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THEME

Existence and stability of some fractional differential equations in the framework of Ulam–Hyers–Rassias

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شكر وعرفان

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*والي خطيبتي وسندي والرفيقة الدائمة التي تشاركني كل تفاصيل حياتي **أية نبق** .

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الى زمالئي و زميالتي الذين ساندوني في مشواري الدراسي في الا بتدائي المتوسط الثانوي و الجامعي

* الى جميع الأساتذة الذين انارو لي دربي خلال سنوات دراستي

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الملخص

الهدف من هذه الأطروحة هو دراسة بعض الخصائص النوعية لحلول فئات مختلفة من المعادلات والانشاءات التفاضلية الكسرية غير الخطية التي تتضمن أنواع مختلفة من المشتقات الكسرية. لهذا الهدف، نقوم بتحويل المشكل المعطي إلى معادلة تكاملية مكافئة ثم تستخدم نظريات النقطة الثابتة المناسبة بحيث تكون النقطة الثابتة التي تم الحصول عليها هي حلول للمشكل المعطي. نقوم أيضا مثلا توضيحيا لكل مشكل مدرس لإظهار فعالية النتائج النظرية.

الكلمات المفتاحية:
المعادلات التفاضلية الكسرية، المعادلات التفاضلية الكسرية الهجينة، المعادلات التكاملية التفاضلية الكسرية، المشتق الكسري، المشكل الحديثة، فضاء بناءً النقطة الثابتة، قياس كوراتوشمكي لعدم الترامس، الوحيد، الوحدانية، استقرأ ولايم.وفي دراسة مسائل التحكم الأمثل .

Abstract

The objective of this thesis is to study some qualitative properties of solutions for various classes of nonlinear fractional , we begin by convert the problem umber considered into an equivalent integral equation and using the fixed point theorem . We finish by given example for each problem consider.

Keywords: Fractional differential equations, hybrid fractional differential equations, fractional integro-differential equations, fractional differential inclusions, fractional derivatives, Banach space, fixed point.

Résumé

L'objectif de cette mémoir est d'étudier certaines propriétés qualitatives des solutions de diverses classes d'équations différentielles fractionnaires non linéaires. on convertissons par couvertir problème à une équation intégrale et en utilisons le théorème de points fixe.on trmine de domner un exemple pour chaque probléme considéré.

Mots-clés : Équations différentielles fractionnaires, équations différentielles fractionnaires fractionnaires, inclusions , problèmes aux limites, espace de Banach, point fixe, mesure de non-compacité de Kuratowski, existence, unicité, stabilité d'Ulam.

List of Symbols

$\Gamma(z)$	Gamma function
$B(z,w)$	Beta function
$E_\alpha(z)$	Mittag-Leffler function
$E_{\alpha,\beta}(z)$	Two-parameter Mittag-Leffler function
$C(J,\mathbb{R})$	Space of continuous functions on J
$L^p(J,\mathbb{R})$	Lebesgue integrable functions on J
$AC(J,\mathbb{R})$	Absolutely continuous functions on J
RLI^α_{a+}	Riemann-Liouville fractional integral (left)
RLD^α_{a+}	Riemann-Liouville fractional derivative (left)
CD^α_{a+}	Caputo fractional derivative (left)
HI^α_{a+}	Hadamard fractional integral
HD^α_{a+}	Hadamard fractional derivative
CHD^α_{a+}	Caputo-Hadamard fractional derivative
$I^{\alpha;\psi}_{a+}$	ψ-fractional integral
$RLD^{\alpha,\psi}_{a+}$	ψ-Riemann-Liouville derivative
$CD^{\alpha,\psi}_{a+}$	ψ-Caputo derivative
$HD^{\alpha,\beta;\psi}_{a+}$	ψ-Hilfer derivative
$\varphi_p(u)$	p-Laplacian operator : $\|u\|^{p-2}u$
$\mu_k(Q)$	Kuratowski measure of noncompactness
$Fix\, F$	Fixed point set of F
$S_{F,y}$	Selections of multivalued map F at point y
$P_{cl}(X)$	Set of closed subsets of X
I^α_{a+}	Riemann–Liouville fractional integral
$EKI_{\eta,\alpha,a+,\delta}$	Erdélyi–Kober fractional integral
$CD^\alpha y(t)$	Caputo fractional derivative of function $y(t)$
$I^\alpha CD^\alpha f(t)$	Identity for Caputo derivative via integral
$\theta_f(t)$	Perturbation function for Ulam-type stability

Contents

Introduction

In recent decades, the study of fractional differential equations has gained significant attention due to their ability to model various complex phenomena in physics, engineering, and biology more accurately than classical differential equations. These equations, which involve derivatives of arbitrary (non-integer) order, offer a powerful framework for describing systems with memory and hereditary properties.

Among the various tools employed in the study of such systems, the theory of fractional calculus has proven particularly useful. This theory extends the concepts of integrals and derivatives to non-integer orders, thereby allowing for more flexible and realistic mathematical models.

A central aspect of this memory is the investigation of fractional differential equations involving non-linear operators, such as the p-Laplacian operator, which arises naturally in nonlinear elasticity, fluid mechanics, and non-Newtonian fluid theory. In addition, the memory incorporates the concept of Ulam stability, which concerns the stability of solutions under small perturbations and is essential in ensuring the reliability of mathematical models.

This work is also concerned with the existence and uniqueness of solutions to certain classes of fractional differential equations. These analytical results are often obtained using fixed point theorems and functional analysis techniques, including measures of noncompactness and multivalued mappings.

The main objectives of this memory are to present new existence and stability results for mixed-type and implicit fractional differential equations by applying and extending existing mathematical tools. Particular emphasis is placed on demonstrating Ulam-type stability, which contributes to a deeper understanding of the robustness of the studied systems.

The fractional calculus is a field of mathematical analysis that embraces the integrals and derivatives of functions of any real or complex order. For the past few decades, this field has been one of the handover fist sprawling fields of mathematics by the virtue of the amazing findings obtained when researchers enrolled the fractional operators in their attempts to construe some problems that arise in the nature. See [7, 13, 3, 4, 1, 23].

At the beginning of the fractional calculus in 1695 [12], it was consisted of one main integral operator, namely the Riemann-Liouville fractional integral and two fractional derivatives, namely the Riemann-Liouville and Caputo derivatives. Because of penurious number of operators, researchers were compelled to discover and develop new fractional operators that allow them better comprehend the world around them. In this purpose, new derivatives and fractional integrals has been arising. The kernel of these integrals and fractional derivatives differs, resulting in a large number of definitions, see [21, 23, 10, 18, 8, 14].

Due to the large number of integral and fractional derivative definitions, it was necessary to create a fractional derivative of a function f with respect to another function, which is called ψ-Riemann-Liouville, using the fractional derivative in the Riemann-Liouville sense, is given by [12]

$$^{RL}D_{a^+}^{\alpha,\psi}f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a^+}^{n-\alpha,\psi}f(t),$$

where $\alpha \in (n-1, n)$, $n = [\alpha]+1$ for $\alpha \notin \mathbb{N}$ and $n = \alpha$ for $\alpha \in \mathbb{N}$. However, such a definition only encompasses the possible fractional derivatives that contain the differentiation operator acting on the integral operator. On the other hand, In subsection 1.3.2, We will mention a corresponding fractional integral which generalized the Riemann-Liouville fractional integrals and some special cases of this operator.

In the same way, recently, Almeida [9] using the idea of the fractional derivative in the Caputo sense, proposes a new fractional derivative called ψ-Caputo derivative with respect to another function ψ, which generalizes a class of fractional derivatives, whose definition is given by

$$^{C}D_{\alpha+}^{\alpha,\psi}f\left(t\right)=I_{\alpha+}^{(n-\alpha),\psi}f^{[n]}\left(t\right),$$

where $\alpha\in(n-1,n)$, $n=[\alpha]+1$ for $\alpha\notin\mathbb{N}$ and $n=\alpha$ for $\alpha\in\mathbb{N}$.

Despite that the ψ-Riemann-Liouville and ψ-Caputo definitions of fractional derivatives are very broad, there is the possibility of proposing a fractional differentiable operator that combines these operators and overcomes the wide range of definitions. Motivated by the Hilfer [39] fractional derivative definition, which includes the classical Riemann-Liouville and Caputo fractional derivatives as special cases. Depending on ψ-Riemann-Liouville and ψ-Caputo fractional derivatives in Hilfer's sense of definition, Sousa and Oliviera [89] introduced a new fractional derivative of a function with respect to another ψ function so-called ψ-Hilfer derivative. Which unify a wide class of fractional derivatives. The definition of ψ-Hilfer fractional derivative, its relation with the ψ-Riemann-Liouville fractional integral and some special cases of this derivative are will presented in subsection 1.3.4.

The advantage of the fractional operator ψ-Hilfer proposed here is the freedom of choice of the classical differentiation operator and the choice of the function ψ, i.e., from the choice of the function ψ, the operator of classical differentiation, can act on the fractional integration operator or else the fractional integration operator can act on the classical differentiation operator. As a result, the properties of the two fractional operators mentioned above can be unified and obtained.

this memry is amanged follws : In chapter 1, we collect definition auniliary resuts . chapter 2 , is devorted

Chapitre 1

Preliminaries and Background Materials

In this chapter, we introduce the necessary concepts for the good understanding of this memory. We present some fundamental notions, definitions, and lemmas related to fractional calculus, measures of noncompactness, multivalued analysis, Ulam stability and some fixed point theorems which play an important role in the achievement of the desired results in this memory.

1.1 Functional spaces

Let $J = [a, b]$, the compact intervals of $\mathbb{R}$. We present the following functional spaces :

Definition 1.1 *Denote by $C(J, \mathbb{R})$ the Banach space of all continuous functions $f : J \to \mathbb{R}$ endowed with the norm*

$$\|f\|_\infty = \sup\{|f(t)| : t \in J\},$$

and $C^n(J, \mathbb{R})$ denotes the class of all real valued functions defined on J which have a continuous nth order derivative.

Definition 1.2 *Denote by $L^1(J, \mathbb{R})$ the Banach space of measurable functions $f : J \to \mathbb{R})thatareLebesgueintegr$*

$$\|f\|_{L^1} = \int_J |f(t)|dt,$$

and by $L^p(J, \mathbb{R})$ we denote the space of Lebesgue integrable functions in J where $|f|^p$ belongs to $L^1(J, \mathbb{R})$ endowed with norm

$$\|f\|_{L^p} = \left(\int_J |f(t)|^p dt\right)^{\frac{1}{p}}.$$

Definition 1.3 ***A** function $f : J \to \mathbb{R}$ is said absolutely continuous on J if for all $\epsilon > 0$ there exists a number δ_ϵ such that ; for all finite partition $[a_i, b_i]$ in J, then*

$$\sum_{i=1}^{p}(b_i - a_i) < \delta_\epsilon.$$

implies that

$$\sum_{i=1}^{p} |f(b_i) - f(a_i)| < \epsilon.$$

Definition 1.4 *Let $AC(J,\mathbb{R})$ be the space of absolutely continuous functions on J. For $n \in \mathbb{N}$, we denote by $AC^n(J,\mathbb{R})$ the space of functions $f : J \to \mathbb{R}$ which have continuous derivatives up to order $n-1$ on J such that $f^{(n-1)} \in AC(J,\mathbb{R})$, defined by*

$$AC^n(J,\mathbb{R}) = \left\{ f : J \to \mathbb{R} \,|\, f, f', f'', \ldots, f^{n-1} \in C(J,\mathbb{R}),\ f^{n-1} \in AC(J,\mathbb{R}) \right\}.$$

for more details $AC(J,\mathbb{R})$ and $AC^n(J,\mathbb{R})$,see the book of kolmogorov and fomin .

1.2 Special functions

In what follows, we recall three types of functions that are important in fractional calculus : the Gamma, Beta, and Mittag-Leffler functions. More details about these functions can be found in [21, 15, 5].

1.2.1 Gamma Function

Definition 1.5 ***(Gamma function [11])*** *The Gamma function, denoted by $\Gamma(z)$, is a generalization of the factorial function $n!$, i.e., $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. For complex arguments with positive real part it is defined as*

$$\Gamma(z) = \int_0^{\infty} t^{z-1} exp(-t)\, dt, \quad \Re(z) > 0.$$

By analytic continuation the function is extended to the whole complex plane except for the points $0, -1, -2, -3, \ldots$, where it has simple poles. Thus,

$$\Gamma : \mathbb{C} \setminus \{0, -1, -2, \ldots\} \to \mathbb{C}.$$

Some of the most known properties are :

$$\Gamma(1) = \Gamma(2) = 1, \quad \Gamma(z+1) = z\Gamma(z),$$

$$\Gamma(n) = (n-1)!, \quad n \in \mathbb{N}, \quad \Gamma\left(\frac{1}{2}\right) = \pi. \tag{1.1}$$

The Gamma function is studied by many mathematicians. There is a long list of well-known properties (see, for example, [22]), but in this survey, formulas (1.1) are sufficient.

1.2.2 Beta Function

Definition 1.6 ***(Beta function [11])*** *The Beta function is defined by the integral*

$$B(z,w)=\int_0^1 t^{z-1}(1-t)^{w-1}dt, \quad \Re(z)>0, \quad \Re(w)>0.$$

The functions $\Gamma(\cdot)$ and $B(\ldots)$ are related by the formula

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

To demonstrate this relationship, we use the Laplace transform, see.

1.2.3 Mittag-Leffler Function

Definition 1.7 ***(Mittag-Leffler function [11])*** *The Mittag-Leffler function in one parameter is defined by*

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+1)}, \quad \alpha>0, \ z\in\mathbb{C},$$

where it was introduced by Mittag-Leffler [11].

The two-parameter Mittag-Leffler function is defined by the series expansion

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \quad \alpha, \ \beta>0, \ z\in\mathbb{C},$$

which is of great importance for the fractional calculus. In particular,

$$E_{1,1}(z)=\exp(z), \quad E_{2,1}(z)=\cosh(\sqrt{z}), \quad E_{\alpha,1}(z)=E_\alpha(z).$$

1.3 Elements from fractional calculus theory

There are several definitions in fractional calculus that are widely used and important in showing different fractional calculus outcomes. In this section, we will present some definitions of classical fractional integrals and fractional derivatives and their properties. Next, we introduce a new class of fractional integrals and fractional derivatives, because there are so many different fractional operator definitions. The following definition is a special approach when the kernel is unknown, involving a function ψ, making this new operator a generalization of the fractional operators that we use throughout this thesis.

1.3.1 Fractional integrals and fractional derivatives

Let $J=[a,b]$, $(-\infty<a<b<\infty)$, be a finite interval on $\mathbb{R}$. In this subsection, we present some definitions of classical fractional integrals, fractional derivatives, and its properties.

Definition 1.8 ***[Cauchy formula [6]]*** *The Cauchy formula of n-th integral of a locally integrable function f on $\mathbb{R}^+$ is given by*

$$I^n f(t)=\frac{1}{(n-1)!}\int_0^t (t-s)^{n-1} f(s)\,ds.$$

Definition 1.9 ***[Riemann-Liouville fractional integral [21]]*** *for $\alpha>0$. The left-side (right-side resp.) of Riemann-Liouville fractional integral of the function $f\in L^1(J,\mathbb{R})$ of order α is defined by*

$$^{RL}I^{\alpha}_{a+}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)\,ds, \quad t\in J,$$

$$^{RL}I^{\alpha}_{b-}f(t)=\frac{1}{\Gamma(\alpha)}\int_t^b (s-t)^{\alpha-1}f(s)\,ds, \quad t\in J,$$

resp., where $t\in J$.

Riemann-Liouville fractional derivatives are defined depending on their fractional integral and integer order derivative as follows.

Definition 1.10 ***[Riemann-Liouville fractional derivative [16]]*** *For $\alpha>0$. The left-side (right-side resp.) of Riemann-Liouville fractional order derivative of order α of $f\in L^1(J,\mathbb{R})$, is given by*

$$^{RL}D^{\alpha}_{a+}f(t) = \left(\tfrac{d}{dt}\right)^n\left(^{RL}I^{n-\alpha}_{a,t}f(t)\right)=\tfrac{1}{\Gamma(n-\alpha)}\left(\tfrac{d}{dt}\right)^n\int_a^t (t-s)^{n-\alpha-1} f(s)\,ds,$$

$$^{RL}D^{\alpha}_{b-}f(t) = \left(-\tfrac{d}{dt}\right)^n\left(^{RL}I^{n-\alpha}_{t,b}f(t)\right)=\tfrac{1}{\Gamma(n-\alpha)}\left(-\tfrac{d}{dt}\right)^n\int_t^b (s-t)^{n-\alpha-1} f(s)\,ds,$$

resp., where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of real number α.

Definition 1.11 ***Caputo fractional derivative [16]*** *For $\alpha > 0$. The left-side (right-side resp.) of Caputo fractional order derivative of order α of $f \in AC^n(J,\mathbb{R})$, is defined by*

$$\begin{aligned} {}^C D^{\alpha}_{a+} f(t) &= {}^{RL} I^{n-\alpha}_{a+}\left(f^{(n)}(t)\right) = \tfrac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} f^{(n)}(s)\, ds,\ t \in J, \\ {}^C D^{\alpha}_{b-} f(t) &= {}^{RL} I^{n-\alpha}_{b-}\left(f^{(n)}(t)\right) = \tfrac{1}{\Gamma(n-\alpha)} \int_t^b (s-t)^{n-\alpha-1} f^{(n)}(s)\, ds,\ t \in J, \end{aligned}$$

resp., where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of real number α.

In what follows, we consider some properties of the Riemann-Liouville and Caputo fractional integral and derivatives. In particular, we are interested by the left-side fractional derivatives and integrals.

Lemma 1.1 *(Relation between Riemann-Liouville and Caputo derivatives [2])*
Let $\alpha \in (n-1, n]$. If the function $f \in C^n(J)$, then

$$^C D^{\alpha}_{a+} f(t) = {}^{RL} D^{\alpha}_{a+} f(t) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha}.$$

Lemma 1.2 *([2]) For $\alpha, \beta > 0$ and $f \in L^1(J)$. Then, we have*

1. *The integral operator ${}^{RL} I^{\alpha}_{a+}$ is linear,*
2. ${}^{RL} I^{\alpha}_{a+}\, {}^{RL} I^{\beta}_{a+} f(t) = {}^{RL} I^{\beta}_{a+}\, {}^{RL} I^{\alpha}_{a+} f(t) = {}^{RL} I^{\alpha+\beta}_{a+} f(t)$,
3. ${}^{RL} D^{\alpha}_{a+}\, {}^{RL} I^{\alpha}_{a+} f(t) = f(t)$,
4. ${}^{RL} D^{\beta}_{a+}\, {}^{RL} I^{\alpha}_{a+} f(t) = {}^{RL} I^{\alpha-\beta}_{a+} f(t)$.

Lemma 1.3 *([2]) For $\alpha \geq 0$ and $\beta > 0$, we have*

$$\begin{aligned} ({}^{RL} I^{\alpha}_{a+} (t-a)^{\beta-1})(x) &= \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (x-a)^{\beta+\alpha-1}, \quad \alpha > 0, \\ ({}^{RL} D^{\alpha}_{a+} (t-a)^{\beta-1})(x) &= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (t-a)^{\beta-\alpha-1}, \quad \alpha \geq 0. \end{aligned}$$

Lemma 1.4 *([2]) Let $n-1 < \alpha \leq n$, $n \in \mathbb{N}$. and $f \in C(J)$, then the Riemann-Liouville fractional differential equation*

$${}^{RL} D^{\alpha}_{a+} f(t) = 0,$$

has a general solution

$$f(t) = c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n}, \quad c_i \in \mathbb{R}, \quad i = 0,1,2,\ldots,n.$$

From the above lemma, it follows that

$${}^{RL} I^{\alpha}_{a+}\, {}^{RL} D^{\alpha}_{a+} f(t) = f(t) - c_1(t-a)^{\alpha-1} - c_2(t-a)^{\alpha-2} - \cdots - c_n(t-a)^{\alpha-n} \quad c_i \in \mathbb{R}, \quad i = 1,2,\ldots,n.$$

Lemma 1.5 *([2]) Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$. If $f\in AC^n(J)$, then the Caputo fractional differential equation*

$$^{C}D^{\alpha}_{a+}f(t)=0,$$

has a general solution

$$f(t)=c_0+c_1(t-a)+c_2(t-a)^2+\cdots+c_{n-1}(t-a)^{n-1}.$$

From the above lemma, it follows that

$$^{RL}I^{\alpha}_{0+}\,{}^{C}D^{\alpha}_{0+}f(t)=f(t)-c_0-c_1(t-a)-c_2(t-a)^2-\ldots-c_{n-1}(t-a)^{n-1},$$

for some $c_i\in\mathbb{R}$, $i=0,1,2,\ldots,n-1$.

Definition 1.12 ***[Hadamard fractional integral[2]]*** *Let $a>0$. The Hadamard fractional integral of order $\alpha>0$ for a function $f\in L^1(J,\mathbb{R})$ is defined as*

$$^{H}\Im^{\alpha}_{a+}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{a-1}f(s)\frac{ds}{s},\quad t\in J.$$

Set $\delta=(t\frac{d}{dt})$, $a,\alpha>0$, $n=[\alpha]+1$, where α denotes the integer part of α. Define the space

$$AC^n_{\delta}(J,\mathbb{R})=\left\{f:J\to\mathbb{R}:\delta^{n-1}f(t)\in AC(J,\mathbb{R})\right\}.$$

Definition 1.13 ***[Hadamard fractional derivative[2]]*** *Let $a>0$. The Hadamard fractional derivative of order $\alpha>0$ for a function $f\in AC^n_{\delta}(J,\mathbb{R})$ is defined as*

$$^{H}\mathfrak{D}^{\alpha}_{a+}f(t)=\delta^n\left(^{H}\mathfrak{I}^{n-\alpha}_{a+}f\right)(t)=\frac{1}{\Gamma(n-\alpha)}\left(t\frac{d}{dt}\right)^n\int_a^t\left(\log\frac{t}{s}\right)^{n-a-1}f(s)\frac{ds}{s}.$$

Definition 1.14 ***[Caputo-Hadamard fractional derivative[2, 7]]*** *Let $a>0$. The Caputo-Hadamard fractional derivative of order $\alpha>0$ for a function $f\in AC^n_{\delta}(J,\mathbb{R})$ is defined as*

$$^{C}_{H}\mathfrak{D}^{\alpha}_{a+}f(t)=\left(^{H}\mathfrak{I}^{n-\alpha}_{a+}\delta^n f\right)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\left(\log\frac{t}{s}\right)^{n-a-1}\delta^n f(s)\frac{ds}{s}.$$

Lemma 1.6 *([2, 7]) Let $\alpha,\beta>0$ and $n=[\alpha]+1$. Then, we have*

1. *The integral operator $^{H}\mathfrak{I}^{\alpha}_{a+}$ is linear,*
2. *$^{H}\mathfrak{I}^{\alpha}_{a+}(\log t)^{\beta-1}(x)=\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(\log\frac{x}{a})^{\beta+\alpha-1}$,*
3. *$^{C}_{H}\mathcal{D}^{\alpha}_{a+}(\log t)^{\beta-1}(x)=\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}(\log\frac{x}{a})^{\beta-\alpha-1}$, $\quad\beta>n$,*
4. *$^{C}_{H}\mathcal{D}^{\alpha}_{a+}(\log t)^{k}=0$, $\quad k=0,1,...,n-1$.*

Lemma 1.7 *([2, 7]) Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$, the general solution of the fractional differential equation*

$$^{H}\mathfrak{D}_{a+}^{\alpha}f\left(t\right)=0,$$

is given by

$$f\left(t\right)=\sum_{k=1}^{n}c_{k}\left(\log\frac{t}{a}\right)^{\alpha-k},$$

where $c_k\in\mathbb{R}$, $k=1,2,\ldots,n$ are arbitrary constants.

From the above lemma, it follows that

$$^{H}\mathfrak{J}_{a+}^{\alpha}\,{}^{H}\mathfrak{D}_{a+}^{\alpha}f\left(t\right)=f\left(t\right)-\sum_{k=1}^{n}c_{k}\left(\log\frac{t}{a}\right)^{\alpha-k},$$

for some $c_k\in\mathbb{R}$, $k=1,2,\ldots,n$ are arbitrary constants.

Lemma 1.8 *Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$. If $f\in AC_{\delta}^{n}\left(J,\mathbb{R}\right)$, then the Caputo-Hadamard fractional differential equation*

$${}_{H}^{C}\mathfrak{D}_{a+}^{\alpha}f\left(t\right)=0,$$

has a solution

$$f\left(t\right)=\sum_{k=0}^{n-1}c_{k}\left(\log\frac{t}{a}\right)^{k},$$

and the following formula holds

$$^{H}\mathfrak{I}_{a+}^{\alpha}\left({}_{H}^{C}\mathfrak{D}_{a+}^{\alpha}f\left(t\right)\right)=f(t)-\sum_{k=0}^{n-1}c_{k}\left(\log\frac{t}{a}\right)^{k},$$

where $c_k\in\mathbb{R}$, $k=0,1,2,\ldots,n-1$.

1.3.2 Fractional ψ-integral

Definition 1.15 *([23]) Let (a,b), $(-\infty\leq a<b\leq\infty)$ be a finite or infinite interval of the real line $\mathbb{R}$ and $a>0$. Also let $\psi\left(t\right)$ be an increasing and positive monotone function on (a,b), having a continuous derivative $\psi'\left(t\right)$ on (a,b). The left sided fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by*

$$I_{a+}^{\alpha;\psi}f\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{a}^{t}\psi'\left(s\right)\left(\psi\left(t\right)-\psi\left(s\right)\right)^{\alpha-1}f\left(s\right)ds. \tag{1.2}$$

Lemma 1.9 *([7, 23]) Let $\alpha, \beta, \delta > 0$. Then, the left-sided ψ-fractional integral satisfies the following properties :*

1. *The integral operator $I_{a+}^{\alpha;\psi}$ is linear,*
2. *The semigroup property of the fractional integration operator $I_{a+}^{\alpha;\psi}$ is given by the following result*

$$I_{a+}^{\alpha;\psi} I_{a+}^{\beta;\psi} f(t) = I_{a+}^{\alpha+\beta,\psi} f(t),$$

   holds almost everywhere if $f \in L^1(J, \mathbb{R})$.

3. *Commutativity*

$$I_{a+}^{\alpha;\psi} \left(I_{a+}^{\beta;\psi} f(t) \right) = I_{\alpha+}^{\beta;\psi} \left(I_{a+}^{\alpha;\psi} f(t) \right).$$

Lemma 1.10 *([23]) Let $\alpha, \beta > 0$. Then*

$$I_{a+}^{\alpha;\psi} (\psi(t) - \psi(a))^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} (\psi(t) - \psi(a))^{\alpha+\beta-1}.$$

The fractional integral operator with respect to another function defined in (1.2) is a general operator, in the sense that it is enough to choose a function ψ and obtain an existing fractional integral operator. In the following, we present a class of fractional integrals, based on the choice of the arbitrary ψ function.

1. *Choosing $\psi(t) = t$ and replacing in equation (1.2), we get*

$$I_{a+}^{\alpha;t} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds = {}^{RL}I_{a+}^{\alpha} f(t),$$

   the Riemann-Liouville fractional integral.

2. *If we consider $\psi(t) = \log(t)$ and $a > 0$ in equation (1.2), we have*

$$I_{a+}^{\alpha;t} h(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{1}{s} (\log t - \log s)^{\alpha-1} f(s)\, ds$$

$$= \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s} \right)^{a-1} f(s) \frac{ds}{s} = {}^{H}\mathfrak{I}_{a+}^{\alpha} f(t),$$

   the Hadamard fractional integral.

3. Choosing 1 $\psi(t) = t^{\delta}$ *and* $g(t) = t^{m} f(t)$ *and substituting in equation (1.2), we get*

$$t^{-\delta(\alpha+\eta)} I_{a+}^{\alpha,\psi} g(t) = t^{-\delta(\alpha+n)} I_{a+}^{\alpha;t^{\delta}} + t^{\alpha\eta} f(t)$$
$$= \frac{\delta t^{-\delta(\alpha+\eta)}}{\Gamma(\alpha)} \int_a^t s^{\delta\eta+\delta-1} \left(t^{\delta} - s^{\delta}\right)^{\alpha-1} f(s)\, ds$$
$$= {}^{EK}I_{a+,\delta}^{\eta,\alpha} f(t),$$

the Erdelyi-Kober fractional integral.

1.3.3 Fractional ψ-derivative

We start by evoking two definitions of fractional derivatives with respect to another function, both of which are motivated by the fractional derivatives of Riemann-Liouville and Caputo, in that order, choosing a specific function ψ.

Definition 1.16 *([7]) Let* $\psi'(t) \neq 0$ $(-\infty \leq a < t < b \leq \infty)$ *and* $\alpha > 0$, $n \in \mathbb{N}$. *The Riemann-Liouville derivative of a function* f *with respect to* ψ *of order* α *correspondent to the Riemann-Liouville, is defined by*

$$^{RL}D_{a+}^{\alpha,\psi} f(t) = \left(\frac{1}{\psi'(l)} \frac{d}{dt}\right)^n I_{a+}^{(n-\alpha),\psi} f(t),$$
$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n \int_a^t \psi'(s) (\psi(t) - \psi(s))^{n-\alpha-1} f(s)\, ds,$$

where $n = \lceil \alpha \rceil$ *and* $\lceil \alpha \rceil$ *denotes the smallest integer greater than or equal to* α.

Definition 1.17 *([18]) Let* $\alpha > 0$, $n \in \mathbb{N}$, $J = [a, b]$ *is the interval* $-\infty \leq a < b \leq \infty$, $f, \psi \in C^n(J, \mathbb{R})$ *two functions such that* ψ *is increasing and* $\psi'(t) \neq 0$, *for any* $t \in J$. *The left sided* ψ*-Caputo fractional derivative of a function* f *of order* α *is given by*

$$^{C}D_{a+}^{\alpha,\psi} f(t) = I_{a+}^{(n-\alpha),\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt}\right)^n f(t).$$

Lemma 1.11 *([10]) Let* $\alpha, \beta > 0$. *Then*

$$^{C}D_{a+}^{\alpha,\psi} (\psi(t) - \psi(a))^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\beta\alpha)} (\psi(t) - \psi(a))^{\alpha+\beta-1}.$$

Lemma 1.12 *([10]) If* $f \in C^n(J, \mathbb{R})$ *and* $\alpha \in (n-1, n)$, *then*

$$I_{a+}^{\alpha,\psi}\, {}^{C}D_{a+}^{\alpha,\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{[k]}(a^+)}{k!} (\psi(t) - \psi(a))^k.$$

In particular, given $\alpha \in (0,1)$, we have

$$I_{a+}^{\alpha,\psi}\, {}^{C}D_{a+}^{\alpha,\psi} f(t) = f(t) - f(a).$$

1.3.4 Fractional ψ-Hilfer derivative

From the definition of fractional derivative in the Riemann-Liouville sense and the Caputo sense, was introduced the Hilfer fractional derivative, which combines both derivatives. Motivated by the definition of Hilfer, we present a new generalized operator the so-called ψ-Hilfer fractional derivative of a function f with respect to another function. From the fractional derivative ψ-Hilfer, we introduce some relations between the ψ-fractional integral and the fractional derivative ψ-Hilfer.

Definition 1.18 *([10]) Let $\alpha \in (n-1,n)$ with $n \in \mathbb{N}$, $J=[a,b]$ is the interval $-\infty \leq a < b \leq \infty$, $f,\psi \in C^{n}(J,\mathbb{R})$ two functions such that ψ is increasing and $\psi'(t) \neq 0$, for any $t \in J$. The left sided ψ-Hilfer fractional derivative ${}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} f(t)$ of function f of order α and type $\beta \in [0,1]$ is defined by*

$$ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} f(t) = I_{a+}^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(t), \ n \in \mathbb{N}. \tag{1.3}$$

Lemma 1.13 *([10]) For $\delta > 0$, $\alpha \in (n-1,n)$ and $\beta \in [0,1]$, we have*

$$ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} (\psi(t)-\psi(a))^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta-\alpha)} (\psi(t)-\psi(a))^{\delta-\alpha-1}, \ \delta > n.$$

Lemma 1.14 *([10]) In particular, given $n \leq k \in \mathbb{N}$ and as $\delta > n$ we have*

$$ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} (\psi(t)-\psi(a))^{k} = \frac{k!}{\Gamma(k+1-\alpha)} (\psi(t)-\psi(a))^{k-\alpha}.$$

On the other hand, for $n > k \in \mathbb{N}_0$, we have

$$ {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} (\psi(t)-\psi(a))^{k} = 0.$$

Lemma 1.15 *([18]) Let $n-1 < \alpha < n$, $\beta \in [0,1]$ and $\gamma = \alpha + \beta(n-\alpha)$. If $f \in C^{n}(J,\mathbb{R})$, then*

1. $I_{a+}^{\alpha,\psi}\, {}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} f(t) = f(t) - \sum\limits_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\psi}^{[n-k]} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(a)$

   where $f_{\psi}^{[n-k]} f(t) = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n-k} f(t)$,

2. ${}^{\mathcal{H}}D_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha,\psi} f(t) = f(t)$.

In the following, using the ψ-Hilfer fractional derivative operator defined in equation (1.3), we can combine in this derivative different types of fractional derivatives by changing the value for ψ and taking the limit of the parameter β. Some of them are presented below.

1. *Consider $\psi(t) = t$ and taking the limit $\beta \to 1$ on both sides of equation (1.3), we get*

$$^{\mathcal{H}}D_{a+}^{\alpha,1;t} f(t) = I_{a+}^{(n-\alpha);t} \left(\frac{d}{dt}\right)^n f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} \left(\frac{d}{dt}\right)^n f(s)\, ds$$

$$= {}^{C}D_{a+}^{\alpha} f(t),$$

the Caputo fractional derivative.

2. *For $\psi(t) = t$ and taking the limit $\beta \to 0$ on both sides of equation (1.3), we have*

$$^{\mathcal{H}}D_{a+}^{\alpha,0;t} f(t) = \left(\frac{d}{dt}\right)^n I_{a+}^{(n-\alpha);t} f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} f(s)\, ds$$

$$= {}^{RL}D_{a+}^{\alpha} f(t),$$

the Riemann-Liouville fractional derivative.

3. *For $\psi(t) = \log t$, $a > 0$ and taking the limit $\beta \to 0$ on both sides of equation (1.3), we have*

$$^{\mathcal{H}}D_{a+}^{\alpha,0;logt} f(t) = \left(t\frac{d}{dt}\right)^n I_{a+}^{(n-\alpha)t} f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log\frac{t}{s}\right)^{n-\alpha-1} f(s) \frac{ds}{s}$$

$$= {}^{H}D_{a+}^{\alpha} f(t),$$

the Hadamard fractional derivative.

4. *For $\psi(t) = \log t$, $a > 0$ and taking the limit $\beta \to 1$ on both sides of equation (1.3), we have*

$$^{\mathcal{H}}D_{a+}^{\alpha,1;t} f(t) = I_{a+}^{(n-\alpha);t} \left(t\frac{d}{dt}\right)^n f(t)$$

$$= \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log\frac{t}{s}\right)^{n-a-1} \left(s\frac{d}{ds}\right)^n f(s) \frac{ds}{s}$$

$$= {}^{C}D_{a+}^{\alpha;\psi} f(t),$$

the Caputo-Hadamard fractional derivative.

5. *For $\psi(t) = t$ and replacing in equation (1.3), we have*

$$^{\mathcal{H}}D_{a+}^{\alpha,\beta} f(t) = I_{a+}^{\beta(n-\alpha)} \left(\frac{d}{dt}\right)^n I_{a+}^{(1-\beta)(n-\alpha)} f(t)$$
$$= {}^{\mathcal{H}}I_{a+}^{\beta(n-\alpha)} D^{\alpha \mathcal{H}} I_{a+}^{(1-\beta)(n-\alpha)} f(t)$$
$$= {}^{\mathcal{H}}D_{a+}^{\alpha,\beta} f(t),$$

the Hilfer fractional derivative.

1.4 Functional tools

In what follows, we present some concepts of functional analysis that will we use throughout this memory.

Theorem 1.1 *(Ascoli-Arzela)([6]) Let $A \subset C([0,T], \mathbb{R})$. A is relatively compact ($\overline{A}$ is compact) if :*

1. *A is uniformly bounded, i.e., there exists $M > 0$ such that*

$$|f(t)| \leq M \text{ for every } f \in A \text{ and } t \in [0,T],$$

2. *A is equicontinuous, i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $t_1, t_2 \in [0,T]$, $|t_1 - t_2| \leq \delta$ implies $|f(t_1) - f(t_2)| \leq \epsilon$ for every $f \in A$.*

Definition 1.19 *([6]) A map $f : [0,T]X \to X$ is said to be Caratheodory if :*

1. *$t \to f(t,x)$ is measurable for each $x \in X$,*

2. *$x \to f(t,x)$ is continuous for almost all $t \in [0,T]$.*

Moreover, f is called L^1-Caratheodory if $\forall \rho > 0$, there exists $\varphi_\rho \in L^1([0,T], \mathbb{R}^+)$ such that

$$|f(t,x)| \leq \varphi_\rho(t), \text{ for all } |x| \leq \rho \text{ and for a.e. } t \in [0,T].$$

Lemma 1.16 *(Standard Gronwall inequality)([6]) Let $f : [0,T] \to \mathbb{R}^+$ be real function and w is a nonnegative locally integrable function on $[0,T]$.*
Assume that there is a constant $a > 0$ such that for $0 < \alpha < 1$

$$f(t) \leq w(t) + a \int_0^t (t-s)^{\alpha-1} f(s) ds.$$

Then, there exist a constant $k = k(\alpha)$ such that

$$f(t) \leq w(t) + ka \int_0^t (t-s)^{\alpha-1} w(s) ds.$$

Lemma 1.17 *(Standard Gronwall inequality 02)([22]) Let $f : [1, T] \to [0, \infty)$ be a real function and w is a nonnegative locally integrable function on $[1, T]$. Assume that there is a constant $a > 0$ such that for $0 < a < 1$*

$$f(t) \leq w(t) + a \int_1^t \left(\log \frac{t}{s}\right)^{a-1} f(s) \frac{ds}{s}.$$

Then, there exists a constant $k = k(a)$ such that

$$f(t) \leq w(t) + ka \int_1^t \left(\log \frac{t}{s}\right)^{a-1} w(s) \frac{ds}{s}$$

for every $t \in [1, T]$.

Lemma 1.18 *(Generalization of Gronwall inequality)([8]) Let f, g be two integrable functions and h be continuous with domain $[a, b]$. Let $\Psi \in C^1([a, b], \mathbb{R})$ be an increasing function such that $\Psi'(t) \neq 0$, $\forall t \in [a, b]$. Assume that :*

1. *f and g are nonnegative functions,*
2. *h is nonnegative and nondecreasing.*

If

$$f(t) \leq g(t) + h(t) \int_a^t \Psi'(s) (\Psi(t) - \Psi(s))^{\alpha-1} f(s)\, ds,$$

then

$$f(t) \leq g(t) + \int_a^t \sum_{k=1}^{\infty} \frac{[h(t) \Gamma(\alpha)]^k}{\Gamma(k\alpha)} \Psi'(s) (\Psi(t) - \Psi(s))^{k\alpha-1} g(s)\, ds.$$

Lemma 1.19 *([8]) Under the hypotheses of Lemma 1.18, assume further that $g(t)$ is nondecreasing function for $t \in [a, b]$. Then*

$$f(t) \leq g(t) E_\alpha \left(h(t) \Gamma(\alpha) (\Psi(t) - \Psi(a))^\alpha\right),$$

where E_α is the Mittag-Leffler function.

1.5 Background about measures of non-compactness

1.5.1 The general notion of a measure of noncompactness

Firstly, we need to fix the notation. In what follows, (E, d) will be a metric space, and $(X, \|\cdot\|)$ a Banach space. Let Q be a non-empty subset of X, then $\overline{Q}$ and $\operatorname{conv} Q$ denote the closure and the closed convex hull of Q, respectively. When Q is a bounded subset, $\operatorname{Diam}(Q)$ denotes the diameter of Q. Also, we denote by $\mathfrak{B}_E$ (resp. $\mathfrak{B}_X$) the class of nonempty and bounded subsets of E (resp. of X). We begin with the following general definition.

Definition 1.20 *([15, 14]) A mapping $\mu : \mathfrak{B}_E \to \mathbb{R}^+$ will be called a measure of noncompactness in E if it satisfies the following conditions :*

1. ***Regularity*** *: $\mu(Q) = 0$ if, and only if, Q is a precompact set.*
2. ***Invariant under closure*** *: $\mu(Q) = \mu\left(\overline{Q}\right)$, for all $Q \in \mathfrak{B}_E$.*
3. ***Semi-additivity*** *: $\mu(Q_1 \cup Q_2) = \max\{\mu(Q_1), \mu(Q_2)\}$, for all $Q_1, Q_2 \in \mathfrak{B}_E$.*

For a Banach space X, we additionally require :

4. **Semi-homogeneity** : $\mu(\lambda Q) = |\lambda| \mu(Q)$ for $\lambda \in \mathbb{R}$ and $Q \in \mathfrak{B}_X$.
5. **Invariant under translations** : $\mu(x + Q) = \mu(Q)$, for all $x \in X$ and $Q \in \mathfrak{B}_X$.

The three main measures of noncompactness are :

- Kuratowski MNC
- Hausdorff MNC
- De Blasi Measure of Weak Noncompactness

In this thesis, we focus on the Kuratowski MNC.

1.5.2 The Kuratowski Measure of Noncompactness

Definition 1.21 *([14, 16]) Let (E, d) be a metric space and Q be a bounded subset of E. The Kuratowski measure of noncompactness of Q, denoted by $\mu_{\mathbf{k}}(Q)$, is the infimum of the set of all numbers $\epsilon > 0$ such that Q can be covered by a finite number of sets with diameters $< \epsilon$, i.e.,*

$$\mu_{\mathbf{k}}(Q) = \inf\left\{\epsilon > 0 : Q \subseteq \bigcup_{i=1}^{n} S_i,\ S_i \subset E,\ \operatorname{diam}(S_i) < \epsilon,\ i = 1, 2, \ldots, n,\ n \in \mathbb{N}\right\}.$$

1. **Regularity** : $\mu_{\mathbf{k}}(Q) = 0$ if and only if Q is precompact.

2. **Closure invariance** : $\mu_{\mathbf{k}}(Q)=\mu_{\mathbf{k}}(\overline{Q})$.

3. **Semi-additivity** : $\mu_{\mathbf{k}}(Q_1\cup Q_2)=\max\{\mu_{\mathbf{k}}(Q_1),\mu_{\mathbf{k}}(Q_2)\}$.

4. **Monotonicity** : $Q_1\subset Q_2\Rightarrow \mu_{\mathbf{k}}(Q_1)\leq\mu_{\mathbf{k}}(Q_2)$.

5. **Algebraic semi-additivity** : $\mu_{\mathbf{k}}(Q_1+Q_2)\leq\mu_{\mathbf{k}}(Q_1)+\mu_{\mathbf{k}}(Q_2)$.

6. **Semi-homogeneity** : $\mu_{\mathbf{k}}(\lambda Q)=|\lambda|\,\mu_{\mathbf{k}}(Q)$.

7. **Convex hull invariance** : $\mu_{\mathbf{k}}(\operatorname{conv} Q)=\mu_{\mathbf{k}}(Q)$.

8. **Intersection property** : $\mu_{\mathbf{k}}(Q_1\cap Q_2)\leq\min\{\mu_{\mathbf{k}}(Q_1),\mu_{\mathbf{k}}(Q_2)\}$.

Lemma 1.20 *([21])*
Let $J=[0,T]$ and D be a bounded, closed and convex subset of the Banach space $C(J,X)$. Let G be continuous on JJ and $f: JX\to X$ satisfy Caratheodory conditions. Assume there exists $p\in L^1(J,\mathbb{R}^+)$ such that for each $t\in J$ and each bounded $B\subset X$,

$$\lim_{h\to 0^+}\mu_{\mathbf{k}}(f(J_{t,h}B))\leq p(t)\,\mu_{\mathbf{k}}(B),\ \text{where } J_{t,h}=[t-h,t]\cap J.$$

If V is an equicontinuous subset of D, then

$$\mu_{\mathbf{k}}\left(\left\{\int_J G(s,t)\,f(s,y(s))\,ds : y\in V\right\}\right)\leq\int_J\|G(s,t)\|\,p(s)\,\mu_{\mathbf{k}}(V(s))\,ds.$$

1.6 Multivalued Analysis

In this section, we introduce some definitions, notations, and preliminary facts for multivalued analysis, which are used throughout this thesis.

Definition 1.22 *([6]) Let $(X,\|\cdot\|)$ and $(Y,\|\cdot\|)$ be two Banach spaces. A multivalued function $\mathcal{F}$ (or a set-valued map, multivalued map) from X into $\mathcal{P}(Y)$ is a correspondence that associates to each element $x\in X$ a subset $\mathcal{F}(x)$ of Y.*
We denote this correspondence by $\mathcal{F}: X\to\mathcal{P}(Y)$. We define :

1. *The **effective domain** $\operatorname{Dom}\mathcal{F}=\{x\in X\mid\mathcal{F}(x)\neq\emptyset\}$.*

2. *The **graph** $\operatorname{Gr}(\mathcal{F})=\{(x,y)\in XY\mid x\in\operatorname{Dom}\mathcal{F}, y\in\mathcal{F}(x)\}$.*

3. *The **image** of a set $A\in\mathcal{P}(X)$: $\mathcal{F}(A)=\bigcup_{x\in A}\mathcal{F}(x)$.*

4. *The **inverse image** of a set $B\in\mathcal{P}(Y)$: $\mathcal{F}^{-1}(B)=\{x\in X\mid\mathcal{F}(x)\cap B\neq\emptyset\}$.*

5. *The multivalued map $\mathcal{F} : X \to \mathcal{P}(Y)$ is* ***convex (closed, compact) valued*** *if $\mathcal{F}(x)$ is convex (closed, compact) for all $x \in X$.*

6. *$\mathcal{F}$ is* ***bounded on bounded sets*** *if $\mathcal{F}(B) = \bigcup_{x\in B} \mathcal{F}(x)$ is bounded in Y for all bounded sets B of X, i.e.,*
$$\sup_{x\in B} [\sup\{\|y\| : y \in \mathcal{F}(x)\}] < \infty.$$

7. *$\mathcal{F}$ is called* ***upper semi-continuous*** *(u.s.c.) on X if $\mathcal{F}^{-1}(A)$ is closed in X whenever $A \subset Y$ is closed.*

8. *$\mathcal{F}$ is said to be* ***completely continuous*** *if $\mathcal{F}(B)$ is relatively compact for every bounded subset B of X.*

9. *A multivalued map $\mathcal{F} : X \to \mathcal{P}_0(Y)$ (where $\mathcal{P}_0(Y) = \{A \in \mathcal{P}(Y) \mid A \neq \emptyset\}$) is said to be* ***measurable*** *if for every open $U \subset Y$, the set $\mathcal{F}^{-1}(U)$ is measurable in X.*

10. *$\mathcal{F}$ has a* ***fixed point*** *if there exists $x \in X$ such that $x \in \mathcal{F}(x)$. The fixed point set of $\mathcal{F}$ is denoted by $\operatorname{Fix}\mathcal{F}$.*

For each $y \in C([a, b], \mathbb{R})$, the set of selections of $\mathcal{F}$ at point y is defined by
$$S_{\mathcal{F},y} = \{v \in L^1([a, b], \mathbb{R}) \mid v(t) \in \mathcal{F}(t, y) \text{ for a.e. } t \in [a, b]\}.$$

We denote by $\mathcal{P}_p$ the set of all nonempty subsets of X which have property "p", where "p" can be :

- bounded (b)
- closed (cl)
- convex (c)
- compact (cp)

Thus :
$$\begin{aligned}
\mathcal{P}_b(X) &= \{A \in \mathcal{P}(X) \mid A \text{ is bounded}\}\\
\mathcal{P}_{cl}(X) &= \{A \in \mathcal{P}(X) \mid A \text{ is closed}\}\\
\mathcal{P}_{cp}(X) &= \{A \in \mathcal{P}(X) \mid A \text{ is compact}\}\\
\mathcal{P}_{cpc}(X) &= \{A \in \mathcal{P}(X) \mid A \text{ is compact and convex}\}
\end{aligned}$$

Definition 1.23 *([6]) A multivalued map $\mathcal{F} : [a, b]\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be* ***Carathéodory*** *if :*

1. *$t \mapsto \mathcal{F}(t, x)$ is measurable for each $x \in \mathbb{R}$,*

2. *$x \mapsto \mathcal{F}(t, x)$ is upper semi-continuous for almost all $t \in [a, b]$.*

Further, a Carathéodory function $\mathcal{F}$ is called L^1-Carathéodory if

3. *For each $r > 0$, there exists $\varphi_r \in L^1([a,b],\mathbb{R}^+)$ such that*

$$\|\mathcal{F}(t,x)\| = \sup\{|v| : v \in \mathcal{F}(t,x)\} \leq \varphi_r(t)$$

for all $\|x\| \leq r$ and for a.e. $t \in [a,b]$.

Lemma 1.21 *([21]) Let (E,d) be a metric space induced from the normed space $(X,\|\cdot\|)$. Consider $H_d : \mathcal{P}(X)\mathcal{P}(X) \to \mathbb{R}^+ \cup \{\infty\}$ given by*

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$ and $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{cl,d}(X), H_d)$ is a metric space.

Definition 1.24 *([6]) A multivalued operator $\mathcal{N} : X \to \mathcal{P}_{cl}(X)$ is called :*

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$H_d(\mathcal{N}(x),\mathcal{N}(y)) \leq \gamma d(x,y) \text{ for each } x,y \in X.$$

(b) A contraction if and only if it is γ-Lipschitz with $\gamma < 1$.

Lemma 1.22 *([10]) If $\mathcal{F} : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $\mathrm{Gr}(\mathcal{F})$ is a closed subset of XY, i.e., for every sequence $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{y_n\}_{n\in\mathbb{N}} \subset Y$, if when $n \to \infty$, $x_n \to x_*$, $y_n \to y_*$ and $y_n \in \mathcal{F}(x_n)$, then $y_* \in \mathcal{F}(x_*)$. Conversely, if $\mathcal{F}$ is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 1.23 *([18]) Let X be a separable Banach space. Let $\mathcal{F} : [a,b]X \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multivalued map and let Θ be a linear continuous mapping from $L^1([a,b],X)$ to $C([a,b],X)$. Then the operator*

$$\Theta \circ S_{\mathcal{F}} : C([a,b],X) \to \mathcal{P}_{cp,c}(C([a,b],X)),\ x \mapsto (\Theta \circ S_{\mathcal{F}})(x) = \Theta(S_{\mathcal{F},x}),$$

is a closed graph operator in $C([a,b],X)C([a,b],X)$.

For more details on multivalued maps and the proof of the known results cited in this section, we refer the interested reader to the books by Deimling, Górniewicz , and Hu and Papageorgiou .

1.7 Fixed Point Theorems

The theory of fixed points is one of the most powerful tools of modern mathematics, which is used to show the existence and uniqueness of fixed points of various kinds of equations. Throughout this study, we convert the given problem into an equivalent integral equation and then use the appropriate fixed point theorem such that the fixed points obtained are thus the solutions of the given problem. In this section we collect the fixed point theorems which are used in the proofs of our main results. We start with the definition of a fixed point.

Definition 1.25 *For a mapping Φ of a set E into itself, an element x of E is a fixed point of Φ, if $\Phi(x) = x$.*

Theorem 1.2 ***(Banach's fixed point theorem)*** *[5] Let Ω be a non-empty closed subset of a Banach space $(X, \|\cdot\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 1.3 ***(Schauder's fixed point theorem)*** *[5] Let Ω be a nonempty closed bounded convex subset of a Banach space X and $\Phi : \Omega \to \Omega$ be a continuous compact operator. Then Φ has a fixed point in Ω.*

Theorem 1.4 ***(Schaefer's fixed point theorem)*** *[5] Let X be a Banach space, and $\Phi : X \to X$ is completely continuous operator. If the set $B_\lambda = \{x \in X : x = \lambda \Phi x, \lambda \in (0, 1)\}$ is bounded, then Φ has fixed point in X.*

Theorem 1.5 ***(Krasnoselskii's fixed point theorem)*** *[5] Let Ω be a non-empty closed bounded convex subset of a Banach space $(X, \|\cdot\|)$. Suppose that F_1 and F_2 map Ω into X such that :*

1. *$F_1x + F_2y \in \Omega$ for all $x, y \in \Omega$,*
2. *F_1 is continuous and compact,*
3. *F_2 is a contraction with constant $l < 1$.*

Then there is a $x \in \Omega$ with $F_1x + F_2x = x$.

Theorem 1.6 ***(Mönch's fixed point theorem)*** *[13] Let Ω be a bounded, closed and convex subset of the Banach space such that $0 \in \Omega$, and let Φ be a continuous mapping of Ω into itself. If the implication*

$$V = \operatorname{conv} \Phi(V) \quad \text{or} \quad V = \Phi(V) \cup \{0\} \Rightarrow \mu(V) = 0,$$

holds for every $V \subset \Omega$, then Φ has a fixed point.

Theorem 1.7 ***(Nonlinear alternative of Kakutani maps)*** *[3] Let Ω be a closed convex subset of a Banach space X and $\mathcal{U}$ be an open subset of Ω with $0 \in \mathcal{U}$. Suppose that $N : \overline{\mathcal{U}} \to \mathcal{P}_{cp,c}(\Omega)$ is an upper semi-continuous compact map. Then either :*

(i) N has a fixed point in $\mathcal{U}$, or

(ii) there is $x \in \partial\mathcal{U}$ and $\mu \in (0, 1)$ with $x \in \mu N(x)$.

Theorem 1.8 ***(Covitz and Nadler fixed point theorem)*** *[9] Let (E, d) be complete metric space. If $N : E \to \mathcal{P}_{cl}(E)$ is a contraction, then $\operatorname{Fix} N \neq \emptyset$.*

1.8 Ulam's stability

The stability of the Ulam can be viewed as a special kind of data dependence which was initiated by the Ulam in [15]. Rassias in [19] extended the concept of Ulam-Hyers stability. To define Ulam's stability, we consider the following fractional differential equation

$$H_{0+}^{D^{\alpha,\beta;\psi}} x(t) = f(t, x(t)), \quad t \in [0, T]. \tag{1.4}$$

Definition 1.26 *[10] The equation (1.4) is said to be Ulam-Hyers stable if there exists a real number $k > 0$ such that for each $\epsilon > 0$ and for each $y \in C([0, T], \mathbb{R})$ solution of the inequality*

$$\left| H_{0+}^{D^{\alpha,\beta;\psi}} y(t) - f(t, y(t)) \right| \leq \epsilon, \quad t \in [0, T], \tag{1.5}$$

there exists a solution $x \in C([0, T], \mathbb{R})$ of the equation (1.4) with

$$|y(t) - x(t)| \leq k\epsilon, \quad t \in [0, T].$$

Definition 1.27 *[10] Assume that $y \in C([0, T], \mathbb{R})$ satisfies the inequality in (1.5) and $x \in C([0, T], \mathbb{R})$ is a solution of the equation (1.4). If there is a function $\phi_{\mathcal{F}} \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\phi_{\mathcal{F}}(0) = 0$ satisfying*

$$|y(t) - x(t)| \leq \phi_{\mathcal{F}}(\epsilon), \quad t \in [0, T].$$

Then the equation (1.4) is said to be generalized Ulam-Hyers stable.

Definition 1.28 *[10] The equation (1.4) is said to be Ulam-Hyers-Rassias stable with respect to $\phi_{\mathcal{F}} \in C([0, T], \mathbb{R}^+)$ if there exists a real number $k > 0$ such that for each $\epsilon > 0$ and for each $y \in C([0, T], \mathbb{R})$ solution of the inequality*

$$\left| H_{0+}^{D^{\alpha,\beta;\psi}} y(t) - f(t, y(t)) \right| \leq \epsilon \phi_{\mathcal{F}}(t), \quad t \in [0, T], \tag{1.6}$$

there exists a solution $x \in C([0, T], \mathbb{R})$ of the equation (1.4) with

$$|y(t) - x(t)| \leq k\phi_{\mathcal{F}}(t)\epsilon, \quad t \in [0, T].$$

Definition 1.29 *[10] Assume that $y \in C([0, T], \mathbb{R})$ satisfies the inequality in (1.6) and $x \in C([0, T], \mathbb{R})$ is a solution of the equation (1.4). If there exists a constant $k > 0$ such that*

$$|y(t) - x(t)| \leq k\phi_{\mathcal{F}}(t), \quad t \in [0, T],$$

then the equation (1.4) is said to be generalized Ulam-Hyers-Rassias stable.

Remark 1.1 If there is a function $v \in C([0, T], \mathbb{R})$ (dependent on y), such that

1. $|v(t)| \leq \epsilon$, for all $t \in [0, T]$,
2. $H_{0+}^{D^{\alpha,\beta;\psi}} y(t) = f(t, y(t)) + v(t), \quad t \in [0, T]$,

then a function $y \in C([0, T], \mathbb{R})$ is a solution of the inequality (1.5).

Chapitre 2

EXISTENCE , UNIQUNESS AND ULAM STABILITY RESULTS FOR A MIXED -TYPE FRACTIONAL DIFFEREN TIAL EQUATIONS WITH P-LAPLACIAN OPERATOR

2.1 Introduction

$$(P)\begin{cases} \mathcal{C}D_t^{\beta-}(\phi_p(D_{0t}^{\alpha}, u(t))) + f(t, u(t)) = 0, \quad t \in [0,\, 1], \\ \phi_p(D_0^{\alpha}, u(1)) = 0, \\ t^{2-\alpha}u(t)\big|_{t=0} = 0, \\ u(1) = \lambda \int_0^{\eta} u(s)ds, \end{cases}$$

where $1 < \alpha < 2$, $0 < \beta < 1$, $\lambda > 0$, $0 < \eta < 1$, $0 < \lambda\eta^{\alpha} < \alpha$ and $\phi_p(u) = |u|^{\alpha-2}$, $u, \phi_p^{-1} = \phi_p$, $\frac{1}{p} + \frac{1}{q} = 1$. $\mathcal{C}D_t^{\beta-}$ and $D_{0t}^{\alpha-}$ denote the right Caputo fractional derivative of order β and the left Riemann-Liouville fractional derivative of order α and f ; $[0,\, 1]\mathbb{R} \to \mathbb{R}$ is a continuous function. The left and right Riemann-Liouville fractional integrals of order $\mu > 0$ of a function f are defined by

$$I_{0t}^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(s)ds,$$

$$I_{1-}^{\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_t^1 (s-t)^{\mu-1} f(s)ds.$$

The left Riemann-Liouville fractional derivative and the right Caputo fractional derivative of order $\mu > 0$ of a function f are respectively :

$$D_{0t}^{\mu}f(t) = \frac{d^n}{dt^n}\left(I_{0t}^{n-\mu}f\right)(t),$$
$$\mathcal{C}D_{1-}^{\mu}f(t) = (-1)^n I_{1-}^{n-\mu}f^{(n)}(t),$$

where $n = [\mu] + 1$, $[\mu]$ denotes the integer part of the real number μ.

Lemma 2.1 *([22]) Let ϕ_p be p-Laplacian operator. Then,*

(i) If $1 < p \leq 2$, $uv > 0$ and $|u|$, $|v| \geq m > 0$, then

$$|\phi_p(u) - \phi_p(v)| \leq (p-1)m^{p-2}|u - v|.$$

(ii) If $p > 2$ and $|u|$, $|v| \leq M$, then

$$|\phi_p(u) - \phi_p(v)| \leq (p-1)M^{p-2}|u - v|.$$

The problem (P) is Ulam-Hyers (UH) stable if there exists a real number $k > 0$ such that for each $\epsilon > 0$ and for each $v \in C^1([0,\ 1], \mathbb{R})$ solution of the inequality

$$|\mathcal{C}D_t^{\beta-}(\phi_p(D_{0t}^{\alpha}, v(t))) + f(t, v(t))| \leq \epsilon, \quad t \in [0,\ 1],$$

there exists a solution $u \in C^1([0,\ 1], \mathbb{R})$ of the problem (P) such that

$$|v(t) - u(t)| \leq k\epsilon, \quad t \in [0,\ 1].$$

2.2 Properties

[Generalized Ulam-Hyers-Rassias stability]([10]) Assume that $v \in C^1([0,1], \mathbb{R})$ satisfies the inequality in (2.2) and $u \in C([0,1], \mathbb{R})$ is a solution of the problem (P). If there exists a real number $k_{\theta_f} > 0$ such that

$$|v(t) - u(t)| \leq k_{\theta_f}\theta_f(t), \quad t \in [0,1],$$

then the problem (P) is said to be generalized Ulam-Hyers-Rassias stable (GUHR).

Remark 2.1 *A function $v \in C^1([0,1], \mathbb{R})$ is a solution of the inequality (2.1) if there is a function $\Phi \in C([0,1], \mathbb{R})$, which depends on v, such that*

1. *$|\Phi(t)| \leq \epsilon$, for all $t \in [0,1]$,*
2. *$\mathcal{C}D_t^{\beta}(\phi_p(D_{0t}^{\alpha}v(t))) = -f(t, v(t)) + \Phi(t), \quad t \in [0,1]$.*

Remark 2.2 *A function $v \in C^1([0,1], \mathbb{R})$ is a solution of the inequality (2.2) if there is a function $\Phi \in C([0,1], \mathbb{R})$, which depends on v, such that*

1. $|\Phi(t)| \leq \epsilon\theta_f(t)$, *for all* $t \in [0,1]$,

2. $\mathcal{C}D_t^{\beta}(\phi_p(D_{0t}^{\alpha}v(t))) = -f(t, v(t)) + \Phi(t), \quad t \in [0,1]$.

Theorem 2.1 *[Arzela-Ascoli Theorem]([20]) Let* $X \subset C([0,1],\mathbb{R})$, X *is relatively compact if and only if* X *is uniformly bounded and equicontinuous.*

Theorem 2.2 *[Schauder fixed point theorem]([20]) If* Ω *is a nonempty closed bounded convex subset of a Banach space* X *and* $T : \Omega \to \Omega$ *is completely continuous, then* T *has a fixed point in* Ω.

Theorem 2.3 *[Banach's fixed point theorem]([20]) Let* Ω *be a non empty closed convex subset of a Banach space* X, *then any contraction mapping* $T : \Omega \to \Omega$ *has a unique fixed point.*

2.3 Existence and uniqueness

Lemma 2.2 *([14]) The boundary value problem*

$$(P1)\begin{cases} \mathcal{C}D_t^{\beta}(\phi_p(D_{0t}^{\alpha}u(t))) + h(t) = 0, \quad t \in [0,1], \\ \phi_p(D_{0t}^{\alpha}u(1)) = 0, \\ t^{2-\alpha}u(t)\big|_{t=0} = 0, \\ u(1) = \lambda\int_0^1 u(s)ds, \end{cases}$$

has a unique solution, which is given by

$$u(t) = \int_0^1 G(t,s)g(s)ds, \quad t \in [0,1], \tag{3.1}$$

where $G(t,s)$ *and* $H(s,\tau)$ *are the Green's functions associated with the problem.*
where

$$G(t,s) = G_1(t,s) + G_2(t,s), \quad t,s \in [0,1], \tag{3.2}$$

$$G_1(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.3}$$

$$G_2(t,s) = \frac{t^{\alpha-1}\alpha\lambda}{(\alpha-\lambda\eta^{\alpha})}\int_0^{\eta} G_1(t,s)dt, \quad t,s \in [0,1], \tag{3.4}$$

$$g(s) = \phi_p\left(J_{1-}^{\beta}h(s)\right), \quad s \in [0,1]. \tag{3.5}$$

Proof. Applying the fractional integral operator J_{1-}^{β} to the first equation of (P1), we get

$$\phi_p(D_{0t}^{\alpha}u(s)) = \frac{-1}{\Gamma(\beta)}\int_s^1 (t-s)^{\beta-1}h(s)dt + c. \tag{3.6}$$

By the boundary value condition $\phi_p(D_{0t}^{\alpha}u(1)) = 0$, we have $c = 0$, consequently,

$$\phi_p(D_{0t}^{\alpha}u(t)) = -J_{1-}^{\beta}h(t),$$

and then,

$$D_{0t}^{\alpha}u(s) = -\phi_q\left(\frac{1}{\Gamma(\beta)}\int_s^1 (t-s)^{\beta-1}h(s)dt\right).$$

Letting

$$g(s) = \phi_q\left(\frac{1}{\Gamma(\beta)}\int_s^1 (t-s)^{\beta-1}h(s)dt\right),$$

so

$$D_{0t}^{\alpha}u(t) = -g(t),$$

we arrive at

$$u(t) = -\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds + c_1t^{\alpha-1} + c_2t^{\alpha-2}. \tag{3.7}$$

Condition $t^{2-\alpha}u(t)\big|_{t=0} = 0$ implies that $c_2 = 0$, i.e.,

$$u(t) = -I_{0+}^{\alpha}g(t) + c_1t^{\alpha-1}. \tag{3.8}$$

By using the condition $u(1) = \lambda\int_0^{\eta}u(s)ds$, we obtain

$$u(1) = -I_{0+}^{\alpha}g(1) + c_1 = \lambda\int_0^{\eta}u(s)ds, \tag{3.9}$$

which implies

$$c_1 = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}g(s)ds + \lambda\int_0^{\eta}u(s)ds.$$

Hence

$$\begin{aligned} u(t) = &-\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds + \frac{t^{\alpha-1}}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}g(s)ds \\ &+ t^{\alpha-1}\lambda\int_0^{\eta}u(s)ds. \end{aligned} \tag{3.10}$$

As a result,

$$u(t) = \int_0^1 G_1(t,s)g(s)ds + t^{\alpha-1}\lambda\int_0^{\eta}u(s)ds. \tag{3.11}$$

where G_1 is defined by (3.3). From (3.11), we have

$$\int_0^\eta u(t)dt = \frac{\alpha}{\alpha - \lambda\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)g(s)dsdt. \tag{3.12}$$

Substituting (3.12) into (3.11), we obtain

$$\begin{aligned} u(t) &= \int_0^1 G_1(t,s)g(s)ds + \frac{\alpha\lambda t^{\alpha-1}}{\alpha - \lambda\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)g(s)dsdt \\ &= \int_0^1 G_1(t,s)g(s)ds + \int_0^1 G_2(t,s)g(s)ds \\ &= \int_0^1 G(t,s)g(s)ds, \end{aligned}$$

where G and G_2 are defined by (3.2) and (3.4) respectively.

Conversely, we prove that the function $u \in C([0,1],\mathbb{R})$ defined by (3.1) is a solution of problem (P1). It is easy to verify that the function u satisfies the first equation of the problem (P1), and it is straightforward to prove the first and second boundary conditions. For the last condition we have,

$$\begin{aligned} \lambda \int_0^\eta u(t)dt &= \lambda \int_0^\eta \int_0^1 G_1(t,s)g(s)dsdt \\ &\quad + \frac{\alpha\lambda^2}{\alpha - \lambda\eta^\alpha} \int_0^\eta t^{\alpha-1} \left(\int_0^\eta \int_0^1 G_1(t,s)g(s)dsdt \right) dt \\ &= \lambda \int_0^\eta \int_0^1 G_1(t,s)g(s)dsdt \\ &\quad + \frac{\lambda^2\eta^\alpha}{\alpha - \lambda\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)g(s)dsdt \\ &= \frac{\alpha\lambda}{\alpha - \lambda\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)g(s)dsdt. \end{aligned}$$

On the other hand, we have

$$G_1(1,s) = 0, \text{ for all } s \in [0,1],$$

and

$$u(t) = \int_0^1 G_1(t,s)g(s)ds + \frac{\alpha\lambda t^{\alpha-1}}{\alpha - \lambda\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)g(s)dsdt,$$

then

$$u(1) = \frac{\alpha\lambda}{\alpha - \lambda\eta^\alpha} \int_0^\eta \int_0^1 G_1(t,s)g(s)dsdt,$$

we conclude that

$$u(1) = \lambda \int_0^\eta u(t)dt.$$

Lemma 2.3 *The function G defined by (3.2) is continuous on $[0,1][0,1]$ and satisfies*

1. $G(t,s) \geq 0$ for $t, s \in [0,1]$.
2. $G(t,s) \leq \delta$, where $\delta = \frac{\alpha - \eta^\alpha \lambda + \alpha\lambda\eta}{\Gamma(\alpha)(\alpha - \lambda\eta^\alpha)}$ for $t, s \in [0,1]$.

Proof. Firstly, observing the expression of function G, it is easy to see that $G(t,s) \geq 0$ for $t, s \in [0,1]$. Secondly, by definition of function G_1, it is clear that $G_1(t,s) \leq \frac{1}{\Gamma(\alpha)} forall (t, s \in [0,1]), on the other hand,$ $G_2(t,s) = \frac{t^{\alpha-1}\alpha\lambda}{(\alpha-\lambda\eta^\alpha)} \int_0^\eta G_1(t,s)dt \leq \frac{\alpha\lambda\eta}{\Gamma(\alpha)(\alpha-\lambda\eta^\alpha)}$,
then

$$G(t,s) = G_1(t,s) + G_2(t,s) \leq \frac{\alpha - \lambda\eta^\alpha + \alpha\lambda\eta}{\Gamma(\alpha)(\alpha - \lambda\eta^\alpha)},$$

for all $t, s \in [0,1]$.
Let the Banach space $E = C([0,1], \mathbb{R})$ be endowed with the norm

$$\|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

We define the operator $T : E \to E$ by

$$Tu(t) = \int_0^1 G(t,s)\Phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u(\tau))d\tau \right) ds.$$

To prove the uniqueness of the solution of the problem (P), we used the Banach contraction mapping principle. For this, we make the following assumptions.

(H1) There exists a positive continuous function $w(t)$ such that

$$|f(t, u(t))| \leq w(t), \quad (t, u) \in [0,1]\mathbb{R}.$$

(H2) There exists a positive constant L such that for all $u, v \in \mathbb{R}$

$$|f(t,u) - f(t,v)| \leq L \|u - v\|, \text{ for all } t \in [0,1].$$

Theorem 2.4 *Suppose that $1 < p \leq 2$. Assume (H1) and (H2) holds. If*

$$\frac{LM^{q-2}\delta(q-1)}{\Gamma(\beta+2)} < 1,$$

where

$$M = \frac{1}{\Gamma(\beta+1)} \max_{t\in[0,1]} w(t),$$

then the problem (P) has a unique solution.

Proof. By condition (H1), we have that

$$\begin{aligned}\frac{1}{\Gamma(\beta)}\int_s^1(\tau-s)^{\beta-1}f(\tau,u(\tau))d\tau &\leq \frac{1}{\Gamma(\beta)}\int_s^1(\tau-s)^{\beta-1}w(\tau)d\tau \\ &\leq \frac{\max_{t\in[0,1]}(w(t))}{\Gamma(\beta)}\int_s^1(\tau-s)^{\beta-1}d\tau \\ &\leq M.\end{aligned} \tag{3.13}$$

By Lemma , for any $u, v \in E$, we obtain

$$\begin{aligned}|Tu(t)-Tv(t)| &\leq \left|\int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_s^1(\tau-s)^{\beta-1}f(\tau,u(\tau))d\tau\right)ds\right. \\ &\quad \left.-\int_0^1 G(t,s)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_s^1(\tau-s)^{\beta-1}f(\tau,v(\tau))d\tau\right)ds\right| \\ &\leq \frac{M^{q-2}\delta(q-1)}{\Gamma(\beta)}\int_0^1\left(\int_s^1(\tau-s)^{\beta-1}\left|f(\tau,u(\tau))-f(\tau,v(\tau))\right|d\tau\right)ds \\ &\leq \frac{M^{q-2}\delta(q-1)}{\Gamma(\beta)}\int_0^1\left(\int_s^1(\tau-s)^{\beta-1}\left|f(\tau,u(\tau))-f(\tau,v(\tau))\right|d\tau\right)ds \\ &\leq \frac{M^{q-2}\delta(q-1)}{\Gamma(\beta)}\int_0^1\left(\int_s^1(\tau-s)^{\beta-1}L\left\|u-v\right\|_\infty d\tau\right)ds \\ &\leq \frac{LM^{q-2}\delta(q-1)}{\Gamma(\beta+2)}\left\|u-v\right\|_\infty.\end{aligned}$$

This implies that $T : E \to E$ is a contraction mapping. By means of the Banach contraction mapping principle, we get that T has a unique fixed point which is a solution of problem (P). ■

Now, we use the Schauder's fixed point theorem to investigate the existence results for the problem (P). To prove the main result, we make the following assumption :

(H3) There exist $a_1, a_2 \in C([0,1],\mathbb{R}^+)$, $0 \leq \nu < p-1$ such that

$$|f(t,u(t))| \leq a_1(t) + a_2(t)|u|^\nu, \quad (t,u) \in [0,1]\mathbb{R}.$$

Let

$$\Omega = \{u \in E, \|u\|_\infty < R\},$$

where R is chosen such that

$$R \geq \max\left\{\left(\frac{2A_1\delta}{\Gamma(\beta+1)^{q-1}(\beta(q-1)+1)}\right)^{q-1}, \left(\frac{2A_2\delta}{\Gamma(\beta+1)^{q-1}(\beta(q-1)+1)}\right)^{\frac{q-1}{1-\nu(q-1)}}\right\},$$

where $A_1 = \max_{t\in[0,1]} a_1(t)$, $A_2 = \max_{t\in[0,1]} a_2(t)$.

Theorem 2.5 *Assume (H3) hold. Then the problem(P) has at least one solution in* Ω.

Proof. For convenience, the proof will be done in several steps.

Claim 1. We shall show that T is continuous.

Consider the sequence $(u_n)_n$ converges to u in E, then for every $t \in [0,1]$ we have

$$\begin{aligned}
|Tu_n(t) - Tu(t)| &= \left| \int_0^1 G(t,s)\phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u_n(\tau)) d\tau \right) ds \right. \\
&\quad \left. - \int_0^1 G(t,s)\phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u(\tau)) d\tau \right) ds \right| \\
&\leq \int_0^1 G(t,s) \left| \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u_n(\tau)) d\tau \right) \right. \\
&\quad \left. - \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u(\tau)) d\tau \right) \right| ds \\
&\leq \delta \int_0^1 \left| \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u_n(\tau)) d\tau \right) \right. \\
&\quad \left. - \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u(\tau)) d\tau \right) \right| ds.
\end{aligned}$$

From the continuity of f, ϕ_q and by the dominated convergence theorem, we get $\|Tu_n - Tu\|_\infty \to 0$ as $n \to \infty$. So, the operator T is continuous.

Claim 2. We show that $T(\Omega) \subset \Omega$.

By condition (H3), for all $t \in [0,1]$ and $u \in \Omega$, we have

$$\begin{aligned}
|Tu(t)| &\leq \int_0^1 G(t,s) \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u(\tau)) d\tau \right)^{q-1} ds \\
&\leq \int_0^1 G(t,s) \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} (a_1(t) + a_2(t)|u|^\nu) d\tau \right)^{q-1} ds \\
&\leq \frac{\delta}{\Gamma(\beta+1)^{q-1}[\beta(q-1)+1]} [A_1 + A_2 R^\nu]^{q-1} \\
&\leq R,
\end{aligned}$$

which implies that $T(\Omega) \subset \Omega$ and the set $T(\Omega)$ is uniformly bounded.

Claim 3. We show that the $T(\Omega)$ is an equicontinuous set.

Let $u \in \Omega$, $0 \leq t_1 \leq t_2 \leq 1$. We have

$$\begin{aligned}
|Tu(t_2) - Tu(t_1)| &= \left| \int_0^1 G(t_2, s)\phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u(\tau)) d\tau \right) ds \right. \\
&\quad \left. - \int_0^1 G(t_1, s)\phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u(\tau)) d\tau \right) ds \right| \\
&\leq \int_0^1 |G(t_2, s) - G(t_1, s)| \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} |f(\tau, u(\tau))| d\tau \right)^{q-1} ds \\
&\leq \frac{(A_1 + A_2 R^{\nu})^{q-1}}{\Gamma(\alpha+1)(\Gamma(\beta+1)^{q-1})} \left[(1 + \lambda^*)(t_2^{\alpha-1} - t_1^{\alpha-1}) - (t_2^{\alpha} - t_1^{\alpha}) \right],
\end{aligned}$$

where $\lambda^* = \frac{\lambda\alpha}{(\alpha - \lambda\eta^{\alpha})\Gamma(\alpha-1)}$. As $t_2 \to t_1$, the right-hand side of the above inequality tends to zero, therefore $T(\Omega)$ is an equicontinuous set. Therefore by the Arzela-Ascoli implies that T is completely continuous. According to the Schauder's fixed point theorem, the operator T has at least one fixed point $u \in \Omega$ which is a solution of the problem 2.5.

2.4 Ulam stability results

Theorem 2.6 *Assume $1 < p \leq 2$. Suppose that (H1) and (H2) holds. If*

$$\frac{LM^{q-2}\delta(q-1)}{\Gamma(\beta+2)} < 1,$$

then the problem (p) is UH stable.

Proof. Assume $1 < p \leq 2$. Suppose v is a solution of the following inequality for $\epsilon \in (0, 1]$

$$\left| {}^{C}D_{1^-}^{\beta}(\phi_p(D_{0^+}^{\alpha} v(t))) + f(t, v(t)) \right| \leq \epsilon, \text{ for all } t \in [0, 1]. \tag{4.1}$$

By Remark 2.8, we have

$$v(t) = \int_0^1 G(t, s)\phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} (f(\tau, v(\tau)) - \Phi(\tau)) d\tau \right) ds.$$

Other hand, from (H1) we obtain

$$\begin{aligned}
\left| \frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} (f(\tau, v(\tau)) - \Phi(\tau)) d\tau \right| &\leq \frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} |f(\tau, v(\tau)) - \Phi(\tau)| d\tau \\
&\leq \frac{1}{\Gamma(\beta+1)} \max_{t \in [0,1]} w(t) + \frac{\varepsilon}{\Gamma(\beta+1)} \\
&\leq M + \frac{1}{\Gamma(\beta+1)} = M_0,
\end{aligned}$$

then, by Remark 2.8 and by Lemma (2.3)(i), we have

$$|v(t) - Tv(t)| \leq \frac{M_0^{q-2}\delta(q-1)}{\Gamma(\beta+2)}\epsilon.$$

Suppose v is a solution of the inequality (4.1), we obtain

$$\begin{aligned}
|v(t) - u(t)| &= \left| v(t) - \int_0^1 G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_s^1 (\tau-s)^{\beta-1} f(\tau,u(\tau))d\tau\right) ds \right| \\
&= \left| v(t) - \int_0^1 G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_s^1 (\tau-s)^{\beta-1} f(\tau,v(\tau))d\tau\right) ds \right. \\
&\quad + \int_0^1 G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_s^1 (\tau-s)^{\beta-1} f(\tau,v(\tau))d\tau\right) ds \\
&\quad \left. - \int_0^1 G(t,s)\varphi_q\left(\frac{1}{\Gamma(\beta)}\int_s^1 (\tau-s)^{\beta-1} f(\tau,u(\tau))d\tau\right) ds \right| \\
&\leq |v(t) - Tv(t)| + |Tv(t) - Tu(t)| \\
&\leq \frac{M_0^{q-2}\delta(q-1)}{\Gamma(\beta+2)}\varepsilon + \frac{LM^{q-2}\delta(q-1)}{\Gamma(\beta+2)}\|v-u\|_\infty,
\end{aligned}$$

then

$$|v(t) - u(t)| \leq \frac{M_0^{q-2}\delta(q-1)}{\Gamma(\beta+2) - LM^{q-2}\delta(q-1)}\varepsilon, \quad t \in [0,1].$$

where $M_0 = M + \frac{1}{\Gamma(\beta+1)}$. Setting

$$k = \frac{M_0^{q-2}\delta(q-1)}{\Gamma(\beta+2) - LM^{q-2}\delta(q-1)},$$

we obtain

$$|v(t) - u(t)| \leq k\epsilon. \tag{4.2}$$

Therefore, the problem (P) is UH stable.

Remark 2.3 *By setting $\theta_f(\epsilon) = k\epsilon$ in (4.2), we obtain $\theta_f(0) = 0$ and then Problem (P) is GUH stable.*

Theorem 2.7 *Assume that $1 < p \leq 2$. Suppose that the conditions $(H1)$ and $(H2)$ holds. If*

$$\frac{LM^{q-2}\delta(q-1)}{\Gamma(\beta+2)} < 1,$$

and if there exists a constant $k_{\theta f} > 0$ such that

$$\frac{M_0^{q-2}(q-1)}{\Gamma(\beta)} \int_0^1 G(t,s) \left[\int_s^1 (\tau - s)^{\beta-1} |\Phi(\tau)| d\tau \right] ds \leq \epsilon k_{\theta f} \theta_f(t), \quad t \in [0,1],$$

where $\theta_f \in C([0,1], \mathbb{R}^+)$. Then the problem (P) is UHR stable.

Proof. Suppose v is a solution of the following inequality

$$\left| {}^C D_1^\beta - (\phi_p(D_{0+}^\alpha v(t))) + f(t, v(t)) \right| \leq \epsilon \theta_f(t), \text{ for all } t \in [0,1]. \tag{4.3}$$

From Remark 2.9, we have

$$v(t) = \int_0^1 G(t,s) \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} (f(\tau, v(\tau)) - \Phi(\tau)) \, d\tau \right) ds.$$

On the other hand, for $\varepsilon \in (0,1]$, we have

$$\left| \frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} (f(\tau, v(\tau)) - \Phi(\tau)) \, d\tau \right| \leq M_0,$$

then

$$\begin{aligned}
|v(t) - Tv(t)| &\leq \int_0^1 G(t,s) \left| \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} (f(\tau, v(\tau)) - \Phi(\tau)) \, d\tau \right) - \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, v(\tau)) \, d \right. \\
&\leq \frac{M_0^{q-2}(q-1)}{\Gamma(\beta)} \int_0^1 G(t,s) \int_s^1 (\tau - s)^{\beta-1} |\Phi(\tau)| \, d\tau ds \\
&\leq k_{\theta_f} \varepsilon \theta_f(t).
\end{aligned}$$

Suppose v is a solution of the inequality (4.1). We have :

$$\begin{aligned}
|v(t) - u(t)| &= \left| v(t) - \int_0^1 G(t,s) \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u(\tau)) \, d\tau \right) ds \right| \\
&\left| \begin{aligned} &= v(t) - \int_0^1 G(t,s) \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, v(\tau)) \, d\tau \right) ds \\
&\quad + \int_0^1 G(t,s) \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, v(\tau)) \, d\tau \right) ds \\
&\quad - \int_0^1 G(t,s) \phi_q \left(\frac{1}{\Gamma(\beta)} \int_s^1 (\tau - s)^{\beta-1} f(\tau, u(\tau)) \, d\tau \right) ds \end{aligned} \right. \\
&\leq |v(t) - Tv(t)| + |Tv(t) - Tu(t)| \\
&\leq k_{\theta_f} \epsilon \theta_f(t) + \frac{L M^{q-2} \delta (q-1)}{\Gamma(\beta + 2)} \|v - u\|_\infty,
\end{aligned}$$

then :

$$|v(t) - u(t)| \leq \frac{\Gamma(\beta+2)k_{\theta_f}}{\Gamma(\beta+2) - LM^{q-2}\delta(q-1)}\epsilon\theta_f(t), \quad t \in [0,1].$$

setting

$$d = \frac{\Gamma(\beta+2)k_{\theta f}}{\Gamma(\beta+2) - LM^{q-2}\delta(q-1)},$$

we obtain

$$|v(t) - u(t)| \leq \epsilon d\theta_f(t), \quad t \in [0,1]. \tag{4.4}$$

Therefore, the problem (P) is UHR stable.

Remark 2.4 *By putting $\epsilon = 1$ in (4.4), we deduce that problem (P) is GUHR stable.*

5 Illustrative examples

5.1 Example

Consider the fractional boundary value problem

$$(P2) \begin{cases} CD_1^{3/5}(\phi_2(D_{0+}^{4/3}u(t))) + \frac{1}{100}(1+t)(1+|u(t)|^{1/5}) = 0, & t \in [0,1], \\ \phi_2(D_{0+}^{4/3}u(1)) = 0, & t^{2/3}u(t)|_{t=0} = 0, \\ u(1) = \lambda \int_0^{1/2} u(s)ds. \end{cases}$$

We have $\alpha = \frac{4}{3}, \beta = \frac{3}{5}, p = 3, q = \frac{3}{2}, \eta = \frac{1}{2}, \lambda = 2, \delta = 3.8868$ and

$$f(t,u) = \frac{1}{100}(1+t)(1+|u(t)|^{1/5}), \quad t \in [0,1], u \in \mathbb{R},$$

satisfy

$$|f(t,u)| \leq \frac{1}{100}(1+t) + \frac{1}{100}(1+t)|u|^{\frac{1}{5}}.$$

Then, we get $v = \frac{1}{5}, A_1 = A_2 = \frac{1}{50}$ and $R = \frac{1}{2} \geq \max\{1.0683, 1.1163\}$ such that condition (H3) hold. By Theorem (3.4), we get that the problem $(P2)$ has at least one solution.

5.2 Example

Consider the following boundary value problem

$$(P3) \begin{cases} CD_1^{1/2}(\phi_3/2(D_{0+}^{3/2}u(t))) + \frac{1}{10}(1-t)^2\left(\frac{1+|u(t)|}{2+|u(t)|}\right) = 0, & t \in [0,1], \\ \phi_3/2(D_{0+}^{3/2}u(1)) = 0, & t^{1/2}u(t)|_{t=0} = 0, \\ u(1) = \lambda \int_0^{1/2} u(s)ds, \end{cases}$$

Here, we have $\alpha = \frac{3}{2}, \beta = \frac{1}{2}, p = \frac{3}{2}, q = 3, \eta = \frac{1}{2}, \lambda = 4, \delta = 40.588$ and

$$f(t,u) = \frac{1}{10}(1-t)^2\left(\frac{1+|u|}{2+|u|}\right), \quad t \in [0,1], u \in \mathbb{R}.$$

There exists a function $w(t) = \frac{1}{10}(1-t)^2 \geq 0$, for all $t \in [0,1]$. Such that

$$|f(t,u(t))| \leq w(t), \quad t \in [0,1],$$

and

$$|f(t,u) - f(t,v)| = \left|\frac{1}{10}(1-t)^2(|u|-|v|)\right| \leq \frac{1}{10}|u-v|,\ t \in [0,1],\ u,v \in \mathbb{R}.$$

In view of Theorem (2.4)

$$\frac{LM^{q-2}\delta(q-1)}{\Gamma(\beta+2)} = 0.45936 < 1.$$

Therefore, we conclude that the problem $(P3)$ has a unique solution. Similarly, this implies that the solution of problem $(P3)$ is UH stable with $k = 474.35$ and hence GUH stable. By setting

$$\theta_f(t) = 1.1284 + 39.46t^{1/2},\ t \in [0,1],$$

we ensure that the conditions of Theorem (2.6) are satisfied. So, the problem $(P3)$ is UHR stable by respect to θ_f with $d = 474.35$ and also, GUHR stable.

Chapitre 3

ON STABILITY FOR NONLINEAR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS

3.1 Introduction

$$^{c}D^{\alpha}y(t) = f(t, y(t),^{c} D^{\alpha}y(t)),\ \forall\, t \in J,\ 0 < \alpha \leq 1, \tag{1}$$

$$y(0) = y_0, \tag{2}$$

where $^{c}D^{\alpha}$ is the Caputo fractional derivative, $f : J\mathbb{R}\mathbb{R} \to \mathbb{R}$ is a given function space, $y_0 \in \mathbb{R}$ and $J = [0, T]$, $T > 0$.

Lemma 3.1 ([2]) *Let $\alpha \geq 0$ and $n = \lfloor \alpha \rfloor + 1$. Then*

$$I^{\alpha}(^{c}D^{\alpha}f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{f^{k}(0)}{k!}t^{k}.$$

We state the following generalization of Gronwall's lemma for singular kernels.

Lemma 3.2 ([8]) *Let $\nu\colon [0, T] \to [0, +\infty)$ be a real function and $w(\cdot)$ is a nonnegative, locally integrable function on $[0, T]$ and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$\nu(t) \leq w(t) + a\int_0^t (t-s)^{-\alpha}\nu(s)\, ds,$$

Then, there exists a constant $K = K(\alpha)$ such that

$$\nu(t) \leq w(t) + Ka\int_0^t (t-s)^{-\alpha}w(s)\, ds,\ \textit{for every } t \in [0, T].$$

We adopt the definitions in Rus [10] : Ulam-Hyers stability, generalized Ulam-Hyers stability, Ulam-Hyers-Rassias stability and generalized Ulam-Hyers-Rassias stability for the equation, for the implicit fractional-order differential equation (1).

Definition 3.1 *The equation (1) is* Ulam-Hyers stable *if there exists a real number* $c_f > 0$ *such that for each* $\epsilon > 0$ *and for each solution* $z \in C^1(J, \mathbb{R})$ *of the inequality*

$$|^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \epsilon, \quad t \in J, (3)$$

there exists a solution $y \in C^1(J, \mathbb{R})$ *of equation (1) with*

$$z(t) - y(t) \leq c_f \epsilon, \quad t \in J.$$

3.2 Properties

01/The equation (1) is *generalised Ulam-Hyers stable* if there exists $\psi_f \in C(\mathbb{R}_+, \mathbb{R}_+)$, $\psi_f(0) = 0$, such that for each solution $z \in C^1(J, \mathbb{R})$ of the inequality (3) there exists a solution $y \in C^1(J, \mathbb{R})$ of the equation (1) with

$$z(t) - y(t) \leq \psi_f(\epsilon), \quad t \in J.$$

02/ The equation (1) is *Ulam-Hyers-Rassias stable* with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_f > 0$ such that for each $\epsilon > 0$ and for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$${}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t)) \leq \epsilon\varphi(t), \quad t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1) with

$$|z(t) - y(t) \leq c_f \epsilon \varphi(t)|, \quad t \in J.$$

03/ The equation (1) is *generalised Ulam-Hyers-Rassias stable* with respect to $\varphi \in C(J, \mathbb{R}_+)$ if there exists a real number $c_{f,\varphi} > 0$ such that for each solution $z \in C^1(J, \mathbb{R})$ of the inequality

$$|^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \varphi(t), \ t \in J,$$

there exists a solution $y \in C^1(J, \mathbb{R})$ of equation (1) with

$$|z(t) - y(t)| \leq c_{f,\varphi} \varphi(t), \ t \in J.$$

Remark 3.1 *A function* $z \in C^1(J, \mathbb{R})$ *is a solution of the inequality (3) if and only if there exists a function* $g \in C(J, \mathbb{R})$ *(which depends on y) such that*

(i) $|g(t)| \leq \epsilon, \forall t \in J.$

(ii) ${}^cD^\alpha z(t) = f(t, z(t), {}^cD^\alpha z(t)) + g(t), \ t \in J.$

Remark 3.2 *Clearly,*

(i) Definition 2.5 $\Longrightarrow$ *Definition 2.6.*

(ii) Definition 2.7 $\Longrightarrow$ *Definition 2.8.*

Remark 3.3 *A solution of the implicit differential inequation (3) with fractional order is called a fractional* ϵ*-solution of the implicit fractional differential equation (1).*

So, the Ulam stabilities of the implicit differential equations with fractional order are some special types of data dependence of the solutions of fractional implicit differential equations.

3.3 Existence and Ulam-Hyers Stability

Definition 3.2 *A function* $u \in C^1(J)$ *is said to be a solution of the problem (1)–(2) if* u *satisfies equation (1) and condition (2) on* J.

Lemma 3.3 *Let a function* $f(t, u, v)\colon J\mathbb{R}\mathbb{R} \to \mathbb{R}$ *be continuous. Then the problem (1)–(2) is equivalent to the problem*

$$y(t) = y_0 + I^\alpha g(t),$$

where $g \in C(J, \mathbb{R})$ *satisfies the functional equation*

$$g(t) = f(t, y_0 + I^\alpha g(t), g(t)).$$

Proof. If ${}^{\varepsilon}D^\alpha y(t) = g(t)$ then $I^\alpha\, {}^{\varepsilon}D^\alpha y(t) = I^\alpha g(t)$. So we obtain $y(t) = y_0 + I^\alpha g(t)$.

Lemma 3.4 *([17]) Assume*

(H1) The function $f : J\mathbb{R}\mathbb{R} \to \mathbb{R}$ *is continuous.*

(H2) There exist constants $K > 0$ *and* $0 < L < 1$ *such that*

$$|f(t, u, v) - f(t, \bar{u}, \bar{v})| \leq K|u - \bar{u}| + L|v - \bar{v}| \text{ for any } u, v, \bar{u}, \bar{v} \in \mathbb{R} \text{ and } t \in J.$$

If

$$\frac{KT^\alpha}{(1 - L)\Gamma(\alpha + 1)} < 1,$$

then there exists a unique solution for the IVP (1)-(2) on J.

Theorem 3.1 *Assume that the assumptions (H1), (H2) and (7) hold. Then the equation (1) is Ulam-Hyers stable.*

Proof. Let $z \in C(J, \mathbb{R})$ be a solution of the inequation (3), i.e.

$$|{}^{\varepsilon}D^{\alpha}z(t) - f(t, z(t), {}^{\varepsilon}D^{\alpha}z(t))| \le \varepsilon,\ \ t \in J.$$

Let us denote by $y \in C(J, \mathbb{R})$ the unique solution of the Cauchy problem

$$
\begin{gathered}
{}^{\varepsilon}D^{\alpha}y(t) = f(t, y(t), {}^{\varepsilon}D^{\alpha}y(t)),\ \forall\, t \in J,\ 0 < \alpha \le 1, \\
y(0) = z(0).
\end{gathered}
$$

By using Lemma 3.2, we have

$$y(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s) ds,$$

where $g_y \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_y(t) = f(t, y(0) + t^{\alpha} g_y(t), g_y(t)).$$

But, by integration of the formula (8) we obtain

$$
\begin{aligned}
\left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s) ds \right| &\le \frac{\varepsilon t^{\alpha}}{\Gamma(\alpha+1)} \\
&\le \frac{\varepsilon \tau^{\alpha}}{\Gamma(\alpha+1)},
\end{aligned}
$$

where $g_z \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_z(t) = f(t, z(0) + I^{\alpha} g_z(t), g_z(t)).$$

On the other hand, we have, for each $t \in J$

$$
\begin{aligned}
|z(t) - y(t)| &= \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)\, ds \right| \\
&= \Bigg| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left(g_z(s) - g_y(s)\right)\, ds \Bigg| \\
&\le \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)\, ds \right| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |g_z(s) - g_y(s)|\, ds,
\end{aligned}
$$

where

$$g_y(t) = f(t, y(t), g_y(t)),$$

and

$$g_z(t) = f(t, z(t), g_z(t)).$$

By (H2), we have, for each $t \in J$

$$\begin{aligned} |g_z(t) - g_y(t)| &= |f(t, z(t), g_z(t)) - f(t, y(t), g_y(t))| \\ &\leq K|z(t) - y(t)| + L|g_z(t) - g_y(t)|. \end{aligned}$$

Then

$$|g_z(t) - g_y(t)| \leq \frac{K}{1-L}|z(t) - y(t)|. \tag{11}$$

Thus, by (9), (10), and (11) we get

$$|z(t) - y(t)| \leq \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} + \frac{K}{(1-L)\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}|z(s) - y(s)|\, ds.$$

Then Lemma 2.4 implies that for each $t \in J$

$$|z(t) - y(t)| \leq \frac{\epsilon T^\alpha}{\Gamma(\alpha+1)} \left[1 + \frac{\gamma K T^\alpha}{(1-L)\Gamma(\alpha+1)}\right] := c\epsilon, \tag{12}$$

where $\gamma = \gamma(\alpha)$ is a constant. So, the equation (1) is Ulam-Hyers stable. This completes the proof.

By putting $\psi(\epsilon) = c\epsilon$, $\psi(0) = 0$ yields that the equation (1) is generalized Ulam-Hyers stable.

3.4 Ulam-Hyers-Rassias Stability

Theorem 3.2 *Assume (H1), (H2) and*

(H3)The function $\varphi \in C(J, \mathbb{R}_+)$ is increasing and there exists $\lambda_\varphi > 0$ such that, for each $t \in J$, we have

$$I^\alpha \varphi(t) \leq \lambda_\varphi \varphi(t).$$

Then the equation (1) is Ulam-Hyers-Rassias stable with respect to φ.

Proof. : Let $z \in C(J, \mathbb{R})$ be a solution of the inequation (4), i.e.

$$|{}^cD^\alpha z(t) - f(t, z(t), {}^cD^\alpha z(t))| \leq \varepsilon\varphi(t),\ t \in J,\ \varepsilon > 0. \tag{13}$$

Let us denote by $y \in C(J, \mathbb{R})$ the unique solution of the Cauchy problem

$$\begin{gathered} {}^cD^\alpha y(t) = f(t, y(t), {}^cD^\alpha y(t)),\ \forall\, t \in J,\ 0 < \alpha \leq 1, \\ y(0) = z(0). \end{gathered}$$

By using Lemma 3.2, we have

$$y(t) = z(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)ds,$$

where $g_y \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_y(t) = f(t, y(0) + I^\alpha g_y(t), g_y(t)).$$

But, by integration of the formula (13) and by (H3), we obtain

$$\begin{aligned} \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)ds \right| &\le \frac{\varepsilon}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \varphi(s)ds \\ &\le \varepsilon \lambda_\varphi \varphi(t), \end{aligned} \tag{14}$$

where $g_z \in C(J, \mathbb{R})$ satisfies the functional equation

$$g_z(t) = f(t, z(0) + I^\alpha g_z(t), g_z(t)).$$

On the other hand, we have, for each $t \in J$

$$\begin{aligned} |z(t) - y(t)| &= \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_y(s)ds \right| \\ &= \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)ds \right. \\ &\quad \left. + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} (g_z(s) - g_y(s))ds \right| \\ &\le \left| z(t) - z(0) - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g_z(s)ds \right| \end{aligned} \tag{15}$$

where

$$g_y(t) = f(t, y(t), g_y(t)),$$

and

$$g_z(t) = f(t, z(t), g_z(t)).$$

By $(H2)$, we have

$$\begin{aligned} |g_z(t) - g_y(t)| &= |f(t, z(t), g_z(t)) - f(t, y(t), g_y(t))| \\ &\le K|z(t) - y(t)| + L|g_z(t) - g_y(t)|. \end{aligned}$$

Then

$$|g_z(t) - g_y(t)| \le \frac{K}{1-L}|z(t) - y(t)|. \tag{16}$$

Thus, by (14), (15), and (16)

$$|z(t) - y(t)| \le \varepsilon\lambda_\varphi\varphi(t) + \frac{K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|z(s) - y(s)|\,ds.$$

Then Lemma 2.4 implies that, for each $t \in J$

$$|z(t) - y(t)| \le \varepsilon\lambda_\varphi\varphi(t) + \frac{\gamma_1\varepsilon\lambda_\varphi K}{(1-L)\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\varphi(s)\,ds, \tag{17}$$

where constant $\gamma_1 = \gamma_1(\alpha)$ is a constant.
Thus, by (H3) and (17), we obtain

$$\begin{aligned} |z(t) - y(t)| &\le \varepsilon\lambda_\varphi\varphi(t) + \frac{\gamma_1 K\varepsilon\lambda_\varphi^2\varphi(t)}{(1-L)} \\ &= \left(1 + \frac{\gamma_1 K\lambda_\varphi}{1-L}\right)\lambda_\varphi\varepsilon\varphi(t). \end{aligned}$$

Then, for each $t \in J$

$$|z(t) - y(t)| \le \left[\left(1 + \frac{\gamma_1 K\lambda_\varphi}{1-L}\right)\lambda_\varphi\right]\varepsilon\varphi(t) := c\,\varepsilon\varphi(t). \tag{18}$$

So, the equation (1) is Ulam-Hyers-Rassias stable. This completes the proof.

5 Examples

Example 5.1

Consider the following Cauchy problem

$${}^{\varepsilon}D^{\frac{1}{2}}y(t) = \frac{1}{100}(t\cos y(t) - y(t)\sin(t)) + \frac{|{}^{\varepsilon}D_t^{\frac{1}{2}}y(t)|}{50 + |{}^{\varepsilon}D_t^{\frac{1}{2}}y(t)|}, \quad \forall\, t \in [0,1], \tag{19}$$

$$y(0) = 1. \tag{20}$$

Set

$$f(t,u,v) = \frac{1}{100}(t\cos u - u\sin(t)) + \frac{v}{50+v}, \quad t \in [0,1],\ u,v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$:

$$\begin{aligned} |f(t,u,v) - f(t,\bar{u},\bar{v})| &\leq \frac{1}{100}|t||\cos u - \cos \bar{u}| \\ &+ \frac{1}{100}|\sin t||u-\bar{u}| + \frac{50|v-\bar{v}|}{(50+v)(50+\bar{v})} \\ &\leq \frac{1}{100}|u-\bar{u}| + \frac{1}{100}|u-\bar{u}| + \frac{1}{50}|v-\bar{v}| \\ &\leq \frac{1}{50}|u-\bar{u}| + \frac{1}{50}|v-\bar{v}|. \end{aligned}$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{50}$.
Thus condition

$$\frac{KT^{\alpha}}{(1-L)\Gamma(\alpha+1)} = \frac{\frac{1}{50}}{(1-\frac{1}{50})\Gamma(\frac{3}{2})} = \frac{2}{49\sqrt{\pi}} < 1,$$

is satisfied. It follows from Theorem 3.3 that the problem (19)-(20) has a unique solution on J, and from Theorem 3.4, equation (19) is Ulam-Hyers stable.

Example 5.2

Consider the following Cauchy problem

$$^{\varepsilon}D^{\frac{1}{2}}y(t) = \frac{2+|y(t)|+|^{\varepsilon}D^{\frac{1}{2}}y(t)|}{120e^{t+10}(1+|y(t)|+|^{\varepsilon}D^{\frac{1}{2}}y(t)|)}, \quad \forall t \in [0,1], \tag{21}$$

$$y(0) = 1. \tag{22}$$

Set

$$f(t,u,v) = \frac{2+|u|+|v|}{120e^{t+10}(1+|u|+|v|)}, \quad t \in [0,1],\ u, v \in \mathbb{R}.$$

Clearly, the function f is jointly continuous.
For any $u, v, \bar{u}, \bar{v} \in \mathbb{R}$ and $t \in [0,1]$

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \leq \frac{1}{120e^{10}}\big(|u-\bar{u}| + |v-\bar{v}|\big).$$

Hence condition (H2) is satisfied with $K = L = \frac{1}{120e^{10}}$.
Let $\varphi(t) = t^2$. We have

$$I^{\alpha}\varphi(t) \leq \frac{2}{\Gamma(\frac{7}{2})}t^2 := \lambda_{\varphi}\varphi(t).$$

Thus condition (H3) is satisfied with $\varphi(t) = t^2$ and $\lambda_\varphi = \frac{2}{\Gamma(\frac{7}{2})} = \frac{16}{15\sqrt{\pi}}$. It follows from Theorem 3.3 that the problem (21)-(22) has a unique solution on J, and from Theorem 4.1 equation (21) is Ulam-Hyers-Rassias stable.

Conclusion

In this thesis, we have explored various aspects of fractional differential equations, particularly those involving nonlinear operators such as the p-Laplacian. By utilizing advanced tools from fractional calculus, fixed point theory, and the concept of measures of noncompactness, we established significant results concerning the existence, uniqueness, and stability of solutions.

The study began with a comprehensive overview of the necessary mathematical background, including special functions, fractional integral and derivative operators, and foundational concepts from functional analysis. We then focused on the analytical treatment of mixed-type and nonlinear implicit fractional differential equations, where new results regarding Ulam-Hyers and Ulam-Hyers-Rassias stability were proven.

Through the application of well-known fixed point theorems—such as Banach's and Schauder's theorems—we demonstrated the effectiveness of these techniques in establishing robust results for complex fractional models. Moreover, we showed that the inclusion of the -Hilfer derivative and other generalized operators enriches the analytical framework and allows for broader applicability in modeling real-world phenomena.

Overall, the findings presented in this work contribute to the growing body of literature in the field of fractional calculus and its applications to nonlinear systems. The results obtained not only enhance theoretical understanding but also lay the groundwork for future studies involving numerical methods, further generalizations, or applications in physical and engineering problems.

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