ &- \frac{bl^2 \rho_2}{k_0} \int_0^1 \psi \psi_t dx + \frac{bl\rho_1}{k_0} \int_0^1 \psi w_t dx, \end{aligned} \tag{1.47}$$

*satisfies, for any* $\varepsilon_4, \varepsilon_5, \delta_3 > 0$, *the estimate*

$$\begin{aligned}
F_7'(t) \leq & -\left(\frac{k}{2} - \frac{b\eta}{\gamma\alpha\varepsilon_{10}} + \frac{\gamma}{4\varepsilon_1} + \frac{kb\rho_3}{\gamma 4\varepsilon_2\rho_1}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) + \frac{b}{4\varepsilon_3}\right)\int_0^1 (\varphi_x + \psi + lw)^2\, dx \\
& + \varepsilon_8 \int_0^1 w_t^2 dx + \left(\frac{b^2l^2}{k} + \frac{bl^2\rho_2\delta_3}{k_0} + b\varepsilon_3 + \frac{bl^2}{k_0}c_1 + 2\left(\frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1}\right) + c_2\right)\int_0^1 \psi_x^2 dx \\
& + \varepsilon_9 \int_0^1 (w_x - l\varphi)^2\, dx + c\left(1 + \frac{1}{\varepsilon_8} + \frac{b\rho_1\varepsilon_4}{k}\right)\int_0^1 \psi_t^2 dx \\
& + c\left(1 + \frac{1}{\varepsilon_8}\right)\int_0^1 q^2 dx + c\left(1 + \frac{1}{\varepsilon_9} + \frac{b\rho_3\mu_1}{\gamma\rho_1}\varepsilon_5\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)\int_0^1 \theta^2 dx \\
& + \left(\frac{b\eta}{\gamma\alpha}\varepsilon_{10} + \gamma\varepsilon_1 + \frac{kb\rho_3}{\gamma\rho_1}\varepsilon_2\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)\int_0^1 \theta_x^2 dx \\
& + \left(\frac{b\rho_1}{k4\varepsilon_4} + \frac{b\rho_3\mu_1}{4\varepsilon_5\gamma\rho_1}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)\int_0^1 \varphi_t^2 dx + \frac{l\varepsilon_1}{b^2}\int_0^1 (w + \psi)^2\, dx \\
& + \frac{\varepsilon}{2}\int_0^1 (\varphi_x + \psi)^2 + \frac{g_0 bl^2\rho_2}{k_0 4\delta_3}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx. \qquad (1.48)
\end{aligned}$$

*Proof.* Taking the derivate of $F_7$, we obtain

$$\begin{aligned}
F_7'(t) = & \rho_2 \int_0^1 \psi_{tt}(\varphi_x + \psi + lw)\, dx + \rho_2 \int_0^1 \psi_t (\varphi_x + \psi + lw)_t\, dx \\
& + \frac{b\rho_1}{k}\int_0^1 \varphi_{tt}\psi_x dx - \frac{b\rho_1}{k}\int_0^1 \varphi_t\psi_{xt} dx + \frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 \theta_t\varphi_t dx \\
& + \frac{b\rho_3}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 \theta\varphi_{tt} dx - \frac{b}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 q_t(\varphi_x + \psi + lw)\, dx \\
& - \frac{b}{\gamma}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\int_0^1 q(\varphi_x + \psi + lw)_t\, dx - \frac{bl^2\rho_2}{k_0}\int_0^1 \psi_t^2 dx - \frac{bl^2\rho_2}{k_0}\int_0^1 \psi_{tt}\psi dx \\
& + \frac{bl\rho_1}{k_0}\int_0^1 w_{tt}\psi dx + \frac{bl\rho_1}{k_0}\int_0^1 w_t\psi_t dx. \qquad (1.49)
\end{aligned}$$

Now, we work out the terms in the right-hand side of (1.49), using the equations in (1.1)-(1.3) and integration by parts.

$$\begin{aligned}\rho_2 \int_0^1 \psi_{tt} (\varphi_x + \psi + lw)\, dx &= -k \int_0^1 (\varphi_x + \psi + lw)^2\, dx \\ &\quad -\gamma \int_0^1 \theta_x (\varphi_x + \psi + lw)\, dx \\ &\quad -b \int_0^1 \psi_x (\varphi_x + \psi + lw)_x\, dx \\ &\quad -\int_0^1 \int_0^\infty g(s) \psi_{xx}(x, t-s)\, ds\, (\varphi_x + \psi + lw)\, dx \\ &\quad -\int_0^1 f(\psi) (\varphi_x + \psi + lw)\, dx, \end{aligned} \tag{1.50}$$

$$\begin{aligned}\rho_1 \int_0^1 \varphi_{tt} \psi_x dx &= k \int_0^1 \psi_x (\varphi_x + \psi + lw)_x\, dx \\ &\quad + k_0 l \int_0^1 (w_x - l\varphi)\, dx - \mu_1 \int_0^1 \varphi_t \psi_x dx, \end{aligned} \tag{1.51}$$

$$\rho_3 \int_0^1 \theta_t \varphi_t dx = k \int_0^1 q \varphi_{xt} dx + \gamma \int_0^1 \psi_t \varphi_{xt} dx, \tag{1.52}$$

$$\begin{aligned}\int_0^1 \theta \varphi_{tt} dx &= -\frac{k}{\rho_1} \int_0^1 \theta_x (\varphi_x + \psi + lw)\, dx \\ &\quad + \frac{lk_0}{\rho_1} \int_0^1 \theta (w_x - l\varphi)\, dx - \frac{\mu_1}{\rho_1} \int_0^1 \theta \varphi_t dx, \end{aligned} \tag{1.53}$$

$$\begin{aligned}-\int_0^1 q_t (\varphi_x + \psi + lw)\, dx &= \frac{\beta}{\alpha} \int_0^1 q (\varphi_x + \psi + lw)\, dx \\ &\quad + \frac{k}{\alpha} \int_0^1 \theta_x (\varphi_x + \psi + lw)\, dx, \end{aligned} \tag{1.54}$$

$$\begin{aligned}-\rho_2 \int_0^1 \psi_{tt} \psi dx &= b \int_0^1 \psi_x^2 dx + k \int_0^1 \psi (\varphi_x + \psi + lw)\, dx - \gamma \int_0^1 \theta \psi_x dx \\ &\quad + \int_0^1 \int_0^\infty g(s) \psi_{xx} ds \psi dx + \int_0^1 f(\psi) \psi dx, \end{aligned} \tag{1.55}$$

$$\rho_1 \int_0^1 w_{tt} \psi dx = -k_0 \int_0^1 \psi_x (w_x - l\varphi)\, dx - kl \int_0^1 \psi (\varphi_x + \psi + lw)\, dx. \tag{1.56}$$

The substitution of (1.50)-(1.56) into (1.49), bearing (1.5) in mind, gives

$$\begin{aligned} F_7'(t) &= -k\int_0^1 (\varphi_x+\psi+lw)^2\,dx + \left(\rho_2 - \frac{bl^2\rho_2}{k_0}\right)\int_0^1 \psi_t^2 dx \\ &+\left(l\rho_2+\frac{bl\rho_1}{k_0}\right)\int_0^1 \psi_t w_t dx + \frac{b\eta}{\alpha\gamma}\int_0^1 \theta_x(\varphi_x+\psi+lw)\,dx \\ &-\frac{b}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 q\psi_t dx - \frac{bl}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 q w_t dx \\ &+\frac{blk_0\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 \theta(w_x-l\varphi)\,dx - \frac{\gamma bl^2}{k_0}\int_0^1 \theta\psi_x dx \\ &+\frac{b\beta}{\alpha\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 q(\varphi_x+\psi+lw)\,dx + \frac{b^2l^2}{k_0}\int_0^1 \psi_x^2 dx \\ &-bl\int_0^1 \psi_x(w_x-l\varphi)\,dx - \gamma\int_0^1 \theta_x(\varphi_x+\psi+lw)\,dx \\ &-\frac{kb\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 \theta_x(\varphi_x+\psi+lw)\,dx \\ &-b\int_0^1 \psi_x(\varphi_x+\psi+lw)_x\,dx + \frac{b\rho_1}{k}\int_0^1 \psi_t\varphi_{xt} dx \\ &-\frac{b\rho_3\mu_1}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{b}\right)\int_0^1 \theta\varphi_t dx \\ &-\int_0^1 f(\psi)(\varphi_x+\psi+lw)\,dx + \frac{bl^2}{k_0}\int_0^1 f(\psi)\psi dx \\ &+\frac{bl^2\rho_2}{k_0}\int_0^1 \psi_x(x)\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds dx. \end{aligned} \tag{1.57}$$

And we have

$$\begin{aligned} \int_0^1 |\varphi_x f(\psi)|\,dx &\le \|\varphi_x\|\,\|\psi\|_{2(\theta+1)}^{\theta}\,\|\psi\|_{2(\theta+1)} \\ &\le \frac{\varepsilon}{2b^2}\int_0^1 \varphi_x^2 dx + \frac{b^2}{2\varepsilon\lambda_1}\int_0^1 \psi_x^2 dx \\ &\le \frac{\varepsilon}{b^2}\int_0^1 (\varphi_x+\psi)^2\,dx + \frac{\varepsilon}{b^2}\int_0^1 \psi^2 dx + \frac{b^2}{2\varepsilon\lambda_1}\int_0^1 \psi_x^2 dx \\ &\le \frac{\varepsilon}{b^2}\int_0^1 (\varphi_x+\psi)^2\,dx + \left(\frac{\varepsilon}{b^2\lambda_1}+\frac{b^2}{2\varepsilon\lambda_1}\right)\int_0^1 \psi_x^2 dx. \end{aligned} \tag{1.58}$$

Estimate (1.48) follows thanks to Young's inequality and the fact that $k = k_0$. □

**Lemma 1.13.** *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (1.8)-(1.10). Then the functional*

$$F_8(t) := \int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx. \tag{1.59}$$

*Then we have the following estimate, for any $\varepsilon_{11} > 0$,*

$$\begin{aligned} F_8'(t) \leq & \left(-K + \varepsilon_{11}\left(\frac{K}{2} + \frac{\mu_2 c}{2}\right)\right)\int_0^1 \varphi_x^2 dx + \frac{K}{2\varepsilon_{11}}\int_0^1 \psi_x^2 dx \\ & + \frac{\mu_2}{2\varepsilon_{11}}\int_0^1 z^2(x,1,t)\,dx + \rho_1\int_0^1 \varphi_t^2 dx, \end{aligned} \tag{1.60}$$

*where $c = 1/\pi^2$ is the Poincaré constant.*

*Proof.* Taking the derivative of (1.59) with respect to t, we have

$$F_8'(t) = \rho_1\int_0^1 \varphi_{tt}\varphi dx + \rho_1\int_0^1 \varphi_t^2 dx + \mu_1\int_0^1 \varphi_t\varphi dx. \tag{1.61}$$

Then, by using the first equation in (1.1), we find

$$F_8'(t) = k\int_0^1 (\varphi_x + \psi + lw)_x\,\varphi dx - \mu_2\int_0^1 \varphi z(x,1,t)\,dx + \rho_1\int_0^1 \varphi_t^2 dx. \tag{1.62}$$

Consequently, we arrive at

$$F_8'(t) = -k\int_0^1 (\varphi_x + \psi + lw)\,\varphi_x dx - \mu_2\int_0^1 \varphi z(x,1,t)\,dx + \rho_1\int_0^1 \varphi_t^2 dx. \tag{1.63}$$

Applying Young's inequality and Poincaré's inequality, we find (1.59). □

**Lemma 1.14.** *Let $(\varphi,\psi,w,\theta,q,z,\eta^t)$ be the solution of (1.8)-(1.10). Then, we define the functional*

$$F_9(t) := \int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\,d\rho dx. \tag{1.64}$$

*Then the following result holds.*

$$F_9'(t) \leq -F_9(t) - \frac{c_1}{2\tau}\int_0^1 z^2(x,1,t)\,dx + \frac{1}{2\tau}\int_0^1 \psi_t^2(x,t)\,dx, \tag{1.65}$$

*where c is a positive constant.*

*Proof.* Taking the derivative of (1.64) with respect to $t$ and using the equation (1.7), we get

$$\begin{aligned} \frac{d}{dt}\left(\int_0^1 \int_0^1 e^{-2\tau\rho} z^2 (x,\rho,t)\, d\rho dx\right) &= -\frac{1}{\tau}\int_0^1 \int_0^1 e^{-2\tau\rho} z z_\rho (x,\rho,t)\, d\rho dx \\ &= -\int_0^1 \int_0^1 e^{-2\tau\rho} z^2 (x,\rho,t)\, d\rho dx \\ &\quad -\frac{1}{2\tau}\int_0^1 \int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho} z^2 (x,\rho,t)\right) d\rho dx. \end{aligned} \tag{1.66}$$

Making use of the estimate above, implies that there exists a positive constant $c_1$ such that (1.65) holds.

□

Now, we are ready to state and prove the main result of this section.

**Theorem 1.15.** *Assume that $\eta = 0$ and $k = k_0$. Then $(\varphi, \psi, w, \theta, q, z, \eta^t)$ the solution of (1.8)-(1.10) satisfies*

$$E(t) \leq c_0 e^{-c_1 t}, \qquad t \geq 0, \tag{1.67}$$

*where the positive constant $c_0$ is directly depending on initial data and the uniform constant $c_1$ is depending only on the coefficients of the system. For $N, N_i > 0$,*

$$\mathcal{L}(t) := NE(t) + \sum_{i=1}^{i=9} N_i F_i(t), \tag{1.68}$$

Then, from(1.28), (1.30), (1.33), (1.36), (1.39), (1.42), (1.45), (1.48),(1.60) and (1.65) we have

$$
\begin{aligned}
\mathcal{L}'(t) \leq\ & \left[-\beta N + c_1\left(1+\frac{1}{\varepsilon_1}\right) + cN_2 + c\left(1+\frac{1}{\varepsilon_8}\right)N_7\right]\int_0^1 q^2 dx \\
& -N\left[\mu_1 - \frac{\xi}{2} - \frac{|\mu_2|}{2}\right]\|\varphi_t\|_2^2 - N\left[\frac{\xi}{2} - \frac{|\mu_2|}{2}\right]\|z(x,1,t)\|_2^2 \\
& -\left[\frac{N_1\rho_3}{2} - N_2\left(C_1 + 1 + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right) - cN_4\right. \\
& \left. -c\left(1 + \frac{1}{\varepsilon_9} + \frac{b\rho_3\mu_1}{\gamma\rho_1 c}\varepsilon_5\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\right)N_7\right]\int_0^1 \theta^2 dx \\
& +\left[\varepsilon_1 N_1 - CN - N_2\frac{\rho_2}{\gamma} + \rho_2 N_4 + cN_5 + \frac{\rho_1}{2}N_6\right. \\
& \left. +c\left(1 + \frac{1}{\varepsilon_8} + \frac{b\rho_1\varepsilon_4}{k}\right)N_7 + \rho_1 N_8 + \frac{1}{2\tau}N_9\right]\int_0^1 \psi_t^2 dx \\
& +\left[\varepsilon_2 N_2 + cN_3 + \frac{k^2}{b}N_4 + lkN_5 + kN_6\right. \\
& \left. -\left(\frac{k}{2} - \frac{b\tilde{\eta}}{\alpha\gamma\varepsilon_{10}} + \frac{\gamma}{4\varepsilon_1} + \frac{kb\rho_3}{\gamma 4\varepsilon_2\rho_1}\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) + \frac{b}{4\varepsilon_3}\right)N_7\right]\int_0^1 (\varphi_x + lw + \psi)^2 dx \\
& +\left[\left(\varepsilon_3 + \frac{\rho_3}{\gamma}\left(\frac{\varepsilon_2}{b^2\lambda_2} + \frac{b^2}{2\varepsilon_2\lambda_2}\right)\right)N_2 + cN_3 + (\delta_2 + \frac{b}{2} + C_2)N_4\right. \\
& +(\frac{bl^2\rho_2\delta_3}{k_0} + \frac{b^2l^2}{k} + b\varepsilon_3 + \frac{bl^2}{k_0}c_1 + 2\left(\frac{\varepsilon}{b^2\lambda_1} + \frac{b^2}{2\varepsilon\lambda_1}\right) + c_2)N_7 \\
& \left. +\left(\frac{k}{2\varepsilon_{11}} - k + \varepsilon_{11}\frac{k}{2} + \varepsilon_{11}\frac{\mu_2 c}{2}\right)N_8\right]\int_0^1 \psi_x^2 dx \\
& +\left[-\rho_1 N_3 - \frac{l\rho_1}{2}N_5 + \rho_1 N_6 + \varepsilon_8 N_7\right]\int_0^1 w_t^2 dx \\
& +\left[k_0 N_3 - \left(lk_0 - \frac{\mu_1}{4\varepsilon_6} - \frac{\mu_2}{4\varepsilon_7}\right)N_5 - k_0 N_6 + \varepsilon_9 N_7\right]\int_0^1 (w_x - l\varphi)^2 dx \\
& +\left[\frac{\mu_2}{4\varepsilon_5}N_3 + \varepsilon_7\mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}}N_8 - \frac{c_1}{2\tau}N_9\right]\int_0^1 z^2(x,1,t)\, dx
\end{aligned}
$$

$$
\begin{aligned}
&+ [\varepsilon_5\mu_2 + \varepsilon_4\mu_1] N_3 \int_0^1 \varphi^2 dx \\
&+ \left[ -\left(1 - \frac{1}{4\varepsilon_4}\right) N_3 + (l\rho_1 + \varepsilon_6\mu_1) N_5 - \frac{\rho_1}{2} N_6 + \rho_1 N_8 \right. \\
&+ \left. \left( \frac{b\rho_1}{k4\varepsilon_4} + \frac{b\rho_3\mu_1}{4\varepsilon_5\gamma\rho_1} \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right) N_7 \right] \int_0^1 \varphi_t^2 dx \\
&+ \left[ \left( -k + \varepsilon_{11} \left( \frac{k}{2} + \frac{\mu_2 c}{2} \right) \right) N_8 \right] \int_0^1 \varphi_x^2 dx \\
&+ \left[ \left( \frac{b\tilde{\eta}\varepsilon_{10}}{\alpha\gamma} + \gamma\varepsilon_1 + \frac{kb\rho_3}{\gamma\rho_1} \varepsilon_2 \left( \frac{\rho_1}{k} - \frac{\rho_2}{b} \right) \right) N_7 + N_2\delta_1 \right] \int_0^1 \theta_x^2 dx \\
&- N_9 F_9(t) + \left( \frac{N_2 g_0}{4\delta_1} + \frac{N_4 g_0}{4\delta_2} + \frac{N_7 g_0 b l^2 \rho_2}{k_0 4\delta_3} - N\frac{\zeta}{2} \right) \int_0^1 \int_0^\infty g(s) \left| \eta_x^t(x,s) \right|^2 ds dx.
\end{aligned} \tag{1.69}
$$

At this point, we have to choose our constants very carefully. First, choosing $\varepsilon_i$ $i = 1, ..., 10$ small enough such that

$$\varepsilon_1 \le \frac{N_2 \frac{\rho_2}{\gamma} + \rho_2 N_4 + c N_5 + \frac{\rho_1}{2} N_6}{N_1}.$$

Moreover, we pick $N_9$ large enough so that

$$\frac{\mu_2}{4\varepsilon_5} N_3 + \varepsilon_7\mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}} N_8 - \frac{c_1}{2\tau} N_9 \le 0,$$

and

$$N_9 \ge \frac{\frac{\mu_2}{4\varepsilon_5} N_3 + \varepsilon_7\mu_2 N_5 + \frac{\mu_2}{2\varepsilon_{11}} N_8}{\frac{c_1}{2\tau}},$$

we take $\varepsilon_{11}$ small enough such that

$$\varepsilon_{11} \le \frac{k}{\left( \frac{k}{2} + \frac{\mu_2 c}{2} \right) N_8}.$$

Next, choosing $N_5$ large enough such that

$$\frac{N_5 \rho_3 \kappa}{4} \ge N_4 \left( \gamma\rho_3 + \frac{\rho_3}{2\varepsilon_4} (b + 2\kappa) \right).$$

After that, we can choose $N$ large enough such that

$$N \ge \frac{c_1 \left(1 + \frac{1}{\varepsilon_1}\right) + c N_2 + c \left(1 + \frac{1}{\varepsilon_8}\right) N_7}{\beta},$$

and

$$\frac{N_2 g_0}{4\delta_1} + \frac{N_4 g_0}{4\delta_2} + \frac{g_0 b l^2 \rho_2}{k_0 4\delta_3} - N\frac{\zeta}{2} \le 0.$$

Consequently, there exists a positive constant $\eta_1$ such that (1.69) becomes

$$\frac{d}{dt}\mathcal{L}(t) \leq -\eta_1 \int_0^1 (\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x + lw + \psi)^2 + \theta^2 + q^2)dx - \eta_1 \int_0^1 \int_0^1 z^2(x, \rho, t)\, d\rho dx, \quad (1.70)$$

which implies by (1.27) that there exists also $\eta_2$, such that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\eta_2 E(t), \quad \forall t \geq 0. \quad (1.71)$$

**Lemma 1.16.** *For N large enough, there exists two positive constants $\beta_1$ and $\beta_2$ depending on $N_i$, $i = 1, ..., 9$ and $\varepsilon_i$, $i = 1, ..., 11$ such that*

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \quad (1.72)$$

*Proof.* We consider the functional

$$H(t) = \sum_{i=1}^{i=9} N_i F_i(t),$$

and show that

$$|H(t)| \leq CE(t), \quad C > 0.$$

From (1.29), (1.32), (1.35), (1.38), (1.41), (1.44), (1.47), (1.59) and (1.64), we obtain

$$\begin{aligned}
|H(t)| \leq\; & N_1 \left|\alpha\rho_3 \int_0^1 \theta \int_0^x q(y)\, dydx\right| + N_2 \left|-\frac{\rho_2\rho_3}{\gamma} \int_0^1 \theta dx \int_0^x \psi_t(y)\, dydx\right| \\
& + N_3 \left|\rho_1 \int_0^1 (\varphi\varphi_t + ww_t)\, dx\right| + N_4 \left|\rho_2 \int_0^1 \int_0^x \psi\psi_t(t, x)\, dx\right| \\
& + N_5 \left|-\rho_1 \int_0^1 \varphi_t(w_x - l\varphi)\, dx - \rho_1 \int_0^1 w_t(\varphi_x + \psi + lw)\, dx\right| \\
& + N_6 \left|-\rho_1 \int_0^1 (w_x - l\varphi) \int_0^x w_t(y)\, dydx - \rho_1 \int_0^1 \varphi_t \int_0^x (\varphi_x + \psi + lw)\, dydx\right| \\
& + N_7 \left|\rho_2 \int_0^1 (\varphi_x + \psi + lw)\, dx + \frac{b\rho_1}{k} \int_0^1 \varphi_t\psi_x dx\right| \\
& + N_8 \left|\int_0^1 \rho_1\varphi\varphi_t dx + \frac{\mu_1}{2} \int_0^1 \varphi^2 dx\right| + N_9 \int_0^1 \int_0^1 e^{-2\tau\rho} z^2(x, \rho, t)\, d\rho dx.
\end{aligned}$$

By using, the trivial relation

$$\int_0^1 (\varphi + lw)^2\, dx \leq 2c \int_0^1 (\varphi_x + lw + \psi)^2\, dx + 2c \int_0^1 \psi_x^2 dx,$$

Young's and Poincaré's inequalities, we get

$$
\begin{aligned}
|H(t)| \leq\ & \alpha_1 \int_0^1 \varphi_t^2 dx + \alpha_2 \int_0^1 \psi_t^2 dx + \alpha_3 \int_0^1 w_t^2 + \alpha_4 \int_0^1 \psi_x^2 + \alpha_5 \int_0^1 \theta^2 dx \\
& + \alpha_6 \int_0^1 q^2 dx + \alpha_7 \int_0^1 \left( (\varphi_x + lw + \psi)^2 + (w_x - l\varphi)^2 \right) dx \\
& + \int_0^1 \int_0^1 z^2(x,\rho,t)\, d\rho dx, \qquad (1.73)
\end{aligned}
$$

where the positive constants $\alpha_1, ..., \alpha_7$ are determined as follows:

$$
\begin{cases}
\alpha_1 := \frac{1}{2}(N_3\rho_1 + N_8\rho_1), \\
\alpha_2 := \frac{1}{2}\left(N_4\rho_2 + N_2\frac{\rho_2\rho_3}{\gamma}\right), \\
\alpha_3 = \frac{1}{2}(N_3\rho_1 + N_6\rho_1), \\
\alpha_4 := \frac{b\rho_1}{2k}, \\
\alpha_5 := \frac{1}{2}\left(N_1\rho_3 + \frac{\rho_2\rho_3}{\gamma}\right), \\
\alpha_6 := \frac{1}{2}(N_1\rho_3 + N_5\tau_0\rho_3), \\
\alpha_7 := \frac{1}{2}(N_7\rho_2 + 3\rho_1).
\end{cases}
$$

According to (1.73), we have

$$
|H(t)| \leq \widehat{C}E(t),
$$

for

$$
\widehat{C} = \frac{\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}}{\min\{\rho_1, \rho_2, \rho_3, k, b, \kappa, \gamma, \delta, \tau_0\}}.
$$

Therefor we get

$$
|\mathcal{L}(t) - NE(t)| \leq \widehat{C}E(t).
$$

Then, we can choose $N$ large enough so that $\beta_1 = N - \widehat{C} > 0$. Then (1.72) holds true for $\beta_2 = N + \widehat{C} > 0$, this concludes the proof of the Lemma. □

Combining now (1.71) and (1.72), we conclude that there exists some $\Lambda > 0$ such that

$$
\frac{d}{dt}\mathcal{L}(t) \leq -\Lambda\mathcal{L}(t), \quad \forall t \geq 0. \qquad (1.74)
$$

Integration of (1.74) yields (1.75)

$$
\mathcal{L}(t) \leq \mathcal{L}(0)\, e^{-\Lambda t}, \quad \forall t \geq 0. \qquad (1.75)
$$

Finally, using (1.72) and (1.75), so (1.67) is satisfed. Hence the proof of Theorem (1.15) is completed.

# Chapter 2

## A stability result for a Timoshenko system with infinite history and distributed delay term

### 2.1 Introduction

In this chapter, we investigated (with Ouchenane) [28], the following Timoshenko system with infinite history and distributed delay term

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + \int_0^\infty g(s)\,\psi_{xx}(x,t-s)\,ds \\ +K(\varphi_x+\psi)(x,t) + \mu_1\psi_t(x,t) + \int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds = 0, \end{cases} \tag{2.1}$$

where $(x,t)\in(0,1)\times(0,\infty)$, $s>0$ represents the time delay, $\mu_1$ is a positive constant and $\mu_2:[\tau_1,\tau_2]\to\mathbb{R}$ is a bounded function satisfying

$$\int_{\tau_1}^{\tau_2}|\mu_2(s)|\,ds < \mu_1, \tag{2.2}$$

where $\tau_1$ and $\tau_2$ two real numbers satisfying $0\le\tau_1\le\tau_2$, and the relaxation function $g$ satisfies the folowing assumptions

(G1) $g:\mathbb{R}_+\to\mathbb{R}_+$ is a $C^1$ function satisfying

$$g(0)>0, \qquad b-\int_0^\infty g(s)\,ds = b-g_0 = L>0.$$

(G2) There exists a positive constant $\zeta$ such that

$$g'(t) \leq -\zeta g(t), \qquad \forall t \geq 0. \tag{2.3}$$

System (2.1) is provided with the following initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \psi_t(x,-t) = f_0(x,t) \text{ in } (0,1) \times (0,\tau_2), \end{cases} \tag{2.4}$$

and

$$\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \qquad \forall t \geq 0, \tag{2.5}$$

where $f_0$ is the history function.

Our goal in this chapter is to investigate the system (2.1) under suitable assumptions and show that even in the presence of the viscoelastic term ($g \neq 0$), we can establish a general energy decay regardless also of the speeds of wave propagation. to achieve our goals we make use the energy method combined with some properties of convex functions. The arguments of convexity were introduced by Lasiecka and Tataru [32] and used by Liu and Zuazua [38] and others.

## 2.2 Preliminaries

The aim main in this section is to present some materials needed in the proof of our result. We also state, without proof, a local existence result for problem (2.1). The proof can be established by using Faedo-Galerkin method as in [53]. Let us introduce the following new dependent variable

$$z(x,\rho,s,t) = \psi_t(x,t-s\rho), \qquad \text{in } (0,1)\times(0,1)\times(\tau_1,\tau_2)\times(0,\infty).$$

Then, we get the following system

$$\begin{cases} sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ z(x,0,\tau,t) = \psi_t(x,t). \end{cases}$$

We then set an auxiliary variable as in [10]

$$\eta^t(x,s) = \psi(x,t) - \psi(x,t-s), \qquad s \geq 0.$$

Then

$$\eta^t_t(x,s) + \eta^t_s(x,s) = \psi_t(x,t).$$

Hence, we can rewrite the problem (2.1) as

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\varphi_x+\psi) + \int_0^\infty g(s)\,\eta^t_{xx}(x,s)\,ds \\ +\mu_1\psi_t(x,t) + \int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds = 0, \\ sz_t(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, \\ \eta^t_t(x,s) + \eta^t_s(x,s) = \psi_t(x,t), \end{cases} \tag{2.6}$$

where $x \in (0,1)$, $\rho \in (0,1)$, and $t > 0$. System (2.6) subjected to the following initial conditions

$$\begin{cases} \left.\begin{array}{l} \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \end{array}\right. & x \in (0,1), \\ z(x,\rho,s,0) = f_0(x,\rho s), & \text{in } (0,1)\times(0,1)\times(0,\tau_2), \\ \eta^t(x,0) = 0, & \forall t \geq 0, \\ \eta^0(x,s) = \eta_0(s) = 0, & \forall s \geq 0. \end{cases} \tag{2.7}$$

In addition, we consider the following boundary conditions

$$\begin{aligned} \varphi(0,t) &= \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \qquad \forall t \geq 0, \\ \eta^t(0,s) &= \eta^t(1,s) = 0, \qquad \forall s \geq 0. \end{aligned} \tag{2.8}$$

We now define the energy space

$$\boldsymbol{H} := \left[H_0^1(0,1)\times L^2(0,1)\right]^2 \times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \times L_g^2\left(\mathbb{R}^+, H_0^1(0,1)\right), \tag{2.9}$$

where $L_g^2\left(\mathbb{R}^+, H_0^1(0,1)\right)$ denotes the Hilbert space of $H_0^1$−valued functions on $R^+$.

## 2.3 Exponential stability

The functional energy of the problem (2.6)-(2.8) is given by

$$\begin{aligned} E(t) &= E\left(t,\varphi,\psi,z,\eta^{t}\right) \\ &= \frac{1}{2}\int_0^1\left(\rho_1\varphi_t^2+\rho_2\psi_t^2\right)dx+\frac{1}{2}\int_0^1\left\{K\left(\varphi_x+\psi\right)^2+b\psi_x^2\right\}dx \\ &\quad +\frac{1}{2}\int_0^1\int_0^\infty g\left(s\right)\left|\eta_x^t\left(x,s\right)\right|^2dsdx \\ &\quad +\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\left|\mu_2\left(s\right)\right|z^2\left(x,\rho,s,t\right)dsd\rho dx. \end{aligned} \tag{2.10}$$

We multiply $(2.6)_1$ by $\varphi_t$, $(2.6)_2$ by $\psi_t$ and $(2.6)_3$ by $|\mu_2(s)|\,z$, integratiny by parts over $(0,1)$, using Young and cauchy-schwarz's inéquality we get

$$\begin{aligned} \frac{dE\left(t\right)}{dt} &\leq \frac{1}{2}\int_0^1\int_0^\infty g'\left(s\right)\left|\eta_x^t\left(x,s\right)\right|^2dsdx \\ &- C\left\{\int_0^1\psi_t\int_{\tau_1}^{\tau_2}\mu_2\left(s\right)z\left(x,1,s,t\right)+\int_0^1\psi_t^2\left(x,t\right)dx\right\}, \end{aligned} \tag{2.11}$$

where $C>0$, which implies that the energy $E$ is a non-increasing function with respect to $t$.

Our main stability result reads as follows

**Theorem 2.1.** *Let $U_0\in D\left(A\right)$. Assume that $\int_{\tau_1}^{\tau_2}\left|\mu_2\left(s\right)\right|ds<\mu_1$ and*

$$\frac{K}{\rho_1}=\frac{b}{\rho_2}. \tag{2.12}$$

*Then there exist two positive constants $C$ and $\gamma$ independent of $t$ such that*

$$E\left(t\right)\leq Ce^{-\gamma t},\qquad \forall t>0. \tag{2.13}$$

*Remark* 2.2. To derive the exponential decay of the solution, it is enough to construct a functional $L(t)$, equivalent to the energy $E(t)$, satisfying

$$\frac{dL\left(t\right)}{dt}\leq-\Lambda L\left(t\right),\qquad \forall t>0,$$

where $\Lambda$ is a positive constant. In order to obtain such a functional $L$, we need several Lemmas.

Let us first define the following functional.

$$I_1(t) := -\int_0^1 (\rho_1\varphi_t\varphi + \rho_2\psi_t\psi)\,dx - \frac{\mu_1}{2}\int_0^1 \psi^2 dx. \tag{2.14}$$

Then we have the following estimate.

**Lemma 2.3.** *Let $(\varphi, \psi, z, \eta^t)$ be the solution of (2.6)-(2.8), then for any $\varepsilon, \delta_1 > 0$, we have*

$$\begin{aligned}
\frac{dI_1(t)}{dt} &\le -\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx \\
&+ \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx + \frac{c\varepsilon_2}{2}\int_0^1 \psi^2 dx \\
&+ (b+\delta_1)\int_0^1 \psi_x^2 dx + \frac{1}{2\varepsilon_2}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\psi_t^2(x,t-s)\,ds dx + K\int_0^1 (\varphi_x+\psi)^2 dx,
\end{aligned} \tag{2.15}$$

*where $c = 1/\pi^2$ is the Poincaré's constant.*

*Proof.* Taking the derivative of (2.14), integrating by parts, we obtain

$$\begin{aligned}
\frac{dI_1(t)}{dt} &= -\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx - \int_0^1 (\rho_1\varphi_{tt}\varphi_t + \rho_2\psi_{tt}\psi_t)\,dx \\
&\quad -\mu_1\int_0^1 \psi_t\psi dx.
\end{aligned} \tag{2.16}$$

Therefore, by using $(2.6)_1$, $(2.6)_2$, integration by parts, we obtain from (2.16)

$$\begin{aligned}
\frac{dI_1(t)}{dt} &= -\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right) dx + K\int_0^1 (\varphi_x+\psi)^2 dx + b\int_0^1 \psi_x^2 dx \\
&+ \int_0^1 \psi\int_{\tau_1}^{\tau_2} \mu_2(s)\psi_t(x,t-s)\,ds dx \\
&+ \int_0^1 \psi_x(x,t)\int_0^\infty g(s)\left|\eta_x^t(x,s)\right| ds dx.
\end{aligned} \tag{2.17}$$

By exploiting Young and Poincaré's inequalities, we get for any $\varepsilon > 0$,

$$\begin{aligned}
&\int_0^1 \psi\int_{\tau_1}^{\tau_2} \mu_2(s)\psi_t(x,t-s)\,ds dx \\
&\le \frac{c\varepsilon_2}{2}\int_0^1 \psi^2 dx + \frac{1}{2\varepsilon_2}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\psi_t^2(x,t-s)\,ds dx.
\end{aligned} \tag{2.18}$$

Moreover, Young, Hölder's inequalities and (2.3) imply that for any $\delta_1 > 0$

$$\int_0^1 \psi_x(x,t)\int_0^\infty g(s)\left|\eta_x^t(x,s)\right| dsdx$$
$$\leq \delta_1 \int_0^1 \psi_x^2(x,t)\,dx + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \tag{2.19}$$

Inserting the estimates (2.18) and (2.19) into (2.17), then (2.16) is fulfilled. □

Now, let $w$ be the solution of

$$-w_{xx} = \psi_x, \qquad w(0) = w(1) = 0, \tag{2.20}$$

then

$$w(x,t) = -\int_0^x \psi(y,t)\,dy + x\left(\int_0^1 \psi(y,t)\,dy\right).$$

We have the following inequalities.

**Lemma 2.4.** *The solution of (2.20) satisfies*

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx,$$

*and*

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

*Proof.* We multiply (2.20) by $w$, integrate by parts and use the Cauchy-Schwarz's inequality to obtain

$$\int_0^1 w_x^2 dx \leq \int_0^1 \psi^2 dx.$$

Next, we differentiate (2.20) with respect to $t$ and by the same procedure, we obtain

$$\int_0^1 w_t^2 dx \leq \int_0^1 \psi_t^2 dx.$$

□

Let $w$ be the solution of (2.20). We introduce the following functional

$$I_2(t) := \int_0^1 (\rho_2\psi_t\psi + \rho_1\varphi_t w)\,dx + \frac{\mu_1}{2}\int_0^1 \psi^2 dx. \tag{2.21}$$

Then, we have the following estimate.

**Lemma 2.5.** *Let $(\varphi, \psi, z, \eta^t)$ be the solution of (2.6)-(2.8). Then we have for any $\varepsilon_3 > 0$,*

$$\begin{aligned}\frac{dI_2(t)}{dt} &\leq (\delta_1 - b)\int_0^1 \psi_x^2 dx + \rho_1\lambda_2 \int_0^1 \varphi_t^2 dx + \frac{c\varepsilon_3}{2}\int_0^1 \psi^2 dx \\ &+ \left(\rho_2 + \frac{\rho_1}{4\lambda_2}\right)\int_0^1 \psi_t^2 dx + \left(\frac{\gamma\tau_0}{2\kappa\varepsilon_3} + \frac{\delta\gamma}{2\kappa\varepsilon_3}\right)\int_0^1 q^2 dx \\ &+ \frac{1}{2\varepsilon_3}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, ds dx + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx.\end{aligned} \tag{2.22}$$

*Proof.* By taking the derivative of (2.21), we conclude

$$\begin{aligned}\frac{dI_2(t)}{dt} &= -b\int_0^1 \psi_x^2 dx - K\int_0^1 \psi^2 dx + \rho_2\int_0^1 \psi_t^2 dx + K\int_0^1 w_x^2 dx \\ &+\rho_1\int_0^1 \varphi_t w_t dx + \int_0^1 \psi_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\, ds dx \\ &- \int_0^1 \psi \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,s,t)\, ds dx.\end{aligned}$$

We apply Young and Poincaré's inequalities, we find

$$\int_0^1 \psi_x(x,t)\int_0^\infty g(s)\eta_x^t(x,s)\, ds dx \leq \delta_1\int_0^1 \psi_x^2(x,t) + \frac{g_0}{4\delta_1}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 ds dx,$$

and for any $\lambda_2 > 0$ we have

$$\rho_1\int_0^1 \varphi_t\psi_t dx \leq \rho_1\lambda_2\int_0^1 \varphi_t^2 dx + \frac{\rho_1}{4\lambda_2}\int_0^1 \psi_t^2 dx.$$

□

Now, we define the functional $I_3$

$$\begin{aligned}I_3(t) &:= \rho_2\int_0^1 \psi_t(\varphi_x + \psi)\, dx + \frac{\rho_1 b}{K}\int_0^1 \psi_x\varphi_t dx \\ &+\frac{\rho_1}{K}\int_0^1 \varphi_t\int_0^\infty g(s)\eta_x^t(x,s)\, ds dx.\end{aligned} \tag{2.23}$$

**Lemma 2.6.** *Let $(\varphi, \psi, z, \eta^t)$ be the solution of (2.6)-(2.8). Assume that*

$$\frac{\rho_1}{K} = \frac{\rho_2}{b + g_0} = \frac{\rho_2}{b}. \tag{2.24}$$

*Then, for any* $\varepsilon_4 > 0$, *we have*

$$\begin{aligned} \frac{dI_3(t)}{dt} &\leq \left[\varphi_x\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\right)\right]_{x=0}^{x=1} - (K - 2\varepsilon_4)\int_0^1 (\varphi_x + \psi)^2\,dx \\ &+ \left(\rho_2 + \frac{\mu_1^2}{4\varepsilon_4}\right)\int_0^1 \psi_t^2 dx + \varepsilon_4 \int_0^1 \varphi_t^2 dx + \frac{1}{2\varepsilon_4}\int_0^1\int_{\tau_1}^{\tau_2} \mu_2(s)\,z^2(x,1,s,t)\,dsdx \qquad (2.25) \\ &- g_0 C(\varepsilon_4)\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx. \end{aligned}$$

*Proof.* Differentiating $I_3(t)$, we obtain

$$\begin{aligned} \frac{dI_3(t)}{dt} &= \rho_2\int_0^1 \psi_{tt}(\varphi_x + \psi)\,dx + \rho_2\int_0^1 \psi_t(\varphi_x + \psi)_t\,dx \\ &+\frac{\rho_1 b}{K}\int_0^1 \psi_x\varphi_{tt}dx + \frac{\rho_1}{K}\int_0^1 \varphi_t\int_0^\infty g(s)\,\eta_x^t(x,s)\,dsdx \\ &+\frac{\rho_1 b}{K}\int_0^1 \psi_{xt}\varphi_t dx + \frac{\rho_1}{K}\int_0^1 \varphi_{tt}\int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,dsdx. \end{aligned}$$

Then, by using (2.6), we find

$$\begin{aligned} \frac{dI_3(t)}{dt} &= \rho_2\int_0^1 (\varphi_x + \psi)\,(b\psi_{xx}(x,t) - K(\varphi_x + \psi)(x,t) \\ &-\mu_1\psi_t(x,t) - \int_{\tau_1}^{\tau_2} \mu_2(s)\,\psi_t(x,t-s)\,ds\Big)\,dx \\ &+\int_0^1 (\varphi_x + \psi)\int_0^\infty g(s)\,\eta_{xx}^t(x,s)\,dsdx + \rho_2\int_0^1 \psi_t^2 dx \\ &+b\int_0^1 (\varphi_x + \psi)_x\,\psi_x dx + \left(\frac{\rho_1 b}{K} - \rho_2\right)\int_0^1 \psi_{tx}\varphi_t dx \\ &+\frac{\rho_1}{K}\int_0^1 \varphi_t\int_0^\infty g(s)\,(\psi_{tx}(t,x) - \eta_{tx}^t(x,s))\,dsdx \\ &+\frac{\rho_1}{K}\int_0^1 (\varphi_x + \psi)_x\int_0^\infty g(s)\,\eta_x^t(x,s)\,dsdx. \end{aligned}$$

By (2.24), we obtain

$$\begin{aligned} \frac{dI_3(t)}{dt} &= -K\int_0^1 (\varphi_x + \psi)^2\,dx - \mu_1\int_0^1 (\varphi_x + \psi)\,\psi_t dx + \rho_2\int_0^1 \psi_t^2 dx \\ &- \int_0^1 (\varphi_x + \psi)\int_{\tau_1}^{\tau_2} \mu_2(s)\,\psi_t(x,t-s)\,dsdx + \frac{\rho_1}{K}\int_0^1 \varphi_t\int_0^\infty g'(s)\,\eta_x^t(x,s)\,dsdx \qquad (2.26) \\ &+ \left[b\psi_x\varphi_x dx\right]_{x=0}^{x=1} + \left[\varphi_x(x,t)\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right]. \end{aligned}$$

For any $\varepsilon_4 > 0$, Young's inequality leads to

$$\left|\mu_1 \int_0^1 (\varphi_x + \psi)\psi_t(x,t)\right| \leq \varepsilon_4 \int_0^1 (\varphi_x + \psi)^2 dx + \frac{\mu_1^2}{4\varepsilon_4}\int_0^1 \psi_t^2 dx, \tag{2.27}$$

and

$$\begin{aligned}&\left|\int_0^1 (\varphi_x + \psi)\int_{\tau_1}^{\tau_2} \mu_2(s)\psi_t(x,t-s)\,dsdx\right| \\ &\leq \frac{c\varepsilon_4}{2}\int_0^1 (\varphi_x + \psi)^2 dx + \frac{1}{2\varepsilon_4}\int_0^1 \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\,dsdx,\end{aligned} \tag{2.28}$$

and

$$\begin{aligned}&\left|\frac{\rho_1}{K}\int_0^1 \varphi_t \int_0^\infty g'(s)\eta_x^t(x,s)\,dsdx\right| \\ \leq\ & \frac{\rho_1^2}{4K\varepsilon_4}\int_0^1 \left(\int_0^\infty g'(s)\eta_x^t(x,s)\,ds\right)^2 dx + \varepsilon_4 \int_0^1 \varphi_t^2 dx \\ \leq\ & -g(0)\,C(\varepsilon_4)\int_0^1 \int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2 dsdx + \varepsilon_4 \int_0^1 \varphi_t^2 dx.\end{aligned} \tag{2.29}$$

Plugging (2.27), (2.28) and (2.29) into (2.26), then inequality (2.25) holds. □

Next, in order to handle the boundary terms appearing in (2.25) we use, as in [47], the function

$$q(x) = 2 - 4x \qquad x \in (0,1).$$

So, we have the following result.

**Lemma 2.7.** *Let $(\varphi, \psi, z, \eta^t)$ be the solution of (2.6). Then we have that for a positive constant $\varepsilon_6$*

$$\begin{aligned}
&\left[\varphi_x\left(b\psi_x - \int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)\right]_{x=0}^{x=1} \qquad (2.30)\\
&\leq -\frac{\varepsilon_6}{K}\frac{d}{dt}\int_0^1 \rho_1 q(x)\,\varphi_t\varphi_x dx + K^2\varepsilon_6\int_0^1(\varphi_x+\psi)^2\,dx\\
&-\frac{\rho_2}{4\varepsilon_6}\frac{d}{dt}\int_0^1 q(x)\,\psi_t\left(b\psi_x - \int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)dx + 3\varepsilon_6\int_0^1\varphi_x^2dx\\
&+\left(\varepsilon_6+\frac{b}{4\varepsilon_6}\left(4+\frac{3}{2\varepsilon_6^2}\right)\right)\int_0^1\psi_x^2dx + \frac{1}{4\varepsilon_6}\left(2\rho_2(b+g_0)+4\mu_1^2\varepsilon_6^2+\rho_2\varepsilon_6\right)\int_0^1\psi_t^2dx\\
&-\frac{\rho_2 g(0)\,C(\varepsilon_6)}{4\varepsilon_6}\int_0^1\int_0^\infty g'(s)\left|\eta_x^t(x,s)\right|^2dsdx + \frac{2\rho_1\varepsilon_6}{K}\int_0^1\varphi_t^2dx\\
&+\frac{g_0}{4\varepsilon_6}\left(4+\frac{3}{2\varepsilon_6^2}\right)\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx\\
&+\frac{1}{2\varepsilon_4}\int_0^1\int_{\tau_1}^{\tau_2}\mu_2(s)\,z^2(x,1,s,t)\,dsdx.
\end{aligned}$$

*Proof.* By using Young and poincaré inequalities, we obtain for any $\varepsilon_6 > 0$,

$$\begin{aligned}
&\left[\varphi_x\left(b\psi_x + \int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)\right]_{x=0}^{x=1} \qquad (2.31)\\
=\;& \varphi_x(1)\left(b\psi_x(1) + \int_0^\infty g(s)\,\psi_x(1,t-s)\,ds\right)\\
&-\varphi_x(0)\left(b\psi_x(0) + \int_0^\infty g(s)\,\psi_x(0,t-s)\,ds\right)\\
\leq\;& \frac{1}{4\varepsilon_6}\left[\left(b\psi_x(1) + \int_0^\infty g(s)\,\psi_x(1,t-s)\,ds\right)^2\right.\\
&\left.+\left(b\psi_x(0) + \int_0^\infty g(s)\,\psi_x(0,t-s)\,ds\right)^2\right] + \varepsilon_6\left[\varphi_x(1)^2+\varphi_x(0)^2\right].
\end{aligned}$$

On the other hand, it is clear that

$$\begin{aligned}
&\frac{d}{dt}\int_0^1\rho_2 q(x)\,\psi_t\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx \qquad (2.32)\\
&= \int_0^1\rho_2 q(x)\,\psi_{tt}\left(b\psi_x + \int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\\
&+\int_0^1\rho_2 q(x)\,\psi_t\left(b\psi_{tx} + \int_0^\infty g(s)\,\eta_{tx}^t(x,s)\,ds\right)dx.
\end{aligned}$$

Now, using $(2.6)_2$, we find

$$\begin{aligned}
&\frac{d}{dt}\int_0^1 \rho_2 q(x)\left(b\psi_x+\int_0^\infty g(s)\eta_x^t(x,s)\,ds\right)dx \\
&=\int_0^1 q(x)\Big(b\psi_{xx}-k(\varphi_x+\psi)-\mu_1\psi_t \\
&-\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t)\,ds+\int_0^\infty g(s)\eta_{xx}^t(x,s)\,ds\Big) \\
&\times\left(b\psi_x-\int_0^\infty g(s)\eta_x^t(x,s)\,ds\right)dx \\
&+\int_0^1 \rho_2 q(x)\psi_t\left(b\psi_{tx}+\int_0^\infty g(s)\eta_{tx}^t(x,s)\,ds\right)dx.
\end{aligned} \tag{2.33}$$

By the fact that

$$\begin{aligned}
&\int_0^1 q(x)\left(b\psi_{xx}+\int_0^\infty g(s)\eta_{xx}^t(x,s)\,ds\right)\left(b\psi_x+\int_0^\infty g(s)\eta_x^t(x,s)\,ds\right)dx \\
&=-\frac{1}{2}\int_0^1 q'(x)\left(b\psi_x+\int_0^\infty g(s)\eta_x^t(x,s)\,ds\right)^2 dx \\
&+\left[\frac{q(x)}{2}\left(b\psi_x+\int_0^\infty g(s)\eta_x^t(x,s)\,ds\right)^2\right]_{x=0}^{x=1}.
\end{aligned} \tag{2.34}$$

The last term in (2.33) can be treated as follows

$$\begin{aligned}
&\int_0^1 \rho_2 q(x)\psi_t\left(b\psi_{tx}+\int_0^\infty g(s)\eta_{tx}^t(x,s)\,ds\right)dx \\
=\;& \rho_2 b\int_0^1 q(x)\psi_t\psi_{tx}dx+\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g(s)\eta_{tx}^t(x,s)\,dsdx \\
=\;& -\frac{\rho_2 b}{2}\int_0^1 q'(x)\psi_t^2dx+\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g(s)\eta_{tx}^t(x,s)\,dsdx \\
=\;& -\frac{\rho_2 b}{2}\int_0^1 q'(x)\psi_t^2dx+\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g(s)\left(\psi_t-\eta_s^t\right)_x dsdx \\
=\;& -\frac{\rho_2 b}{2}\int_0^1 q'(x)\psi_t^2dx+\rho_2 g_0\int_0^1 q(x)\psi_t\psi_{tx}dx-\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g(s)\eta_{sx}^t dsdx \\
=\;& -\frac{\rho_2(b+g_0)}{2}\int_0^1 q'(x)\psi_t^2dx+\rho_2\int_0^1 q(x)\psi_t\int_0^\infty g'(s)\eta_x^t dsdx.
\end{aligned} \tag{2.35}$$

Inserting (2.34) and (2.35) in (.2.33), we arrive at

$$\begin{aligned}
&\left(b\psi_x(0,t)+\int_0^\infty g(s)\,\eta_x^t(0,s)\,ds\right)^2+\left(b\psi_x(1,t)+\int_0^\infty g(s)\,\eta_x^t(1,s)\,ds\right)^2 \\
=\;& -\frac{d}{dt}\int_0^1 \rho_2 q\psi_t\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx+2\rho_2(b+g_0)\int_0^1\psi_t^2dx \\
&-K\int_0^1 q(\varphi_x+\psi)\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx \\
&+\rho_2\int_0^1 q\psi_t\int_0^\infty g'(s)\,\eta_x^t(x,s)\,dsdx \\
&-\mu_1\int_0^1 q(x)\,\psi_t\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx \\
&+2\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2dx \\
&-\int_0^1 q(x)\int_{\tau_1}^{\tau_2}\mu_2(s)\,\psi_t(x,t-s)\,ds\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx.
\end{aligned} \tag{2.36}$$

Now, we estimate terms in the RHS of (2.36) as follows.
First, using Minkowski and Young's inequalities, we have

$$\begin{aligned}
&2\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2dx \\
\leq\;& 4b^2\int_0^1\psi_x^2dx+4g_0\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx.
\end{aligned} \tag{2.37}$$

Second, by Young's inequality and (2.37), we have for any $\lambda > 0$

$$\begin{aligned}
&\left|K\int_0^1 q(x)(\varphi_x+\psi)\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\right| \\
\leq\;& 2K\left|\int_0^1(\varphi_x+\psi)\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\right| \\
\leq\;& 4K^2\lambda\int_0^1(\varphi_x+\psi)^2dx+\frac{1}{4\lambda}\int_0^1\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)^2dx \\
\leq\;& 4K^2\lambda\int_0^1(\varphi_x+\psi)^2dx+\frac{b^2}{2\lambda}\int_0^1\psi_x^2dx+\frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx.
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\left|\mu_1\int_0^1 q(x)\,\psi_t\left(b\psi_x+\int_0^\infty g(s)\,\eta_x^t(x,s)\,ds\right)dx\right| \\
\leq\;& 4\mu_1\lambda\int_0^1\psi_t^2dx+\frac{b^2}{2\lambda}\int_0^1\psi_x^2dx+\frac{g_0}{2\lambda}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2dsdx,
\end{aligned}$$

and

$$
\begin{aligned}
& \left| -\int_0^1 q(x) \int_{\tau_1}^{\tau_2} \mu_2(s) \psi_t(x,t-s)\, ds \left( b\psi_x + \int_0^\infty g(s) \eta_x^t(x,s)\, ds \right) dx \right| \\
\leq\ & b \int_0^1 q(x) \psi_x \int_{\tau_1}^{\tau_2} \mu_2(s) \psi_t(x,t-s)\, ds dx \\
+\ & \int_0^1 \left( q(x) \int_{\tau_1}^{\tau_2} \mu_2(s) \psi_t(x,t-s)\, ds \int_0^\infty g(s) \eta_x^t(x,s)\, ds \right) dx \\
\leq\ & 4\delta_0 \lambda \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\, ds + \frac{b^2}{2\lambda} \int_0^1 \psi_x^2 dx + \frac{g_0}{2\lambda} \int_0^1 \int_0^\infty g(s) \left| \eta_x^t(x,s) \right|^2 ds dx.
\end{aligned}
$$

For any $\varepsilon_2 > 0$, we have

$$
\begin{aligned}
& \left| \rho_2 \int_0^1 q\psi_t \int_0^\infty g'(s) \eta_x^t(x,s)\, ds dx \right| \\
\leq\ & \rho_2 \varepsilon_2 \int_0^1 \psi_t^2 dx - \rho_2 g(0) C(\varepsilon_2) \int_0^1 \int_0^\infty g'(s) \left| \eta_x^t(x,s) \right|^2 ds dx.
\end{aligned}
$$

Inserting all the above estimates into (2.36), we obtain

$$
\begin{aligned}
& \left( b\psi_x(0,t) + \int_0^\infty g(s) \eta_x^t(0,s)\, ds \right)^2 + \left( b\psi_x(1,t) + \int_0^\infty g(s) \eta_x^t(1,s)\, ds \right)^2 \qquad (2.38) \\
& \leq -\frac{d}{dt} \int_0^1 \rho_2 q \psi_t \left( b\psi_x + \int_0^\infty g(s) \eta_x^t(x,s)\, ds \right) dx \\
& + \left( 2\rho_2 (b + g_0) + 4\mu_1^2 \lambda + \rho_2 \varepsilon_2 \right) \int_0^1 \psi_t^2 dx \\
& + b^2 \left( 4 + \frac{3}{2\lambda} \right) \int_0^1 \psi_x^2 dx + 4K^2 \lambda \int_0^1 (\varphi_x + \psi)^2 dx \\
& - \rho_2 g(0) C(\varepsilon_2) \int_0^1 \int_0^\infty g'(s) \left| \eta_x^t(x,s) \right|^2 ds dx \\
& + g_0 \left( 4 + \frac{3}{2\lambda} \right) \int_0^1 \int_0^\infty g(s) \left| \eta_x^t(x,s) \right|^2 ds dx + 4\delta_0 \lambda \int_{\tau_1}^{\tau_2} \mu_2(s) z^2(x,1,s,t)\, ds.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\left[ \varphi_x^2(1) - \varphi_x^2(0) \right] \ \leq\ & -\frac{d}{dt} \frac{1}{k} \int_0^1 \rho_1 q(x) \varphi_t \varphi_x dx \\
& + \ 3 \int_0^1 \varphi_x^2 dx + \int_0^1 \psi_x^2 dx + \frac{2\rho_1}{k} \int_0^1 \varphi_t^2 dx. \qquad (2.39)
\end{aligned}
$$

Consequently, substituting (2.38) and (2.39) into (2.31), our desired estimate (2.30) holds. □

Now, we define the functional

$$I_4(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z^2(x, 1, s, t)\, dsd\rho dx. \tag{2.40}$$

Then the following result holds.

**Lemma 2.8.** *Let* $(\varphi, \psi, z, \eta^t)$ *be the solution of* (2.6)-(2.8)*. Then for* $C_1 > 0$*, we have*

$$\begin{aligned} \frac{dI_4(t)}{dt} \quad &\leq \quad -C_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z^2(x, 1, s, t)\, dsd\rho dx, \\ &\quad -C_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s, t)\, dsdx + \mu_1 \int_0^1 \psi_t^2 dx, \end{aligned} \tag{2.41}$$

*where* $C_1$ *is a positive constant.*

*Proof.* Differentiating (2.40) and using $z(x, 0, s, t) = \psi_t$, $e^{-s} \leq e^{-s\rho}$, we get for all $\rho \in [0, 1]$

$$\begin{aligned} \frac{dI_4(t)}{dt} \quad &\leq \quad \int_0^1 \int_{\tau_1}^{\tau_2} e^{-s} |\mu_2(s)| z^2(x, 1, s, t)\, dsdx + \int_{\tau_1}^{\tau_2} |\mu_2(s)|\, ds \int_0^1 \psi_t^2 dx \\ &\quad - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s} |\mu_2(s)| z^2(x, 1, s, t)\, dsd\rho dx. \end{aligned}$$

Since $s \to -e^{-s}$ is an increasing function, we have $-e^{-s} \leq -e^{-\tau_2}$ for all $s \in [\tau_1, \tau_2]$. Finally, setting, $C_1 = -e^{-\tau_2}$ and recalling (2.2), we obtain (2.41). □

*Proof.* (Proof of Theorem 2.1)

We are now ready to define the Lyapunov functional $L(t)$ as follows

$$L(t) := NE(t) + \frac{1}{4} I_1(t) + N_2 I_2(t) + I_3(t) + \frac{\varepsilon_2}{K} \int_0^1 \rho_1 q \varphi_t \varphi_x dx \tag{2.42}$$

$$+ \frac{1}{4\varepsilon_2} \int_0^1 \rho_2 q(x) \psi_t \left( b\psi_x + \int_0^\infty g(s) \eta_x^t(x, s)\, ds \right) dx + N_4 I_4(t),$$

where $N$, $N_2$, $N_4$ are positive real numbers which will be chosen later.
Consequently, the estimates (2.11), (2.16), (2.22), (2.25), (2.30) and (2.41) together with (2.3) and the following inequality

$$\int_0^1 \varphi_x^2 dx \leq 2 \int_0^1 (\varphi_x + \psi)^2 dx + 2 \int_0^1 \psi_x^2 dx. \tag{2.43}$$

Lead to

$$
\begin{aligned}
\frac{d}{dt}L(t) \leq & \left\{-MC - \frac{\rho_1}{4} + N_2\left(\rho_2 + \frac{\rho_1}{4\lambda_2}\right) + \left(\rho_2 + \frac{\mu_1^2}{4\varepsilon_1}\right)\right. \\
& \left. + \frac{1}{4\varepsilon_2}\left(2\rho_2(b+g_0) + 4\mu_1^2\varepsilon_2^2 + \rho_2\varepsilon_2\right) + N_4\mu_1 + \frac{1}{2\tau}\right\}\int_0^1 \psi_t^2 dx \\
& + \left\{\frac{1}{8\varepsilon_2} + \frac{N_2}{2\varepsilon_4} + \frac{1}{2\varepsilon_4} - C_1N_4\right\}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(s)|\, z^2(x,1,s,t)\,dsdx \\
& + \left\{-\frac{\rho_1}{4} + N_2\rho_1\lambda_2 + \frac{2\rho_1\varepsilon_2}{K} + \varepsilon_1\right\}\int_0^1 \varphi_t^2 dx \\
& + \left\{-\left(\frac{3K}{4} - 2\varepsilon\right) + K^2\varepsilon_2 + 6\varepsilon_2 + \frac{\varepsilon_4 c}{2}\right\}\int_0^1 (\varphi_x + \psi)^2 dx - I_3(t) \\
& + \left\{\frac{1}{4}(b+\delta_1) + N_2(\delta_1 + \mu_2 C^*\lambda_2 - b) + 7\varepsilon_2\right. \\
& \left. + \frac{b^2}{4\varepsilon_2}\left(4 + \frac{3}{2\varepsilon_2^2}\right)\right\}\int_0^1 \psi_x^2 dx + \left\{\left(\frac{c\varepsilon_2}{8} - \frac{cN_2\varepsilon_3}{2}\right)\right\}\int_0^1 \psi^2 dx \\
& + \left\{\frac{g_0}{4\delta_1}\left(\frac{1}{4} + N_2\right) + \frac{g_0}{4\varepsilon_2}\left(4 + \frac{2}{2\varepsilon_2^2}\right)\right. \\
& \left. - \zeta\left(\frac{M}{2} - g_0 C(\varepsilon_1) - \frac{\rho_2 g(0)\, C(\varepsilon_2)}{4\varepsilon_2}\right)\right\}\int_0^1\int_0^\infty g(s)\left|\eta_x^t(x,s)\right|^2 dsdx \\
& - C_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\,|\mu_2(s)|\, z^2(x,1,s,t)\,dsd\rho dx.
\end{aligned}
\tag{2.44}
$$

At this point, we have to choose our constants very carefully.
First, let us choose $\varepsilon$ small enough such that

$$\varepsilon \leq \frac{3K}{8}.$$

Then, we take $\varepsilon_2 = \varepsilon_1$ and choose $\varepsilon_2$ small enough such that

$$\varepsilon_2 \leq \min\left\{\frac{K/8}{(K^2+6)}, \frac{\rho_1/8}{(2\rho_1/K)+1}\right\}.$$

Then, we choose $\lambda_2 = \delta_1$ and choose $\varepsilon_2$ small enough such that

$$\lambda_2 \leq \frac{b/2}{1+\mu_2 C^*}.$$

Once all the above constants are fixed, we fix $N_2$ large enough such that

$$N_2\frac{b}{4} \geq \frac{1}{4}(b+\delta_1) + 7\varepsilon_2 + \frac{b}{4\varepsilon_2}\left(4 + \frac{3}{2\varepsilon_2^2}\right).$$

After that, we pick $\lambda_2$ so small that

$$\lambda_2 \leq \frac{1}{32N_2}.$$

Finally, we choose M large enough so that, there exists a positive constant $\eta_1$, such that (2.44) becomes

$$\begin{aligned}
&\frac{d}{dt}L(t) \leq -\eta_1 \int_0^1 \left(\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x + \psi)^2 + \psi^2\right) dx \\
&-\eta_1 \int_0^1 \int_0^\infty g(s) \left|\eta_x^t(x,s)\right|^2 dsdx \\
&+\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,s,t)\, dsdx \\
&-\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} se^{-s\rho} |\mu_2(s)| z^2(x,1,s,t)\, dsd\rho dx,
\end{aligned} \tag{2.45}$$

which implies by (2.10), that there exists also $\eta_2 > 0$, such that

$$\frac{d}{dt}L(t) \leq -\eta_2 E(t), \quad \forall t \geq 0. \tag{2.46}$$

In addition, we can choose $M$ large enough so that

$$\beta_1 E(t) \leq L(t) \leq \beta_2 E(t), \quad \forall t \geq 0. \tag{2.47}$$

Combining (2.46) and (2.47), we conclude that there exists $\Lambda > 0$ such that

$$\frac{d}{dt}L(t) \leq -\Lambda L(t), \quad \forall t \geq 0. \tag{2.48}$$

A simple integration of (2.48) leads to.

$$L(t) \leq L(0)e^{-\Lambda t}, \quad \forall t \geq 0. \tag{2.49}$$

Again, the us (2.47) and (2.49) yields the desired result (2.13). This completes the proof of Theorem 2.1 □

# Chapter 3

## Exponential stability for a Timoshenko thermoelastic system with second sound and a time-varying delay term in the internal feedback

### 3.1 Introduction

In this chapter, we consider the Timoshenko thermoelastic system of second sound with a time-varing delay term

$$
\begin{cases}
\rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + \mu_1\varphi_t(x,t) + \mu_2\varphi_t(x,t-\tau(t)) = 0, \\
\rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \gamma\theta_x(x,t) = 0, \\
\rho_3\theta_t(x,t) + \kappa q_x(x,t) + \gamma\psi_{tx}(x,t) = 0, \\
\tau_0 q_t(x,t) + \delta q(x,t) + \kappa\theta_x(x,t) = 0, \\
\varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\
\theta(x,0) = \theta_0(x), q(x,0) = q_0(x), \varphi_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), \\
\varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = q(0,t) = q(1,t) = 0,
\end{cases}
\tag{3.1}
$$

where $(x,t) \in (0,1)\times(0,\infty)$, $\rho_1$, $\rho_2$, $\rho_3$, $\gamma$, $\tau_0$, $\delta$, $\kappa$, $\mu_1$,$\mu_2$, $K$ are positive constants, $\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, f_0$ are the initial functions and $\tau(t) > 0$ is the time-varying delay.

Note that for $\tau(t) = \tau$, the system (3.1) has been studied by Ouchenane [55] where he proved the existence and regularity of the solutions under the assumption $\mu_1 > \mu_2$.

## 3.2 Well-posedness of the system

In this section, we state some preliminaries and assumptions, then we prove the existence and uniqueness of the solutions of our problem. Indeed, in the same spirit of [25], we introduce the auxiliary variable $W(x,\rho,t)$ as follow

$$W(x,\rho,t) = \varphi_t(x, t-\tau(t)\rho),$$

hence, we obtain

$$\tau(t) W_t(x,\rho,t) + (1-\tau'(t)\rho)W_\rho(x,\rho,t) = 0.$$

Then, the problem (2.1) can be reduced to

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - K(\varphi_x+\psi)_x(x,t) + \mu_1\varphi_t(x,t) + \mu_2 W(x,1,t) = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + K(\varphi_x+\psi)(x,t) + \gamma\theta_x(x,t) = 0, \\ \rho_3\theta_t(x,t) + \kappa q_x(x,t) + \gamma\psi_{tx}(x,t) = 0, \\ \tau_0 q_t(x,t) + \delta q(x,t) + \kappa\theta_x(x,t) = 0, \\ \tau(t) W_t(x,\rho,t) + (1-\tau'(t)\rho)W_\rho(x,\rho,t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \quad \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x), q(x,0) = q_0(x), \varphi_t(x,t-\tau(0)) = f_0(x,t-\tau(0)), \\ W(x,0,t) = \varphi_t(x,t), W(x,1,t) = f_0(x,t-\tau(t)), \\ q(0,t) = q(1,t) = \psi(0,t) = \psi(1,t) = \varphi(0,t) = \varphi(1,t) = 0, \end{cases} \tag{3.2}$$

where $(x,\rho,t) \in (0,1)\times(0,1)\times(0,\infty)$ and the function $\tau(t)$ satisfies (15), (16) and the condition

$$0 < \tau_0 \leq \tau(t) \leq \bar{\tau}, \qquad \text{for any } t > 0. \tag{3.3}$$

Let $\mathcal{X} = (\varphi, \varphi_t, \psi, \psi_t, \theta, q, W)^T$, we rewrite (3.2) as

$$\begin{cases} \mathcal{X}' = \mathcal{A}\mathcal{X}, \\ \mathcal{X}(0) = \mathcal{X}_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, f_0(., -\rho\tau(0)))^T, \end{cases} \tag{3.4}$$

with

$$\mathcal{A}(t)\begin{pmatrix}\varphi\\ u\\ \psi\\ v\\ \theta\\ q\\ W\end{pmatrix}=\begin{pmatrix}u\\ \frac{k}{\rho_1}(\varphi_{xx}+\psi_x)-\frac{\mu_1}{\rho_1}u-\frac{\mu_2}{\rho_1}W(.,1)\\ v\\ \frac{b}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}(\varphi_x+\psi)-\frac{\gamma}{\rho_2}\theta_x\\ -\frac{k}{\rho_3}q_x-\frac{\gamma}{\rho_3}v_x\\ -\frac{\delta}{\tau_0}q-\frac{k}{\tau_0}\theta_x\\ (\tau'(t)\rho-1)W_\rho/\tau(t)\end{pmatrix},$$

where

$$D(\mathcal{A}(t))=\left\{(\varphi,\varphi_t,\psi,\psi_t,\theta,q,W)^T\in H: W_\rho\in L^2\left((0,1)\times L^2(0,1)\right), u=W(.,0)\,, \text{in } (0,1)\right\}, \qquad (3.5)$$

with

$$H=\mathcal{V}\times H_0^1(0,1)\times\mathcal{V}\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times L^2\left((0,1)\times H^1(0,1)\right),$$

for $t>0$ and $\mathcal{V}=H^2(0,1)\cap H_0^1(0,1)$.

Notice that, $D(\mathcal{A}(t))$ is independent of $t$.

Now, defing $\mathcal{E}$ as,

$$\begin{aligned}\mathcal{E} &= H_0^1(0,1)\times L^2(0,1)\times H_0^1(0,1)\times\left(L^2(0,1)\right)^2\\ &\quad\times L^2(0,1)\times L^2((0,1)\times(0,1)),\end{aligned}$$

with the inner product

$$\begin{aligned}\left\langle \mathcal{X},\bar{\mathcal{X}}\right\rangle_{\mathcal{E}} &= \int_0^1\left\{\rho_1 u\tilde{u}+\rho_2 v\tilde{v}+k(\varphi_x+\psi)\left(\tilde{\varphi}_x+\tilde{\psi}\right)+b\psi_x\tilde{\psi}_x+\rho_3\theta\tilde{\theta}\right\}dx\\ &\quad+\int_0^1\tau_0 q\tilde{q}dx+\int_0^1\int_0^1 W(x,\rho)\tilde{W}(x,\rho)\,d\rho dx,\end{aligned}$$

for $\mathcal{X}=(\varphi,u,\psi,v,\theta,q,W)^T$ , $\tilde{\mathcal{X}}=\left(\tilde{\varphi},\tilde{u},\tilde{\psi},\tilde{v},\tilde{\theta},\tilde{q},\tilde{W}\right)^T$.

Our first result is

**Theorem 3.1.** *We assume that (14) holds and (15), (16), (3.3) are satisfied. Then, for any* $\mathcal{X}_0 \in \mathrm{D}(\mathcal{A}(0))$ *, there exists a unique solution* $\mathcal{X}$ *of (3.2) satisfying*

$$\mathcal{X} \in C([0,+\infty), \mathrm{D}(\mathcal{A}(0))) \cap C^1([0,+\infty), \mathcal{E}).$$

The proof of the above theorem is based on the theorem below ( for more detail see [22]).

**Theorem 3.2.** *the following hypothesis:*

*(i)* $\mathrm{D}(\mathcal{A}(0))$ *is a dense subset of* $\mathcal{E}$;

*(ii) for any* $t > 0$ *we have* $\mathrm{D}(\mathcal{A}(t)) = \mathrm{D}(\mathcal{A}(0))$;

*(iii)* $\mathcal{A}(t)$ *generates a strongly continuous semigroup on* $\mathcal{E}$ *for all* $t \in [0;T]$, *and the family* $\mathcal{A} = \{\mathcal{A}(t) : t \in [0;T]\}$ *is stable with stability constants C and m independent of t, i.e, the semigroup* $(S_t(s))_{s\geq 0}$ *generated by* $\mathcal{A}(t)$ *satisfies*

$$\|(S_t(s))(u)\|_{\mathcal{E}} \leq Ce^{ms}\|u\|_{\mathcal{E}}, \qquad \text{for all } u \in \mathcal{E}, s \geq 0, C > 0, m > 0;$$

*(iv)* $\partial_t \tilde{\mathcal{A}}(t) \in L^\infty_*([0,T], B(\mathrm{D}(\mathcal{A}(0)), \mathcal{E}))$, *where* $L^\infty_*([0,T], B(\mathrm{D}(\mathcal{A}(0)), \mathcal{E}))$ *is the space of equivalent classes of essentially bounded, strongly measurable functions from* $[0;T]$ *into the set* $B(\mathrm{D}(\mathcal{A}(0)); \mathcal{E})$ *of bounded operators from* $\mathrm{D}(\mathcal{A}(0))$ *into* $\mathcal{E}$,

*are hold.*

*Then given an initial data in* $\mathrm{D}(\mathcal{A}(0))$, *problem (3.4) has a unique solution*

$$\mathcal{X} \in C([0,T), \mathrm{D}(\mathcal{A}(0))) \cap C^1([0,T), \mathcal{E}).$$

*Proof.* **(Proof of theorem 3.1)** Our proof follows the method used in [52].

i) Density of $D(\mathcal{A}(0))$ in $\mathcal{E}$. Let $G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7)^T \in \mathcal{E}$ be orthogonal to all elements of $D(\mathcal{A}(0))$ with the inner product $\langle .,. \rangle_{\mathcal{E}}$ :

$$\begin{aligned} 0 &= \langle \mathcal{X}, G\rangle_{\mathcal{E}} = \int_0^1 \{\rho_1 u g_2 + \rho_2 v g_4 + k(\varphi_x + \psi)(g_{1x} + g_3) + b\psi_x g_{3x} + \rho_3 \theta g_5\}\, dx \\ &\quad + \int_0^1 \tau_0 q g_6 dx + \int_0^1 \int_0^1 W(x,\rho)\, g_7 d\rho dx, \end{aligned} \tag{3.6}$$

for all $\mathcal{X} = (\varphi, u, \psi, v, \theta, q, W)^T \in D(\mathcal{A}(0))$. To show that $g_j = 0$, for all $j = 1, ..., 7$, we take $W \in D((0,1) \times (0,1))$ and $\varphi = u = \psi = v = \theta = q = 0$, so $\mathcal{X} = (0,0,0,0,0,0,W)^T \in D(\mathcal{A}(0))$ therfore, from

(3.6), we deduce that

$$\int_0^1 W(x,\rho)\, g_7 d\rho dx = 0,$$

since $D((0,1)\times(0,1))$ is dense in $L^2((0,1)\times(0,1))$, it follows then that $g_7 = 0$. Similarly, let $u \in D(0,1)$, then $\mathcal{X} = (0, u, 0, 0, 0, 0, 0)^T \in D(\mathcal{A}(0))$, which implies from (3.6) that

$$\int_0^1 u g_2 dx = 0.$$

So, as above, $g_2 = 0$. Moreover let $\mathcal{X} = (\varphi, 0, 0, 0, 0, 0, 0)^T \in D(\mathcal{A}(0))$, then we obtain from (3.6) that

$$\int_0^1 \varphi g_{1x} dx = 0.$$

It is immediate that $\mathcal{X} = (\varphi, 0, 0, 0, 0, 0, 0)^T \in D(\mathcal{A}(0))$ for $\varphi \in \mathcal{V}$ which is dense in $H_0^1(0,1)$, with the inner product

$$\langle g, h\rangle_{H_0^1(0,1)} = \int_0^1 g_x h_x dx.$$

We get $g_1 = 0$. By the same ideas as above, we can also show that $g_3 = 0$,

for $v \in D(0,1)$, we get from (3.6)

$$\int_0^1 v g_4 dx = 0,$$

and by density of $D(0,1)$ in $L^2(0,1)$, we obtain $g_4 = 0$.

For $q \in D(0,1)$, we get from (3.6)

$$\int_0^1 q g_6 dx = 0,$$

and by density of $D(0,1)$ in $L^2(0,1)$, we obtain $g_6 = 0$.

Next, let $\mathcal{X} = (0, 0, 0, 0, \theta, 0, 0)^T \in D(\mathcal{A}(0))$, then we obtain from (3.6) that

$$\int_0^1 \theta g_5 dx = 0,$$

consequently $g_5 = 0$. This completes the proof of (i).

(ii) for any $t > 0$ we have $D(\mathcal{A}(t)) = D(\mathcal{A}(0))$.

(iii) $\mathcal{A}(t)$ generates a $C_0$-semigroup in $\mathcal{E}$ for a fixed $t$. We define the time-dependent inner-product on $\mathcal{E}$, (which is equivalent to the classical inner product)

$$\begin{aligned}\left\langle \mathcal{X}, \bar{\mathcal{X}} \right\rangle_t &= \int_0^1 \left\{ \rho_1 u\bar{u} + \rho_2 v\bar{v} + k\left(\varphi_x + \psi\right)\left(\bar{\varphi}_x + \bar{\psi}\right) + b\psi_x\bar{\psi}_x + \rho_3\theta\bar{\theta} \right\} dx \\ &\quad + \int_0^1 \tau_0 q\bar{q}dx + \xi\tau(t)\int_0^1\int_0^1 W(x,\rho)\,\bar{W}(x,\rho)\,d\rho dx, \end{aligned} \tag{3.7}$$

where $\xi$ satisfies

$$\frac{\mu_2}{\sqrt{1-d}} \leq \xi \leq 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}, \tag{3.8}$$

thanks to hypothesis (14).

Let us set

$$\varkappa(t) = \frac{\left(\tau'(t)^2 + 1\right)^{1/2}}{2\tau(t)}.$$

At this stage we show that the dissipativity of the operator $\tilde{\mathcal{A}}(t) = \mathcal{A}(t) - \varkappa(t)\, I$.

For a fixed $t$ and $\mathcal{X} = (\varphi, u, \psi, v, \theta, q, W)^T \in D(\mathcal{A}(t))$, we have

$$\begin{aligned}\langle \mathcal{A}(t)\,\mathcal{X}, \mathcal{X}\rangle_t &= -\delta\int_0^1 q^2dx - \mu_1\int_0^1 u^2(x)\,dx - \mu_2\int_0^1 W(x,1)\,u(x)\,dx \\ &\quad -\xi\int_0^1\int_0^1 (1-\tau'(t)\rho)\,W(x,\rho)\,W_\rho(x,\rho)\,dxd\rho. \end{aligned} \tag{3.9}$$

Observing that

$$\begin{aligned}&\int_0^1\int_0^1 (1-\tau'(t)\rho)\,W(x,\rho)\,W_\rho(x,\rho)\,dxd\rho \qquad (3.10)\\ =\;& \int_0^1\int_0^1 \frac{1}{2}\frac{\partial}{\partial\rho}W^2(1-\tau'(t)\rho)\,dxd\rho \\ =\;& \frac{\tau'(t)}{2}\int_0^1\int_0^1 W^2(x,\rho)\,dxd\rho + \frac{1}{2}\int_0^1 \left\{W^2(x,1)(1-\tau'(t)) - W^2(x,0)\right\}dx. \end{aligned}$$

Whereupon,

$$\begin{aligned}\langle \mathcal{A}(t)\mathcal{X},\mathcal{X}\rangle_t &= -\delta\int_0^1 q^2dx-\mu_1\int_0^1 u^2(x)\,dx-\mu_2\int_0^1 W(x,1)\,u(x)\,dx\\ &-\frac{\tau'(t)\xi}{2}\int_0^1\int_0^1 W^2(x,\rho)\,dxd\rho\\ &-\frac{\xi}{2}\int_0^1 W^2(x,1)(1-\tau'(t))\,dx+\frac{\xi}{2}\int_0^1 u^2(x)\,dx. \end{aligned}\tag{3.11}$$

By applying the Cauchy-Schwarz inequality and (15), we get

$$\begin{aligned}\langle \mathcal{A}(t)\mathcal{X},\mathcal{X}\rangle_t &\le \left(-\mu_1+\frac{\mu_2}{2\sqrt{1-d}}+\frac{\xi}{2}\right)\int_0^1 u^2(x)\,dx-\delta\int_0^1 q^2dx\\ &+\left(\frac{\mu_2\sqrt{1-d}}{2}-\xi\frac{(1-d)}{2}\right)\int_0^1 W^2(x,1)\,dx+\varkappa(t)\langle U,U\rangle_t\,. \end{aligned}$$

Condition (3.8) allows to write

$$-\mu_1+\frac{\mu_2}{2\sqrt{1-d}}+\frac{\xi}{2}\le 0,\qquad \frac{\mu_2\sqrt{1-d}}{2}-\xi\frac{(1-d)}{2}\le 0.$$

Therefore, the operator $\tilde{\mathcal{A}}(t)$ is dissipative.

We are now in a position to show that the operator $\lambda I-\mathcal{A}(t)$ is surjective. Let $(f_1,f_2,f_3,f_4,f_5,f_6,f_7)^T\in\mathcal{E}$ and $\mathcal{X}=(\varphi,u,\psi,v,\theta,q,W)^T\in D(\mathcal{A}(t))$ be solution of the following system

$$\begin{cases}\lambda\varphi-u=f_1,\\ \lambda u-\frac{k}{\rho_1}(\varphi_{xx}+\psi_x)+\frac{\mu_1}{\rho_1}u+\frac{\mu_2}{\rho_1}W(.,1)=f_2,\\ \lambda\psi-v=f_3,\\ \lambda v-\frac{b}{\rho_2}\psi_{xx}+\frac{k}{\rho_2}(\varphi_x+\psi)+\frac{\gamma}{\rho_2}\theta_x=f_4,\\ \lambda\theta+\frac{k}{\rho_3}q_x+\frac{\gamma}{\rho_3}v_x=f_5,\\ \lambda q+\frac{\delta}{\tau_0}q+\frac{k}{\tau_0}\theta_x=f_6,\\ \lambda W+\frac{(1-\tau'(t)\rho)}{\tau(t)}W_\rho=f_7.\end{cases}\tag{3.12}$$

The first and the third equations in (3.12) give

$$\begin{cases}u=\lambda\varphi-f_1,\\ v=\lambda\psi-f_3.\end{cases}\tag{3.13}$$

Moreover, by (3.12) we find $W$ as

$$W(x,0) = u(x), \qquad \text{for } x \in (0,1). \tag{3.14}$$

We obtain, by using the last equation in (3.12) and the same approach as in [52]

$$W(x,\rho) = u(x)\, e^{-\lambda\rho\tau(t)} + \tau(t)\, e^{-\lambda\rho\tau(t)} \int_0^1 f_7(x,\sigma)\, e^{-\lambda\rho\tau(t)} d\sigma, \quad \text{if } \tau'(t) = 0,$$

and

$$W(x,\rho) = u(x)\, e^{\vartheta_\rho(t)} + e^{\vartheta_\rho(t)} \int_0^1 \frac{f_7(x,\sigma)\,\tau(t)}{1-\tau'(t)\,\sigma} e^{-\vartheta_\rho(t)} d\sigma, \quad \text{if } \tau'(t) \neq 0,$$

where $\vartheta_\rho(t) = \lambda \frac{\tau(t)}{\tau'(t)} \ln(1-\tau'(t)\rho)$. Whereupon, from (3.13), we have

$$W(x,\rho) = \lambda\varphi(x)\, e^{-\lambda\rho\tau(t)} - f_1 e^{-\lambda\rho\tau(t)} + \tau(t)\, e^{-\lambda\rho\tau(t)} \int_0^1 f_7(x,\sigma)\, e^{-\lambda\rho\tau(t)} d\sigma, \quad \text{if } \tau'(t) = 0, \tag{3.15}$$

and

$$W(x,\rho) = \lambda\varphi(x)\, e^{\vartheta_\rho(t)} - f_1 e^{\vartheta_\rho(t)} + e^{\vartheta_\rho(t)} \int_0^1 \frac{f_7(x,\sigma)\,\tau(t)}{1-\tau'(t)\,\sigma} e^{-\vartheta_\rho(t)} d\sigma, \quad \text{if } \tau'(t) \neq 0. \tag{3.16}$$

By using (3.12) and (3.13), we get

$$\begin{cases} \left(\lambda^2 + \frac{\mu_1}{\rho_1}\lambda + \lambda e^{-\lambda\tau}\frac{\mu_2}{\rho_1}\right)\varphi - \frac{k}{\rho_1}(\varphi_{xx}+\psi_x) = f_2 + \left(\lambda + \frac{\mu_1}{\rho_1}\right) f_1 - \frac{\mu_2}{\rho_1} W_0(x), \\ \lambda^2\psi - \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(\varphi_x+\psi) + \frac{\gamma}{\rho_2}\theta_x = f_4 + \lambda f_3, \\ \lambda\theta + \frac{k}{\rho_3} q_x + \frac{\gamma\lambda}{\rho_3}\psi_x = f_5 + \frac{\gamma}{\rho_3} f_{3x}, \\ \lambda q + \frac{\delta}{\tau_0} q + \frac{k}{\tau_0}\theta_x = f_6. \end{cases} \tag{3.17}$$

We obtain, by solving the system (3.17)

$$(\varphi,\psi,\theta,q) \in \mathcal{V} \times \mathcal{V} \times H^2(0,1) \times H_0^1(0,1),$$

such that

$$\begin{cases} \int_0^1 \left(\left(\lambda^2\rho_1 + \mu_1\lambda + \lambda e^{-\lambda\tau(t)}\mu_2\right)\varphi w + k\left(\varphi_x + \psi\right) w_x\right) dx = \int_0^1 \left(\rho_1 f_2 + \left(\lambda\rho_1 + \mu_1\right) f_1 - \mu_2 W_0\left(x\right)\right) w dx, \\ \int_0^1 \left(\rho_2\lambda^2\psi X + b\psi_x X_x + k\left(\varphi_x + \psi\right) X + \gamma\theta_x X\right) dx = \int_0^1 \rho_2\left(f_4 + \lambda f_3\right) X dx, \\ \int_0^1 \left(\rho_3\lambda\theta w_1 + kq_x w_1 + \gamma\lambda\psi_x w_1\right) dx = \int_0^1 \left(\rho_3 f_5 + \gamma f_{3x}\right) w_1 dx, \\ \int_0^1 \left(\left(\tau_0\lambda + \delta\right) q X_1 + k\theta_x X_1\right) dx = \int_0^1 \tau_0 f_6 X_1 dx, \end{cases} \tag{3.18}$$

for all $(w, X, w_1, X_1) \in H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)$. From (3.15) and (3.16), we obtain

$$W(x,1) = \begin{cases} \lambda\varphi(x) e^{-\lambda\tau(t)} + W_0(x), & \text{if } \tau'(t) = 0, \\ \lambda\varphi(x) e^{\vartheta(t)} + W_0(x), & \text{if } \tau'(t) \neq 0, \end{cases}$$

where $x \in (0,1)$ and

$$W_0(x) = \begin{cases} -f_1 e^{-\lambda\rho\tau(t)} + \tau(t) e^{-\lambda\rho\tau(t)} \int_0^1 f_7(x,\sigma) e^{-\lambda\rho\tau(t)} d\sigma, & \text{if } \tau'(t) = 0, \\ -f_1 e^{\vartheta_\rho(t)} + e^{\vartheta_\rho(t)} \int_0^1 \frac{f_7(x,\sigma)\tau(t)}{1-\tau'(t)\sigma} e^{-\vartheta_\rho(t)} d\sigma, & \text{if } \tau'(t) \neq 0. \end{cases} \tag{3.19}$$

Thus, problem (3.18) is equivalent to

$$Y\left(\left(\varphi, \psi, \theta, q\right), \left(w, X, w_1, X_1\right)\right) = Q\left(w, X, w_1, X_1\right), \tag{3.20}$$

where the bilinear from $Y : \left[H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)\right]^2 \to \mathbb{R}$, and the linear form $Q : \left(H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \times H_0^1(0,1)\right) \to \mathbb{R}$ are defined by

$$\begin{aligned} Y\left(\left(\varphi, \psi, \theta, q\right), \left(w, X, w_1, X_1\right)\right) &= \int_0^1 \left(\left(\lambda^2\rho_1 + \mu_1\lambda + \lambda e^{-\lambda\tau(t)}\mu_2\right)\varphi w + k\left(\varphi_x + \psi\right)\left(w_x + X\right)\right) dx \\ &+ \int_0^1 \left(\rho_2\lambda^2\psi X + b\psi_x X_x + \gamma\theta_x w_{1x}\right) dx \\ &+ \int_0^1 \left(\rho_3\lambda\theta w_1 + kq_x w_1 + \gamma\lambda\psi_x w_1\right) dx \\ &+ \int_0^1 \left(\left(\tau_0\lambda + \delta\right) q X_1 + k\theta_x X_{1x}\right) dx, \end{aligned}$$

and

$$Q(w,X,w_1,X_1) = \int_0^1 (\rho_1 f_2 + (\lambda\rho_1+\mu_1) f_1 - \mu_2 W_0(x))\, w dx + \int_0^1 \rho_2 (f_4+\lambda f_3)\, X dx + \int_0^1 (\rho_3 f_5 + \gamma f_{3x})\, w_1 dx + \int_0^1 \tau_0 f_6 X_1 dx, \tag{3.21}$$

if $\tau'(t) = 0$ , where $W_0(x)$ satisfies the first equation in (3.19).

If $\tau'(t) \neq 0$, we define

$$\begin{aligned} Y((\varphi,\psi,\theta,q),(w,X,w_1,X_1)) = & \int_0^1 \left(\left(\lambda^2\rho_1 + \mu_1\lambda + \lambda e^{\vartheta_\rho(t)}\mu_2\right)\varphi w + k(\varphi_x+\psi)(w_x+X)\right) dx \\ & + \int_0^1 \left(\rho_2\lambda^2\psi X + b\psi_x X_x + \gamma\theta_x w_{1x}\right) dx \\ & + \int_0^1 (\rho_3\lambda\theta w_1 + kq_x w_1 + \gamma\lambda\psi_x w_1)\, dx \\ & + \int_0^1 ((\tau_0\lambda+\delta)\, qX_1 + k\theta_x X_{1x})\, dx, \end{aligned}$$

and the operator $Q$ is defined by the above formula (3.21). So by applying the Lax-Milgram theorem, problem (3.20) has a unique solution $(\varphi,\psi,\theta,q)$ for all $(w,X,w_1,X_1) \in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)$. By using the classical elliptic regularity, it follows from (3.18) that $(\varphi,\psi,\theta,q) \in H^2(0,1)\times H^2(0,1)\times H^1(0,1)\times H_0^1(0,1)$.

Consequently, the operator $\lambda I - \mathcal{A}(t)$ is surjective

$$\lambda I - \tilde{\mathcal{A}}(t) = (\lambda + \varkappa(t))\, I - \mathcal{A}(t), \qquad \text{for any fixed } t>0 \text{ and } \lambda>0.$$

Since $\varkappa(t) > 0$, then, we conclude the surjectivity of the operator $\lambda I - \tilde{\mathcal{A}}(t)$.

In this step, it is sufficient to prove

$$\frac{\|\phi\|_t}{\|\phi\|_s} \leq e^{\frac{c}{2\tau_0}|t-s|}, \qquad \text{for any } t,s \in [0,T], \tag{3.22}$$

where $\phi = (\varphi, u, \psi, v, \theta, q, W)^T$. By (3.7), we have

$$\begin{aligned}\|\phi\|_t^2 &= \langle \phi, \phi \rangle_t = \int_0^1 \left\{\rho_1 u^2 + \rho_2 v^2 + k(\varphi_x + \psi)^2 + b\psi_x^2 + \rho_3\theta^2 + \tau_0 q^2\right\} dx \\ &\quad + \xi\tau(t) \int_0^1 \int_0^1 W^2(x,\rho)\, d\rho dx,\end{aligned}$$

$$\begin{aligned}\|\phi\|_s^2 &= \langle \phi, \phi \rangle_s = \int_0^1 \left\{\rho_1 u^2 + \rho_2 v^2 + k(\varphi_x + \psi)^2 + b\psi_x^2 + \rho_3\theta^2 + \tau_0 q^2\right\} dx \\ &\quad + \xi\tau(s) \int_0^1 \int_0^1 W^2(x,\rho)\, d\rho dx,\end{aligned}$$

then,

$$\begin{aligned}\|\phi\|_t^2 - \|\phi\|_s^2 e^{\frac{c}{2\tau_0}|t-s|} &= \left(1 - e^{\frac{c}{2\tau_0}|t-s|}\right) \int_0^1 \left\{\rho_1 u^2 + \rho_2 v^2 + k(\varphi_x + \psi)^2 + b\psi_x^2 + \rho_3\theta^2 + \tau_0 q^2\right\} dx \\ &\quad + \xi\left(\tau(t) - \tau(s) e^{\frac{c}{2\tau_0}|t-s|}\right) \int_0^1 \int_0^1 W^2(x,\rho)\, d\rho dx. \qquad (3.23)\end{aligned}$$

Here, we shwo that $\tau(t) - \tau(s) e^{\frac{r}{2\tau_0}|t-s|} \leq 0$, for $r > 0$, we have

$$\tau(t) = \tau(s) + \tau'(a)(t-s),$$

with $a \in (s,t)$ ,we obtain

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{|\tau'(a)|}{\tau(s)} |t-s|.$$

By using $\tau \in W^{2,\infty}([0,T])$ and $\tau'$ is bounded, we deduce that

$$\frac{\tau(t)}{\tau(s)} \leq 1 + \frac{r}{\tau_0} |t-s| \leq e^{\frac{c}{2\tau_0}|t-s|},$$

which proves (3.22) and therefore (iii).

(iv) We have the operator $\tilde{\mathcal{A}}(t)$ is dissipative, which means that

$$\langle A(t)U, U \rangle_t - \chi(t) \langle U, U \rangle_t \leq 0,$$

where

$$\tilde{\mathcal{A}}(t) = A(t) - \chi(t) I,$$

and

$$\chi(t) = \frac{\left(\tau'(t)^2 + 1\right)^{\frac{1}{2}}}{2\tau(t)}.$$

Moreover

$$\chi'(t) = \frac{\tau'(t)\,\tau''(t)}{2\tau(t)\left(\tau'(t)^2 + 1\right)^{\frac{1}{2}}} - \frac{\tau'(t)\left(\tau'(t)^2 + 1\right)^{\frac{1}{2}}}{2\tau(t)^2},$$

is bounded on $[0, T], \forall T > 0$ then it is easy to check that

$$\frac{d}{dt}\mathcal{A}(t)\mathcal{X} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{(\tau''(t)\tau(t)\rho - \tau'(t)(\tau'(t)\rho - 1))}{\tau^2(t)} W_\rho \end{pmatrix}.$$

Then, by using (16) and (3.3), thus

$$\frac{d}{dt}\tilde{\mathcal{A}}(t) \in L_*^\infty([0, T], B(D(A(0)), \mathcal{H})),$$

where (iv) holds in [52]. Then, we conclude that the problem

$$\begin{cases} \tilde{\mathcal{X}}_t = \tilde{\mathcal{A}}(t)\tilde{\mathcal{X}}, \\ \tilde{\mathcal{X}}(0) = \mathcal{X}_0, \end{cases} \tag{3.24}$$

admits a unique solution $\tilde{\mathcal{X}} \in C([0, +\infty), \mathcal{E})$ and if $\mathcal{X}_0 \in D(\mathcal{A}(0))$, then

$$\tilde{\mathcal{X}} \in C([0, +\infty), D(\mathcal{A}(0))) \cap C^1([0, +\infty), \mathcal{E}).$$

Now, let

$$\mathcal{X}(t) = e^{\beta(t)}\tilde{\mathcal{X}}(t),$$

where $\beta(t) = \int_0^t k(s)\,ds$, then we have by using (3.24)

$$\begin{aligned}
\mathcal{X}_t(t) &= k(t)\,e^{\beta(t)}\tilde{\mathcal{X}}(t) + e^{\beta(t)}\tilde{\mathcal{X}}_t(t) \\
&= k(t)\,e^{\beta(t)}\tilde{\mathcal{X}}(t) + e^{\beta(t)}\tilde{\mathcal{A}}(t)\,\tilde{\mathcal{X}}(t) \\
&= e^{\beta(t)}\left(k(t)\,\tilde{\mathcal{X}}(t) + \tilde{\mathcal{A}}(t)\,\tilde{\mathcal{X}}(t)\right) \\
&= e^{\beta(t)}\mathcal{A}(t)\,\tilde{\mathcal{X}}(t) \\
&= \mathcal{A}(t)\,e^{\beta(t)}\tilde{\mathcal{X}}(t) \\
&= \mathcal{A}(t)\,\mathcal{X}(t).
\end{aligned}$$

Consequently, $\mathcal{X}(t)$ is the unique solution of (3.4). □

## 3.3 Exponential stability

In this section, we give some technical lemma and we prove the stability results, following [55]; we introduce the new variable

$$\bar{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)\,dx.$$

Using (3.1), we get

$$\int_0^1 \bar{\theta}(x,t)\,dx = 0,$$

then, $\left(\varphi, \psi, \bar{\theta}, q, W\right)$ satisfies the system (3.2).

For a positive constant $\xi$ satisfying

$$\frac{\mu_2}{\sqrt{1-d}} \le \xi \le 2\mu_1 - \frac{\mu_2}{\sqrt{1-d}}, \tag{3.25}$$

we define the energy functional as

$$\begin{aligned}
E(t) &= E(t,\varphi,\psi,\theta,q,W) \\
&= \frac{1}{2}\int_0^1 \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2\right)dx + \frac{1}{2}\int_0^1 \left\{k(\varphi_x+\psi)^2 + b\psi_x^2 + \rho_3\theta^2\right\}dx \\
&\quad + \frac{1}{2}\int_0^1 \tau_0 q^2\,dx + \frac{\xi}{2}\tau(t)\int_0^1\int_0^1 W^2(x,\rho,t)\,d\rho\,dx.
\end{aligned} \tag{3.26}$$

By multiplying the system (3.2) by $\varphi_t, \psi_t, \theta, q$, respectively, integrating over $(0,1)$, and summing them up, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\rho_1\varphi_t^2+\rho_2\psi_t^2\right)dx+\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{k\left(\varphi_x+\psi\right)^2+b\psi_x^2+\rho_3\theta^2+\tau_0 q^2\right\}dx\\
=\;& -\delta\int_0^1 q^2dx-\mu_1\int_0^1 \varphi_t^2\left(x,t\right)dx-\mu_2\int_0^1 \varphi_t\left(x,t\right)W\left(x,1,t\right)dx. \qquad (3.27)
\end{aligned}$$

We multiply $(3.2)_3$ by $W$

$$\begin{aligned}
&\frac{\xi}{2}\frac{d}{dt}\int_0^1\int_0^1 \tau\left(t\right)W^2\left(x,\rho,t\right)d\rho dx \qquad (3.28)\\
=\;& -\xi\int_0^1\int_0^1 \left(1-\tau'\left(t\right)\rho\right)W\left(x,\rho,t\right)W_\rho\left(x,\rho,t\right)d\rho dx\\
&+\frac{\xi}{2}\tau'\left(t\right)\int_0^1\int_0^1 W^2\left(x,\rho,t\right)d\rho dx\\
=\;& -\frac{\xi}{2}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(\left(1-\tau'\left(t\right)\rho\right)W^2\left(x,\rho,t\right)\right)d\rho dx\\
=\;& \frac{\xi}{2}\int_0^1 \left(W^2\left(x,0,t\right)-W^2\left(x,1,t\right)\right)dx+\frac{\xi\tau'\left(t\right)}{2}\int_0^1 W^2\left(x,1,t\right)dx.
\end{aligned}$$

From (3.26), (3.27) and (3.28), we obtain

$$\begin{aligned}
\frac{dE\left(t\right)}{dt} =\;& -\left(\mu_1-\frac{\xi}{2}\right)\int_0^1 \varphi_t^2\left(x,t\right)dx+\left(\frac{\xi\tau'\left(t\right)}{2}-\frac{\xi}{2}\right)\int_0^1 W^2\left(x,1,t\right)dx \qquad (3.29)\\
&-\mu_2\int_0^1 \varphi_t\left(x,t\right)W\left(x,1,t\right)dx-\delta\int_0^1 q^2dx.
\end{aligned}$$

Thanks to Young's inequality, the last term in (3.29) can be estimated as follows

$$\begin{aligned}
&\mu_2\int_0^1 \varphi_t\left(x,t\right)W\left(x,1,t\right)dx \qquad (3.30)\\
\le\;& \frac{\mu_2}{2\sqrt{1-d}}\int_0^1 \varphi_t^2\left(x,t\right)dx+\frac{\mu_2\sqrt{1-d}}{2}\int_0^1 W^2\left(x,1,t\right)dx.
\end{aligned}$$

Inserting (3.30) into (3.29), we obtain

$$\begin{aligned}
\frac{dE\left(t\right)}{dt} \le\;& -\left(\mu_1-\frac{\xi}{2}-\frac{\mu_2}{2\sqrt{1-d}}\right)\int_0^1 \varphi_t^2\left(x,t\right)dx-\delta\int_0^1 q^2dx \qquad (3.31)\\
&+\left(\frac{\xi}{2}\left(\tau'\left(t\right)-1\right)+\frac{\mu_2\sqrt{1-d}}{2}\right)\int_0^1 W^2\left(x,1,t\right)dx.
\end{aligned}$$

Then, we deduce that there exists $C > 0$ such that

$$\frac{dE(t)}{dt} \leq -\delta \int_0^1 q^2 dx - C\left\{\int_0^1 \varphi_t^2(x,t)\,dx + \int_0^1 W^2(x,1,t)\,dx\right\}. \tag{3.32}$$

Then, $E$ is a non-increasing function.

Our second result is

**Theorem 3.3.** *Let $U_0 \in D(\mathcal{A}(0))$. Assume that (14) holds. Then under the hypotheses (15), (16) and (3.3), any solution of problem (3.2). satisfies*

$$E(t) \leq Ce^{-\gamma t}, \qquad \forall t \geq 0, \tag{3.33}$$

*for some positive constants $C$ and $\gamma$ independent of $t$.*

To derive the exponential decay of the solution, we build a functional $\mathcal{L}(t)$ which is equivalent to the energy $E(t)$ and satisfies

$$\frac{d\mathcal{L}(t)}{dt} \leq -\Lambda\mathcal{L}(t), \qquad \forall t \geq 0,$$

for some constant $\Lambda > 0$.

First, let us consider the functional $I_1$ given by

$$I_1(t) := \int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx. \tag{3.34}$$

Hence, we obtain

**Lemma 3.4.** *Let $(\varphi, \psi, \theta, q, W)$ be the solution of (3.2). Hence, we have*

$$\begin{aligned}\frac{dI_1}{dt} \leq & \left(-k + \varepsilon_1\left(\frac{k}{2} + \frac{\mu_2 c}{2}\right)\right)\int_0^1 \varphi_t^2 dx + \frac{k}{2\varepsilon_1}\int_0^1 \psi_x^2 dx \\ & + \frac{\mu_2}{2\varepsilon_1}\int_0^1 W^2(x,1,t)\,dx + \rho_1\int_0^1 \varphi_t^2 dx,\end{aligned} \tag{3.35}$$

*where $c = 1/\pi^2$ and $\varepsilon_1 > 0$.*

*Proof.* Differentiating $I_1$, we obtain

$$\frac{dI_1}{dt} = \rho_1\int_0^1 \varphi_{tt}\varphi dx + \rho_1\int_0^1 \varphi_t^2 dx + \mu_1\int_0^1 \varphi\varphi_t dx, \tag{3.36}$$

thanks to $(3.2)_1$, we find

$$\frac{dI_1}{dt} = k\int_0^1 (\varphi_x + \psi)_x \varphi dx - \mu_2 \int_0^1 \varphi\varphi_t (x, t - \tau(t))\, dx + \rho_1 \int_0^1 \varphi_t^2 dx.$$

Thus,

$$\frac{dI_1}{dt} = -k\int_0^1 (\varphi_x + \psi)\varphi_x dx - \mu_2 \int_0^1 \varphi W(x,1,t)\, dx + \rho_1 \int_0^1 \varphi_t^2 dx.$$

The use of Young's and Poincaré's inequalities leads to (3.35). □

Now, let $w$ be the solution of

$$-w_{xx} = \psi_x, w(0) = w(1) = 0. \tag{3.37}$$

Then we get

$$w(x,t) = -\int_0^x \psi(y,t)\, dy + x\left(\int_0^1 \psi(y,t)\, dy\right).$$

**Lemma 3.5.** *The solution of (3.37) satisfies*

$$\int_0^1 w_x^2 dx \le \int_0^1 \psi^2 dx,$$

*and*

$$\int_0^1 w_t^2 dx \le \int_0^1 \psi_t^2 dx.$$

*Proof.* Multiplying (3.37) by $w$, integrating by parts and applying the Cauchy–Schwarz inequality to get

$$\int_0^1 w_x^2 dx \le \int_0^1 \psi^2 dx.$$

We differentiate (3.37), we get

$$\int_0^1 w_t^2 dx \le \int_0^1 \psi_t^2 dx.$$

This complete the proof of the lemma. □

We introduce the following functional

$$I_2(t) := \int_0^1 \left(\rho_2\psi_t\psi + \rho_1\varphi_t w - \frac{\gamma\tau_0}{k}\psi q\right) dx. \tag{3.38}$$

**Lemma 3.6.** *Let $(\varphi, \psi, \theta, q, W)$ be the solution of (3.2). Hence, we get,*

$$\begin{aligned} \frac{dI_2(t)}{dt} \leq & \left(-b + \frac{c\mu_1\varepsilon_2}{2} + \frac{c\mu_2\varepsilon_2}{2} + \frac{\delta\gamma\varepsilon_2 c}{2k}\right)\int_0^1 \psi_x^2 dx + \frac{\mu_2}{2\varepsilon_2}\int_0^1 W^2(x,1,t)\,dx \\ & + \left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2k} + \frac{\rho_1\varepsilon_2}{2}\right)\int_0^1 \psi_t^2 dx + \left(\frac{\mu_1}{2\varepsilon_2} + \frac{\rho_1}{2\varepsilon_2}\right)\int_0^1 \varphi_t^2 dx \\ & + \left(\frac{\gamma\tau_0}{2k\varepsilon_2} + \frac{\delta\gamma}{2k\varepsilon_2}\right)\int_0^1 q^2 dx, \end{aligned} \tag{3.39}$$

*where $\varepsilon_2 > 0$.*

*Proof.* By derivating (3.38), we deduce that

$$\begin{aligned} \frac{dI_2(t)}{dt} = & -b\int_0^1 \psi_x^2 dx + k\int_0^1 \varphi\psi_x dx - k\int_0^1 \psi^2 dx + \rho_2\int_0^1 \psi_t^2 dx - k\int_0^1 \varphi_x w_x dx \\ & -k\int_0^1 \psi w_x dx - \mu_1\int_0^1 \varphi_t w dx - \mu_2\int_0^1 W(x,1,t)\,w dx + \rho_1\int_0^1 \varphi_t w_t dx \\ & -\frac{\gamma\tau_0}{k}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{k}\int_0^1 \psi q dx. \end{aligned}$$

By using (3.37) and Lemma 2, we have

$$\begin{aligned} \frac{dI_2(t)}{dt} \leq & -b\int_0^1 \psi_x^2 dx + k\int_0^1 \varphi\psi_x dx - k\int_0^1 \psi^2 dx + \rho_2\int_0^1 \psi_t^2 dx + k\int_0^1 \varphi_x \psi dx \\ & +k\int_0^1 \psi^2 dx - \mu_1\int_0^1 \varphi_t w dx - \mu_2\int_0^1 W(x,1,t)\,w dx + \rho_1\int_0^1 \varphi_t w_t dx \\ & -\frac{\gamma\tau_0}{k}\int_0^1 \psi_t q dx + \frac{\delta\gamma}{k}\int_0^1 \psi q dx. \end{aligned}$$

Applying Young's and Poincaré's inequalities and using Lemma 2, we obtain (3.39). □

Now, we defined the functional

$$I_3(t) := \xi\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho} W^2(x,\rho,t)\,dx d\rho. \tag{3.40}$$

It satisfies the estimate stated in the following lemma

**Lemma 3.7.** *Let $(\varphi, \psi, \theta, q, W)$ be the solution of (3.2). Then, we have*

$$\frac{dI_3(t)}{dt} \leq -2I_3(t) + \xi\int_0^1 \varphi_t^2(x,t)\,dx. \tag{3.41}$$

*Proof.* Differentiating (3.41), we obtain

$$\begin{aligned}\frac{dI_3(t)}{dt} &= \xi\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}W^2(x,\rho,t)\,dxd\rho \\ &\quad -2\xi\tau(t)\tau'(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}\rho W^2(x,\rho,t)\,dxd\rho \\ &\quad +2\xi\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}W_t(x,\rho,t)\,W(x,\rho,t)\,dxd\rho.\end{aligned} \tag{3.42}$$

Using $(3.2)_3$ and the last term in (3.42) we find

$$\begin{aligned}&\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}W_t(x,\rho,t)\,W(x,\rho,t)\,dxd\rho \\ =& \int_0^1\int_0^1 e^{-2\tau(t)\rho}\left(\tau'(t)\rho-1\right)W_\rho(x,\rho,t)\,W(x,\rho,t)\,dxd\rho.\end{aligned} \tag{3.43}$$

Also, one can see that

$$\begin{aligned}&\int_0^1\int_0^1 e^{-2\tau(t)\rho}\left(\tau'(t)\rho-1\right)W_\rho(x,\rho,t)\,W(x,\rho,t)\,dxd\rho \\ =& \frac{1}{2}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau(t)\rho}\left(\tau'(t)\rho-1\right)W^2(x,\rho,t)\right)dxd\rho \\ &+\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}\left(\tau'(t)\rho-1\right)W^2(x,\rho,t)\,dxd\rho \\ &-\frac{\tau'(t)}{2}\int_0^1\int_0^1 e^{-2\tau(t)\rho}W^2(x,\rho,t)\,dxd\rho.\end{aligned} \tag{3.44}$$

Using (3.44) and (3.43), the equation (3.42) takes the form

$$\begin{aligned}\frac{dI_3(t)}{dt} &= -2\xi\tau(t)\int_0^1\int_0^1 e^{-2\tau(t)\rho}W^2(x,\rho,t)\,dxd\rho+\xi\int_0^1 \varphi_t^2 x(,t)\,dx \\ &\quad -\xi e^{-2\tau(t)}\left(1-\tau'(t)\right)\int_0^1 W^2(x,1,t)\,dx.\end{aligned} \tag{3.45}$$

Then, the proof of the lemma is complete. □

Now, as in [46], we introduce the functional

$$I_4(t) := \rho_2\rho_3\int_0^1\left(\int_0^x \theta(t,y)\,dy\right)\psi_t(t,x)\,dx. \tag{3.46}$$

**Lemma 3.8.** *Let* $(\varphi, \psi, \theta, q, W)$ *be the solution of (3.2). Then, we get*

$$\begin{aligned}\frac{dI_4(t)}{dt} \leq & \left(-\gamma\rho_2 + \frac{\varepsilon_4\rho_2 k}{2}\right)\int_0^1 \psi_t^2 dx + \left(\frac{\varepsilon_4\rho_3}{2}(b+kc)\right)\int_0^1 \psi_x^2 dx \\ & + \frac{\varepsilon_4' k\rho_3 c}{2}\int_0^1 \varphi_x^2 dx + \left(\gamma\rho_3 + \frac{\rho_3}{2\varepsilon_4'}(b+2k)\right)\int_0^1 \theta^2 dx \\ & + \frac{\rho_2 k}{2\varepsilon_4}\int_0^1 q^2 dx,\end{aligned} \tag{3.47}$$

*where* $\varepsilon_4, \varepsilon_4' > 0$.

*Proof.* Differentiating (3.46) and using $(3.2)_3$, we obtain

$$\begin{aligned}\frac{dI_4(t)}{dt} &= \int_0^1 \left(\int_0^x \rho_3\theta_t dy\right)\rho_2\psi_t dx + \int_0^1 \left(\int_0^x \rho_3\theta dy\right)\rho_2\psi_{tt} dx \\ &= -\int_0^1 \left(\int_0^x (kq_x + \gamma\psi_{tx})\, dy\right)\rho_2\psi_t dx \\ &\quad + \int_0^1 \left(\int_0^x \rho_3\theta dy\right)(b\psi_{xx} - k(\varphi_x + \psi) - \gamma\theta_x)\, dx, \\ &= -\gamma\rho_2 \int_0^1 \psi_t^2 dx - \rho_2 k \int_0^1 q\psi_t dx - b\rho_3 \int_0^1 \theta\psi_x dx \\ &\quad + k\rho_3 \int_0^1 \theta\varphi dx - k\rho_3 \int_0^1 \left(\int_0^x \theta dy\right)\psi dx + \gamma\rho_3 \int_0^1 \theta^2 dx.\end{aligned}$$

Applying Young's and Poincaré's inequalities, we find (3.47). □

Here, we introduce the functional

$$I_5(t) := -\tau_0\rho_3 \int_0^1 q(t,x)\left(\int_0^x \theta(t,y)\, dy\right) dx. \tag{3.48}$$

**Lemma 3.9.** *Let* $(\varphi, \psi, \theta, q, W)$ *be the solution of (3.2). Then, we have*

$$\begin{aligned}\frac{dI_5(t)}{dt} \leq & \left(-\rho_3 k + \frac{\varepsilon_5\rho_3\delta c}{2}\right)\int_0^1 \theta^2 dx + \frac{\varepsilon_5'\tau_0\gamma}{2}\int_0^1 \psi_t^2 dx \\ & \left(\tau_0 k + \frac{\rho_3\delta}{2\varepsilon_5} + \frac{\tau_0\gamma}{2\varepsilon_5'}\right)\int_0^1 q^2 dx,\end{aligned} \tag{3.49}$$

*for any* $\varepsilon_5, \varepsilon_5' > 0$.

We refer the reader to the proof in [46].

*Proof.* **(Proof of Theorem 3.3)** We define the Lyapunov functional $\mathcal{L}$ as follows:

$$\mathcal{L}(t) := NE(t) + I_1(t) + N_2 I_2(t) + I_3(t) + N_4 I_4(t) + N_5 I_5(t). \tag{3.50}$$

Where $N, N_2, N_4, N_5 > 0$.

Combining (3.32), (3.35), (3.39), (3.41), (3.47) and (3.49), we get

$$\begin{aligned}
\frac{d\mathcal{L}(t)}{dt} \leq\ & \left[\frac{k}{2\varepsilon_1} + N_2\left(-b + \frac{c\mu_1\varepsilon_2}{2} + \frac{c\mu_2\varepsilon_2}{2} + \frac{\delta\gamma\varepsilon_2 c}{2k}\right) + N_4\left(\frac{\varepsilon_4'\rho_3}{2}(b+kc)\right)\right]\int_0^1 \psi_x^2 dx \\
& + \left[-k + \varepsilon_1\left(\frac{k}{2} + \frac{\mu_2 c}{2}\right) + N_4\frac{\varepsilon_4' k\rho_3 c}{2}\right]\int_0^1 \varphi_x^2 dx - 2I_3(t) \\
& + \left[-NC + \frac{\mu_2}{2\varepsilon_1} + N_2\frac{\mu_2}{2\varepsilon_1}\right]\int_0^1 W^2(x,1,t)\, dx \\
& + \left[-NC + N_2\left(\frac{\mu_1}{2\varepsilon_2} + \frac{\rho_1}{2\varepsilon_2}\right) + \rho_1 + \xi\right]\int_0^1 \varphi_t^2 dx \\
& + \left[N_2\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2k} + \frac{\rho_1\varepsilon_2}{2}\right) + \frac{1}{2\tau} + N_4\left(-\gamma\rho_2 + \frac{\varepsilon_4\rho_2 k}{2}\right) + N_5\frac{\varepsilon_5'\tau_0\gamma}{2}\right]\int_0^1 \psi_t^2 dx \\
& + \left[-N\delta + N_2\left(\frac{\gamma\tau_0}{2k\varepsilon_2} + \frac{\delta\gamma}{2k\varepsilon_2}\right) + N_4\frac{\rho_2 k}{2\varepsilon_4} + N_5\left(\tau_0 k + \frac{\rho_3\delta}{2\varepsilon_5} + \frac{\tau_0\gamma}{2\varepsilon_5'}\right)\right]\int_0^1 q^2 dx \\
& + \left[N_4\left(\gamma\rho_3 + \frac{\rho_3}{2\varepsilon_4'}(b+2k)\right) + N_5\left(-\rho_3 k + \frac{\varepsilon_5\rho_3\delta c}{2}\right)\right]\int_0^1 \theta^2 dx.
\end{aligned} \tag{3.51}$$

Choose $\varepsilon_1, \varepsilon_2, \varepsilon_4$ and $\varepsilon_5$ small enough, such that

$$\begin{aligned}
\varepsilon_2\left(\frac{c\mu_1}{2} + \frac{c\mu_2}{2} + \frac{\delta\gamma c}{2k}\right) &\leq \frac{b}{2}, \quad \varepsilon_1\left(\frac{k}{2} + \frac{\mu_2 c}{2}\right) \leq \frac{k}{2}, \\
\varepsilon_4 &\leq \frac{\gamma}{k}, \quad \varepsilon_5 \leq \frac{k}{\delta c}.
\end{aligned}$$

We can choose $N_2$ large enough, so that

$$N_2 \geq \frac{2K}{b\varepsilon_1}.$$

And also $N_4$ large enough so that

$$N_4\frac{\gamma\rho_2}{4} \geq N_2\left(\rho_2 + \frac{\gamma\tau_0\varepsilon_2}{2k} + \frac{\rho_1\varepsilon_2}{2}\right) + \frac{1}{2\tau}.$$

Fixed $N_2$ and $N_4$ we have

$$\varepsilon_4' \leq \min\left\{\frac{N_2 b}{4N_4\rho_3(b+kc)}, \frac{k}{2N_4 k\rho_3 c}\right\}.$$

Let $N_5$ be large enough such that

$$\frac{N_5\rho_3 k}{4} \geq N_4\left(\gamma\rho_3 + \frac{\rho_3}{2\varepsilon_4'}(b+2k)\right).$$

We fix $\varepsilon_5'$ small enough, we have

$$\varepsilon_5' \leq \frac{N_4\gamma\rho_2}{4N_5\tau_0\gamma}.$$

Now, we have

$$\begin{cases} \frac{CN}{2} \geq \max\left\{\frac{\mu_2}{2\varepsilon_1} + N_2\frac{\mu_2}{2\varepsilon_1}, N_2\left(\frac{\mu_1}{2\varepsilon_2} + \frac{\rho_1}{2\varepsilon_2}\right) + \rho_1\right\}, \\ \frac{N\delta}{2} \geq N_2\left(\frac{\gamma\tau_0}{2k\varepsilon_2} + \frac{\delta\gamma}{2k\varepsilon_2}\right) + N_4\frac{\rho_2 k}{2\varepsilon_4} + N_5\left(\tau_0 k + \frac{\rho_3\delta}{2\varepsilon_5} + \frac{\tau_0\gamma}{2\varepsilon_5'}\right). \end{cases}$$

We see that (3.51) is equivalent to

$$\frac{d}{dt}\mathcal{L}(t) \leq -\eta_1\int_0^1\left(\psi_t^2 + \psi_x^2 + \varphi_t^2 + (\varphi_x+\psi)^2 + \theta^2 + q^2\right)dx - \eta_1\int_0^1\int_0^1 W^2(x,\rho,t)\,d\rho dx, \tag{3.52}$$

then, we have

$$\frac{d}{dt}\mathcal{L}(t) \leq \eta_2 E(t), \qquad \text{for all } t \geq 0, \tag{3.53}$$

where $-\eta_1$ and $\eta_2$ as a positive constant. □

**Lemma 3.10.** *There exist two positive constants $\beta_1, \beta_2$ such that*

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \qquad \textit{for all } t \geq 0. \tag{3.54}$$

*Proof.* We defined the functional

$$H(t) = I_1(t) + N_2 I_2(t) + I_3(t) + N_4 I_4(t) + N_5 I_5(t),$$

and we prove that

$$|H(t)| \leq CE(t), \quad C > 0.$$

From (3.34),(3.38),(3.40),(3.46) and (3.48) we find

$$\begin{aligned} |H(t)| \;\leq\; & \left|\int_0^1 \rho_1\varphi_t\varphi dx + \frac{\mu_1}{2}\int_0^1 \varphi^2 dx\right| + N_2\left|\int_0^1\left(\rho_2\psi_t\psi dx + \rho_1\varphi_t w - \frac{\gamma\tau_0}{k}\psi q\right)dx\right| \\ & + \left|\int_0^1\int_0^1 e^{-2\tau\rho}W^2(x,\rho,t)\,d\rho dx\right| + N_4\left|\rho_2\rho_3\int_0^1\left(\int_0^x \theta(t,y)\,dy\right)\psi_t(t,x)\,dx\right| \\ & + N_5\left|-\tau_0\rho_3\int_0^1 q(t,x)\left(\int_0^x \theta(t,y)\,dy\right)dx\right|. \end{aligned}$$

By using the relation

$$\int_0^1 \varphi^2 dx \leq 2c \int_0^1 (\varphi_x + \psi)^2 \, dx + 2c \int_0^1 \psi_x^2 dx,$$

moreover, by applying Young's and Poincaré's inequalities, we get

$$\begin{aligned} |H(t)| &\leq \alpha_1 \int_0^1 \varphi_t^2 dx + \alpha_2 \int_0^1 \psi_t^2 dx + \alpha_3 \int_0^1 (\varphi_x + \psi)^2 \, dx + \alpha_4 \int_0^1 \psi_x^2 dx + \alpha_5 \int_0^1 \theta^2 dx \\ &\quad + \alpha_6 \int_0^1 q^2 dx + \int_0^1 \int_0^1 W^2(x, \rho, t) \, d\rho dx, \end{aligned} \tag{3.55}$$

where the positive constants $\alpha_1, \ldots, \alpha_6$ are:

$$\begin{cases} \alpha_1 := \frac{1}{2}(\rho_1 + N_2\rho_1), \\ \alpha_2 := \frac{1}{2}(N_2\rho_2 + N_4\rho_2\rho_3), \\ \alpha_3 := \rho_1 c, \\ \alpha_4 := \frac{1}{2}\left(\frac{N_2\gamma\tau_0 c}{k} + N_2\rho_1 c^2 + N_2\rho_2 c\right), \\ \alpha_5 := \frac{1}{2}(N_4\rho_2\rho_3 c + N_5\tau_0\rho_3 c), \\ \alpha_6 := \frac{1}{2}\left(N_2\frac{\gamma\tau_0}{k} + N_5\tau_0\rho_3\right). \end{cases}$$

By (3.55), we obtain

$$|H(t)| \leq \tilde{C} E(t),$$

for

$$\tilde{C} = \frac{\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}}{\min\{\rho_1, \rho_2, \rho_3, K, b, k, \gamma, \delta, \tau_0\}},$$

we get

$$|\mathcal{L}(t) - NE(t)| \leq \tilde{C} E(t).$$

□

Combining (3.53) and (3.54), we deduce that

$$\frac{d}{dt}\mathcal{L}(t) \leq -\Lambda \mathcal{L}(t), \qquad \text{for all } t, \Lambda \geq 0. \tag{3.56}$$

Then

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\Lambda t}, \qquad \text{for all } t \geq 0. \tag{3.57}$$

Thus, the proof of the theorem is complete.

# Chapter 4

## Well-posedness and Stability result of a nonlinear damping Porous-elastic system in Thermoelasticity of second sound with infinite memory and distributed delay terms

### 4.1 Introduction

In this chapter, we treated (with Ouchenane and Choucha) [29] the following system,

$$
\begin{cases}
\rho_1 u_{tt} - \mu u_{xx} - b\phi_x = 0, \\
\rho_2 \phi_{tt} - \delta \phi_{xx} + bu_x + \xi \phi + \displaystyle\int_0^\infty g(s)\phi_{xx}(t-s)ds + \gamma \theta_x \\
+\mu_1 \phi_t + \displaystyle\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \phi_t(x, t-\varrho) d\varrho + \alpha(t) f(\phi_t) = 0, \\
\rho_3 \theta_t + \kappa q + \gamma \phi_{tx} = 0, \\
\rho_4 q_t + dq + \kappa \theta_x = 0,
\end{cases}
\tag{4.1}
$$

where

$$(x, \varrho, t) \in (0, 1) \times (\tau_1, \tau_2) \times (0, \infty),$$

with the Neumann-Dirichlet boundary conditions

$$
\begin{aligned}
u_x(0,t) &= u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, \\
\theta_x(0,t) &= \theta_x(1,t) = q(0,t) = q(1,t) = 0, \quad t \geq 0,
\end{aligned}
$$

and the initial data

$$\begin{aligned} &u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,1), \\ &\phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), \quad x \in (0,1), \\ &\theta(x,0) = \theta_0(x), q(x,0) = q_0(x), \quad x \in (0,1), \\ &\phi_t(x,-t) = f_0(x,t), \quad (x,t) \in (0,1) \times (0,\tau_2). \end{aligned} \tag{4.2}$$

Here $\rho_1, \rho_2, \rho_3, \rho_4, \mu, , b, \delta, \xi, \gamma, d, \kappa$ and $\mu_1$ are positive constants, satisfying $\mu\xi > b^2$, the term $\alpha(t)f(\phi_t)$ is the nonlinear damping term, where the functions $\alpha$ and $f$ are specified later, the term $\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|\phi_t(x,t-\varrho)d\varrho$ is a distributed delay that acts only on the porous equation and $\tau_1, \tau_2$ are tow real numbres with $0 \leq \tau_1 \leq \tau_2$ , where $\mu_2$ is an $L^\infty$function, and the function $g$ is called the relaxation function.

We introduce the following assumptions that has been considered in many works such that
(**H1**) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a $C^1$ function satisfing

$$g(0) > 0, \;\; \delta - \int_0^\infty g(s)ds = l > 0, \;\; \int_0^\infty g(s)ds = g_0. \tag{4.3}$$

(**H2**) There exists a non-increasing differentiable function $\alpha, \eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \leq -\eta(t)g(t), \quad t \geq 0, \tag{4.4}$$

and

$$\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0.$$

(**H3**) $f : \mathbb{R} \to \mathbb{R}$ is a non-decreasing $C^0$ function, such that there exist $v_1, v_2, \varepsilon > 0$, and a strictly increasing function $G \in C^1([0,\infty))$, with $G(0) = 0$, and $G$ is a linear or strictly convex $C^2$-function on $(0,\varepsilon]$ such that

$$\begin{cases} s^2 + f^2(s) \leq sf(s), & \forall |s| < \varepsilon, \\ v_1|s| \leq |f(s)| \leq v_2|s|, & \forall |s| \geq \varepsilon, \end{cases} \tag{4.5}$$

which implies that $sf(s) > 0$ for all $s \neq 0$. $f$ also satisfies the following property:

$$|f(\psi_2) - f(\psi_1)| \leq k_0(|\psi_2|^\beta + |\psi_1|^\beta)|\psi_2 - \psi_1|, \;\; \psi_1, \psi_2 \in \mathbb{R}, \tag{4.6}$$

where $k_0, \beta > 0$.

(**H4**) $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function satisfing

$$\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho < \mu_1. \tag{4.7}$$

Note that condition (4.4) was used by Messaoudi in [41], and the assumptions (4.5) was first introduced by Lasiecka and Tataru [32] in 1993.

Now, as in [53], taking the following new variable

$$y(x, \rho, \varrho, t) = \phi_t(x, t - \varrho\rho),$$

then we obtain

$$\begin{cases} \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \\ y(x, 0, \varrho, t) = \phi_t(x, t), \end{cases}$$

we introduce the following new variable:

$$\eta^t(x, s) = \phi(x, t) - \phi(x, t - s), \quad (x, t, s) \in (0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+,$$

which was adopted in articles ([13],[57]), where $\eta^t$ is the relative history of $\phi$ that satisfies.

$$\eta^t_t + \eta^t_s = \phi_t(x, t), \quad (x, t, s) \in (0, 1) \times (0, 1) \times \mathbb{R}_+ \times \mathbb{R}_+.$$

Consequently, the problem (4.1) is equivalent to

$$\begin{cases} \rho_1 u_{tt} - \mu u_{xx} - b\phi_x = 0, \\ \rho_2 \phi_{tt} - l\phi_{xx} + bu_x + \xi\phi - \int_0^\infty g(s)\eta^t_{xx}(s)ds + \gamma\theta_x \\ +\mu_1\phi_t + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x, 1, \varrho, t) d\varrho + \alpha(t) f(\phi_t) = 0, \\ \rho_3\theta_t + \kappa q + \gamma\phi_{tx} = 0, \\ \rho_4 q_t + dq + \kappa\theta_x = 0, \\ \varrho y_t(x, \rho, \varrho, t) + y_\rho(x, \rho, \varrho, t) = 0, \\ \eta^t_t + \eta^t_s = \phi_t(x, t), \end{cases} \tag{4.8}$$

where

$$(x, \rho, \varrho, t) \in (0, 1) \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty),$$

with the following boundary and initial conditions:

$$\begin{cases} u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, \\ \theta_x(0,t) = \theta_x(1,t) = q(0,t) = q(1,t) = 0, \quad t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \quad x \in (0,1), \\ \phi(x,0) = \phi_0(x), \phi_t(x,0) = \phi_1(x), \quad x \in (0,1), \\ \theta(x,0) = \theta_0(x), q(x,0) = q_0(x), \quad x \in (0,1), \\ y(x,\rho,\varrho,0) = f_0(x,\rho\varrho), \quad x \in (0.1), \rho \in (0.1), \varrho \in (0,\tau_2), \\ \eta^t(x,0) = 0, \eta^0(x,s) = \eta_0(x,s), \quad (x,s) \in (0,1) \times \mathbb{R}_+. \end{cases} \tag{4.9}$$

Meanwhile, from $(4.1)_1$ and (4.2), it follows that

$$\frac{d^2}{dt^2} \int_0^1 u(x,t)\, dx = 0. \tag{4.10}$$

So, by solving (4.10) and using the initial data of $u$, we get

$$\int_0^1 u(x,t)\, dx = t \int_0^1 u_1(x)\, dx + \int_0^1 u_0(x)\, dx.$$

Consequently, if we let

$$\bar{u}(x,t) = u(x,t) - t \int_0^1 u_1(x)\, dx - \int_0^1 u_0(x)\, dx, \tag{4.11}$$

we get

$$\int_0^1 \bar{u}(x,t)\, dx = 0, \quad \forall t \geq 0,$$

and, from $(4.1)_3$ and (4.2), it follows that

$$\frac{d}{dt} \int_0^1 \theta(x,t)\, dx = 0. \tag{4.12}$$

So, by solving (4.12) and using the initial data of $\theta$, we get

$$\int_0^1 \theta(x,t)\, dx = \int_0^1 \theta_0(x)\, dx.$$

Consequently, if we let

$$\bar{\theta}(x,t) = \theta(x,t) - \int_0^1 \theta_0(x)\, dx,$$

we get

$$\int_0^1 \bar{\theta}(x,t)\, dx = 0, \quad \forall t \geq 0.$$

Therefore, the use of Poincare's inequality for $\bar{u}, \bar{\theta}$ is justified. In addition, simple substitution shows that $(\bar{u}, \phi, \bar{\theta}, q, y, \eta^t)$ satisfies system (4.1). Henceforth, we work with $\bar{u}, \bar{\theta}$ instead of $u, \theta$ but write $u, \theta$ for simplicity of notation

## 4.2 Well-posedness

In this section, we prove the existence and uniqueness result of the system (4.8)-(4.9) by using the Semi-group theory.
First, we introduce the vector function

$$U = (u, u_t, \phi, \phi_t, \theta, q, y, \eta^t)^T,$$

and the new dependent variables $v = u_t, \psi = \phi_t, \varphi = \eta^t$, then the system (4.8) can be written as follows:

$$\begin{cases} U_t = \mathcal{A}U + \Gamma(U), \\ U(0) = U_0 = (u_0, u_1, \phi_0, \phi_1, \theta_0, q_0, f_0, \eta_0)^T, \end{cases} \tag{4.13}$$

where $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} :\to \mathcal{H}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} v \\ \frac{\mu}{\rho_1}u_{xx} + \frac{b}{\rho_1}\phi_x \\ \psi \\ \frac{l}{\rho_2}\phi_{xx} - \frac{b}{\rho_2}u_x - \frac{\xi}{\rho_2}\phi - \frac{\gamma}{\rho_2}\theta_x + \frac{1}{\rho_2}\int_0^\infty g(s)\varphi_{xx}(s)ds \\ \quad -\frac{\mu_1}{\rho_2}\psi - \frac{1}{\rho_2}\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)|y(x, 1, \varrho, t)d\varrho \\ -\frac{\kappa}{\rho_3}q_x - \frac{\gamma}{\rho_3}\psi_x \\ -\frac{d}{\rho_4}q - \frac{\kappa}{\rho_4}\theta_x \\ -\frac{1}{\varrho}y_\rho \\ -\varphi_s + \psi \end{pmatrix},$$

and

$$\Gamma(U) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\alpha(t)}{\rho_2} f(\psi) \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.14}$$

and $\mathcal{H}$ is the energy space given by

$$\begin{aligned} \mathcal{H} &= H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times L^2_*(0,1) \times L^2(0,1) \\ &\quad \times L^2((0,1) \times (0,1) \times (\tau_1, \tau_2)) \times L_g, \end{aligned}$$

where

$$\begin{aligned} L^2_*(0,1) &= \{\Phi \in L^2(0,1) \ / \ \int_0^1 \Phi(x)dx = 0\}, \\ H^1_*(0,1) &= H^1(0,1) \cap L^2_*(0,1), \\ L_g &= \{\Phi : \mathbb{R}_+ \to H^1_0(0,1), \ \int_0^1 \int_0^\infty g(s)\Phi^2_x(s)ds < \infty\}, \end{aligned}$$

where the space $L_g$ is endowed with the following inner product:

$$< \Phi_1, \Phi_2 >_{L_g} = \int_0^1 \int_0^\infty g(s)\Phi_{1x}(s)\Phi_{2x}(s)ds.$$

For any

$$U = (u, v, \phi, \psi, \theta, q, y, \varphi)^T \in \mathcal{H}, \quad \widehat{U} = (\widehat{u}, \widehat{v}, \widehat{\phi}, \widehat{\psi}, \widehat{\theta}, \widehat{q}, \widehat{y}, \widehat{\varphi})^T \in \mathcal{H},$$

we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned} < U, \widehat{U} >_{\mathcal{H}} &= \rho_1 \int_0^1 v\widehat{v}dx + \mu \int_0^1 u_x\widehat{u}_xdx + \rho_2 \int_0^1 \psi\widehat{\psi}dx + \xi \int_0^1 \phi\widehat{\phi}dx \\ &\quad + \rho_3 \int_0^1 \theta\widehat{\theta}dx + \rho_4 \int_0^1 q\widehat{q}dx + l \int_0^1 \phi_x\widehat{\phi}_xdx + b \int_0^1 (u_x\widehat{\phi} + \widehat{u}_x\phi)dx \\ &\quad + \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y\widehat{y}d\varrho d\rho dx + < \varphi, \widehat{\varphi} >_{L_g} . \end{aligned} \tag{4.15}$$

The domain of $\mathcal{A}$ is given by

$$\mathcal{D}(\mathcal{A}) = \left\{ \begin{array}{l} U \in \mathcal{H} \ / \ u \in H_*^2 \cap H_*^1, \ \ \phi \in H^2 \cap H_0^1, v \in H_*^1(0,1), \\ \psi \in H_0^1(0,1), \ \ \theta \in H_*^1(0,1), \ \ \ q \in H_0^1(0,1), \ \ \varphi \in L_g, \\ y, y_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)), \ \ y(x,0,\varrho,t) = \psi \end{array} \right\},$$

where

$$H_*^2(0,1) = \left\{ \Phi \in H^2(0,1) \ / \ \Phi_x(1) = \Phi_x(0) = 0 \right\}.$$

Clearly, $\mathcal{D}(\mathcal{A})$ is dense in $\mathcal{H}$. Now, we can give the following existence result.

**Theorem 4.1.** *Let $U_0 \in \mathcal{H}$ and assume that (4.3)-(4.7) holds. Then, there exists a unique soltion $U \in C(\mathbb{R}_+, \mathcal{H})$ of problem (4.13). Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$, then*

$$U \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}_+, \mathcal{H})$$

*Proof.* First, we prove that the operator $\mathcal{A}$ is dissipative. For any $U_0 \in \mathcal{D}(\mathcal{A})$ and by using (4.15), we have

$$\begin{aligned} < \mathcal{A}U, U >_{\mathcal{H}} \ &= \ -\mu_1 \int_0^1 \psi^2 dx - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| \psi y(x,1,\varrho,t) d\varrho dx \\ &\quad -d \int_0^1 q^2 dx - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y_\rho y d\varrho d\rho dx \\ &\quad - \int_0^1 \int_0^\infty g(s) \varphi_{xs}(s) \varphi_x(s) ds dx. \end{aligned} \tag{4.16}$$

For the third term of the right-hand side of (4.16), we have

$$\begin{aligned} - \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y_\rho y d\varrho d\rho dx \ &= \ -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \int_0^1 |\mu_2(\varrho)| \frac{d}{d\rho} y^2 d\rho d\varrho dx \\ &= \ -\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx \\ &\quad +\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,0,\varrho,t) d\varrho dx. \end{aligned} \tag{4.17}$$

By using Young's inequality, we get

$$\begin{aligned} - \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| \psi y(x,1,\varrho,t) d\varrho dx \ &\leq \ \frac{1}{2} (\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho) \int_0^1 \psi^2 dx \\ &\quad +\frac{1}{2} \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx. \end{aligned} \tag{4.18}$$

By integration the last term of the right-hand side of (4.16), we have

$$-\int_0^1\int_0^\infty g(s)\varphi_{xs}(s)\varphi_x(s)dsdx = \frac{1}{2}\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx. \tag{4.19}$$

Substituting (4.17),(4.18) and (4.19) into (4.16), using the fact that $y(x,0,\varrho,t) = \psi(x,t)$ and (4.7), we obtained

$$\begin{aligned} < \mathcal{A}U, U >_{\mathcal{H}} \ &\leq \ -(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\int_0^1\psi^2dx - d\int_0^1 q^2dx \\ &+\frac{1}{2}\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx \leq 0. \end{aligned}$$

Hence, the operator $\mathcal{A}$ is dissipative.

Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $(\lambda I-\mathcal{A})$ is surjective. Indeed, for any $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$, we prove that there exists a unique $V = (u, v, \phi, \psi, \theta, q, y, \varphi) \in \mathcal{D}(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})V = F. \tag{4.20}$$

That is

$$\begin{cases} \lambda u - v = f_1 \in H_*^1(0,1) \\ \rho_1\lambda v - \mu u_{xx} - b\phi_x = \rho_1 f_2 \in L_*^2(0,1) \\ \lambda\phi - \psi = f_3 \in H_0^1(0,1) \\ \rho_2\lambda\psi - l\phi_{xx} + bu_x + \xi\phi + \gamma\theta_x - \int_0^\infty g(s)\varphi_{xx}(s)ds \\ \quad +\mu_1\psi + \int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho = \rho_2 f_4 \in L^2(0,1) \\ \rho_3\lambda\theta + \kappa q_x + \gamma\psi_x = \rho_3 f_5 \in L_*^2(0,1) \\ (\rho_4\lambda + d)q + \kappa\theta_x = \rho_4 f_6 \in L^2(0,1) \\ \lambda\varrho y_t(x,\rho,\varrho,t) + y_\rho(x,\rho,\varrho,t) = \varrho f_7 \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)) \\ \lambda\varphi + \varphi_s - \psi = f_8 \in L_g. \end{cases} \tag{4.21}$$

We note that the equation $(4.21)_5$ with $y(x,0,\varrho,t) = \psi(x,t)$ has a unique solution given by

$$y(x,\rho,\varrho,t) = e^{-\lambda\rho\varrho}\psi + \varrho e^{\lambda\varrho\rho}\int_0^\rho e^{\lambda\varrho\sigma}f_7(x,\sigma,\varrho,t)d\sigma,$$

then

$$y(x,1,\varrho,t) = e^{-\lambda\varrho}\psi + \varrho e^{\lambda\varrho}\int_0^1 e^{\lambda\varrho\sigma}f_7(x,\sigma,\varrho,t)d\sigma, \tag{4.22}$$

and we infer from $(4.21)_6$ that

$$\varphi = e^{\lambda s}\int_0^s e^{\tau}(\psi + f_8(\tau))d\tau, \tag{4.23}$$

and we have

$$v = \lambda u - f_1, \ \ \psi = \lambda \phi - f_3. \tag{4.24}$$

Inserting (4.22), (4.23) and (4.24) in $(4.21)_2$ and $(4.21)_4$, we get

$$\begin{cases} \rho_1 \lambda^2 u - \mu u_{xx} - b\phi_x = h_1 \in L^2_*(0,1), \\ \mu_3 \phi - \mu_4 \phi_{xx} + b u_x + \gamma \theta_x = h_2 \in L^2(0,1), \\ \rho_3 \theta + \frac{\kappa}{\lambda} q_x + \gamma \phi_x = h_3 \in L^2_*(0,1), \\ \frac{(\rho_4 \lambda + d)}{\lambda} q + \frac{\kappa}{\lambda} \theta_x = h_4 \in L^2(0,1), \end{cases} \tag{4.25}$$

where

$$\begin{cases} \mu_3 = \rho_2 \lambda^2 + \xi + \lambda \mu_1 + \lambda \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| e^{-\lambda \varrho} d\varrho \\ \mu_4 = l + \int_0^\infty g(s)(1 - e^{\lambda s}) ds \\ h_1 = \rho_1 (\lambda f_1 + f_2) \\ h_2 = (\rho_2 \lambda + \mu_1 + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| e^{-\lambda \varrho} d\varrho) f_3 + \rho_2 f_4 \\ \quad - \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| e^{\lambda \varrho} \int_0^1 e^{\lambda \varrho \sigma} f_7(x,\sigma,\varrho,t) d\sigma d\varrho \\ \quad + \int_0^\infty g(s) e^{\lambda s} \int_0^s e^{\tau} (f_8(\tau))_{xx} d\tau ds \\ h_3 = \frac{1}{\lambda}(\rho_3 f_5 + \gamma f_{3x}) \\ h_4 = \frac{\rho_4 f_6}{\lambda}. \end{cases}$$

We multiply (4.25) by $\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{q}$, respectively, and integrate their sum over $(0,1)$ to get the following variational formulation:

$$B((u,\phi,\theta,q),(\widehat{u},\widehat{\phi},\widehat{\theta},\widehat{q})) = \Upsilon(\widehat{u},\widehat{\phi},\widehat{\theta},\widehat{q}), \tag{4.26}$$

where

$$B : (H^1_*(0,1) \times H^1_0(0,1) \times L^2_*(0,1) \times L^2(0,1))^2 \to \mathbb{R},$$

is the bilinear form defined by

$$\begin{aligned} B((u,\phi,\theta,q),(\widehat{u},\widehat{\phi},\widehat{\theta},\widehat{q})) \;\; = \;\; & \lambda^2 \rho_1 \int_0^1 u\widehat{u} dx + \mu_3 \int_0^1 \phi \widehat{\phi} dx + \mu \int_0^1 u_x \widehat{u}_x dx \\ & + \mu_4 \int_0^1 \phi_x \widehat{\phi}_x dx + b \int_0^1 (u_x \widehat{\phi} + \phi \widehat{u}_x) dx \\ & + \rho_3 \int_0^1 \theta \widehat{\theta} dx + \frac{\kappa}{\lambda} \int_0^1 q_x \widehat{\theta} dx \\ & + \gamma \int_0^1 \phi_x \widehat{\theta} dx + \frac{(\rho_4 \lambda + d)}{\lambda} \int_0^1 q\widehat{q} dx \\ & + \frac{\kappa}{\lambda} \int_0^1 \theta_x \widehat{q} dx, \end{aligned}$$

and

$$\Upsilon : (H^1_*(0,1) \times H^1_0(0,1) \times L^2_*(0,1) \times L^2(0,1)) \to \mathbb{R},$$

is the linear functional given by

$$\Upsilon(\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{q})) = \int_0^1 h_1 \widehat{u} dx + \int_0^1 h_2 \widehat{\phi} dx + \int_0^1 h_3 \widehat{\theta} dx + \int_0^1 h_4 \widehat{q} dx.$$

Now, for $V = H^1_*(0,1) \times H^1_0(0,1) \times L^2_*(0,1) \times L^2(0,1)$, equipped with the norm

$$\|(u, \phi, \theta, q)\|_V^2 = \|u\|_2^2 + \|\phi\|_2^2 + \|u_x\|_2^2 + \|\phi_x\|_2^2 + \|\theta\|_2^2 + \|q\|_2^2.$$

then, we have

$$\begin{aligned} B((u, \phi, \theta, q), (u, \phi, \theta, q)) &= \lambda^2 \rho \int_0^1 u^2 dx + \mu_3 \int_0^1 \phi^2 dx + \mu \int_0^1 u_x^2 dx \\ &\quad + 2b \int_0^1 u_x \phi dx + \mu_4 \int_0^1 \phi_x^2 dx + \rho_3 \int_0^1 \theta^2 dx \\ &\quad + \frac{(\rho_4 \lambda + d)}{\lambda} \int_0^1 q^2 dx. \end{aligned}$$

On the other hand, we can write

$$\begin{aligned} \mu u_x^2 + 2b u_x \phi + \mu_3 \phi^2 &= \frac{1}{2}[\mu(u_x + \frac{b}{\mu}\phi)^2 + \mu_3(\phi + \frac{b}{\mu_3}u_x)^2 \\ &\quad + (\mu - \frac{b^2}{\mu_3})u_x^2 + (\mu_3 - \frac{b^2}{\mu})\phi^2]. \end{aligned}$$

Since, $\mu\xi > b^2$, we deduce that

$$\mu u_x^2 + 2b u_x \phi + \mu_3 \phi^2 > \frac{1}{2}[(\mu - \frac{b^2}{\mu_3})u_x^2 + (\mu_3 - \frac{b^2}{\mu})\phi^2],$$

then, for some $M_0 > 0$

$$B((u, \phi, \theta, q), (u, \phi, \theta, q)) \geq M_0 \|(u, \phi, \theta, q)\|_V^2.$$

Thus $B$ is coercive, similarly,

$$\Upsilon(\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{q}) \geq M_1 \|(\widehat{u}, \widehat{\phi}, \widehat{\theta}, \widehat{q})\|_V^2.$$

Consequently, using Lax-Milgram theorem, we conclude that (4.8) has a unique solution:

$$(u, \phi, \theta, q) \in H^1_*(0, 1 \times H^1_0(0,1) \times L^2_*(0,1) \times L^2(0,1).$$

Substituting $u, \phi, \theta, q$ into (4.22), (4.23) and (4.24), respectively, we have

$$v \in H^1_*(0,1), \quad \psi \in H^1_0(0,1), \quad \varphi \in L_g$$
$$y, y_\rho \in L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)).$$

Moreover, if we take $(\widehat{u}, \widehat{\theta}, \widehat{q}) = 0 \in H^1_*(0,1) \times L^2_*(0,1) \times L^2(0,1)$ in (4.26) to obtain

$$\mu_3 \int_0^1 \phi\widehat{\phi}dx + b\int_0^1 u_x\widehat{\phi}dx + \mu_4 \int_0^1 \phi_x\widehat{\phi}_x dx + \gamma\int_0^1 \theta_x\widehat{\phi}dx = \int_0^1 h_2\widehat{\phi}dx,$$

we get

$$\mu_4 \int_0^1 \phi_x\widehat{\phi}_x dx = \int_0^1 (h_2 - \mu_3\phi - bu_x - \gamma\theta_x)\widehat{\phi}dx, \;\; \forall\widehat{\phi} \in H^1_0(0,1), \tag{4.27}$$

which yields

$$-\mu_4\phi_{xx} = (h_2 - \mu_3\phi - bu_x - \gamma\theta_x) \in L^2(0,1).$$

Thus

$$\phi \in H^2(0,1) \cap H^1_0(0,1),$$

consequently, (4.27) takes the following form

$$\int_0^1 (-\mu_4\phi_{xx} - h_2 + \mu_3\phi + bu_x + \gamma\theta_x)\widehat{\phi}dx = 0, \;\; \forall\widehat{\phi} \in H^1_0(0,1).$$

Hence, we get

$$-\mu_4\phi_{xx} + \mu_3\phi + bu_x + \gamma\theta_x = h_2.$$

This give $(4.25)_2$. Similarly, if we take $(\widehat{\phi}, \widehat{\theta}, \widehat{q}) = 0 \in H^1_0(0,1) \times L^2_*(0,1) \times L^2(0,1)$ in (4.26) to obtain

$$\mu \int_0^1 u_x\widehat{u}_x dx + b\int_0^1 \phi\widehat{u}_x dx + \lambda^2\rho \int_0^1 u\widehat{u}dx = \int_0^1 h_1\widehat{u}dx,$$

we get

$$\mu \int_0^1 u_x\widehat{u}_x dx = \int_0^1 (h_1 + b\phi_x - \lambda^2\rho u)\widehat{u}dx, \;\; \forall\widehat{u} \in H^1_*(0,1), \tag{4.28}$$

which yields

$$-\mu u_{xx} = (h_1 + b\phi_x - \lambda^2\rho u) \in L^2_*(0,1),$$

consequently, (4.28) takes the following form

$$\int_0^1 (-\mu u_{xx} - h_1 - b\phi_x + \lambda^2\rho u)\widehat{u}dx = 0, \;\; \forall\widehat{u} \in H^1_*(0,1).$$

Hence, we get

$$-\mu u_{xx} - b\phi_x + \lambda^2 \rho u = h_1.$$

This give $(4.25)_1$. Similarly, we get

$$\begin{cases} \kappa q_x = \lambda h_3 - \lambda \rho_3 \theta - \lambda \gamma \phi_x \in L^2_*(0,1), \\ \kappa \theta_x = \lambda h_4 - (\rho_4 \lambda + d) q \in L^2(0,1), \end{cases}$$

thus, we have

$$(\theta, q) \in H^1_*(0,1) \times H^1_0(0,1).$$

Moreover, (4.28) also holds for any $\Phi \in C^1([0,1])$. Then, by using integration by parts, we obtain

$$\mu \int_0^1 u_x \Phi_x dx + \int_0^1 (-h_1 - b\phi_x + \lambda^2 \rho u)\Phi dx = 0, \quad \forall \Phi \in C^1([0,1]).$$

Then, we get for any $\Phi \in C^1([0,1])$

$$u_x(1)\Phi(1) - u_x(0)\Phi(0) = 0.$$

Since $\Phi$ is arbitrary, we get that $u_x(0) = u_x(1) = 0$. Hence, $u \in H^2_*(0,1) \cap H^1_*(0,1)$.
Therefore, the application of regularity theory for the linear elliptic equations guarantees the existence of unique $U \in \mathcal{D}(\mathcal{A})$ such that (4.20) is satisfied.
Consequently, we conclude that $\mathcal{A}$ is a maximal dissipative operator.
Now, we prove that the operator $\Gamma$ defined in (4.14) is locally Lipschitz in $\mathcal{H}$. Let $U = (u, v, \phi, \psi, \theta, q, y, \varphi)^T \in \mathcal{H}$, $\widehat{U} = (\widehat{u}, \widehat{v}, \widehat{\phi}, \widehat{\psi}, \widehat{\theta}, \widehat{q}, \widehat{y}, \widehat{\varphi})^T \in \mathcal{H}$. Then, we have

$$\|\Gamma(U) - \Gamma(\widehat{U})\|_{\mathcal{H}} \leq M_3 \|f(\psi) - f(\widehat{\psi})\|_{L^2}.$$

By using (4.6), and Holder and Poincare inequalities, we can get

$$\begin{aligned} \|f(\psi) - f(\widehat{\psi})\|_{L^2} &\leq k_0 (\|\psi\|^{\beta}_{2\beta} + \|\widehat{\psi}\|^{\beta}_{2\beta}) \|\psi - \widehat{\psi}\| \\ &\leq k_1 \|\psi_x - \widehat{\psi}_x\|_{L^2}, \end{aligned}$$

which gives us

$$\|\Gamma(U) - \Gamma(\widehat{U})\|_{\mathcal{H}} \leq M_4 \|U - \widehat{U}\|_{\mathcal{H}}.$$

Then, the operator $\Gamma$ is locally Lipschitz in $\mathcal{H}$. Consequently, the well-posedness result follows from the Hille-Yosida theorem. This completes the proof. □

## 4.3 Stability result

In this section, we state and prove our decay result for the energy of the system (4.8)-(4.9) using the multiplier technique. We need the following lemmas.

**Lemma 4.2.** *The energy functional E, defined by*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho_1 u_t^2 + \mu u_x^2 + \rho_2 \phi_t^2 + l\phi_x^2 + \xi\phi^2 + \rho_3\theta^2 + \rho_4 q^2 + 2bu_x\phi\right] dx \\ &+\frac{1}{2}\int_0^1\int_0^\infty g(s)\varphi_x^2(s)dsdx \\ &+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\,d\varrho d\rho dx, \end{aligned} \tag{4.29}$$

*satisfies*

$$\begin{aligned} E'(t) &\le -\eta_0\int_0^1 \phi_t^2 dx + \frac{1}{2}\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx \\ &\quad -d\int_0^1 q^2dx - \alpha(t)\int_0^1 \phi_t f(\phi_t)dx \ \le 0, \end{aligned} \tag{4.30}$$

*where* $\eta_0 = \mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho > 0$ *and* $\varphi(s) = \eta^t = \phi(x,t) - \phi(x,t-s)$.

*Proof.* Multiplying the first equation of (4.8) by $u_t$ and the second equation by $\phi_t$, then integration by parts over $(0,1)$, and using (4.9), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left[\rho_1 u_t^2 + \mu u_x^2 + \rho_2\phi_t^2 + \delta\phi_x^2 + \xi\phi^2 + \rho_3\theta^2 + \rho_4 q^2 + 2bu_x\phi\right] dx \\ &-\int_0^1 \phi_{xt}\int_0^\infty g(s)\varphi_x(s)dsdx + \mu_1\int_0^1\phi_t^2dx + d\int_0^1 q^2dx \\ &+\int_0^1\phi_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)\,d\varrho dx + \alpha(t)\int_0^1\phi_t f(\phi_t)dx = 0. \end{aligned} \tag{4.31}$$

The last term in the left hand side of (4.31) is estimated as follows.

$$\begin{aligned} \int_0^1\phi_t\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)\,d\varrho dx &\le \frac{1}{2}\Big(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\int_0^1\phi_t^2dx \\ &+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)\,d\varrho dx, \end{aligned} \tag{4.32}$$

and

$$-\int_0^1 \phi_{xt}\int_0^\infty g(s)\varphi_x(s)\,dsdx \;\leq\; \frac{1}{2}\frac{d}{dt}\int_0^1\int_0^\infty g(s)\varphi_x^2(s)\,dsdx -\frac{1}{2}\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)\,dsdx. \tag{4.33}$$

Now, multiplying the equation (4.8)$_3$ by $y|\mu_2(\varrho)|$, and integrating the result over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$

$$\begin{aligned}
\frac{d}{dt}\;\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)d\varrho d\rho dx
&= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|yy_\rho(x,\rho,\varrho,t)\,d\varrho d\rho dx\\
&= -\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|\frac{d}{d\rho}y^2(x,\rho,\varrho,t)\,d\varrho d\rho dx\\
&= \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|(y^2(x,0,\varrho,t)-y^2(x,1,\varrho,t))d\varrho dx\\
&= \frac{1}{2}(\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho)\int_0^1\phi_t^2dx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)\,d\varrho dx.
\end{aligned}\tag{4.34}$$

Now, using (4.31),(4.32),(4.33), and (4.34), we have

$$\begin{aligned}
E'(t) \;\leq\; & -\Big(\mu_1-\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|d\varrho\Big)\int_0^1\phi_t^2dx+\frac{1}{2}\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)\,dsdx\\
&-\alpha(t)\int_0^1\phi_t f(\phi_t)dx-d\int_0^1 q^2dx,
\end{aligned}$$

then, by (4.3), there exists a positive constant $\eta_0$ such that

$$\begin{aligned}
E'(t) \;\leq\; & -\eta_0\int_0^1\phi_t^2dx+\frac{1}{2}\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)\,dsdx\\
&-\alpha(t)\int_0^1\phi_t f(\phi_t)dx-d\int_0^1 q^2dx,
\end{aligned}$$

hence, by (4.4) − (4.7) we obtain $E$ is a non-increasing function. □

*Remark* 4.3. Using ($\mu\xi > b^2$), we conclude that the energy $E(t)$definie by (4.29) satisfies

$$\begin{aligned}
E(t) \;>\; & \frac{1}{2}\int_0^1\Big[\rho_1u_t^2+\widehat{\mu}u_x^2+\rho_2\phi_t^2+l\phi_x^2+\widehat{\xi}\phi^2+\rho_3\theta^2+\rho_4q^2\Big]dx\\
&+\frac{1}{2}\int_0^1\int_0^\infty g(s)\varphi_x^2(s)dsdx\\
&+\frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\varrho|\mu_2(\varrho)|y^2(x,\rho,\varrho,t)\,d\varrho d\rho dx,
\end{aligned}$$

where

$$\widehat{\mu}=\frac{1}{2}(\mu-\frac{b^2}{\xi})>0,\quad \widehat{\xi}=\frac{1}{2}(\xi-\frac{b^2}{\mu})>0,$$

then $E(t)$ is positive function.

We consider the following lemmas.

**Lemma 4.4.** *The functional*

$$D_1(t):=\rho_2\int_0^1\phi_t\phi dx+\frac{b\rho_1}{\mu}\int_0^1\phi\int_0^x u_t(y)\,dydx+\frac{\mu_1}{2}\int_0^1\phi^2dx,$$

*satisfies*

$$\begin{aligned}D_1'(t) &\le -\frac{l}{2}\int_0^1\phi_x^2dx-\widehat{\mu}\int_0^1\phi^2dx+\varepsilon_1\int_0^1 u_t^2dx+c(1+\frac{1}{\varepsilon_1})\int_0^1\phi_t^2dx\\ &+c\int_0^1\int_0^\infty g(s)\varphi_x^2(s)dsdx+c\int_0^1 f^2(\phi_t)dx+c\int_0^1\theta^2dx\\ &+c\int_0^1\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y^2(x,1,\varrho,t)\,d\varrho dx, \qquad (4.35)\end{aligned}$$

where $\widehat{\mu}=\xi-\frac{b^2}{\mu}>0$.

*Proof.* Direct computation using integration by parts and Young's inequality, for $\varepsilon_1>0$, yields

$$\begin{aligned}D_1'(t) &= -l\int_0^1\phi_x^2dx-\left(\xi-\frac{b^2}{\mu}\right)\int_0^1\phi^2dx+\frac{b\rho}{\mu}\int_0^1\phi_t\int_0^x u_t(y)\,dydx\\ &+\int_0^1\phi_x\int_0^\infty g(s)\,\varphi_x(s)\,dsdx+\alpha(t)\int_0^1\phi f(\phi_t)dx\\ &+\rho_2\int_0^1\phi_t^2dx-\int_0^1\phi\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx-\gamma\int_0^1\theta_x\phi dx\\ &\le -l\int_0^1\phi_x^2dx-\left(\xi-\frac{b^2}{\mu}\right)\int_0^1\phi^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1\phi_t^2dx\\ &+\varepsilon_1\int_0^1\left(\int_0^x u_t(y)\,dy\right)^2dx+\int_0^1\phi_x\int_0^\infty g(s)\,\varphi_x(s)\,dsdx\\ &-\int_0^1\phi\int_{\tau_1}^{\tau_2}|\mu_2(\varrho)|y(x,1,\varrho,t)d\varrho dx+\alpha(t)\int_0^1\phi f(\phi_t)dx\\ &+\gamma\int_0^1\theta\phi_x dx. \qquad (4.36)\end{aligned}$$

By Cauchy-Schwartz inequality, it is clear that

$$\int_0^1 \left(\int_0^x u_t(y)\,dy\right)^2 dx \le \int_0^1 \left(\int_0^1 u_t dx\right)^2 dx \le \int_0^1 {u_t}^2 dx.$$

So, estimate (4.36) becomes

$$\begin{aligned} D_1'(t) \le & -l\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 \phi_t^2 dx \\ & +\varepsilon_1 \int_0^1 {u_t}^2 dx - \int_0^1 \phi \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t) d\varrho dx + \gamma \int_0^1 \theta\phi_x dx \\ & + \int_0^1 \phi_x \int_0^\infty g(s)\varphi_x(s)\,dsdx + \alpha(t)\int_0^1 \phi f(\phi_t) dx. \end{aligned} \tag{4.37}$$

The last term in the RHS of (4.37) is estimated as follows:

$$\int_0^1 \phi_x \int_0^\infty g(s)\varphi_x(s)\,dsdx \le c\delta_1 \int_0^1 \phi_x^2 dx + \frac{c}{4\delta_1}\int_0^1\int_0^\infty g(s)\varphi_x^2(s) dsdx, \tag{4.38}$$

where we have used Cauchy-Schwartz, Young's and poincare's inequalities, for $\delta_1, \delta_2, \delta_3, \delta_4 > 0$. By substituting (4.38) into(4.36), we obtain

$$\begin{aligned} D_1'(t) \le & -(l - c\delta_1 - \mu_1 c\delta_2 - c\delta_3 + \delta_4)\int_0^1 \phi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \phi^2 dx \\ & +\varepsilon_1 \int_0^1 u_t^2 dx + c(1+\frac{1}{\varepsilon_1})\int_0^1 \phi_t^2 dx + \frac{1}{4\delta_4}\int_0^1 \theta^2 dx \\ & +\frac{c}{4\delta_1}\int_0^1\int_0^\infty g(s)\varphi_x^2(s) dsdx + \frac{1}{4\delta_3}\int_0^1 f^2(\phi_t) dx \\ & +\frac{1}{4\delta_2}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t) d\varrho dx. \end{aligned}$$

Bearing in mind that $\mu\xi > b^2$ and letting $\delta_1 = \frac{l}{8c}$, $\delta_2 = \frac{l}{8c\mu_1}$, $\delta_3 = \frac{l}{8c}$, and $\delta_4 = \frac{l}{8}$, we obtain estimate (4.35). □

**Lemma 4.5.** *Then, for any $\varepsilon_2 > 0$ the functional*

$$\begin{aligned} D_2(t) := & \frac{\rho_2\rho_4}{\mu}\int_0^1 \phi_t u_x dx + \frac{\delta\rho_1\rho_4}{\mu^2}\int_0^1 \phi_x u_t dx \\ & \frac{\delta\rho_3\rho_4}{\gamma\mu}\chi \int_0^1 \theta u_t dx + \frac{\delta\kappa\rho_4}{\mu\gamma}\chi\int_0^1 q u_x dx \\ & -\frac{\rho_1\rho_4}{\mu^2}\int_0^1 u_t \int_0^\infty g(s)\phi_x(t-s)\,dsdx, \end{aligned}$$

*satisfies,*

$$\begin{aligned}
D_2'(t) \leq & -\frac{b\rho_4}{2\mu}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx + c\varepsilon_2 \int_0^1 u_t^2 dx \\
& + c\int_0^1 \theta^2 dx + c\int_0^1 \phi_t^2 dx + c\int_0^1 q^2 dx \\
& + c\int_0^1\int_0^\infty g(s)\varphi_x^2(s)\,dsdx - \frac{c}{\varepsilon_2}\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)\,dsdx \\
& + c\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)d\varrho dx + c\int_0^1 f^2(\phi_t)dx \\
& + \left(\frac{\delta\rho_3\rho_4}{\gamma\rho_1}\chi - \frac{\delta\kappa^2}{\gamma\mu}\chi - \frac{\rho_4\gamma}{\mu}\right)\int_0^1 u_x\theta_x dx. \qquad (4.39)
\end{aligned}$$

*Where* $\chi = (\frac{\rho_2}{\delta} - \frac{\rho_1}{\mu})$, *and* $\delta = l + g_0$.

*Proof.* By differentiating $D_2$, then using (4.8), integration by parts, and (4.9) we obtain

$$\begin{aligned}
D_2'(t) = & -\frac{b\rho_4}{\mu}\int_0^1 u_x^2 dx + \frac{\rho_4 lb}{\mu^2}\int_0^1 \phi_x^2 dx - \frac{\rho_4\xi}{\mu}\int_0^1 u_x\phi dx \\
& -\frac{\rho_1\rho_4}{\mu^2}\int_0^1 u_t\int_0^\infty g'(s)\varphi_x(s)\,dsdx - \frac{\rho_4\alpha(t)}{\mu}\int_0^1 u_x f(\phi_t)dx \\
& -\frac{\rho_4\mu_1}{\mu}\int_0^1 \phi_t u_x dx - \frac{d\delta\kappa}{\gamma\mu}\chi\int_0^1 q u_x dx - \frac{\delta\rho_3\rho_4 b}{\rho_1\mu}\chi\int_0^1 \theta\phi_x dx \\
& -\frac{b\rho_4}{\mu^2}\int_0^1 \phi_x\int_0^\infty g(s)\varphi_x(s)\,dsdx \\
& -\frac{\rho_4}{\mu}\int_0^1 u_x\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)d\varrho dx \\
& + \left(\frac{\delta\rho_3\rho_4}{\gamma\rho_1}\chi - \frac{\delta\kappa^2}{\gamma\mu}\chi - \frac{\rho_4\gamma}{\mu}\right)\int_0^1 u_x\theta_x dx. \qquad (4.40)
\end{aligned}$$

In what follows, we estimate the last six terms in the right hand side of ( 4.40), using Young's, Cauchy-Schwartz, and Poincare's inequalities.
For $\delta_4, \delta_5, \delta_6, \delta_7, \delta_8, \delta_9, \varepsilon_2 > 0$, we have

$$-\frac{\rho_4\xi}{\mu}\int_0^1 u_x\phi dx \leq \frac{\rho_4\xi}{\mu}\delta_4\int_0^1 u_x^2 dx + \frac{\rho_4\xi}{4\mu\delta_4}\int_0^1 \phi^2 dx.$$

By letting $\delta_4 = \dfrac{b}{16\xi}$, using Poincar's inequality, we get

$$-\frac{\rho_4\xi}{\mu}\int_0^1 u_x\phi dx \leq \frac{\rho_4 b}{6\mu}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx, \qquad (4.41)$$

and by Young's and Chauchy-Schawrz inequalities, we get

$$-\frac{\rho_4 b}{\mu}\int_0^1 \phi_x \int_0^\infty g(s)\varphi_x(s)\,dsdx \le c\delta_5 \int_0^1 \phi_x^2 dx + \frac{c}{4\delta_5}\int_0^1\int_0^\infty g(s)\varphi_x^2(s)\,dsdx.$$

By letting $\delta_5 = \frac{\rho_4 b}{16c\mu}$, we obtain

$$-\frac{\rho_4 b}{\mu}\int_0^1 \phi_x \int_0^\infty g(s)\varphi_x(s)\,dsdx \le \frac{b}{6\rho_2}\int_0^1 \phi_x^2 dx + c\int_0^1\int_0^\infty g(s)\varphi_x^2(s)\,dsdx.$$

Similarly, $\forall \varepsilon_2 > 0$ we have

$$\frac{\rho_1\rho_4}{\mu^2}\int_0^1 u_t \int_0^\infty g'(s)\varphi_x(s)\,dsdx \le c\varepsilon_2 \int_0^1 u_t^2 dx + \frac{c}{\varepsilon_2}\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)\,dsdx,$$

$$-\frac{\rho_4\mu_1}{\mu}\int_0^1 \phi_t u_x dx \le \frac{\rho_4\mu_1\delta_6}{2\mu}\int_0^1 u_x^2 dx + \frac{\rho_4\mu_1}{2\mu\delta_6}\int_0^1 \phi_t^2 dx,$$

$$\frac{\rho_4}{\mu}\int_0^1 u_x \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y(x,1,\varrho,t)d\varrho dx \le \frac{\delta_7 c}{2}\int_0^1 u_x^2 dx + \frac{c}{2\delta_7}\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)d\varrho,$$

$$-\frac{\alpha(t)\rho_4}{\mu}\int_0^1 u_x f(\phi_t)dx \le \frac{c\delta_8}{2}\int_0^1 u_x^2 dx + \frac{c}{2\delta_8}\int_0^1 f^2(\phi_t)dx,$$

$$-\frac{\delta\rho_3\rho_4 b}{\rho_1\mu}\chi\int_0^1 \theta\phi_x dx \le c\int_0^1 \theta^2 dx + c\int_0^1 \phi_x^2 dx,$$

$$-\frac{d\delta\kappa}{\gamma\mu}\chi\int_0^1 q u_x dx \le \delta_9\int_0^1 u_x^2 dx + \frac{c}{\delta_9}\int_0^1 q^2 dx.$$

The replacement of (4.41)-(4.3) into (4.40), and by letting $\delta_6 = \delta_7 = \frac{\rho_4 b}{8\mu_1\mu}$, $\delta_8 = \frac{\rho_4 b}{8\mu}$, and $\delta_9 = \frac{b\rho_4}{8\mu}$ , yields (4.39). □

**Lemma 4.6.** *The functional*

$$D_3(t) := -\rho_1 \int_0^1 u_t u dx,$$

*satisfies*

$$D_3'(t) \le -\rho_1\int_0^1 u_t^2 dx + \frac{3\mu}{2}\int_0^1 u_x^2 dx + c\int_0^1 \phi_x^2 dx.$$

*Proof.* Direct computations give

$$D_3'(t) = -\rho_1\int_0^1 u_t^2 dx + \mu\int_0^1 u_x^2 dx + b\int_0^1 u_x\phi dx.$$

Estimat (4.6) easily follows by using Young's and Poincaré inequalities.

$$\begin{aligned} D_3'(t) &\leq -\rho_1 \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b\varepsilon \int_0^1 u_x^2 dx + \frac{b}{4\varepsilon} \int_0^1 \phi^2 dx \\ &\leq -\rho_1 \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx + b\varepsilon \int_0^1 u_x^2 dx + \frac{bc}{4\varepsilon} \int_0^1 \phi_x^2 dx, \end{aligned}$$

by letting $\varepsilon = \dfrac{\mu}{2b}$, we obtain (4.6). □

Now, let us introduce the following functional used by

**Lemma 4.7.** *The functional*

$$D_4(t) := \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} |\mu_2(\varrho)| y^2(x, \rho, \varrho, t)\, d\varrho d\rho dx,$$

*satisfies,*

$$\begin{aligned} D_4'(t) \leq\ & -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| y^2(x, \rho, \varrho, t)\, d\varrho d\rho dx + \mu_1 \int_0^1 \phi_t^2 dx \\ & -\eta_1 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x, 1, \varrho, t)\, d\varrho dx, \end{aligned} \tag{4.42}$$

*where $\eta_1$ is a positive constant.*

*Proof.* By differentiating $D_4$, with respect to $t$ and using the equation $(4.8)_3$, we have

$$\begin{aligned} D_4'(t) &= -2 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\varrho\rho} |\mu_2(\varrho)| y y_\rho(x, \rho, \varrho, t)\, d\varrho d\rho dx \\ &= -\int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho\rho} |\mu_2(\varrho)| y^2(x, \rho, \varrho, t)\, d\varrho d\rho dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| [e^{-\varrho} y^2(x, 1, \varrho, t) - y^2(x, 0, \varrho, t)] d\varrho dx. \end{aligned}$$

Using the fact that $y(x, 0, \varrho, t) = \phi_t(x, t)$, and $e^{-\varrho} \leq e^{-\varrho\rho} \leq 1$, for all $0 < \rho < 1$, we obtain

$$\begin{aligned} D_4'(t) &= -\eta_1 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| y^2(x, \rho, \varrho, t)\, d\varrho d\rho dx \\ &\quad - \int_0^1 \int_{\tau_1}^{\tau_2} e^{-\varrho} |\mu_2(\varrho)| y^2(x, 1, \varrho, t)\, d\varrho dx + \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| d\varrho \int_0^1 \phi_t^2 dx. \end{aligned}$$

□

Because $-e^{-\varrho}$ is a increasing function, we have $-e^{-\varrho} \leq -e^{-\tau_2}$, for all $\varrho \in [\tau_1, \tau_2]$.
Finally, setting $\eta_1 = e^{-\tau_2}$ and recalling (4.7), we obtain (4.42). We are now ready to prove the main result.

**Lemma 4.8.** *The functional*

$$D_5(t) := -\rho_3\rho_4 \int_0^1 q \int_0^x \theta(y)\,dydx,$$

*satisfies*

$$D_5'(t) \leq -\frac{\rho_3\kappa}{2} \int_0^1 \theta^2 dx + c \int_0^1 \phi_t^2 dx + c \int_0^1 q^2 dx. \tag{4.43}$$

*Proof.* Direct computation using integration by parts and Young's inequality, we get

$$\begin{aligned} D_5'(t) &= \rho_3 d \int_0^1 q \int_0^x \theta(y)\,dydx - \rho_3\kappa \int_0^1 \theta^2 dx \\ &\quad + \kappa\rho_4 \int_0^1 q^2 dx + \rho_4\gamma \int_0^1 q\phi_t dx. \end{aligned}$$

Applying Young's and Cauchy-Schwartz inequalities, we obtain (4.43). □

**Theorem 4.9.** *Assume (4.3)-(4.7) hold. Let $h(t) = \alpha(t).\eta(t)$ be a positive non-increasing function. Then, for any $U_0 \in \mathcal{D}(\mathcal{A})$, satisfying for some $c_0 > 0$,*

$$max\{\int_0^1 \phi_{0x}^2(x,s)dx, \int_0^1 \phi_{0sx}^2(x,s)dx\} \leq c_0, \quad \forall s > 0, \tag{4.44}$$

*there exist positive constants $\beta_1$, $\beta_2$ and $\beta_3$ such that the energy functional given by (4.29) satisfies*

$$E(t) \leq \beta_1 G_0^{-1}\left(\frac{\beta_2 + \beta_3 \int_0^t h(s)\varpi(s)ds}{\int_0^t h(s)ds}\right), \tag{4.45}$$

*where*

$$G_0(t) = tG'(\varepsilon_0 t), \forall \varepsilon_0 \geq 0, \;\; and \quad \varpi(s) = \int_s^\infty g(\sigma)d\sigma.$$

*Proof.* We define a Lyapunov functional

$$\mathcal{L}(t) := NE(t) + N_1D_1(t) + N_2D_2(t) + D_3(t) + N_4D_4(t) + N_5D_5(t), \tag{4.46}$$

where $N$, $N_1$, $N_2$, $N_4$ and $N_5$ are positive constants to be selected later.

By differentiating (4.46) and using (4.30), (4.35), (4.39), (4.6), (4.42), (4.43), we have

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{lN_1}{2} - cN_2 - c\right]\int_0^1 \phi_x^2 dx - [\rho_1 - N_1\varepsilon_1 - N_2 c\varepsilon_2]\int_0^1 u_t^2 dx \\
& -\left[\frac{b\rho_4 N_2}{2\mu} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx + c\left[N_1 + N_2\right]\int_0^1\int_0^\infty g(s)\varphi_x^2(s)dsdx \\
& -\left[\eta_0 N - cN_1(1+\frac{1}{\varepsilon_1}) - N_2 c - \mu_1 N_4 - cN_5\right]\int_0^1 \phi_t^2 dx \\
& -\left[\frac{\rho_3\kappa N_5}{2} - cN_1 - cN_2\right]\int_0^1 \theta^2 dx - N_1\widehat{\mu}\int_0^1 \phi^2 dx \\
& -\left[N_4\eta_1 - cN_1 - cN_2\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx \\
& -\left[dN - cN_2 - cN_5\right]\int_0^1 q^2 dx + \left[\frac{N}{2} - \frac{cN_2}{\varepsilon_2}\right]\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx \\
& -N_4\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(\varrho)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx \\
& +c\left[N_1 + N_2\right]\int_0^1 f^2(\phi_t)dx + N_2\left(\frac{\delta\rho_3\rho_4}{\gamma\rho_1}\chi - \frac{\delta\kappa^2}{\gamma\mu}\chi - \frac{\rho_4\gamma}{\mu}\right)\int_0^1 u_x\theta_x dx,
\end{aligned}$$

where $\chi = (\frac{\rho_2}{\delta} - \frac{\rho_1}{\mu})$, and by setting

$$\varepsilon_1 = \frac{\rho_1}{4N_1}, \varepsilon_2 = \frac{\rho_1}{4cN_2},$$

we obtain

$$\begin{aligned}
\mathcal{L}'(t) \leq & -\left[\frac{lN_1}{2} - cN_2 - c\right]\int_0^1 \phi_x^2 dx - \frac{\rho_1}{2}\int_0^1 u_t^2 dx \\
& -\left[\frac{b\rho_4 N_2}{2\mu} - \frac{3\mu}{2}\right]\int_0^1 u_x^2 dx + c\left[N_1 + N_2\right]\int_0^1\int_0^\infty g(s)\varphi_x^2(s)dsdx \\
& -\left[\eta_0 N - cN_1(1+N_1) - cN_2 - \mu_1 N_4 - cN_5\right]\int_0^1 \phi_t^2 dx \\
& -\left[\frac{\rho_3\kappa N_5}{2} - cN_1 - cN_2\right]\int_0^1 \theta^2 dx - \left[dN - cN_2 - cN_5\right]\int_0^1 q^2 dx \\
& -\left[N_4\eta_1 - cN_1 - cN_2\right]\int_0^1\int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2(x,1,\varrho,t)\, d\varrho dx \\
& -N_1\widehat{\mu}\int_0^1 \phi^2 dx + \left[\frac{N}{2} - cN_2^2\right]\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx \\
& -N_4\eta_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \varrho|\mu_2(s)| y^2(x,\rho,\varrho,t)\, d\varrho d\rho dx \\
& +c\left[N_1 + N_2\right]\int_0^1 f^2(\phi_t)dx + N_2\left(\frac{\delta\rho_3\rho_4}{\gamma\rho_1}\chi - \frac{\delta\kappa^2}{\gamma\mu}\chi - \frac{\rho_4\gamma}{\mu}\right)\int_0^1 u_x\theta_x dx.
\end{aligned}$$

Next, we carefully choose our constants so that the terms inside the brackets are positive. We choose $N_2$ large enough such that

$$\alpha_1 = \frac{b\rho_4 N_2}{2\mu} - \frac{3\mu}{2} > 0,$$

then we choose $N_1$ large enough such that

$$\alpha_2 = \frac{lN_1}{2} - cN_2\left(1 + N_2\right) - c > 0,$$

then we choose $N_4, N_5$ large enough such that

$$\begin{aligned}
\alpha_3 &= N_4\eta_1 - cN_1 - cN_2 > 0, \\
\alpha_4 &= N_5\frac{\rho_3\kappa}{2} - cN_1 - cN_2 > 0,
\end{aligned}$$

thus, we arrive at

$$\begin{aligned}\mathcal{L}'(t) \leq & -\alpha_2 \int_0^1 \phi_x^2 dx - \alpha_0 \int_0^1 \phi^2 dx - \frac{\rho_1}{2} \int_0^1 u_t^2 dx - \alpha_1 \int_0^1 u_x^2 dx \\ & -[\eta_0 N - c] \int_0^1 \phi_t^2 dx + \left[\frac{N}{2} - c\right] \int_0^1 \int_0^\infty g'(s) \varphi_x^2(s) ds dx \\ & +c \int_0^1 \int_0^\infty g(s) \varphi_x^2(s) ds dx - \alpha_3 \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_2(\varrho)| y^2 (x, 1, \varrho, t)\, d\varrho dx \\ & -[dN - c] \int_0^1 q^2 dx - \alpha_6 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(\varrho)| y^2 (x, \rho, \varrho, t)\, d\varrho d\rho dx \\ & -\alpha_4 \int_0^1 \theta_x^2 dx + c \int_0^1 f^2(\phi_t) dx + \alpha_5 \int_0^1 u_x \theta_x dx, \end{aligned} \tag{4.47}$$

where $\alpha_0 = \widehat{\mu} N_1 = \left(\xi - \frac{b^2}{\mu}\right) N_1$, *and* $\alpha_5 = N_2 \Lambda = N_2 \left(\frac{\delta \rho_3 \rho_4}{\gamma \rho_1} \chi - \frac{\delta \kappa^2}{\gamma \mu} \chi - \frac{\rho_4 \gamma}{\mu}\right)$.
On the other hand, if we let

$$\mathfrak{L}(t) = N_1 D_1(t) + N_2 D_2(t) + D_3(t) + N_4 D_4(t) + N_5 D_5(t),$$

then

$$\begin{aligned}|\mathfrak{L}(t)| \leq & \rho_2 N_1 \int_0^1 |\phi \phi_t|\, dx + \frac{b \rho_1 N_1}{\mu} \int_0^1 \left|\phi \int_0^x u_t(y)\, dy\right| dx \\ & +\frac{\mu_1}{2} N_1 \int_0^1 \phi^2 dx + N_2 \int_0^1 \left|\frac{\delta \rho_1 \rho_4}{\mu^2} \phi_x u_t + \frac{\rho_2 \rho_4}{\mu} u_x \phi_t\right| dx \\ & +N_2 \int_0^1 \left|\frac{\rho_1 \rho_4}{\mu^2} u_t \int_0^\infty g(s)\, \phi_x (t-s)\, ds\right| dx \\ & +N_2 \int_0^1 \left|\frac{\delta \rho_3 \rho_4}{\mu \gamma} \chi \theta u_t + \frac{\delta \kappa \rho_4}{\gamma \mu} \chi q u_x\right| dx + \rho_1 \int_0^1 |u_t u|\, dx \\ & +N_4 \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho e^{-\varrho \rho} |\mu_2(\varrho)| y^2 (x, \rho, \varrho, t)\, d\varrho d\rho dx.\end{aligned}$$

Exploiting Young's, Cauchy-Schwartz, and Poincaré inequalities, we obtain

$$\begin{aligned}|\mathfrak{L}(t)| \leq & c \int_0^1 \left(u_t^2 + \phi_t^2 + \phi_x^2 + u_x^2 + \phi^2 + \theta^2 + q^2\right) dx \\ & +c \int_0^1 \int_0^\infty g(s) \varphi_x^2(s) ds dx + c \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} \varrho |\mu_2(s)| y^2 (x, \rho, \varrho, t)\, d\varrho d\rho \\ \leq & cE(t).\end{aligned}$$

Consequently, we obtain

$$|\mathfrak{L}(t)| = |\mathcal{L}(t) - NE(t)| \leq cE(t),$$

that is

$$(N-c)\,E\,(t) \leq \mathcal{L}\,(t) \leq (N+c)\,E\,(t)\,. \tag{4.48}$$

Now, by choosing $N$ large enough such that

$$\frac{N}{2}-c>0, N-c>0, \quad N\eta_0-c>0, Nd-c>0,$$

and exploiting (4.29), estimates (4.47) and (4.48), respectively, give

$$c_2E\,(t) \leq \mathcal{L}\,(t) \leq c_3E\,(t)\,, \quad \forall t \geq 0, \tag{4.49}$$

and

$$\begin{aligned}\mathcal{L}'\,(t) \;&\leq\; -k_1E\,(t)+k_2\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx\\ &+k_3\int_0^1(\phi_t^2+f^2(\phi_t))dx+\alpha_5\int_0^1 u_x\theta_x dx,\end{aligned} \tag{4.50}$$

for some $k_1, k_2, k_3, c_2, c_3 > 0$.

If $\Lambda = \left(\frac{\delta\rho_3\rho_4}{\gamma\rho_1}\chi - \frac{\delta\kappa^2}{\gamma\mu}\chi - \frac{\rho_4\gamma}{\mu}\right) = 0$, in this case, (4.50) takes the from

$$\begin{aligned}\mathcal{L}'\,(t) \;&\leq\; -k_1E\,(t)+k_2\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx\\ &+k_3\int_0^1(\phi_t^2+f^2(\phi_t))dx.\end{aligned} \tag{4.51}$$

By multiplying (4.51) by $h(t)=\alpha(t).\eta(t)$, we obtain

$$\begin{aligned}h(t)\mathcal{L}'\,(t) \;&\leq\; -k_1h(t)E\,(t)+k_2h(t)\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx\\ &+k_3h(t)\int_0^1(\phi_t^2+f^2(\phi_t))dx.\end{aligned} \tag{4.52}$$

We distinguish two cases:

- $G$ is linear on $[0,\varepsilon]$, In this case, using the assumption $(4.5)_1$ and (4.30), we can write

$$k_3h(t)\int_0^1(\phi_t^2+f^2(\phi_t))dx \leq k_3h(t)\int_0^1 \phi_t f(\phi_t))dx \leq -k_3\eta(t)E'(t), \tag{4.53}$$

and, by (4.4) we have

$$\begin{aligned} h(t)\int_0^1\int_0^t g'(s)\varphi_x^2(s)dsdx &\leq h(t)\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx \\ &\leq 2h(t)E'(t), \end{aligned}$$

and, by (4.44) we obtain

$$\begin{aligned} \int_0^1 \varphi_x^2(s)dx &= 2\int_0^1 \phi_x^2(x,t)dx + 2\int_0^1 \phi_x^2(x,t-s)dx \\ &\leq 4\sup_{s>0}\int_0^1 \phi_x^2(x,s)dx + 2\sup_{\tau>0}\int_0^1 \phi_{0x}^2(x,\tau)dx \\ &\leq \frac{8E(0)}{l} + 2c_0, \end{aligned}$$

then, we get

$$h(t)\int_0^1\int_t^\infty g'(s)\varphi_x^2(s)dsdx \leq (\frac{8E(0)}{l} + 2c_0)h(t)\int_t^\infty g(s)ds,$$

hence

$$\begin{aligned} h(t)\int_0^1\int_0^\infty g'(s)\varphi_x^2(s)dsdx &\leq -2\alpha(t)E'(t) \\ &\quad +(\frac{8E(0)}{l} + 2c_0)h(t)\varpi(t). \end{aligned} \tag{4.54}$$

Inserting (4.53) and (4.54) in (4.52). Since $h'(t) \leq 0, \alpha'(t) \leq 0, \eta'(t) \leq 0$. Then, we have

$$\mathcal{L}_1'(t) \leq -k_1 h(t)E(t) + \gamma_1 h(t)\varpi(t), \tag{4.55}$$

and

$$m_1 E(t) \leq \mathcal{L}_1(t) \leq m_2 E(t),$$

with

$$m_1 = \tau_1, \quad m_2 = c_2 h(0) + k_3\eta(0) + 2k_2\alpha(0) + \tau_1,$$

where

$$\begin{aligned} \mathcal{L}_1(t) &= h(t)\mathcal{L}(t) + (k_3\eta(t) + 2k_2\alpha(t) + \tau_1)E(t) \sim E(t), \\ \gamma_1 &= (\frac{8E(0)}{l} + 2c_0), \ \ \tau_1 > 0 \ \ and \ \ \varpi(t) = \int_t^\infty g(s)ds. \end{aligned}$$

Because, $E'(t) \leq 0, \forall t \geq 0$. By using (4.55), we have

$$E(T)\int_0^T h(t)dt \leq \left(\frac{\mathcal{L}_1(0)}{k_1} + \frac{\gamma_1}{k_1}\int_0^T h(t)\varpi(t)dt\right).$$

Using the fact that $G_0^{-1}$ is linear. Then

$$E(T) \leq \zeta G_0^{-1}\left(\frac{\frac{\mathcal{L}_1(0)}{k_1} + \frac{\gamma_1}{k_1}\int_0^T h(t)\varpi(t)dt}{\int_0^T h(t)dt}\right),$$

with $\beta_1 = \zeta, \quad \beta_2 = \frac{\mathcal{L}_1(0)}{k_1}, \quad \beta_3 = \frac{\gamma_1}{k_1}$. This completes the proof.

- $G$ is nonlinear on $[0, \varepsilon]$, we choose $0 \leq \varepsilon_1 \leq \varepsilon$, and we consider

$$I_1(t) = \{x \in (0,1), \quad |\phi_t| \leq \varepsilon_1\}, \quad I_2 = \{x \in (0,1), \quad |\phi_t| > \varepsilon_1\},$$

we define

$$I = \int_{I_1} \phi_t f(\phi_t)dt.$$

Using Jensen's inequality and the assumption $(4.5)_1$, we have

$$\begin{aligned} k_3 h(t)\int_0^1 (\phi_t^2 + f^2(\phi_t))dx &\leq k_3 h(t)\int_0^1 \phi_t f(\phi_t))dx \\ &\leq k_3' h(t)G^{-1}(I(t)) - k_3'\eta(t)E'(t). \end{aligned} \tag{4.56}$$

Inserting (4.56) in (4.52), since $\alpha'(t) \leq 0, \eta'(t) \leq 0$, and $E'(t) \leq 0$, we obtain

$$\mathcal{L}_2'(t) \leq -k_1 h(t)E(t) + \gamma_1 h(t)\varpi(t) + k_3' h(t)G^{-1}(I(t)),$$

and

$$m_3 E(t) \leq \mathcal{L}_2(t) \leq m_4 E(t),$$

with

$$m_3 = \tau_1, \quad m_4 = c_2 h(0) + k_3'\eta(0) + 2k_2\alpha(0) + \tau_1,$$

where

$$\mathcal{L}_2(t) = h(t)\mathcal{L}(t) + (k_3'\eta(t) + 2k_2\alpha(t) + \tau_1)E(t) \sim E(t).$$

Now, for $\varepsilon_0 < \varepsilon_1$ and by using $E'(t) \leq 0, G' > 0$, and $G'' > 0$ on $(0, \varepsilon]$, we define the functional $\mathcal{L}_3(t)$ by,

$$\mathcal{L}_3(t) = G'(\varepsilon_0 E(t))\mathcal{L}_2(t) + \tau_2 E(t) \sim E(t), \quad \tau_2 > 0,$$

satisfies

$$\begin{aligned}\mathcal{L}_3'(t) &= E'(t)(\varepsilon_0 G'(\varepsilon_0 E(t))\mathcal{L}_2(t)+\tau_2)+\mathcal{L}_2'(t)G'(\varepsilon_0 E(t))\\ &\le -k_1 h(t)G_0(E(t))+\gamma_1 G'(\varepsilon_0 E(t))h(t)\varpi(t)\\ &\quad +k_3' h(t)G'(\varepsilon_0 E(t))G^{-1}(I(t)).\end{aligned} \tag{4.57}$$

To estimate the last term of (4.57), using the general Young's inequality:

$$AB \le G^*(A)+G(B), \quad if \;\; A\in(0,G'(\varepsilon)), \;\; B\in(0,\varepsilon),$$

where

$$G^*(A) = s(G'^{-1}(s)-G((G'^{-1}(s)), \quad if \;\; s\in(0,G'(\varepsilon)),$$

satisfies

$$k_3' h(t)G'(\varepsilon_0 E(t))G^{-1}(I(t)) \le k_3'\varepsilon_0 h(t)G_0(E(t)) - k_3'\eta(t)E'(t). \tag{4.58}$$

Inserting (4.58) in (4.57) and letting $\varepsilon_0 = \dfrac{k_1}{2k_3'}$ , we get

$$\mathcal{L}_3'(t)+k_3'\eta(t)E'(t) \le -k_1 h(t)G_0(E(t))+\gamma_1 G'(\varepsilon_0 E(t))h(t)\varpi(t).$$

Since $\eta'(t)\le 0$, then

$$\mathcal{L}_4'(t) \le -k_1 h(t)G_0(E(t))+\gamma_1 G'(\varepsilon_0 E(t))h(t)\varpi(t),$$

where

$$\mathcal{L}_4(t) = \mathcal{L}_3(t)+k_3'\eta(t)E(t)\sim E(t).$$

Since $\alpha(t), G_0(E(t)), G'(\varepsilon_0 E(t))$ are non-increasing functions,
then, for any $T>0$

$$\begin{aligned}k_1 G_0(E(T))\int_0^T h(t)dt &\le k_1\int_0^T h(t)G_0(E(t))dt\\ &\le \mathcal{L}_4(0)+\gamma_1 G'(\varepsilon_0 E(0))\int_0^T h(t)\varpi(t)dt,\end{aligned}$$

which gives (4.45) with $\beta_1=1,\;\beta_2=\dfrac{\mathcal{L}_4(0)}{k_1}$, and $\beta_3=\dfrac{\gamma_1 G'(\varepsilon_0 E(0))}{k_1}$.
The proof is complete.

If $\Lambda = \left(\frac{\delta\rho_3\rho_4}{\gamma\rho_1}\chi - \frac{\delta\kappa^2}{\gamma\mu}\chi - \frac{\rho_4\gamma}{\mu}\right)\neq 0$, and $|\Lambda| < \dfrac{k_1\kappa\mu\rho_4}{2N_2\rho_4(\rho_4+\mu)}$
This case is more important from the physical point of view, where waves are not necessarily of equal

speeds. Let

$$E(t) = E(u, \phi, \theta, q, y, \varphi) = E_1(t),$$

denotes the first-order energy defined in (4.29), and

$$E_2(t) = E(u_t, \phi_t, \theta_t, q_t, y_t, \varphi_t),$$

denotes the second-order energy, then, we have

$$\begin{aligned} E_2'(t) &\leq -\eta_0 \int_0^1 \phi_{tt}^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(s)\varphi_{tx}^2(s) ds \\ &\quad -d \int_0^1 q_t^2 dx - \alpha'(t) \int_0^1 \phi_{tt} f(\phi_t) dx - \alpha(t) \int_0^1 \phi_{tt}^2 f'(\phi_t) dx \\ &= -\eta_0 \int_0^1 \phi_{tt}^2 dx + \frac{1}{2} \int_0^1 \int_0^\infty g'(s)\varphi_{tx}^2(s) ds - d \int_0^1 q_t^2 dx \\ &\quad +\alpha(t)\left(\frac{-\alpha'(t)}{\alpha(t)} \int_0^1 \phi_{tt} f(\phi_t) dx - \int_0^1 \phi_{tt}^2 f'(\phi_t) dx\right). \end{aligned} \tag{4.59}$$

Since $f, g$ are non-decreasing functions, $\alpha(t)$ is a positive function,
and $\lim_{t\to\infty} \frac{-\alpha'(t)}{\alpha(t)} = 0$, we deduce that

$$E_2'(t) \leq -d \int_0^1 q_t^2 dx.$$

The last term in (4.50), by using $(4.8)_1$, Young's inequality,
and by setting $K = \dfrac{\Lambda N_2 \rho_4}{\kappa} = \dfrac{\alpha_5 \rho_4}{\kappa}$, and $\alpha_5 = \Lambda N_2$ . From $(4.8)_4$, we have

$$\begin{aligned} \alpha_5 \int_0^1 u_x \theta_x dx &= -\frac{\alpha_5 \rho_4}{\kappa} \int_0^1 u_x q_t dx - \frac{\alpha_5 d}{\kappa} \int_0^1 u_x q dx \\ &\leq \frac{|K|}{2} \int_0^1 q_t^2 dx + \frac{|K|}{2} \int_0^1 u_x^2 dx \\ &\quad + \frac{|K|}{2} \int_0^1 q^2 dx + \frac{|K|}{2} \int_0^1 u_x^2 dx, \end{aligned}$$

then (4.50)

$$\begin{aligned}\mathcal{L}'(t) &\leq -k_1E_1(t) + k_2\int_0^1\int_0^\infty g(s)\varphi_s^2dsdx + \frac{|K|}{2}\int_0^1 q_t^2dx\\ &\quad +|K|\int_0^1 u_x^2dx + \frac{|K|}{2}\int_0^1 q^2dx + k_3\int_0^1(\phi_t^2 + f^2(\phi_t))dx\\ &\leq -k_4E_1(t) + k_2\int_0^1\int_0^\infty g(s)\varphi_s^2dsdx\\ &\quad +\frac{|K|}{2}\int_0^1 q_t^2dx + k_3\int_0^1(\phi_t^2 + f^2(\phi_t))dx, \end{aligned} \tag{4.60}$$

where

$$k_4 = k_1 - 2\frac{|K|}{\mu} - \frac{|K|}{\rho_4} > 0.$$

Let

$$\mathcal{R}(t) = \mathcal{L}(t) + N_6(E_1(t) + E_2(t)). \tag{4.61}$$

By (4.49) and (4.61), we get

$$|\mathcal{R}(t) - N_6(E_1(t) + E_2(t))| \leq c(E_1(t) + E_2(t))$$

$$(N_6 - c)(E_1(t) + E_2(t)) \leq \mathcal{R}(t) \leq (N_6 + c)(E_1(t) + E_2(t)),$$

and by using (4.59),(4.60) and (4.61), we obtain

$$\begin{aligned}\mathcal{R}'(t) &= \mathcal{L}'(t) + N_6(E_1'(t) + E_2'(t))\\ &\leq -k_4E_1(t) + k_2\int_0^1\int_0^\infty g(s)\varphi_s^2dsdx\\ &\quad +k_3\int_0^1(\phi_t^2 + f^2(\phi_t))dx - (dN_6 - \frac{|K|}{2})\int_0^1 q_t^2dx,\end{aligned}$$

we choose $N_6$ large enough, such that

$$dN_6 - \frac{|K|}{2} > 0,\ \ N_6 - c > 0,$$

we obtain

$$\mathcal{R}(t) \sim (E_1(t) + E_2(t)),$$

and

$$\begin{aligned} \mathcal{R}'(t) &\leq -k_4 E_1(t) + k_2 \int_0^1 \int_0^\infty g(s)\varphi_s^2 ds dx \\ &+k_3 \int_0^1 (\phi_t^2 + f^2(\phi_t))dx. \end{aligned} \tag{4.62}$$

By multiplying (4.62) by $h(t) = \alpha(t).\eta(t)$, we obtain

$$\begin{aligned} h(t)\mathcal{R}'(t) &\leq -k_4 h(t)E(t) + k_2 h(t) \int_0^1 \int_0^\infty g'(s)\varphi_x^2(s) ds dx \\ &+k_3 h(t) \int_0^1 (\phi_t^2 + f^2(\phi_t))dx. \end{aligned}$$

We distinguish two cases:

- $G$ is linear on $[0, \varepsilon]$. In the same way that in the previous case, we obtain

$$\mathcal{R}_1'(t) \leq -k_4 h(t)E(t) + \gamma_1 h(t)\varpi(t), \tag{4.63}$$

and

$$m_1(E_1(t) + E_2(t)) \leq \mathcal{R}_1(t) \leq m_2(E_1(t) + E_2(t)),$$

with

$$m_1 = \tau_1, \quad m_2 = c_2 h(0) + k_3\eta(0) + 2k_2\alpha(0) + \tau_1,$$

where

$$\begin{aligned} \mathcal{R}_1(t) &= h(t)\mathcal{R}(t) + (k_3\eta(t) + 2k_2\alpha(t) + \tau_1)E(t) \sim (E_1(t) + E_2(t)), \\ \gamma_1 &= (\frac{8E(0)}{l} + 2c_0), \quad \tau_1 > 0 \ \textit{and} \ \varpi(t) = \int_t^\infty g(s)ds. \end{aligned}$$

Because, $E'(t) \leq 0, \forall t \geq 0$. By using (4.63), we have

$$E(T)\int_0^T h(t)dt \leq \left(\frac{\mathcal{R}_1(0)}{k_4} + \frac{\gamma_1}{k_4}\int_0^T h(t)\varpi(t)dt\right).$$

Using the fact that $G_0^{-1}$ is linear. Then

$$E(T) \leq \zeta G_0^{-1}\left(\frac{\frac{\mathcal{R}_1(0)}{k_4} + \frac{\gamma_1}{k_4}\int_0^T h(t)\varpi(t)dt}{\int_0^T h(t)dt}\right),$$

with $\beta_1 = \zeta, \quad \beta_2 = \frac{\mathcal{R}_1(0)}{k_4}, \quad \beta_3 = \frac{\gamma_1}{k_4}$. This completes the proof.

- $G$ is nonlinear on $[0,\varepsilon]$, we choose $0 \leq \varepsilon_1 \leq \varepsilon$. And in a similar way to that in the previous case, we get

$$\mathcal{R}_2'(t) \leq -k_1 h(t)E(t) + \gamma_1 h(t)\varpi(t) + k_3' h(t)G^{-1}(I(t)),$$

and

$$m_3(E_1(t)+E_2(t)) \leq \mathcal{R}_2(t) \leq m_4(E_1(t)+E_2(t)), \tag{4.64}$$

with

$$m_3 = \tau_1, \quad m_4 = c_2 h(0) + k_3'\eta(0) + 2k_2\alpha(0) + \tau_1,$$

where

$$\mathcal{R}_2(t) = h(t)\mathcal{R}(t) + (k_3'\eta(t) + 2k_2\alpha(t) + \tau_1)E(t) \sim (E_1(t)+E_2(t)).$$

Now, for $\varepsilon_0 < \varepsilon_1$ and by using $E'(t) \leq 0, G' > 0$, and $G'' > 0$ on $(0,\varepsilon]$, we define the functional $\mathcal{L}_3(t)$ by,

$$\mathcal{R}_3(t) = G'(\varepsilon_0 E(t))\mathcal{R}_2(t) + \tau_2 E(t) \sim (E_1(t)+E_2(t)), \quad \tau_2 > 0,$$

satisfies

$$\begin{aligned}\mathcal{R}_3'(t) &= E'(t)(\varepsilon_0 G'(\varepsilon_0 E(t))\mathcal{R}_2(t) + \tau_2) + \mathcal{R}_2'(t)G'(\varepsilon_0 E(t))\\ &\leq -k_4 h(t)G_0(E(t)) + \gamma_1 G'(\varepsilon_0 E(t))h(t)\varpi(t)\\ &\quad + k_3' h(t)G'(\varepsilon_0 E(t))G^{-1}(I(t)).\end{aligned} \tag{4.65}$$

To estimate the last term of (4.65), using the general Young's inequality:

$$AB \leq G^*(A) + G(B), \quad if \;\; A \in (0, G'(\varepsilon)), \;\; B \in (0,\varepsilon),$$

where

$$G^*(A) = s(G'^{-1}(s) - G((G'^{-1}(s)), \quad if \;\; s \in (0, G'(\varepsilon)),$$

satisfies

$$k_3' h(t)G'(\varepsilon_0 E(t))G^{-1}(I(t)) \leq k_3'\varepsilon_0 h(t)G_0(E(t)) - k_3'\eta(t)E'(t). \tag{4.66}$$

Inserting (4.66) in (4.65) and letting $\varepsilon_0 = \dfrac{k_1}{2k_3'}$, we get

$$\mathcal{R}_3'(t) + k_3'\eta(t)E'(t) \leq -k_4 h(t)G_0(E(t)) + \gamma_1 G'(\varepsilon_0 E(t))h(t)\varpi(t).$$

Since $\eta'(t) \leq 0$, then

$$\mathcal{R}_4'(t) \leq -k_4 h(t)G_0(E(t)) + \gamma_1 G'(\varepsilon_0 E(t))h(t)\varpi(t),$$

where

$$\mathcal{R}_4(t) = \mathcal{R}_3(t) + k_3'\eta(t)E(t) \sim (E_1(t) + E_2(t)).$$

Since $\alpha(t), G_0(E(t)), G'(\varepsilon_0 E(t))$ are non-increasing functions,
then, for any $T > 0$

$$\begin{aligned} k_4 G_0(E(T)) \int_0^T h(t)dt &\leq k_4 \int_0^T h(t)G_0(E(t))dt \\ &\leq \mathcal{R}_4(0) + \gamma_1 G'(\varepsilon_0 E(0)) \int_0^T h(t)\varpi(t)dt, \end{aligned}$$

which gives (4.45) with $\beta_1 = 1,\ \beta_2 = \dfrac{\mathcal{R}_4(0)}{k_4}$, and $\beta_3 = \dfrac{\gamma_1 G'(\varepsilon_0 E(0))}{k_4}$.
The proof is complete.

□

*Remark* 4.10. We give some examples to illustrate the energy decay rates obtained by Theorem 4.9. We consider the three different examples

$$\begin{aligned} \text{If } g(t) &= \beta_1 e^{-\beta_2 t}, \text{ then } g'(t) = -\eta(t)g(t), \text{ where } \eta(t) = \beta_2. \\ \text{If } g(t) &= \frac{\beta_1}{(1+t)^{\beta_2+1}}, \text{ then } g'(t) = -\eta(t)g(t), \text{ where } \eta(t) = \frac{\beta_2+1}{1+t}. \\ \text{If } g(t) &= \frac{\beta_1}{(e^{t(\frac{\pi}{2}-arctgt)}\sqrt{1+t^2})^{\beta_2}}, \text{ then } g'(t) = -\eta(t)g(t), \eta(t) = \beta_2(\frac{\pi}{2} - arctgt), \\ \textit{and} \quad & \alpha(t) = \frac{1}{1+t}, \quad \alpha(t) = \frac{1}{ln(2+t)}. \end{aligned}$$

# Chapter 5

## Global nonexistence of solution for coupled nonlinear Klein-Gordon with degenerate damping and source terms

### 5.1 Introduction

In this chapter, we investigated (with Ouchenane and Yazid), the following system:

$$\begin{cases} u_{tt} - \Delta u_t - div\left(|\nabla u|^{\alpha-2}\nabla u\right) - div\left(|\nabla u_t|^{\beta_1-2}\nabla u_t\right) \\ \quad +a_1|u_t|^{m-2}u_t + m_1u^2 = f_1(u,v), \\ v_{tt} - \Delta v_t - div\left(|\nabla v|^{\alpha-2}\nabla v\right) - div\left(|\nabla v_t|^{\beta_2-2}\nabla v_t\right) \\ \quad +a_2|v_t|^{r-2}v_t + m_2v^2 = f_2(u,v), \end{cases} \tag{5.1}$$

where $u = u(t,x)$, $v = v(t,x)$, $x \in \Omega$, a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $t > 0$ and $a_1$, $a_2$, $b_1$, $b_2$, $m_1$, $m_2 > 0$ and $\beta_1$, $\beta_2$, $m$, $r \geq 2$, $\alpha > 2$, and the two functions $f_1(u,v)$ and $f_2(u,v)$ given by

$$\begin{aligned} f_1(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= b_1|u+v|^{2(\rho+1)}(u+v) + b_2|u|^{(\rho+2)}|v|^{\rho}v. \end{aligned} \tag{5.2}$$

The System (5.1) is supplemented by the following initial and boundary conditions

$$\begin{cases} (u(0),v(0)) = (u_0,v_0), (u_t(0),v_t(0)) = (u_1,v_1), \; x \in \Omega \\ u(x) = v(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{5.3}$$

## 5.2 Preliminaries

In this section, we introduce some notations and some technical lemmas to be used throughout this section. By $\|.\|_q$, we denote the usual $L^q(\Omega)$-norm. The constants $C, c, c_1, c_2, \ldots, c_6$ are positive generic constants, which may be different in various occurrences. We define

$$F(u,v) = \frac{1}{2(\rho+2)}\left[b_1 |u+v|^{2(\rho+2)} + 2b_2 |uv|^{\rho+2}\right].$$

Then , it is clear that, from (5.2),we have

$$uf_1(u,v) + vf_2(u,v) = 2(\rho+2)F(u,v). \tag{5.4}$$

The following lemma was introduced and proved in [45]

**Lemma 5.1.** *There exist two positive constants $c_0$ and $c_1$ such that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right), \tag{5.5}$$

and the energy functional

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\ &\quad + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 - \int_\Omega F(u,v)\,dx. \end{aligned} \tag{5.6}$$

Let us know define a constant $r_\alpha$ as follows :

$$r_\alpha = \frac{N\alpha}{N-\alpha}, \quad if\ N > \alpha,\ r_\alpha > \alpha \quad if\ N = \alpha,\ and\ r_\alpha = \infty \quad if\ N < \alpha. \tag{5.7}$$

The inequality below is a key element in proving the global existence of solution. A similar version of this lemma was first introduced in [64]

**Lemma 5.2.** *Suppose that $\alpha > 2$, and $2 < 2(\rho+2) < r_\alpha$. Then there exists $\eta > 0$ such that the inequality*

$$\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \leq \eta\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right)^{\frac{2(\rho+2)}{\alpha}}, \tag{5.8}$$

*holds.*

*Proof.* It is clear that by using the Minkowski inequality, we get

$$\|u+v\|_{2(\rho+2)}^2 \leq 2(\|u\|_{2(\rho+2)}^2 + \|v\|_{2(\rho+2)}^2),$$

the embedding $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$, gives

$$\|u\|_{2(\rho+2)}^2 \leq C\|\nabla u\|_\alpha^2 \leq C(\|\nabla u\|_\alpha^\alpha)^{\frac{2}{\alpha}} \leq C(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}},$$

and similary , we have

$$\|v\|_{2(\rho+2)}^2 \leq C\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}.$$

Thus, we deduce from the above estimates that

$$\|u + v\|_{2(\rho+2)}^2 \leq C(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}, \tag{5.9}$$

also, Hölder's and Young's inequalities give us

$$\begin{aligned} \|uv\|_{(\rho+2)} &\leq \|u\|_{2(\rho+2)}\|v\|_{2(\rho+2)} \\ &\leq C(\|\nabla u\|_{2(\rho+2)}^2 + \|\nabla v\|_{2(\rho+2)}^2) \\ &\leq C(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha)^{\frac{2}{\alpha}}. \end{aligned} \tag{5.10}$$

Collecting the estimates (5.9) and (5.10), then (5.8) holds. This completes the proof of lemma (5.2) □

**Lemma 5.3.** *Let $v > 0$ be a real positive number and $L$ be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+v}(t), \tag{5.11}$$

*defined in* $[0, \infty)$
*If* $L(0) > 0$, *then the solution cease to exist for* $t \geq L(0)^{-v}\xi^{-1}v^{-1}$.

*Proof.* Direct integration of (5.11) gives:

$$L^{-v}(0) - L^{-v}(t) \geq \xi vt.$$

Thus we obtain the following estimate:

$$L^v(t) \geq \left[L^{-v}(0) - \xi vt\right]^{-1}. \tag{5.12}$$

It is clear that the right-hand side of (5.12) is unbounded when

$$\xi vt = L^{-v}(0).$$

This completes the proof. □

In the following lemma, we show that the total energy of our system is a non-increasing function of $t$, thus, we have

**Lemma 5.4.** *Let* $(u, v)$ *be the solution of system* (5.1)-(5.3) *then the energy functional is a non-increasing function, satisfies*

$$\begin{aligned} \frac{dE(t)}{dt} &= -\|\nabla u_t\|_2^2 - \|\nabla v_t\|_2^2 - \|\nabla u_t\|_{\beta_1}^{\beta_1} - \|\nabla v_t\|_{\beta_2}^{\beta_2} \\ &\quad -a_1\|u_t\|_m^m - a_2\|v_t\|_r^r - m_1^2\|u\|_2^2 - m_2^2\|v\|_2^2, \end{aligned} \tag{5.13}$$

*for all* $t \geq 0$.

*Proof.* We multiply the first equation in (5.1) by $u_t$ and second equation by $v_t$ and integrate over $\Omega$, using integration by parts, we obtain (5.13) □

## 5.3 Global nonexistence result

In this section, we prove under some restrictions on the initial data and under some restrictions on the parameter $\alpha$,$\beta_1$,$\beta_2$,$m$, $r$ the lifespan of solution of problem (5.1)- (5.3) is finite

**Theorem 5.5.** *Suppose that* $\beta_1, \beta_2, m, r \geq 2$, $\alpha > 2$, $\rho > -1$ *such that* $\beta_1, \beta_2 < \alpha$, *and* $\max\{m, r\} < 2(\rho + 2) < r_\alpha$, *where* $r_\alpha$ *is the Sobolev critical exponent of* $W_0^{1,\alpha}(\Omega)$. *defined in* (5.7). *Assume further that*

$$E(0) < E_1, \qquad (\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \zeta_1.$$

*Then, any weak solution of* (5.1)-(5.3) *cannot exist for all time .Here the constants* $E_1$ *and* $\zeta_1$ *are defined in* (5.5).

In order to prove our result and for the sake of simplicity , we take $b_1 = b_2 = 1$ and introduce the following :

$$B = \eta^{\frac{1}{2(\rho+2)}}, \qquad \zeta_1 = B^{\frac{-2(\rho+2)}{2(\rho+2)-\alpha}}, \qquad E_1 = \left(\frac{1}{\alpha} - \frac{1}{2(\rho+2)}\right)\zeta_1^\alpha, \tag{5.14}$$

where $\eta$ is the optimal constant in (5.8).

The following lemma allows us to prove a blow up result for a large class of initial data. This lemma is similar to the one in [64] and has its origin in [67]

**Lemma 5.6.** *Let* $(u, v)$ *be a solution of* (5.1)-(5.3). *Assume that* $\alpha > 2$,
$\rho > -1$. *Assume further that* $E(0) < E_1$ *and*

$$(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha)^{\frac{1}{\alpha}} + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \zeta_1. \tag{5.15}$$

*Then there exists a constant $\zeta_2 > \zeta_1$ such that*

$$\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 > \zeta_2, \tag{5.16}$$

*and*

$$\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]^{\frac{1}{2(\rho+2)}} \geq B\zeta_2,\ \forall t \geq 0. \tag{5.17}$$

*Proof.* We first note that, by (5.6) and the definition of $B$, we have

$$\begin{aligned} E(t) &\geq \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 \\ &\quad - \frac{1}{2(\rho+2)}\left[|u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}\right] \\ &\geq \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 \\ &\quad - \frac{\eta}{2(\rho+2)}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right)^{\frac{2(\rho+2)}{\alpha}} \\ &\geq \frac{1}{\alpha}\zeta^\alpha - \frac{\eta}{2(\rho+2)}\zeta^{2(\rho+2)}, \end{aligned} \tag{5.18}$$

where $\zeta = \left[\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha + m_1^2 \|u\|_\alpha^\alpha + m_2^2 \|v\|_\alpha^\alpha\right]^{\frac{1}{\alpha}}$. It is not hard to verify that $g$ is increasing for $0 < \zeta < \zeta_1$, decreasing for $\zeta > \zeta_1$, $g(\zeta) \to -\infty$ as $\zeta \to +\infty$, and

$$g(\zeta_1) = \frac{1}{\alpha}\zeta_1^\alpha - \frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_1^{2(\rho+2)} = E_1,$$

where $\zeta_1$ is given in (5.14). Therefore, since $E(0) < E_1$, there exists $\zeta_2 > \zeta_1$ such that $g(\zeta_2) = E(0)$. If we set $\zeta_0 = \left[\|\nabla u(0)\|_\alpha^\alpha + \|\nabla v(0)\|_\alpha^\alpha\right]^{\frac{1}{\alpha}} + m_1^2 \|u(0)\|_2^2 + m_2^2 \|v(0)\|_2^2$, then by (5.18) we have $g(\zeta_0) \leq E(0) = g(\zeta_2)$, which implies that $\zeta_0 \geq \zeta_2$.

Now, establish (5.16), we suppose by contradiction that

$$\left(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2 \|u_0\|_2^2 + m_2^2 \|v_0\|_2^2 < \zeta_2,$$

for some $t_0 > 0$; by the continuity of $\|\nabla u(.)\|_\alpha^\alpha + \|\nabla v(.)\|_\alpha^\alpha + m_1^2 \|u(.)\|_2^2 + m_2^2 \|v(.)\|_2^2$ we can choose $t_0$ such that

$$\left(\|\nabla u(t_0)\|_\alpha^\alpha + \|\nabla v(t_0)\|_\alpha^\alpha\right)^{\frac{1}{\alpha}} + m_1^2 \|u(t_0)\|_2^2 + m_2^2 \|v(t_0)\|_2^2 > \zeta_1.$$

Again, the use of (5.18) leads to

$$E(t_0) \geq g\left(\|\nabla u(t_0)\|_\alpha^\alpha + \|\nabla v(t_0)\|_\alpha^\alpha\right) + m_1^2 \|u(t_0)\|_2^2 + m_2^2 \|v(t_0)\|_2^2 > g(\zeta_2) = E(0).$$

This is impossible since $E(t) \leq E(0)$, for all $t \in [0,T)$. Hence, (5.16) is established. To prove (5.17), we make use of (5.6) to get

$$\begin{aligned}
&\frac{1}{\alpha}\left(\|\nabla u_0\|_\alpha^\alpha + \|\nabla v_0\|_\alpha^\alpha\right) + m_1^2 \|u_0\|_2^2 + m_2^2 \|v_0\|_2^2 \\
&\leq E(0) + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right].
\end{aligned}$$

Consequently, (5.16) yields

$$\begin{aligned}
\frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] &\geq \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) - E(0) \\
&\geq \frac{1}{\alpha}\zeta_2^\alpha - E(0) \\
&\geq \frac{1}{\alpha}\zeta_2^\alpha - g(\zeta_2) \\
&= \frac{B^{2(\rho+2)}}{2(\rho+2)}\zeta_2^{2(\rho+2)}.
\end{aligned} \tag{5.19}$$

Therefore, (5.19) and (5.14) yield the desired result. □

*Proof.* Proof of Theorem 5.5 We suppose that the solution exists for all time and set

$$H(t) = E_1 - E(t). \tag{5.20}$$

By using (5.6) and (5.20) we get

$$\begin{aligned}
H'(t) &= \|\nabla u_t\|_2^2 + \|\nabla v_t\|_2^2 + \|\nabla u_t\|_{\beta_1}^{\beta_1} + \|\nabla v_t\|_{\beta_2}^{\beta_2} \\
&\quad + a_1\|u_t\|_m^m + a_2\|v_t\|_r^r + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2.
\end{aligned}$$

From (5.13) , It is clear that for all $t \geq 0$, $H'(t) > 0$. Therefore , we have

$$\begin{aligned}
0 < H(0) \leq H(t) &= E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\
&\quad - \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\
&\quad + \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right].
\end{aligned} \tag{5.21}$$

From (5.6) and (5.16), we obtain, for all $t \geq 0$,

$$\begin{aligned}
&E_1 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) - \frac{1}{\alpha}\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\
&< E_1 - \frac{1}{\alpha}\zeta_1^\alpha = -\frac{1}{2(\rho+2)}\zeta_1^\alpha < 0.
\end{aligned}$$

Hence,

$$0 < H(0) \leq H(t) \leq \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right], \ \forall t \geq 0.$$

Then by (5.5), we have

$$0 < H(0) \leq H(t) \leq \frac{c_1}{2(\rho+2)}\left[\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right], \ \forall t \geq 0. \tag{5.22}$$

We then define

$$L(t) = H^{1-\sigma}(t) + \varepsilon \int_{\Omega} (uu_t + vv_t))dx, \tag{5.23}$$

for $\varepsilon$ small to be chosen later and

$$\begin{aligned} 0 \ < \ \sigma \leq \min\left\{\frac{1}{2}, \frac{\alpha-m}{2(\rho+2)(m-1)}, \frac{\alpha-r}{2(\rho+2)(r-1)},\right. \\ \left.\frac{(\alpha-2)}{2(\rho+2)}, \frac{\alpha-\beta_1}{2(\rho+2)(\beta_1-1)}, \frac{\alpha-\beta_2}{2(\rho+2)(\beta_2-1)}\right\}. \end{aligned} \tag{5.24}$$

Our goal is to show that $L(t)$ satisfies the differential inequality (25). Indeed, taking the derivative of (5.23), using (5.1) and adding subtracting $\varepsilon k H(t)$, we obtain

$$\begin{aligned} L'(t) \ = \ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon k H(t) \\ & +\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ & +\varepsilon(1-k)\int_{\Omega} F(u,v) - \varepsilon k E_1 \\ & -\varepsilon\int_{\Omega}\nabla u \nabla u_t dx - \varepsilon\int_{\Omega}\nabla v\nabla v_t dx \\ & +\varepsilon\left(\frac{k}{\alpha}-1\right)(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha) \\ & -\varepsilon\int_{\Omega}|\nabla u_t|^{\beta_1-2}\nabla u_t\nabla u dx - \varepsilon\int_{\Omega}|\nabla v_t|^{\beta_2-2}\nabla v_t \nabla v dx \\ & -\varepsilon a_1\int_{\Omega}|u_t|^{m-2}u_t u dx - \varepsilon a_2\int_{\Omega}|v_t|^{r-2}v_t v dx. \end{aligned} \tag{5.25}$$

We then exploit Young's inequality to get for $\mu_i, \lambda_i, \delta_i > 0$ $i = 1, 2$

$$\begin{aligned} \int_{\Omega}\nabla u\nabla u_t dx \leq \frac{1}{4\mu_1}\|\nabla u\|_2^2 + \mu_1\|\nabla u_t\|_2^2, \\ \int_{\Omega}\nabla v\nabla v_t dx \leq \frac{1}{4\mu_2}\|\nabla v\|_2^2 + \mu_2\|\nabla v_t\|_2^2, \end{aligned} \tag{5.26}$$

and

$$\int_{\Omega}|\nabla u_t|^{\beta_1-1}\nabla u dx \leq \frac{\lambda_1^{\beta_1}}{\beta_1}\|\nabla u\|_{\beta_1}^{\beta_1} + \frac{\beta_1-1}{\beta_1}\lambda_1^{-\beta_1/(\beta_1-1)}\|\nabla u_t\|_{\beta_1}^{\beta_1},$$

$$\int_{\Omega} |\nabla v_t|^{\beta_2 - 1} \nabla v dx \leq \frac{\lambda_2^{\beta_2}}{\beta_2} \|\nabla v\|_{\beta_2}^{\beta_2} + \frac{\beta_2 - 1}{\beta_2} \lambda_2^{-\beta_2/(\beta_2-1)} \|\nabla v_t\|_{\beta_1}^{\beta_1}, \tag{5.27}$$

and also

$$\int_{\Omega} |u_t|^{m-2} u_t u dx \leq \frac{\delta_1^m}{m} \|u\|_m^m + \frac{m-1}{m} \delta_1^{-m/(m-1)} \|u_t\|_m^m$$

$$\int_{\Omega} |v_t|^{r-2} v_t v dx \leq \frac{\delta_2^r}{r} \|v\|_r^r + \frac{r-1}{r} \delta_2^{-r/(r-1)} \|v_t\|_r^r. \tag{5.28}$$

A substitution of (5.26)-(5.28)) in (5.25) and using (5.5) yields

$$\begin{aligned} L'(t) \geq\ & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon k H(t) \\ & + \varepsilon \left(1 + \frac{k}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\ & + \varepsilon \left(\frac{c_0}{2(\rho+2)} - \frac{k c_1}{2(\rho+2)}\right) \left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) - \varepsilon k E_1 \\ & - \frac{\varepsilon}{4\mu_1} \|\nabla u\|_2^2 - \mu_1 \varepsilon \|\nabla u_t\|_2^2 - \frac{\varepsilon}{4\mu_2} \|\nabla v\|_2^2 - \varepsilon \mu_2 \|\nabla v_t\|_2^2 \\ & + \varepsilon \left(\frac{k}{\alpha} - 1\right) \left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\ & - \varepsilon \frac{\lambda_1^{\beta_1}}{\beta_1} \|\nabla u\|_{\beta_1}^{\beta_1} - \varepsilon \frac{\beta_1 - 1}{\beta_1} \lambda_1^{-\beta_1/(\beta_1-1)} \|\nabla u_t\|_{\beta_1}^{\beta_1} \\ & - \varepsilon \frac{\lambda_2^{\beta_2}}{\beta_2} \|\nabla v\|_{\beta_2}^{\beta_2} - \varepsilon \frac{\beta_2 - 1}{\beta_2} \lambda_2^{-\beta_2/(\beta_2-1)} \|\nabla v_t\|_{\beta_1}^{\beta_1} \\ & - a_1 \varepsilon \frac{\delta_1^m}{m} \|u\|_m^m - a_1 \varepsilon \frac{m-1}{m} \delta_1^{-m/(m-1)} \|u_t\|_m^m \\ & - a_2 \varepsilon \frac{\delta_2^r}{r} \|v\|_r^r - a_2 \varepsilon \frac{r-1}{r} \delta_2^{-r/(r-1)} \|v_t\|_m^m. \end{aligned} \tag{5.29}$$

Let us choose $\delta_1$, $\delta_2$, $\mu_1$, $\mu_2$, $\lambda_1$, and $\lambda_2$ such that

$$\begin{cases} \delta_1^{-m/(m-1)} = M_1 H^{-\sigma}(t) \\ \delta_2^{-r/(r-1)} = M_2 H^{-\sigma}(t) \\ \mu_1 = M_3 H^{-\sigma}(t) \\ \mu_2 = M_4 H^{-\sigma}(t) \\ \lambda_1^{-\beta_1/(\beta_1-1)} = M_5 H^{-\sigma}(t) \\ \lambda_2^{-\beta_2/(\beta_2-1)} = M_6 H^{-\sigma}(t), \end{cases} \tag{5.30}$$

for $M_1$, $M_2$, $M_3$, $M_4$, $M_5$ and $M_6$ large constants to be fixed later. Thus, by using (5.30), and for $M = M_3 + M_4 + (\beta_1 - 1)M_5/\beta_1 + (\beta_2 - 1)M_6/\beta_2 + (m-1)M_1/m + (r-1)M_2/r$,

then, inequality (5.29) takes the form

$$\begin{aligned} L'(t) \geq\ & ((1-\sigma) - \varepsilon M) H^{-\sigma}(t) H'(t) + \varepsilon k H(t) \\ & + \varepsilon\left(1 + \frac{k}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) \\ & + \varepsilon\left(\frac{c_0}{2(\rho+2)} - \frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) \\ & - \varepsilon k E_1 + \varepsilon\left(\frac{k}{\alpha} - 1\right)\left(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha\right) \\ & - \frac{\varepsilon}{4M_3} H^\sigma(t)\|\nabla u\|_2^2 - \frac{\varepsilon}{4M_4} H^\sigma(t)\|\nabla v\|_2^2 \\ & - \frac{a_1\varepsilon}{m} M_1^{-(m-1)} H^{\sigma(m-1)}(t)\|u\|_m^m \\ & - \frac{a_2\varepsilon}{r} M_2^{-(r-1)} H^{\sigma(r-1)}(t)\|v\|_r^r \\ & - \varepsilon\frac{M_5^{-(\beta_1-1)}}{\beta_1} H^{\sigma(\beta_1-1)}(t)\|\nabla u\|_{\beta_1}^{\beta_1} \\ & - \varepsilon\frac{M_6^{-(\beta_2-1)}}{\beta_2} H^{\sigma(\beta_2-1)}(t)\|\nabla u\|_{\beta_2}^{\beta_2}, \end{aligned} \tag{5.31}$$

we then use the two embedding

$$L^{2(\rho+2)}(\Omega) \hookrightarrow L^m(\Omega), W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega),$$

and (5.22) to get

$$\begin{aligned} H^{\sigma(m-1)}(t)\|u\|_m^m \leq\ & c_2(\|u\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)+m} \\ & + \|v\|_{2(\rho+2)}^{2\sigma(m-1)(\rho+2)}\|u\|_{2(\rho+2)}^m) \\ \leq\ & c_2(\|\nabla u\|_\alpha^{2\sigma(m-1)(\rho+2)+m} \\ & + \|\nabla v\|_\alpha^{2\sigma(m-1)(\rho+2)}\|\nabla u\|_\alpha^m). \end{aligned} \tag{5.32}$$

Similarly, the embedding $L^{2(\rho+2)}(\Omega) \hookrightarrow L^{r}(\Omega)$, $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$ and (5.22) give

$$\begin{aligned} H^{\sigma(r-1)}(t)\|v\|_r^r &\leq c_3(\|v\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)+r} \\ &\quad + \|u\|_{2(\rho+2)}^{2\sigma(r-1)(\rho+2)}\|v\|_{2(\rho+2)}^{r}) \\ &\leq c_3(\|\nabla v\|_{\alpha}^{2\sigma(r-1)(\rho+2)+r} \\ &\quad + \|\nabla u\|_{\alpha}^{2\sigma(r-1)(\rho+2)}\|\nabla v\|_{\alpha}^{r}). \end{aligned} \tag{5.33}$$

Furthermore, the two embedding $W_0^{1,\alpha} \hookrightarrow L^{2(\rho+2)}(\Omega)$, $L^{\alpha}(\Omega) \hookrightarrow L^{2}(\Omega)$, yields

$$\begin{aligned} H^{\sigma}(t)\|\nabla u\|_2^2 &\leq c_4\left(\|u\|_{2(\rho+2)}^{2\sigma(\rho+2)}\|\nabla u\|_2^2 + \|v\|_{2(\rho+2)}^{2\sigma(\rho+2)}\|\nabla u\|_2^2\right) \\ &\leq c_4\left(\|\nabla u\|_{\alpha}^{2\sigma(\rho+2)+2} + \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)}\|\nabla u\|_{\alpha}^2\right), \end{aligned} \tag{5.34}$$

and

$$\begin{aligned} H^{\sigma}(t)\|\nabla v\|_2^2 &\leq c_5\left(\|\nabla u\|_{\alpha}^{2\sigma(\rho+2)}\|\nabla v\|_{\alpha}^2 + \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)}\|\nabla v\|_{\alpha}^2\right) \\ &= c_5\left(\|\nabla u\|_{\alpha}^{2\sigma(\rho+2)}\|\nabla v\|_{\alpha}^2 + \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)+2}\right). \end{aligned} \tag{5.35}$$

Since $max(\beta_1, \beta_2) < \alpha$ then we have

$$\begin{aligned} H^{\sigma(\beta_1-1)}(t)\|\nabla u\|_{\beta_1}^{\beta_1} &\leq c_6(\|\nabla u\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_{\alpha}^{\beta_1} \\ &\quad + \|\nabla v\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_{\alpha}^{\beta_1}) \\ &= c_6(\|\nabla u\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)+\beta_1} \\ &\quad + \|\nabla v\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)}\|\nabla u\|_{\alpha}^{\beta_1}), \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} H^{\sigma(\beta_2-1)}(t)\|\nabla v\|_{\beta_2}^{\beta_2} &\leq c_7(\|\nabla u\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_{\alpha}^{\beta_2} \\ &\quad + \|\nabla v\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_{\alpha}^{\beta_2}) \\ &= c_7(\|\nabla u\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)}\|\nabla v\|_{\alpha}^{\beta_2} \\ &\quad + \|\nabla v\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)+\beta_2}), \end{aligned} \tag{5.37}$$

for some positive constants $c_2$, $c_3$, $c_4$, $c_5$, $c_6$ and $c_7$. By using (5.24) and the algebraic inequality

$$z^{\nu} \leq (z+1) \leq \left(1 + \frac{1}{a}\right)(z+a), \quad \forall z \geq 0,\ 0 < \nu \leq 1,\ a \geq 0, \tag{5.38}$$

we have, for all $t \geq 0$,

$$\begin{cases} \|\nabla u\|_{\alpha}^{2\sigma(m-1)(\rho+2)+m} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(0)\right) \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right), \\ \\ \|\nabla v\|_{\alpha}^{2\sigma(r-1)(\rho+2)+r} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \\ \\ \|\nabla u\|_{\alpha}^{2\sigma(\rho+2)+2} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right), \\ \\ \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)+2} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \\ \\ \|\nabla u\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)+\beta_1} \leq d\left(\|\nabla u\|_{\alpha}^{\alpha} + H(t)\right), \\ \\ \|\nabla v\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)+\beta_2} \leq d\left(\|\nabla v\|_{\alpha}^{\alpha} + H(t)\right), \end{cases} \tag{5.39}$$

where $d = 1 + 1/H(0)$.

Also keeping in mind the fact that $max(m, r) < \alpha$ , using Yong's inequality, the inequality (5.38) together withe (5.24), we conclude

$$\begin{cases} \|\nabla v\|_{\alpha}^{2\sigma(m-1)(\rho+2)} \|\nabla u\|_{\alpha}^{m} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha} + \|\nabla u\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla u\|_{\alpha}^{2\sigma(r-1)(\rho+2)} \|\nabla v\|_{\alpha}^{r} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla v\|_{\alpha}^{2\sigma(\rho+2)} \|\nabla u\|_{\alpha}^{2} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha} + \|\nabla u\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla u\|_{\alpha}^{2\sigma(\rho+2)} \|\nabla v\|_{\alpha}^{2} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla v\|_{\alpha}^{2\sigma(\beta_1-1)(\rho+2)} \|\nabla u\|_{\alpha}^{\beta_1} \leq C\left(\|\nabla v\|_{\alpha}^{\alpha} + \|\nabla u\|_{\alpha}^{\alpha}\right), \\ \\ \|\nabla u\|_{\alpha}^{2\sigma(\beta_2-1)(\rho+2)} \|\nabla v\|_{\alpha}^{\beta_2} \leq C\left(\|\nabla u\|_{\alpha}^{\alpha} + \|\nabla v\|_{\alpha}^{\alpha}\right), \end{cases} \tag{5.40}$$

where $C$ is a generic positive constant. Taking into account (5.32)- (5.40) , then, (5.31) takes the form

$$\begin{aligned} L^{'}(t) \geq & ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H^{'}(t) \\ & +\varepsilon\left(1+\frac{k}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\ & +\varepsilon([k/\alpha-1-kE_1\zeta_2^{-a}]-CM_1^{-(m-1)}-CM_2^{-(r-1)} \\ & -\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}-CM_5^{-(\beta_1-1)} \\ & -CM_6^{-(\beta_2-1)}-1)(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha) \\ & +\varepsilon\left(k-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}\right. \\ & \left.-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}\right)H(t) \\ & +\varepsilon\left(\frac{c_0}{2(\rho+2)}-\frac{kc_1}{2(\rho+2)}\right)\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \end{aligned} \tag{5.41}$$

for some constant $k$. Using $k = c_0/c_1$, we arrive at

$$\begin{aligned} L^{'}(t) \geq & ((1-\sigma)-\varepsilon M) H^{-\sigma}(t) H^{'}(t) \\ & +\varepsilon\left(1+\frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\ & +\varepsilon\left(\overline{c}-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}\right. \\ & \left.-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}-1\right)(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha) \\ & +\varepsilon\left(c_0/c_1-CM_1^{-(m-1)}-CM_2^{-(r-1)}-\frac{C}{4}M_3^{-1}-\frac{C}{4}M_4^{-1}\right. \\ & \left.-CM_5^{-(\beta_1-1)}-CM_6^{-(\beta_2-1)}\right)H(t), \end{aligned} \tag{5.42}$$

where $\overline{c} = k/\alpha - 1 - kE_1\zeta_2^{-2} = c_0/(c_1\alpha) - 1 - (c_0/c_1)E_1\zeta_2^{-2} > 0$ since $\zeta_2 > \zeta_1$.
At this point, and for large values of $M_1$, $M_2$, $M_3$, $M_4$, $M_5$ and $M_6$, we can find positive constants $\Lambda_1$ and $\Lambda_2$ such that (5.42) becomes

$$\begin{aligned} L^{'}(t) \geq & ((1-\sigma)-M\varepsilon) H^{-\sigma}(t) H^{'}(t) \\ & +\varepsilon\left(1+\frac{c_0}{2c_1}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\ & +\varepsilon\Lambda_1(\|\nabla u\|_\alpha^\alpha+\|\nabla v\|_\alpha^\alpha)+\varepsilon\Lambda_2 H(t). \end{aligned} \tag{5.43}$$

Once $M_1$, $M_2$, $M_3$, $M_4$, $M_5$ and $M_6$ are fixed (hence, $\Lambda_1$ and $\Lambda_2$), we pick $\varepsilon$ small enough so that $((1-\sigma)-M\varepsilon) \geq 0$ and

$$L(0) = H^{1-\sigma}(0) + \int_\Omega [u_0.u_t + v_0.v_t]\,dx > 0.$$

From these and (5.43) becomes

$$\begin{aligned} L^{'}(t) \geq\ & \varepsilon\Gamma(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 \\ & + \|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha). \end{aligned} \tag{5.44}$$

Thus, we have $L(t) \geq L(0) > 0$, for all $t \geq 0$. Next, by Holder's and Young's inequalities, we estimate

$$\begin{aligned} & \left(\int_\Omega u.u_t(x,t)\,dx + \int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ \leq\ & C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right) \\ \leq\ & C\left(\|\nabla u\|_\alpha^{\frac{\tau}{1-\sigma}} + \|u_t\|_2^{\frac{s}{1-\sigma}} + \|\nabla v\|_\alpha^{\frac{\tau}{1-\sigma}} + \|v_t\|_2^{\frac{s}{1-\sigma}}\right), \end{aligned} \tag{5.45}$$

for $\frac{1}{\tau} + \frac{1}{s} = 1$. We take $s = 2(1-\sigma)$, to get $\frac{\tau}{1-\sigma} = \frac{2}{1-2\sigma}$. By using (5.24) and (5.38) we get

$$\|\nabla u\|_\alpha^{\frac{2}{(1-2\sigma)}} \leq d\left(\|\nabla u\|_\alpha^\alpha + H(t)\right),$$

and

$$\|\nabla v\|_\alpha^{\frac{2}{(1-2\sigma)}} \leq d\left(\|\nabla v\|_\alpha^\alpha + H(t)\right), \qquad \forall t \geq 0.$$

Therefore, (5.45) becomes

$$\begin{aligned} & \left(\int_\Omega u.u_t(x,t)\,dx + \int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ & \leq C(\|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha + \|u_t\|_2^2 + \|v_t\|_2^2 \\ & \quad + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 + H(t)), \qquad \forall t \geq 0. \end{aligned} \tag{5.46}$$

Also, since

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) =\ & \left(H^{1-\sigma}(t) + \varepsilon\int_\Omega (u.u_t + v.v_t)(x,t)\,dx\right)^{\frac{1}{(1-\sigma)}} \\ \leq\ & C\left(H(t) + \left|\int_\Omega (u.u_t(x,t) + v.v_t(x,t))\,dx\right|^{\frac{1}{(1-\sigma)}}\right) \\ \leq\ & C[H(t) + \|\nabla u\|_\alpha^\alpha + \|\nabla v\|_\alpha^\alpha + \|u_t\|_2^2 + \|v_t\|_2^2 \\ & + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2], \qquad \forall t \geq 0, \end{aligned} \tag{5.47}$$

combining withe (5.47) and (5.44), we arrive at

$$L^{'}(t) \geq a_0 L^{\frac{1}{1-\sigma}}(t), \qquad \forall t \geq 0. \tag{5.48}$$

Finally, a simple integration of (5.48) gives the desired result. This completes the proof of Theorem (5.5) □

**Conclusion**

In this thesis research, our main scientific contributions focused on the existence/nonexistence and uniqueness of solutions and also the stability of solutions for various classes of initial value problem and boundary value problem for some elastic, thermoelastic and viscoelastic systems with the presence of different mechanisms of dissipation damping as well as we studied different types of Timoshenko, Bresse and Porous systems, we have shown in chapter three the interest of a time-varing delay term in the internal feedback and in chapter four the interest of nonlinear damping term, this results are based on the argument of the semi-group theory and energy method. For the perspective and the possible generalization, it would be interesting to extend the results by the numerical method to solve these problems. These suggestions will be treated in the future.

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