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Blow-up of solutions of a logarithmic flexible structure system with second sound in the presence of delay term in time

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شكر وعرفان

نحمد الله ونشكره شكراً جزيلًا، إذ هو خالقنا ومعيننا، فهو الأول نلجأ إليه في كل الأوقات والظروف.
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*الى كل افراد عائلتي عطية و هواري كبير او صغيرا
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* الى جميع االساتذة الذين انارو لي دربي خلال سنوات دراستي
الى كل من وسعتهم ذاكرتي ولم تسعهم مذكرتي الى من سيتصفح هذه المذكرة يوما ما الى كل هؤلاء اهدي ثمرة جهدي

ملخص

إنفجار الحل لنظام هيكلي مرن. الهدف الأساسي من هذا العمل هو دراسة لوغارتمي مع الصوت الثاني في وجود حد تأخر زمني. يتكون العمل من فصلين مع مقدمة عامة وقائمة مراجع.

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الكلمات المفتاحية: معادلة الطاقة -التاخير الزمني - انفجار الحل -الوجود المحلي

Abstract

The main objective of this work is the study of the Blow-up of solutions to a Logarithmic Flexible Structure system with Second Sound in the presence of delay term in time. This work is composed of two chapters with general introduction and bibliography.

The first chapter deals with the preliminary notions on the theory of functional spaces, and some theorems and inequalities used in this research.

In the second chapter: we first use the multiplier method to find the energy functional then by using the direct method of Georgiev and Todorova we proved that the solution blows-up in a finite time.

Key words: energy functional - time delay - blow-up of solution - local existence.

Résumé

L'objectif essentiel de ce travail est l'étude de l'explosion de la solution d'un système de structure flexible logarithmique avec deuxième son en présence du terme de retard de temps. Le travail est composé de deux chapitres avec introduction générale et bibliographie.

Le premier chapitre est consacré aux notions préliminaires sur la théorie des espaces fonctionnels, et quelques théorèmes et inégalités utilisées dans cette mémoire.

Dans le deuxième chapitre : nous avons d'abord utilisé la méthode du multiplicateur pour trouver la fonctionnelle de l'énergie, ensuite on applique la méthode directe de Georgiev et Todorova. On a montré que la solution explose dans un temps fini.

Mots clés : Fonctionnelle de l'énergie - retard de temps - explosion de la solution - existence locale.

Contents

Chapter 1

General introduction

Nonlinear evolution equations are defined by the partial differential equations PDE's with time t as one of the independent variables, originate not only from many fields of mathematics, but also from other branches of science such as physics, mechanics and materials science, these equations appeared in the wake of the birth of the Differential calculation during 1890 occurred in the middle of the 17th century by brothers Bernoulli and Newton. More specifically ,the second order evolution problem can be encountered in several scientific fields and many engineering models, Within this context, we will explore a flexible structure system with second sound is one of the nonlinear evolution problems that refers to a type of material or structure where thermal energy is transported via wave-like phenomena, known as second sound, within a flexible or deformable medium. Second sound is a quantum mechanical phenomenon where heat transfer occurs as a wave, rather than through traditional diffusion. This effect is typically observed in superfluids like helium II and certain low-dimensional materials see [16, 21]

The direct method introduced by Georgiev and Todorova is a powerful technique for establishing finite-time blow-up results for solutions to certain nonlinear wave equations. This approach involves analyzing the energy functional associated with the system and demonstrating that, under specific conditions, the energy decreases monotonically, leading to a blow-up in finite time For a comprehensive understanding of this method and its applications, you may refer to the following resources:"Blow Up of Solutions of the Cauchy Problem for a Wave Equation with Nonlinear Damping and Source Terms" by Georgiev and Todorova[13] This foundational paper presents the direct method and applies it to prove finite-time blow-up results for solutions to wave equations with nonlinear damping and source terms. "Blow up of solution for the Kelvin-Voigt type wave equation with Balakrishnan-Taylor damping and acoustic boundary" by Sarra and Zarai This work utilizes the Georgiev-Todorova method to establish finite-time blow-up results for a specific wave equation, providing insights into the application of the direct method in more complex scenarios."Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation" by Messaoudi in [14] this paper investigates the blow-up properties of solutions to a nonlinear viscoelastic hyperbolic equation, employing the direct method to handle cases with negative initial energy. These resources offer detailed explanations and applications of the Georgiev-Todorova direct method, enhancing the understanding of blow-up phenomena in nonlinear wave equations.

A **delay term in time**, also referred to as a *time-delay term*, is a functional component within a differential equation in which the evolution of the system at a given time depends explicitly on its **past states**. Mathematically, a time-delay term involves arguments of the form $u(t-\tau)$, where $\tau > 0$ denotes the **delay parameter**. In the context of partial differential equations (PDEs), such terms often appear as $u(x, t-\tau)$ or $u_t(x, t-\tau)$, and are used to model **memory effects**, **signal propagation delays**, or **feedback mechanisms** where the response is not instantaneous. Time delays introduce **infinite-dimensional dynamics**, even in systems originally described by finite-dimensional ordinary differential equations (ODEs), and can significantly influence the **stability**, **controllability**, and **long-term behavior** of solutions , We may refer to some related studies in this context[17, 12, 18]

In this work([2]), we consider the vibrations of an inhomogeneous flexible structure system with a constant internal delay and logarithmic nonlinear source term

$$\begin{cases} m\left(x\right)u_{tt} - \left(p(x)u_x + 2\delta\left(x\right)u_{tx}\right)_x + \eta\theta_x \\ \quad +\mu u_t(x, t-\tau_0) = u\left|u\right|^{p-2}\ln\left|u\right|^{\gamma}, & x \in \left(0, L\right), t > 0, \\ \theta_t + kq_x + \eta u_{tx} = 0 & x \in \left(0, L\right), t > 0, \\ \tau q_t + \beta q + k\theta_x = 0 & x \in \left(0, L\right), t > 0, \end{cases} \tag{1.0.1}$$

with boundary conditions

$$u\left(0, t\right) = u\left(L, t\right) = 0; \theta\left(0, t\right) = \theta\left(L, t\right) = 0, t \geq 0, \tag{1.0.2}$$

and initial conditions

$$u\left(x, 0\right) = u_0\left(x\right), u_t\left(x, 0\right) = u_1\left(x\right); \theta\left(x, 0\right) = \theta_0\left(x\right); q\left(x, 0\right) = q_0\left(x\right), x \in [0, L] \tag{1.0.3}$$

Where $u\left(x, t\right)$is the displacement of a particle at position $x \in [0, L]$ and the time $t > 0.\eta > 0$ is the coupling constant depending on the heating effect,$p \geq 2$,γ, β and k

are positive constants,μ is a real number.$\tau > 0$ is the relaxation time describing the time lag in the response for the temperature and $\tau_0 > 0$ represents the time delay, in particular if $\tau = 0, (1.1)$ reduces to the viscothermoelastic system with delay, in which the heat flux is given by Fourier's law instead of Cattaneo's law, where$q = q(x,t)$ is the heat flux,$m(x), \delta(x)$ and $p(x)$ are responsible for the inhomogeneous structure of the beam,and, respectively, denote mass per unit length of structure,coefficient of internal material damping (viscoelastic property) and a positive function related to the stress acting on the body at a point x.The model of heat condition, originally due to Cattaneo, is of hyperbolic type. We recall the assumptions of $m(x), \delta(x)$ and $p(x)$ in [3, 7]such that:

$$m, \delta, p \in W^{1,\infty}(0, L), m(x), \delta(x) \text{ and } p(x) > 0, \forall x \in [0, L] \tag{1.0.4}$$

In these kinds of problems, G. C. Gorain [8] in 2013 has established uniform exponential stability of the problem

$$m(x) u_{tt} - (p(x)u_x + 2\delta(x) u_{tx})_x = f(x), on \ (0, L) \times \mathbb{R}^+,$$

which describes the vibrations of an inhomogeneous flexible structure with an exterior disturbing force f.More recently,[15], showed the exponential stability of the vibrations of a inhomogeneous flexible structure with thermal effect governed by the Fourier law,

$$\begin{aligned} m(x) u_{tt} - (p(x)u_x + 2\delta(x) u_{tx})_x - k\theta_x &= f(x) \\ \theta_t - \theta_{xx} - ku_{xt} &= 0 \end{aligned}$$

In addition,in[19],R., Racke studied the exponential stability in linear and nonlinear 1d of Thermoelasticity system with second sound given by

$$\left\{ \begin{array}{c} m(x) u_{tt} - (p(x)u_x + 2\delta(x) u_{tx})_x - k\theta_x = 0, on \ (0, L) \times \mathbb{R}^+ \\ \theta_t + kq_x + \eta u_{tx} = 0, on \ (0, L) \times \mathbb{R}^+ \\ \tau q_t + \beta q + k\theta_x = 0, on \ (0, L) \times \mathbb{R}^+ \end{array} \right. \tag{1.0.5}$$

See in this regard,Refs.[1, 4, 6, 10, 20],for the same problem above,Alves,Gamboa and Gorain proved that system (1.5) is polynomial decaysee[3])with boundary and initial conditions

$$\begin{array}{c} u(0,t) = u(L,t) = 0; \theta(0,t) = \theta(L,t) = 0, t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x); \\ \theta(x,0) = \theta_0(x); q(x,0) = q_0(x), x \in [0, L] \end{array}$$

We know that the dynamic systems with delay terms have become a major research subject in differential equation since the 1970s of the last century. The delay effect that is similar to memory processes is important in the research of applied mathematics such as physics, non-instant transmission phenomena and biological motivations.the model(1.5) is related to the following problem with delay terms

$$\begin{cases} \begin{array}{ll} m(x)u_{tt} - (p(x)u_x + 2\delta(x)u_{tx})_x \\ +\eta\theta_x + \mu u_t(x, t-\tau_0) = 0 & x \in (0, L), t > 0, \\ \theta_t + kq_x + \eta u_{tx} = 0 & x \in (0, L), t > 0, \\ \tau q_t + \beta q + k\theta_x = 0 & x \in (0, L), t > 0, \end{array} \\ u(0, t) = u(L, t) = 0; \theta(0, t) = \theta(L, t) = 0, t \geq 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x); \\ \theta(x, 0) = \theta_0(x); q(x, 0) = q_0(x), x \in [0, L] \end{cases} \tag{1.0.6}$$

The authors prove that the system (1.0.6) is wellposed, and exponential decay under a small condition on time delaysee[7]. .Now in the presence of source terme,the system(1.0.6) become the system studied in this work with logarithmic source term,this type of problems is encountered in many branches of physics such as NuclearPhysics, Optics and Geophysics. It is well known, from the Quantum Field Theory, that such kind of logarithmic nonlinearity appears naturally in inflation cosmology and in supersymmetric field theories (see [5, 9, 11]).

The thes is organized as follows.General introduction with two chapter In the first one ,we introduced and stated without proofs some important materials must be need in the proof.In chaptre 3,we proved the blow up of solutions of an inhomogeneous flexible structure system with a constant internal delay and logarithmic nonlinear source term and bibliography.

Chapter 2

Preliminaries

2.1 Functional Spaces

2.1.1 Normed spaces and Banach spaces

Definition 2.1 *The linear space V is endowed by a binary operation $(v_1, v_1) \to v_1 + v_1 : V \times V \to V$ which makes it a commutative group and furthermore it is equipped with a multiplication $(a, x) \to ax : R \times V \to V$ satisfying*

$$\begin{aligned} (a_1 + a_2) v &= av_1 + av_2, a(v_1 + v_2) \\ &= av_1 + av_2, (a_1 a_2) v \\ &= a_1 (a_2 v) \end{aligned}$$

and

$$1.v = v.$$

Definition 2.2 *Let V be linear space. A non-negative, degree-1 homogeneous, subadditive functional $\|.\|_v : V \to \mathbb{R}$ called a norm if it vanishes only at 0, often, we will write briefly $\|.\|$ instead of$\|.\|_v$, if the following properties are satisfing respectively:*

$$\begin{aligned} \|v\| &\geq 0 \\ \|av\| &= |a| \|v\| \\ \|u + v\| &\leq \|u\| \|v\| \end{aligned}$$

for any $v \in V$ and $a \in \mathbb{R}$ and $\|v\| = 0 \Rightarrow v = 0$. A linear space equipped with a norm is called a normed linear space . If the last property (i.e.$\|v\|_v = 0 \Rightarrow v = 0$) is missing, we call such a functional a seminorm.

Definition 2.3 *A Banach space is a complete normed linear spaceV. Its dual space V' is the linear space of all continuous linear functional $u : V \to \mathbb{R}$. Notation 1.1.4 We can consider the linear space $\ell(V, \mathbb{R})$, being also denoted by V' and called the dual space to V. The original space V is then called predual to V'.*

Proposition 2.1 *V' equipped with the norm $\|.\|_{v'}$ defined by*

$$\|u\|_{v'} = \sup \{|u(x)| : \|x\| \leq 1\} \tag{2.1.1}$$

is also a Banach space. If V is a Banach space such that, for any

$$v \in V, V \to \mathbb{R} : u \Rightarrow \|u + v\|^2 - \|u - v\|^2$$

is linear, then V is called a Hilbert space. In this case, we define the inner product (also called scalar product) by

$$(u,v) = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 \tag{2.1.2}$$

Definition 2.4 *Since u is linear we see that*

$$u : X \to X'' \tag{2.1.3}$$

is a linear isometry of V onto a closed subspace of V'' , we denote this by

$$V \to V''. \tag{2.1.4}$$

when u cover V', we obtain a family $(\varphi_u)_{u\subset V'}$ of applications to V in $\mathbb{R}$

Proposition 2.2 *The weak star topology on V' is the weakest topology on $V'$for which every $(\varphi_x)_{x\in v}$ is continuous.*

Theorem 2.1 *Let V be Banach space. Then, V is reflexive, if and only if,*

$$B_V = \{x \in V : \|x\| \le 1\},$$

is compact with the weak topology $\sigma(V, V')$.

Corollary 2.1 *Every weakly convergent sequence in V' must be bounded if V is a Banach space. In particular, every weakly convergent sequence in a reflexive Banach V must be bounded.*

Definition 2.5 *Let V be a Banach space and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in V. Then u_n converges strongly to u in V if and only if*

$$\lim_{t\to\infty} \|u_n - u\|_v = 0, \tag{2.1.5}$$

and this is denoted by $u_n \to u$, or $\lim_{t\to\infty} u_n = u$.

2.1.2 Hilbert space

Definition 2.6 *A Hilbert space H is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let H complete.*

Theorem 2.2 ***(Riesz)*** *If $(H; \langle ., .\rangle)$ is a Hilbert space, $\langle ., .\rangle$ being a scalar product on H; then $H' = H$ in the following sense: to each $f \in H'$ there corresponds a unique $x \in H$ such that $f = \langle x, .\rangle$and $\|f\|'_H = \|x\|_H$.*

Remark 2.1 *From this theorem we deduce that $H'' = H$. This means that a Hilbert space is reflexive.*

Theorem 2.3 *Let $(u_n)_{n\in N}$ is a bounded sequence in the Hilbert space H; it posses a subsequence which converges in the weak topology of H*

Theorem 2.4 *In the Hilbert space, all sequence which converges in the weak topology is bounded.*

Theorem 2.5 *Let $(u_n)_{n\in N}$ be a sequence which converges to u, in the weak topology and $(v_n)_{n\in N}$ is an other sequence which converge weakly tov ; then*

$$\lim_{n\to\infty} \langle v_n, u_n\rangle \tag{2.1.6}$$

Theorem 2.6 *Let X be a normed space, then the unit ball*

$$B' \equiv \{x \in X : \|x\| \le 1\} \tag{2.1.7}$$

of X' is compact in $\sigma(X', X)$.

2.1.3 The $L^p(\Omega)$ spaces

Definition 2.7 *Let $1 \leq p \leq \infty$ and let Ω be an open domain in $\mathbb{R}^n$;$n \in \mathbb{N}$ Define the standard Lebesgue weth space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |f(x)|^p \, dx < \infty \right\} \tag{2.1.8}$$

Notation 2.1 *If $p = \infty$; we have*

$$L^\infty(\Omega) = \{ f : \Omega \to \mathbb{R} \text{ is measurable and there exists a constant } C \text{ suchthat } |f(x)| \leq C \text{that is } \in \Omega \}.$$

Also, we denote by

$$\|f\|_\infty = \inf \{ C, |f(x)| \leq C \text{ a.e} \in \Omega \} \tag{2.1.9}$$

Notation 2.2 *For $p \in \mathbb{R}$ and $1 \leq p \leq \infty$; we denote by q the conjugate of p i.e. $\frac{1}{P} + \frac{1}{q} = 1$.*

Theorem 2.7 *$L^p(\Omega)$ is a Banach space for all $1 \leq p \leq \infty$.*

Remark 2.2 *In particular, when $P = 2$; $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g \rangle_{L^2(\Omega)} = \int_\Omega f(x) \, g(x) \, dx \tag{2.1.10}$$

is a Hilbert space.

Theorem 2.8 *For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space*

The $L^p(0, T, X)$ spaces

Let X be a Banach space, denote by $L^p(0, T, X)$ the space of measurable functions such that

$$\left(\int_0^T \|f(t)\|_X^p \, dt \right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X) < \infty, \text{for } 1 \leq p \leq \infty} \tag{2.1.11}$$

If $p = \infty$

$$\|f\|_{L^p(0,T,X)} = \sup_{t \in]0,T[} ess \, \|f(t)\|_X \tag{2.1.12}$$

Theorem 2.9 *The space $L^p(0, T, X)$ is complete.We denote by $D'(0, T, X)$ the space of distributions in $]0, T[$ which take its values in X and us define*

$$D'(0, T, X) = \pounds(D]0, T[, X) \tag{2.1.13}$$

where $\pounds(\phi, \varphi)$ is the space of the linear continuous applications of ϕ to φ Since $u \in D'(0, T, X)$; we define the distribution derivation as

$$\frac{\partial u}{\partial t}(\varphi) = -u\left(\frac{d\varphi}{dt}\right), \forall \varphi \in D(]0, T[) \tag{2.1.14}$$

and since, we have $u \in L^p(0, T, X)$

$$u(\varphi) = \int_0^T u(t) \, \varphi(t) \, dt, \forall \varphi \in D(]0, T[) \tag{2.1.15}$$

Lemma 2.1 *Let*

$$f \in L^p(0,T,X) \text{ and } \frac{\partial f}{\partial t} \in L^p(0,T,X), (1 \le p \le \infty)$$

then the function f is continuous from $[0,T]$ to X: i.e $f \in C^1(0,T,X)$

and $g_\mu \to g$ in φ ; then $g_\mu \to g$ in $L^q(\varphi)$

Theorem 2.10 *$L^p(0,T,X)$ equipped with the norm $\|.\|_{L^q(]0,T[,X)}$,$1 \le p \le \infty$ is a Banach space.*

Proposition 2.3 *Let X be a reflexive Banach space, X' it's dual, and $1 \le p\ q \le \infty \frac{1}{P} + \frac{1}{q} = 1$. Then the dual of $L^P(0,T,X)$ is identify algebraically and topologically with$L^q(0,T,X')$.*

Proposition 2.4 *Let X ; Y be Banach space, $X \subset Y$ with continuous embedding, then we have*

$$L^P(0,T,X) \subset L^P(0,T,Y) \tag{2.1.16}$$

with continuous embedding. The following compactness criterion will be useful for nonlinear evolution problem, especially in the limit of the nonlinear term

Definition 2.8 *Let $\Omega \subset \mathbb{R}^N$ be open and let $1 \le p \le \infty$, we say that a function $f : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if $f_{\chi k} \in L^p(\Omega)$ for every compact set K contained in Ω . Note that if $f \in L^p_{loc}(\Omega)$, then $f \in L^1_{loc}(\Omega)$.*

The spaces $\mathbf{C}^k(\Omega)$ et $\mathbf{C}^\infty(\Omega), \mathbf{0 \le k \le \infty}$

Definition 2.9 *We denote by $C(\Omega)$ where $C^0(\Omega)(resp.C^1(\Omega))$,the space of continuous functions (resp. continuously differentiable) on Ω with numerical values (i.e real or complex). For $k \in \mathbb{N}, k \ge 2$, we pose*

$$C^k(\Omega) = \left\{ u \in C^{k-1}(\Omega) : \frac{\partial u}{\partial x_i} \in C^{k-1}(\Omega); i = 1, ..., n \right\},$$

it is the space of k times continuously differentiable functions on Ω. Finally we note

$$C^\infty(\Omega) = \cap_{k \in \mathbb{N}} C^k(\Omega),$$

c'est l'espace des fonctions indéffiniment différentiables sur Ω.

2.1.4 Sobolev spaces

Modern theory of differential equations is based on spaces of function whose derivatives exist in a generalized sense and enjoy a suitable integrability.

Proposition 2.5 *Let Ω be an open domain in $\mathbb{R}^n$, Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)\,dx, forall \varphi \in D(\Omega)$$

where $1 \le p \le \infty$, and it's well-known that f is unique.

Definition 2.10 *Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. The$W^{m,p}(\Omega)$,p(W) is the space of all $f \in L^p(\Omega)$, defined as $W^{m,p}(\Omega)$, such that $\partial^\alpha f \in L^p(\Omega)$ for all $\alpha \in \mathbb{N}^m$ such that*

$$|\alpha| = \sum_{j=1}^{n} \alpha_j \leq m, where, \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} .. \partial_n^{\alpha_n}.$$

Theorem 2.11 *$W^{m,p}(\Omega)$ is a Banach space with their usual norm*

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p}, 1 \leq p \leq \infty, for\ all\ f \in W^{m,p}(\Omega)$$

Definition 2.11 *Denote by $W_0^{m,p}(\Omega)$ the closure of $D(\Omega)$ in $W^{m,p}(\Omega)$.*

Definition 2.12 *When $p = 2$, we prefer to denote by $W^{m,2}(\Omega) = H^m(\Omega)$ and $W_0^{m,2}(\Omega) = H_0^m(\Omega)$ supplied with the norm*

$$\|f\|_{H^m(\Omega)} = \left(\sum_{|\alpha| \leq m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}} \tag{2.1.17}$$

which do at $H^m(\Omega)$ a real Hilbert space with their usual scalar product

$$\langle u, v \rangle_{H^m(\Omega)} = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx$$

Theorem 2.12 *1.*

2. $H^m(\Omega)$ supplied with inner product $\langle .,. \rangle_{H^m(\Omega)}$ is a Hilbert space.

3. If $m \geq m'$, $H^m(\Omega) \to H^{m'}(\Omega)$, with continuous imbedding .

Lemma 2.2 *Since $D(\Omega)$ is dense in $H_0^m(\Omega)$, we identify a dual$H^{-m}(\Omega)$ of $H_0^m(\Omega)$ in a weak subspace on Ω, and we have*

$$D(\Omega) \to H_0^m(\Omega) \to L^2(\Omega) \to H^{-m}(\Omega) \to D'(\Omega)$$

The next results are fundamental in the study of partial differential equations

Theorem 2.13 *Assume that Ω is an open domain in $\mathbb{R}^n$ $(N \geq 1)$, with smooth boundary $\partial\Omega$. Then,*

1. If $1 \leq p \leq \infty$, we have $W^{1,p}$, for every $q \in [p, p^*]$, where $p^* = \frac{np}{n-p}$.
2. If $p = n$ we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, \infty)$.
3. If $p > n$ we have $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, where $\alpha = \frac{p-n}{p}$.
4. If Ω is a bounded, the embedding (2) and (3) of theorem 1.1.4 are compacts. The embedding (1) is compact for all $q \in [p, p^*)$

The $W^{m,p}(\Omega)$ spaces

Proposition 2.6 *Let Ω be an open domain in $\mathbb{R}^n$. Then the distribution $T \in D'(\Omega)$ is in $L^p(\Omega)$ if there exists a function $f \in L^p(\Omega)$ such that*

$$\langle T, \varphi \rangle = \int_{\Omega} f(x)\varphi(x)\,dx, \text{ for all } \varphi \in D(\Omega)$$

where $1 \leq p \leq \infty$ and it's well-known that f is unique. Now, we will introduce the Sobolev spaces: The Sobolev space $W^{K,p}(\Omega)$ is defined to be the subset of L^p such that function f and its weak derivatives up to some order K have a finite L^p norm, for given $p \geq 1$.

$$W^{k,p}(\Omega) = \{f \in L^p; D^\alpha f \in L^p(\Omega).\forall\alpha; |\alpha| \leq k\}.$$

With this definition, the Sobolev spaces admit a natural norm:

$$f \to \|f\|_{W^{k,p}(\Omega)} = \left(\sum_{|\alpha|\leq m} \|D^\alpha f\|^p_{L^p(\Omega)}\right)^{\frac{1}{p}}, for\ p < +\infty$$

and

$$f \to \|f\|_{W^{k,\infty}(\Omega)} = \sum_{|\alpha|\leq m} \|D^\alpha f\|_{L^\infty(\Omega)}, for\ p = +\infty$$

Space $W^{k,p}(\Omega)$ equipped with the norm $\|.\|_{W^{k,p}}$ is a Banach space. Moreover is a reflexive space for $1 \leq p \leq \infty$ and a separable space for $1 \leq p \leq \infty$. Sobolev spaces with $p = 2$ are especially important because of their connection with Fourier series and because they form a Hilbert space. A special notation has arisen to cover this case:

$$W^{k,2}(\Omega) = H^k(\Omega)$$

the H^k inner product is defined in terms of the L^2 inner product:

$$(f, g)_{H^k(\Omega)} = \sum_{|\alpha|\leq k} (D^\alpha f, D^\alpha g)_{L^2(\Omega)}$$

The space $H^m(\Omega)$ and $W^{k,p}(\Omega)$ contain $C^\infty\left(\bar{\Omega}\right)$ and $C^m\left(\bar{\Omega}\right)$. The closure of $D(\Omega)$ for the $H^m(\Omega)$ norm (respectively $W^{m,p}(\Omega)$ norm) is denoted by $H_0^m(\Omega)$ (respectively $W_0^{K,P}(\Omega)$). Now, we introduce a space of functions with values in a space X (a separable Hilbert space).The space$L^2(a, b; X)$ is a Hilbert space for the inner product

$$(f, g)_{L^2(a,b;X)} = \int_a^b (f(t), g(t))_X\,dt$$

wenote that$L^\infty(a, b; X) = \left(L^1(a, b; X)\right)'$. Now, we define the Sobolev spaces with values in a Hilbert space X. For $k \in N$, $p \in [1, \infty]$, we set:

$$W^{k,p}(a, b; X) = \left\{v \in L^p(a, b; X)\,\frac{\partial v}{\partial x_i} \in L^p(a, b; X).\forall i \leq k\right\}$$

The Sobolev space $W^{k,p}(a, b; X)$ is a Banach space with the norm

$$\|f\|_{W^{k,p}(a,b;X)} = \left(\sum_{i=0}^{k} \left\|\frac{\partial f}{\partial x_i}\right\|^p_{L^P(a,b;X)}\right)^{\frac{1}{p}} .for\ p < +\infty$$

and

$$\|f\|_{W^{k,\infty}(a,b;X)} = \left(\sum_{i=0}^{k} \left\| \frac{\partial v}{\partial x_i} \right\|_{L^{\infty}(a,b;X)} \right)^{\frac{1}{p}} .for\ p = +\infty$$

The spaces $W^{k,2}(a,b;X)$ form a Hilbert space and it is noted $H^k(0,T;X)$. The $H^k(0,T;X)$ inner product is defined by:

$$(u,v)_{H^k(a,b;X)} = \sum_{i=0}^{k} \int_a^b \left(\frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^i} \right)_X dt.$$

Theorem 2.14 *Let$1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{p^*}(\mathbb{R}^n)$$

where p^* is given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$ (where $p = n$; $p = 1$). Moreover there exists a constant $C = C(p,n)$ such that

$$\|u\|_{L^p} \leq C \|\nabla u\|_{L^p(\mathbb{R}^n)}, \quad \forall u \in W^{1,P}(\mathbb{R}^n).$$

Corollary 2.2 *Let $1 \leq p \leq n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [P, P^*]$$

with continuous imbedding. For the case $p = n$, we have

$$W^{1,n}(\mathbb{R}^n) \subset L^q(\mathbb{R}^n), \forall q \in [n, +\infty[$$

Theorem 2.15 *Let $p > n$, then*

$$W^{1,p}(\mathbb{R}^n) \subset L^{\infty}(\mathbb{R}^n)$$

with continuous imbedding.

Corollary 2.3 *Let Ω a bounded domain in $\mathbb{R}^n$ of C^1 class with $\Gamma = \partial\Omega$ and $1 \leq p \leq \infty$. We have*

1. *If $1 \leq p \leq \infty$, then $W^{1,p}(\Omega) \subset L^{p^*}(\Omega)$ where $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$*
2. *If $p = n$; then $W^{1,p}(\Omega) \subset L^q(\Omega), \forall q \in [p, +\infty[$*
3. *If $p > n$; then $W^{1,p}(\Omega) \subset L^{\infty}(\Omega)$ with continuous imbedding. Moreover, if $p > n$ we have:*

$$\forall u \in W^{1,p}(\Omega), |u(x) - u(y)| \leq C |x-y|^{\alpha} \|u\|_{W^{1,p}(\Omega)} \ a.e\ x, y \in \Omega$$

2.2 Some integral inequalities

we will give here some important inequalities. These inequalities play an important role in applied mathematics and also, it is very usefull in our next chaptre.

Young's inequality

Let p and q strictly positifs real numbers we define the Young's enequality by

$$\frac{1}{p}+\frac{1}{q}=1.$$

Then

$$\forall(a,b)\in IR^2 : |ab|\leq\frac{|a|^p}{p}+\frac{|b|^q}{q}.$$

Hölder's inequalities

Theorem 2.16 *Assume that $f\in L^p$ and $g\in L^q$ with $1\leq p\leq\infty$. Then $(fg)\in L^1$ and*

$$\|fg\|\leq\|f\|_p\|g\|_q$$

Corollary 2.4 *Let $f_1,f_2,...f_k$ be k functions such that, $f_i\in L^p(\Omega)$,$1\leq i\leq k$*

$$\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+...+\frac{1}{p_k}\leq 1 \tag{2.2.1}$$

Then, the produc $f_1f_2...f_k\in L^p(\Omega)$ and $\|f_1f_2...f_k\|_p\leq\|f_1\|_{p_1}...\|f_k\|_{p_k}$.

Minkowski inequality

Lemma 2.3 *For$1\leq p\leq\infty$, we have*

$$\|u+v\|_{L^p}\leq\|u\|_{L^p}+\|v\|_{L^p}. \tag{2.2.2}$$

Cauchy-Schwarz inequality

Lemma 2.4 *Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1,x_2\rangle\leq\|x_1\|\|x_2\|. \tag{2.2.3}$$

The equality sign holds if and only if x_1 and x_2 are dependent. We will give here some integral inequalities. These inequalities play an important role in applied mathematics and also, it is very useful in our next chapter.

Lemma 2.5 *Let$1\leq p\leq r\leq q$, $\frac{1}{r}=\frac{\alpha}{p}+\frac{1-\alpha}{q}$, and$1\leq\alpha\leq 1$. Then*

$$\|u\|_{L^r}\leq\|u\|_{L^p}^{\alpha}\|u\|_{L^q}^{1-\alpha}.$$

Lemma 2.6 *For all$a,b\in\mathbb{R}^+$, we have*

$$ab\leq\delta a^2+\frac{b^2}{4\delta'},$$

where δ is any positive constant.

Lemma 2.7 *For all $a, b \geq 0$, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q'}, \tag{2.2.4}$$

where, $\frac{1}{P} + \frac{1}{q} = 1$. With $\alpha = 1 - \frac{n}{p} > 0$ and C is a constant which depend on p, n and Ω In particular $W^{1,p}(\Omega) \subset C(\bar{\Omega})$

Lemma 2.8 ***(Sobolev-Poincaré's inequality)***

$$\text{If } \ 2 \leq q \leq \frac{2n}{n-2}, n \geq 3; and, n = 1, 2; \tag{2.2.5}$$

then

$$\|u\|_q \leq C(q, \Omega) \|\nabla u\|_2, \ \forall u \in H_0^1(\Omega) \tag{2.2.6}$$

Remark 2.3 *For all $\varphi \in H^2(\Omega), \Delta\varphi \in L^2(\Omega)$ and for Γ sufficiently smooth, we have*

$$\|\varphi(t)\|_{H^2(\Omega)} \leq C \|\Delta\varphi(t)\|_{L^2(\Omega)}$$

Proposition 2.7 ***(Green's formula)****. For all $u \in H^1(\Omega)$ we have*

$$-\int_\Omega \Delta u v dx = \int_\Omega \nabla u \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \eta} v d\sigma \tag{2.2.7}$$

where $\frac{\partial u}{\partial \eta}$ is a normal derivation of u at Γ

Chapter 3

Blow-up of solution of nonlinear flexible structure system with second sound in the presence of delay term in time and logarithmic source term

3.1 statement of problem

In this work, we consider the vibrations of an inhomogeneous flexible structure system with a constant internal delay and logarithmic nonlinear source term

$$\begin{cases} m(x)\,u_{tt} - (p(x)u_x + 2\delta(x)\,u_{tx})_x + \eta\theta_x \\ +\mu u_t(x, t-\tau_0) = u\,|u|^{p-2}\ln|u|^{\gamma}, & x \in (0,L), t > 0, \\ \theta_t + kq_x + \eta u_{tx} = 0 & x \in (0,L), t > 0, \\ \tau q_t + \beta q + k\theta_x = 0 & x \in (0,L), t > 0, \end{cases} \tag{3.1.1}$$

with boundary conditions

$$u(0,t) = u(L,t) = 0; \theta(0,t) = \theta(L,t) = 0, t \geq 0, \tag{3.1.2}$$

and initial conditions

$$u(x,0) = u_0(x), u_t(x,0) = u_1(x); \theta(x,0) = \theta_0(x); q(x,0) = q_0(x), x \in [0,L] \tag{3.1.3}$$

Where $u(x,t)$is the displacement of a particle at position $x \in [0,L]$ and the time $t > 0.\eta > 0$ is the coupling constant depending on the heating effect,$p \geq 2$,γ, β and k are positive constants,μ is a real number.$\tau > 0$ is the relaxation time describing the time lag in the response for the temperature and $\tau_0 > 0$ represents the time delay.

Let us introduce the function

$$z(x,\rho,t) = u_t(x, t-\rho\tau_0), x \in (0,L), \rho \in (0,1), t > 0$$

Thus, we have

$$\tau_0 z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, x \in (0,L), \rho \in (0,1), t > 0$$

Then, problem $(3.1.1)-(3.1.3)$ is equivalent to

$$\begin{cases} m(x)\,u_{tt}-(p(x)u_x+2\delta(x)\,u_{tx})_x+\eta\theta_x & \\ +\mu z(x,1,t)=u\,|u|^{p-2}\ln|u|^{\gamma}, & x\in(0,L),t>0, \\ \tau_0 z_t(x,\rho,t)+z_\rho(x,\rho,t)=0 & x\in(0,L),\rho\in(0,1),t>0, \\ \theta_t+kq_x+\eta u_{tx}=0 & x\in(0,L),t>0, \\ \tau q_t+\beta q+k\theta_x=0 & x\in(0,L),t>0, \end{cases} \tag{3.1.4}$$

With

$$\begin{cases} u(0,t)=u(L,t)=0;\theta(0,t)=\theta(L,t)=0,t\geq 0, \\ u(x,0)=u_0(x),u_t(x,0)=u_1(x);\theta(x,0)=\theta_0(x),x\in[0,L] \\ q(x,0)=q_0(x),x\in[0,L] \\ z(x,0,t)=u_t(x,t),x\in(0,L),t>0, \\ z(x,\rho,0)=f_0(x,-\rho\tau_0),x\in(0,L),\rho\in(0,1) \end{cases} \tag{3.1.5}$$

We first state a local existence theorem that can be established by combining the arguments of related works[10,6].

Let $v=u_t$ and denote by

$$\Phi=(u,v,\theta,q,z)^T,\Phi(0)=\Phi_0=(u_0,u_1,\theta_0,q_0,f_0)^T$$

The state space of Φ is the Hilbert space

$$\mathcal{F}=H_0^1(0,L)\times L^2(0,L)\times L_*^2(0,L)\times L^2((0,1)\times(0,L))$$

Theorem 3.1 *Assume that*

$$2<p\leq\frac{2n}{n-2},\ \text{if}\ n\geq 3.$$

Then for every $\Phi_0\in\mathcal{F}$,there exists a unique local solution in the class $\Phi\in C([0,T],\mathcal{F})$.

3.2 Blow-up of solution

In this section, we prove that the solutions for the problem $(3.1.1)-(3.1.3)$blows-up in a finite time when the initial energy is negative..We use the improved method of Salim and Messaoudi [20]. We define the energy associated with problem $(3.1.1)-(3.1.3)$ by

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\|m(x)\|_\infty\|u_t(t)\|_2^2\right)+\frac{1}{2}\left(\|p(x)\|_\infty\|u_x\|_2^2\right)+\frac{\tau}{2}\|q\|_2^2 \\ &+\frac{1}{2}\|\theta\|_2^2+\frac{\tau_0|\mu|}{2}\int_0^1\|z(x,\rho,t)\|^2\,d\rho+\frac{\gamma}{p^2}\|u\|_p^p \\ &-\frac{1}{p}\int_0^L|u|^p\ln|u|^{\gamma}\,dx \end{aligned} \tag{3.2.1}$$

Proof. In order to define the energy function $E(t)$ of problem $(3.1.4)-(3.1.5)$, we give the following computation. Multiplying the equations in $(3.1.4)$ by $u_t,\mu z(x,\rho,t),\theta,q$ respectively and integrating the resulting equation over $[0,L]$, we obtain

$$\begin{array}{c} \int_0^L m(x)u_{tt}u_t\,dx-\int_0^L(p(x)u_x+2\delta(x)u_{tx})_x u_t\,dx \\ +\int_0^L\eta\theta_x u_t\,dx+\int_0^L\mu z(x,1,t)u_t\,dx=\int_0^L u\,|u|^{p-2}\ln|u|^{\gamma}\,u_t\,dx \end{array} \tag{3.2.2}$$

$$\int_0^L\int_0^1\mu\,\tau_0\,z(x,\rho,t)\,.\,z_t(x,\rho,t)dx\,d\rho+\int_0^L\int_0^1\mu\,z(x,\rho,t)\,.\,z_\rho(x,\rho,t)dx\,d\rho=0 \tag{3.2.3}$$

$$\int_0^L \theta_t\,\theta\,dx + k\int_0^L q_x\theta\,dx + \eta\int_0^L u_{xt}\theta\,dx = 0 \tag{3.2.4}$$

$$\int_0^L q\,\tau\,q_t\,dx + \beta\int_0^L q^2 dx + k\int_0^L \theta_x\,q\,dx = 0 \tag{3.2.5}$$

we integrating by part over $[0, L]$ and using (3.1.5) and by using

$$\left\{ u_t u_{tt} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|u_t\right|^2 \right. \tag{$*$}$$

on the first term on the left side (3.2.2)we obtain

$$\int_0^L m(x)u_{tt}u_t dx = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \left|u_t\right|^2 m(x)dx \tag{3.2.6}$$

and by the same way by parts on the second and third terms on the left hand side of (3.2.2)

$$\begin{aligned}
-\int_0^L \left(p(x)u_x + 2\delta(x)u_{tx}\right)_x u_t dx &= \int_0^L u_t\left(p(x)u_x + 2\delta(x)u_{tx}\right)_x dx \\
&= -\left(\left[u_t\left(p(x)u_x + 2\delta(x)u_{tx}\right)\right]_0^L - \int_0^L u_{tx}\left(p(x)u_x + 2\delta(x)u_{tx}\right)dx\right) \\
&= -\left[\frac{\mathrm{d}}{\mathrm{d}t}u\left(L,t\right)\left(p(x)u_x + 2\delta(x)u_{tx}\right) - \frac{\mathrm{d}}{\mathrm{d}t}u\left(0,t\right)\left(p(x)u_x + 2\delta(x)u_{tx}\right)\right] \\
&\quad + \int_0^L u_{tx}p(x)u_x + 2u_{tx}\delta(x)u_{tx}dx \\
&= \int_0^L \nabla u_t\,p(x)\nabla u\,dx + 2\int_0^L \delta(x)\left|\nabla u_t\right|^2 dx
\end{aligned} \tag{3.2.7}$$

$$\begin{aligned}
-\int_0^L u_t\left(p(x)u_x + 2\delta(x)u_{tx}\right)_x dx &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L p(x)\left|\nabla u\right|^2 dx + 2\int_0^L \delta(x)\left|\nabla u_t\right|^2 dx \\
&= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|p(x)\|_\infty . \|\nabla u\|_2^2\right) + 2\left(\|\delta(x)\|_\infty . \|\nabla u_t\|_2^2\right)
\end{aligned} \tag{3.2.8}$$

$$\begin{aligned}
\int_0^L \eta\theta_x u_t dx &= \eta\int_0^L \theta_x u_t dx \\
&= \eta\left[\frac{\mathrm{d}}{\mathrm{d}t}u(L,t)\theta(L,t) - \frac{\mathrm{d}}{\mathrm{d}t}u(0,t)\theta(0,t)\right] - \eta\int_0^L u_{tx}\theta dx
\end{aligned}$$

with $u(L,t) = u(0,t) = \theta(L,t) = \theta(0,t) = 0$

$$\begin{aligned}
\int_0^L \eta\theta_x u_t dx &= \underbrace{\eta\left[\frac{\mathrm{d}}{\mathrm{d}t}u(L,t)\theta(L,t) - \frac{\mathrm{d}}{\mathrm{d}t}u(0,t)\theta(0,t)\right]}_{=0} - \eta\int_0^L \nabla u_t\theta dx \\
&= -\eta\int_0^L \nabla u_t\theta dx
\end{aligned} \tag{3.2.9}$$

With

$$\int_0^L \mu z(x,1,t)u_t dx \tag{3.2.10}$$

By using young's inequality in (3.2.6) we obtain

$$\int_0^L m(x)u_{tt}u_t dx \leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\|m(x)\|_\infty \cdot \|u_t\|^2\right) \tag{3.2.11}$$

$$|a.b| \leqslant \frac{|a|^2}{2} + \frac{|b|^2}{2}$$

$$\begin{aligned}
\int_0^L \mu z(x,1,t)u_t\,dx &\leqslant \int_0^L |\mu z(x,1,t)u_t|\,dx \\
&\leqslant |\mu| \int_0^L |z(x,1,t)u_t|\,dx \\
&\leqslant |\mu| \int_0^L \left[\frac{|z(x,1,t)|^2}{2} + \frac{|u_t|^2}{2}\right] dx \\
&\leqslant \frac{|\mu|}{2}\int_0^L |z(x,1,t)|^2\,dx + \frac{|\mu|}{2}\int_0^L |u_t|^2\,dx \\
&\leqslant \frac{|\mu|}{2}\int_0^L |z(x,1,t)|^2\,dx + \frac{|\mu|}{2}\|u_t(x,t)\|_2^2
\end{aligned} \tag{3.2.12}$$

now we treat the source terme

$$\begin{aligned}
\int_0^L u_t u\,|u|^{p-2}\ln|u|^\gamma\,dx &= \int_0^L \frac{1}{p}\frac{d}{dt}\{|u|^p\}\ln|u|^\gamma\,dx \\
&= \int_0^L \frac{1}{p}\frac{d}{dt}\{|u|^p\ln|u|^\gamma\}\,dx \\
&= \int_0^L \frac{1}{p}\left(\frac{d}{dt}|u|^p\ln|u|^\gamma + |u|^p\frac{d}{dt}\ln|u|^\gamma\right)dx \\
&= \int_0^L \frac{1}{p}\frac{d}{dt}\{|u|^p\}\ln|u|^\gamma\,dx + \int_0^L \frac{1}{p}|u|^p\frac{d}{dt}\{\ln|u|^\gamma\}\,dx
\end{aligned}$$

$$\begin{aligned}
\int_0^L \frac{1}{p}\frac{d}{dt}\{|u|^p\}\ln|u|^\gamma\,dx &= \int_0^L \frac{1}{p}\frac{d}{dt}\{|u|^p\ln|u|^\gamma\}\,dx - \int_0^L \frac{1}{p}|u|^p\frac{d}{dt}\{\ln|u|^\gamma\}\,dx \\
&= \frac{1}{p}\frac{d}{dt}\int_0^L |u|^p\ln|u|^\gamma\,dx - \frac{1}{p}\int_0^L \gamma\,|u|^p\cdot\frac{|u_t|}{|u|}dx \\
&= \frac{1}{p}\frac{d}{dt}\int_0^L |u|^p\ln|u|^\gamma\,dx - \frac{1}{p}\int_0^L \gamma\,|u|^{p-1}|u_t|\,dx \\
&= \frac{1}{p}\frac{d}{dt}\int_0^L |u|^p\ln|u|^\gamma\,dx - \frac{1}{p}\int_0^L \gamma\cdot\frac{1}{p}\cdot\frac{d}{dt}|u|^p\,dx \\
&= \frac{1}{p}\frac{d}{dt}\int_0^L |u|^p\ln|u|^\gamma\,dx - \frac{\gamma}{p^2}\frac{d}{dt}\|u\|_p^p
\end{aligned} \tag{3.2.13}$$

Using $(*)$ on the first term on the left side (3.2.3)we obtain

$$\begin{aligned}\int_0^L\int_0^1 \mu\,\tau_0\, z(x,\rho,t)\cdot z_t(x,\rho,t)\,dx\,d\rho &= \int_0^L\int_0^1 \mu\,\tau_0 \frac{1}{2}\frac{d}{dt}\left|z(x,\rho,t)\right|^2\,dx\,d\rho \\ &= \frac{|\mu|\tau_0}{2}\frac{d}{dt}\int_0^L\int_0^1 \left|z(x,\rho,t)\right|^2\,dx\,d\rho \\ &= \frac{|\mu|\tau_0}{2}\frac{d}{dt}\int_0^L\int_0^1 \left|z(x,\rho,t)\right|^2\,d\rho\,dx \end{aligned} \tag{3.2.14}$$

Integrate the second term from the left of (3.2.14)

$$\begin{aligned}\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)\,d\rho &= \left[\frac{1}{2}\left|z(x,1,t)\right|^2 - \frac{1}{2}\left|z(x,0,t)\right|^2\right] \\ &= \left[\frac{1}{2}\left|z(x,1,t)\right|^2 - \frac{1}{2}\left|u_t(x,t)\right|^2\right]\end{aligned}$$

$$\begin{aligned}\int_0^L\int_0^1 \mu z(x,\rho,t)\,z_\rho(x,\rho,t)\,d\rho\,dx &= |\mu|\int_0^L\left[\frac{1}{2}\left|z(x,1,t)\right|^2 - \frac{1}{2}\left|u_t(x,t)\right|^2\right]dx \\ &= |\mu|\int_0^L \frac{1}{2}\left|z(x,1,t)\right|^2dx - |\mu|\int_0^L\frac{1}{2}\left|u_t(x,t)\right|^2dx \\ &= \frac{|\mu|}{2}\int_0^L \left|z(x,1,t)\right|^2dx - \frac{|\mu|}{2}\int_0^L\left|u_t(x,t)\right|^2dx \\ &= \frac{|\mu|}{2}\int_0^L \left|z(x,1,t)\right|^2dx - \frac{|\mu|}{2}\left\|u_t(x,t)\right\|_2^2 \end{aligned} \tag{3.2.15}$$

Using $(*)$ on the first term on the left side (3.2.4)we obtain

$$\begin{aligned}\int_0^L \theta_t\theta\,dx &= \int_0^L\frac{1}{2}\frac{d}{dt}\left|\theta\right|^2dx \\ &= \frac{1}{2}\frac{d}{dt}\int_0^L\left|\theta\right|^2dx \\ &= \frac{1}{2}\frac{d}{dt}\left\|\theta\right\|_2^2\end{aligned} \tag{3.2.16}$$

Using integration by parts over the second term on the left-hand side of (3.2.4) we obtain

$$k\int_0^L \theta\, q_x dx = -k\int_0^L \theta_x q\,dx \tag{3.2.17}$$

$$\eta\int_0^L u_{xt}\theta\,dx = \eta\int_0^L \theta\,u_{xt}dx \tag{3.2.18}$$

substituting(3.2.16)-(3.2.18)using in the equation(3.2.4)

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left\|\theta\right\|_2^2 - k\int_0^L\theta_x q\,dx + \eta\int_0^L\theta\,u_{xt}dx \tag{3.2.19}$$

Using (∗) on the first term on the left side (3.2.5)we obtain

$$\begin{aligned}\int_0^L q\,\tau\,q_t\,dx &= \int_0^L \tau\,q\,q_t\,dx\\ &= \int_0^L \tau\,\frac{1}{2}\frac{d}{dt}\,|q|^2\,dx\\ &= \frac{\tau}{2}\frac{d}{dt}\int_0^L |q|^2\,dx\\ &= \frac{\tau}{2}\frac{d}{dt}\,\|q\|_2^2 \end{aligned} \tag{3.2.20}$$

$$\beta\int_0^L q^2 dx = \beta\int_0^L |q|^2\,dx = \beta\,\|q\|_2^2 \tag{3.2.21}$$

$$k\int_0^L \theta_x\,q\,dx \tag{3.2.22}$$

By collecting (3.2.6)(3.2.7)and(3.2.9)(3.2.12)-(3.2.18)and (3.2.20)-(3.2.22)The following equality is reached

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L m(x)\,|u_t|^2\,dx + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L p(x)\,|\nabla u|^2\,dx\\ &\quad + 2\int_0^L \delta(x)\,|\nabla u_t|^2\,dx - \eta\int_0^L \nabla u_t\,\theta\,dx + \frac{|\mu|}{2}\int_0^L |z(x,1,t)|^2\,dx\\ &\quad + \frac{|\mu|}{2}\int_0^L |u_t(x,t)|^2\,dx + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|q\|_2^2 + \beta\,\|q\|_2^2\\ &\quad + k\int_0^L \theta_x q\,dx + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\,\|\theta\|_2^2 - k\int_0^L \theta_x q\,dx + \eta\int_0^L \theta\,u_{xt}dx\\ &\quad + \frac{|\mu|\,\tau_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\int_0^1 |z(x,\rho,t)|^2\,d\rho dx\\ &\quad + \frac{|\mu|}{2}\int_0^L |z(x,1,t)|^2\,dx - \frac{|\mu|}{2}\int_0^L |u_t(x,t)|^2\,dx\\ &= \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u|^p\ln|u|^\gamma\,dx - \frac{\gamma}{p^2}\int_0^L \frac{\mathrm{d}}{\mathrm{d}t}\,|u|^p\,dx \end{aligned} \tag{3.2.23}$$

Substituting (3.2.8)(3.2.11)into(3.2.23)

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|m(x)\|_\infty\cdot\|u_t\|_2^2\right) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|p(x)\|_\infty\cdot\|\nabla u\|_2^2\right)\\ &\quad + 2\left(\|\delta(x)\|_\infty\cdot\|\nabla u_t\|_2^2\right) + |\mu|\int_0^L |z(x,1,t)|^2dx\\ &\quad + \frac{\tau}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_2^2 + \beta\|q\|_2^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_2^2\\ &\quad + \frac{|\mu|\tau_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L\int_0^1 |z(x,\rho,t)|^2 d\rho dx\\ &\quad + \frac{\gamma}{p^2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_p^p - \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L |u|^p\ln|u|^\gamma dx = 0.\end{aligned}$$

The above computation inspires us to define energy derivative functional as follows

$$
\begin{aligned}
E'(t) &= \frac{1}{2}\frac{d}{dt}\Bigg\{\|m(x)\|_\infty\cdot\|u_t\|_2^2+\|p(x)\|_\infty\cdot\|\nabla u\|_2^2+\tau\|q\|_2^2+\|\theta\|_2^2 \\
&\quad +\tau_0|\mu|\int_0^L\int_0^1|z(x,\rho,t)|^2\,d\rho\,dx\Bigg\} \\
&\quad -\frac{1}{p}\frac{d}{dt}\int_0^L|u|^p\ln|u|^\gamma\,dx+\frac{\gamma}{p^2}\frac{d}{dt}\|u\|_p^p \\
&= -2\|\delta(x)\|_\infty\cdot\|\nabla u_t\|_2^2-\beta\|q\|_2^2-|\mu|\int_0^L|z(x,1,t)|^2\,dx
\end{aligned}
\tag{3.2.24}
$$

and the energy functional given by

$$
\begin{aligned}
E(t) &= \frac{1}{2}\left(\|m(x)\|_\infty\cdot\|u_t\|_2^2\right)+\frac{1}{2}\left(\|p(x)\|_\infty\cdot\|\nabla u\|_2^2\right) \\
&\quad +\frac{\tau}{2}\|q\|_2^2+\frac{1}{2}\|\theta\|_2^2+\frac{\tau_0|\mu|}{2}\int_0^L\int_0^1|z(x,\rho,t)|^2\,d\rho\,dx \\
&\quad -\frac{1}{p}\int_0^L|u|^p\ln|u|^\gamma\,dx+\frac{\gamma}{p^2}\|u\|_p^p
\end{aligned}
\tag{3.2.25}
$$

by using an estimation we find

$$
E'(t)\leqslant-\left[2\left(\|\delta(x)\|_\infty\,.\,\|\nabla u_t\|_2^2\right)+\beta\|q\|_2^2+|\mu|\int_0^L|z(x,1,t)|^2\,dx\right]
\tag{3.2.26}
$$

■

Lemma 3.1 *Suppose that*

$$
2<p\leq\frac{2n}{n-2},n\geq 3.
\tag{3.2.27}
$$

Then there exists a positive constant $C>0$ depending on$[0.L]$ only such that

$$
\left[\int_0^L|u|^p\ln|u|^\gamma\,dx\right]^{\frac{s}{p}}\leq C\left[\int_0^L|u|^p\ln|u|^\gamma\,dx+\|u_x\|_2^2\right]
\tag{3.2.28}
$$

for any $u\in H_0^1(0,L)$and $2\leq s\leq p$, provided that $\int_0^L|u|^p\ln|u|^\gamma\,dx\geq 0$

Proof. if $\int_0^L|u|^p\ln|u|^\gamma\,dx>1$ then

$$
\left[\int_0^L|u|^p\ln|u|^\gamma\,dx\right]^{\frac{s}{p}}\leq\int_0^L|u|^p\ln|u|^\gamma\,dx
\tag{3.2.29}
$$

if $\int_0^L|u|^p\ln|u|^\gamma\,dx\leq 1$ then we set

$$
\Gamma_1=\{x\in[0,L]\,||u|>1\}
$$

and, for any$\beta \leq 2$, we have

$$\begin{aligned}
\left[\int_0^L |u|^p \ln|u|^\gamma \, dx\right]^{\frac{s}{p}} &\leq \left[\int_0^L |u|^p \ln|u|^\gamma \, dx\right]^{\frac{\beta}{p}} \leq \left[\int_{\Gamma_1} |u|^p \ln|u|^\gamma \, dx\right]^{\frac{\beta}{p}} \\
&\leq \left[\int_{\Gamma_1} |u|^{p+1} \, dx\right]^{\frac{\beta}{p}} \leq \left[\int_0^L |u|^{p+1} \, dx\right]^{\frac{\beta}{p}} \\
&= \|u\|_{p+1}^{\frac{\beta(p+1)}{p}}.
\end{aligned}$$

We choose$\beta = \frac{2p}{(p+1)} < 2$ to get

$$\left[\int_0^L |u|^p \ln|u|^\gamma \, dx\right]^{\frac{s}{p}} \leq \|u\|_{p+1}^2 \leq C \|u_x\|_2^2 \tag{3.2.30}$$

Combining (3.2.29) and (3.2.30), the result was obtained. ■

Lemma 3.2 *There exists a positive constant $C > 0$depending on $[0, L]$ only such that*

$$\|u\|_p^p \leq C \left[\int_0^L |u|^p \ln|u|^\gamma \, dx + \|u_x\|_2^2\right] \tag{3.2.31}$$

for any $u \in H_0^1(0, L)$, provided that $\int_0^L |u|^p \ln|u|^\gamma \, dx \geq 0$.

Proof. We set

$$\Gamma_+ = \{x \in [0, L] \, | \, |u| > e\} \text{ and } \Gamma_- = \{x \in [0, L] \, | \, |u| \leq e\}$$

thus

$$\begin{aligned}
\|u\|_p^p &= \int_{\Gamma_+} |u|^p \, dx + \int_{\Gamma_-} |u|^p \, dx \\
&\leq \int_{\Gamma_+} |u|^p \ln|u|^\gamma \, dx + \int_{\Gamma_-} e^p \left|\frac{u}{e}\right|^p dx \\
&\leq \int_{\Gamma_+} |u|^p \ln|u|^\gamma \, dx + e^p \int_{\Gamma_-} \left|\frac{u}{e}\right|^2 dx \\
&\leq \int_0^L |u|^p \ln|u|^\gamma \, dx + e^{p-2} \int_0^L |u|^2 \, dx \\
&\leq C \left\{\int_0^L |u|^p \ln|u|^\gamma \, dx + \|u_x\|_2^2\right\}
\end{aligned}$$

Using the fact that $\|u\|_2^2 \leq C \|u\|_p^2 \leq C \left(\|u\|_p^p\right)^{\frac{2}{p}}$,we have ■

Corollary 3.1 *There exists a positive constant $C > 0$ depending on$[0, L]$ only such that*

$$\|u\|_2^2 \leq C \left[\left(\int_0^L |u|^p \ln|u|^\gamma \, dx\right)^{\frac{2}{p}} + \|u_x\|_2^{\frac{4}{p}}\right] \tag{3.2.32}$$

provided that $\int_0^L |u|^p \ln|u|^\gamma \, dx \geq 0$.

Lemma 3.3 *There exists a positive constant $C > 0$ depending on $[0, L]$ only such that*

$$\|u\|_p^s \leq C\left[\|u\|_p^p + \|u_x\|_2^2\right] \tag{3.2.33}$$

for any $u \in H_0^1(0, L)$ and $2 \leq s \leq p$

Proof. if $\|u\|_p \geq 1$ then

$$\|u\|_p^s \leq \|u\|_p^p$$

if $\|u\|_p \leq 1$ then, $\|u\|_p^s \leq \|u\|_p^2$.Using Sobolev embedding theorems, we have

$$\|u\|_p^s \leq \|u\|_p^2 \leq C\|u_x\|_2^2$$

this purpose, we define

$$H(t) = -E(t) = -\left[\frac{1}{2}\left(\|m(x)\|_\infty\|u_t\|_2^2\right) + \frac{1}{2}\left(\|p(x)\|_\infty\|\nabla u\|_2^2\right) + \frac{\tau}{2}\|q\|_2^2 + \frac{1}{2}\|\theta\|_2^2\right.$$
$$\left. + \frac{\tau_0|\mu|}{2}\int_0^L\int_0^1 |z(x,\rho,t)|^2 d\rho dx - \frac{1}{p}\int_0^L |u|^p \ln|u|^\gamma dx + \frac{\gamma}{p^2}\|u\|_p^p\right]$$

$$\begin{aligned} H(t) &= -E(t) = -\frac{1}{2}\left(\|m(x)\|_\infty \|u_t(t)\|_2^2\right) - \frac{1}{2}\left(\|p(x)\|_\infty \|u_x\|_2^2\right) - \frac{\tau}{2}\|q\|_2^2 \\ &-\frac{1}{2}\|\theta\|_2^2 - \frac{\tau_0|\mu|}{2}\int_0^1 \|z(x,\rho,t)\|^2 d\rho - \frac{\gamma}{p^2}\|u\|_p^p \\ &+\frac{1}{p}\int_0^L |u|^p \ln|u|^\gamma dx \end{aligned} \tag{3.2.34}$$

$$\|u\|_p^s \leqslant C\left\{\|u\|_p^p + \|u_x\|_2^2\right\} \tag{**}$$

■

Corollary 3.2 *Assume that*(3.2.27) *holds. Then*

$$\|u\|_p^s \leq C\left\{\begin{array}{c} \left(1 - \frac{\gamma}{p\|p(x)\|_\infty}\right)\|u\|_p^p - \left(\frac{2}{\|p(x)\|_\infty}\right)H(t) \\ -\left(\frac{\|m(x)\|_\infty}{\|p(x)\|_\infty}\right)\|u_t(t)\|_2^2 - \left(\frac{\tau}{\|p(x)\|_\infty}\right)\|q\|_2^2 \\ -\left(\frac{1}{\|p(x)\|_\infty}\right)\|\theta\|_2^2 - \frac{\tau_0|\mu|}{\|p(x)\|_\infty}\int_0^1\|z(x,\rho,t)\|^2 d\rho \\ +\frac{2}{p\|p(x)\|_\infty}\int_0^L |u|^p \ln|u|^\gamma dx \end{array}\right\} \tag{3.2.35}$$

for any $u \in \left(H_0^1(0, L)\right)^n$ and $2 \leq s \leq p$.

Proof. we have

$$H(t) = -\frac{1}{2}\left(\|m(x)\|_\infty\|u_t\|_2^2\right) - \frac{1}{2}\left(\|p(x)\|_\infty\|u_x\|_2^2\right) - \frac{\tau}{2}\|q\|_2^2 - \frac{1}{2}\|\theta\|_2^2$$
$$-\frac{\tau_0|\mu|}{2}\int_0^L \|z(x,\rho,t)\|^2 d\rho + \frac{1}{p}\int_0^L |u|^p \ln|u|^\gamma dx - \frac{\gamma}{p^2}\|u\|_p^p.$$

and

$$\begin{aligned}\frac{1}{2}\|p(x)\|_\infty \|u_x\|_2^2 =& - H(t) - \frac{1}{2}\|m(x)\|_\infty \|u_t\|_2^2 - \frac{\tau}{2}\|q\|_2^2 - \frac{1}{2}\|\theta\|_2^2 \\ & - \frac{\tau_0 \, |\mu|}{2} \int_0^L \|z(x,\rho,t)\|^2 \, d\rho \\ & + \frac{1}{p}\int_0^L |u|^p \ln |u|^\gamma dx - \frac{\gamma}{p^2}\|u\|_p^p. \end{aligned}$$

with

$$\begin{aligned}\|u_x\|_2^2 = \frac{2}{\|p(x)\|_\infty}\Bigg[& - H(t) - \frac{1}{2}\|m(x)\|_\infty \|u_t\|_2^2 - \frac{\tau}{2}\|q\|_2^2 - \frac{1}{2}\|\theta\|_2^2 \\ & - \frac{\tau_0|\mu|}{2}\int_0^L \int_0^1 |z(x,\rho,t)|^2 d\rho dx \\ & + \frac{1}{p}\int_0^L |u|^p \ln |u|^\gamma dx - \frac{\gamma}{p^2}\|u\|_p^p \Bigg]. \end{aligned}$$

we obtain

$$\begin{aligned}\|u_x\|_2^2 =& -\left(\frac{2}{\|p(x)\|_\infty}\right) H(t) - \frac{\|m(x)\|_\infty \|u_t(t)\|_2^2}{\|p(x)\|_\infty} - \frac{\tau}{\|p(x)\|_\infty}\|q\|_2^2 - \frac{\|\theta\|_2^2}{\|p(x)\|_\infty} \\ & - \frac{\tau_0|\mu|}{\|p(x)\|_\infty}\int_0^1 \|z(x,\rho,t)\|^2 d\rho - \frac{2\gamma}{p^2\|p(x)\|_\infty}\|u\|_p^p \\ & + \frac{2}{p\|p(x)\|_\infty}\int_0^L |u|^p \ln |u|^\gamma dx. \end{aligned}$$

then

$$\begin{aligned}\|u\|_p^s \leqslant C\Bigg[& \|u\|_p^p - \left(\frac{2}{\|p(x)\|_\infty}\right) H(t) - \frac{\|m(x)\|_\infty \|u_t(t)\|_2^2}{\|p(x)\|_\infty} - \frac{\tau}{\|p(x)\|_\infty}\|q\|_2^2 - \frac{\|\theta\|_2^2}{\|p(x)\|_\infty} \\ & - \frac{\tau_0|\mu|}{\|p(x)\|_\infty}\int_0^1 \|z(x,\rho,t)\|^2 d\rho - \frac{2\gamma}{p^2\|p(x)\|_\infty}\|u\|_p^p \\ & + \frac{2}{p\|p(x)\|_\infty}\int_0^L |u|^p \ln |u|^\gamma dx \Bigg]. \end{aligned}$$

with (∗∗) we find

$$\begin{aligned}\|u\|_p^s \leqslant C\Bigg[& \|u\|_p^p - \left(\frac{2}{\|p(x)\|_\infty}\right) H(t) - \frac{\|m(x)\|_\infty \|u_t(t)\|_2^2}{\|p(x)\|_\infty} - \frac{\tau}{\|p(x)\|_\infty}\|q\|_2^2 - \frac{\|\theta\|_2^2}{\|p(x)\|_\infty} \\ & - \frac{\tau_0|\mu|}{\|p(x)\|_\infty}\int_0^1 \|z(x,\rho,t)\|^2 d\rho - \frac{2\gamma}{p^2\|p(x)\|_\infty}\|u\|_p^p \\ & + \frac{2}{p\|p(x)\|_\infty}\int_0^L |u|^p \ln |u|^\gamma dx \Bigg]. \end{aligned}$$

$$\|u\|_p^s \leqslant C\Bigg[\left(1-\frac{\gamma}{p^2\|p(x)\|_\infty}\right)\|u\|_p^p-\left(\frac{2}{\|p(x)\|_\infty}\right)H(t)-\left(\frac{\|m(x)\|_\infty}{\|p(x)\|_\infty}\right)\|u_t(t)\|_2^2$$
$$-\left(\frac{\tau}{\|p(x)\|_\infty}\right)\|q\|_2^2-\left(\frac{1}{\|p(x)\|_\infty}\right)\|\theta\|_2^2-\frac{\tau_0|\mu|}{\|p(x)\|_\infty}\int_0^1\|z(x,\rho,t)\|^2d\rho$$
$$+\frac{2}{p\|p(x)\|_\infty}\int_0^L|u|^p\ln|u|^\gamma\Bigg]dx.$$

for any $u\in\left(H_0^1(0,L)\right)^n$ and $2\le s\le p$. ■

Theorem 3.2 *Assume that* (3.2.27) *holds. Assume further that*

$$\begin{aligned}E(0) &= \frac{1}{2}\left(\|m(x)\|_\infty\|u_1(t)\|_2^2\right)+\frac{1}{2}\left(\|p(x)\|_\infty\|\nabla u_0\|_2^2\right)+\frac{\tau}{2}\|q_0\|_2^2\\ &+\frac{1}{2}\|\theta_0\|_2^2+\frac{\tau_0|\mu|}{2}\int_0^L\int_0^1|f_0(x,-\rho\tau_0)|^2d\rho dx+\frac{\gamma}{p^2}\|u_0\|_p^p\\ &-\frac{1}{p}\int_0^L|u_0|^p\ln|u_0|^\gamma dx<0 \end{aligned}\tag{3.2.36}$$

Then the of solution of(3.1.4)*blows up of in finite time.*

Proof. we have

$$E(t)\le E(0)<0$$

and

$$\begin{aligned}H'(t) &= -E'(t)=2\left(\|\delta(x)\|_\infty\|u_{xt}(t)\|_2^2\right)+\beta\|q\|_2^2\\ &+|\mu|\int_0^L|z(x,1,t)|^2dx\end{aligned}\tag{3.2.37}$$

Hence,

$$\begin{aligned}H'(t) &\ge C_0\left\{\begin{array}{l}\left(\|\delta(x)\|_\infty\|u_{xt}(t)\|_2^2\right)\\ +|\mu|\int_0^L z^2(x,1,t)\,dx\end{array}\right\}\\ &\ge 0;\forall t\in[0,T)\end{aligned}\tag{3.2.38}$$

consequently we get

$$0<H(0)\le H(t)\le\int_0^L|u|^p\ln|u|^\gamma dx;\forall t\in[0,T)\tag{3.2.39}$$

of (3.2.1) and (3.2.34). We then introduce

$$\begin{aligned}L(t) &= H^{1-\alpha}(t)+\varepsilon\int_0^L\left[m(x)u_t(t)\,u(t)+4\delta(x)|u_x|^2\right]dx\\ &+\varepsilon\int_0^L\frac{n\tau}{k}uqdx\end{aligned}\tag{3.2.40}$$

Where $\varepsilon>0$ to be specified later and

$$\frac{2(p-2)}{p^2}<\alpha<\frac{p-2}{2p}<1\tag{3.2.41}$$

A direct differentiation of $L(t)$ gives

$$\begin{aligned}
L'(t) &= (1-\alpha)H'(t)H^{-\alpha}(t) + \varepsilon \int_0^L \Big[m(x)\big(u_{tt}(t)u(t) + u_t(t)u_t(t)\big) \\
&\quad + 4\delta(x)\frac{\mathrm{d}}{\mathrm{d}t}|u_x|^2\Big]\,dx + \varepsilon \int_0^L \frac{\eta\tau}{k}\left[u_t q + q_t u\right]dx \\
&= (1-\alpha)H'(t)H^{-\alpha}(t) + \varepsilon \int_0^L \Big[m(x)u_{tt}u + m(x)|u_t|^2 + 4\delta(x)\frac{\mathrm{d}}{\mathrm{d}t}|u_x|^2\Big]\,dx \\
&\quad + \varepsilon \int_0^L \Big[\frac{\eta\tau}{k}u_t q + \frac{\eta\tau}{k}q_t u\Big]\,dx \\
&= (1-\alpha)H'(t)H^{-\alpha}(t) + \varepsilon \int_0^L m(x)u_{tt}u\,dx + \varepsilon \int_0^L m(x)|u_t|^2\,dx \\
&\quad + \varepsilon \int_0^L 4\delta(x)\frac{\mathrm{d}}{\mathrm{d}t}|u_x|^2\,dx + \varepsilon\frac{\eta\tau}{k}\int_0^L u_t q\,dx + \varepsilon\frac{\eta\tau}{k}\int_0^L q_t u\,dx.
\end{aligned} \tag{3.2.42}$$

From (3.2.2) we find that

$$\begin{aligned}
\varepsilon \int_0^L m(x)u_{tt}u\,dx =& \varepsilon \int_0^L \left(p(x)u_x + 2\delta(x)u_{tx}\right)_x u\,dx - \varepsilon \int_0^L \eta\theta_x u\,dx \\
&- \varepsilon \int_0^L \mu z(x,1,t)u\,dx + \varepsilon \int_0^L u^2|u|^{p-2}\ln|u|^\gamma\,dx.
\end{aligned} \tag{3.2.43}$$

$$\begin{aligned}
\varepsilon \int_0^L \left(p(x)u_x + 2\delta(x)u_{tx}\right)_x u\,dx &= \varepsilon\left[u(x,t)\left(p(x)u_x + 2\delta(x)u_{tx}\right)\right]_0^L \\
&\quad - \varepsilon \int_0^L u_x\left(p(x)u_x + 2\delta(x)u_{tx}\right)dx \\
&= -\varepsilon \int_0^L u_x\left(p(x)u_x + 2\delta(x)u_{tx}\right)dx \\
&= -\varepsilon \int_0^L p(x)\left|u_x\right|^2\,dx - 2\varepsilon \int_0^L \delta(x)u_{tx}u_x\,dx.
\end{aligned} \tag{3.2.44}$$

$$\begin{aligned}
-\varepsilon\eta \int_0^L \theta_x u\,dx &= -\varepsilon\eta\left[u(x,t)\theta(x,t)\right]_0^L + \varepsilon\eta \int_0^L u_x\theta\,dx \\
&= \varepsilon\eta \int_0^L u_x\theta\,dx
\end{aligned} \tag{3.2.45}$$

$$\begin{aligned}
\varepsilon \int_0^L m(x)u_{tt}u\,dx &= -\varepsilon \int_0^L p(x)\left|u_x\right|^2 dx - 2\varepsilon \int_0^L \delta(x)u_{tx}u_x dx \\
&\quad + \varepsilon\eta \int_0^L u_x\theta\,dx - \varepsilon \int_0^L \mu z(x,1,t)u\,dx \\
&\quad + \varepsilon \int_0^L |u|^p \ln|u|^\gamma\,dx
\end{aligned} \tag{3.2.46}$$

From (3.1.4) and by multiplying U we find that

$$\tau \int_0^L u\, q_t\, dx + \beta \int_0^L q\, u\, dx + k \int_0^L \theta_x u dx = 0 \tag{3.2.47}$$

Starting from equation (3.2.47), and after transferring the limits to the other side, we get

$$\tau \int_0^L u\, q_t\, dx = -\beta \int_0^L q\, u\, dx - k \int_0^L \theta_x u dx$$

Multiply by $\frac{\varepsilon\eta}{k}$ both sides to get

$$\frac{\varepsilon\eta}{k} \int_0^L u\, \tau q_t\, dx = -\frac{\varepsilon\eta}{k} \int_0^L \beta\, q\, u\, dx - \varepsilon\eta \int_0^L \theta_x u dx \tag{3.2.48}$$

Integration by parts

$$-\varepsilon\eta \int_0^L \theta_x u dx = \varepsilon\eta \int_0^L u_x \theta\, dx \tag{3.2.49}$$

By substituting (3.2.49) into (3.2.50) we find that

$$\frac{\varepsilon\eta}{k} \int_0^L u\, \tau q_t\, dx = -\frac{\varepsilon\eta}{k} \int_0^L \beta\, q\, u\, dx + \varepsilon\eta \int_0^L \theta\, u_x\, dx \tag{3.2.50}$$

By substituting (3.2.44)-(3.2.46) and (3.2.50)into (3.2.42) we find that

$$\begin{aligned} L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) &+ \varepsilon \int_0^L m(x)\, |u_t|^2\, dx - \varepsilon \int_0^L p(x)\, |u_x|^2\, dx \\ &- 2\varepsilon \int_0^L \delta(x) u_{tx} u_x\, dx + \varepsilon\eta \int_0^L u_x \theta\, dx - \varepsilon \int_0^L \mu z(x,1,t) u\, dx \\ &+ \varepsilon \int_0^L |u|^p \ln |u|^\gamma\, dx + 2\varepsilon \int_0^L \delta(x) u_{tx} u_x\, dx \\ &- \frac{\varepsilon\eta}{k} \int_0^L \beta q u\, dx + \varepsilon\eta \int_0^L \theta u_x\, dx + \frac{\varepsilon\eta\tau}{k} \int_0^L q_t u\, dx \end{aligned}$$

$$\begin{aligned} L'(t) = (1-\alpha)H^{-\alpha}(t)H'(t) &+ \varepsilon \int_0^L m(x)\, |u_t|^2\, dx - \varepsilon \int_0^L p(x)\, |u_x|^2\, dx \\ &+ \frac{\varepsilon\eta\tau}{k} \int_0^L q_t u\, dx + 2\varepsilon\eta \int_0^L \theta u_x\, dx - \varepsilon \int_0^L \mu z(x,1,t) u\, dx \\ &+ \varepsilon \int_0^L |u|^p \ln |u|^\gamma\, dx - \frac{\varepsilon\eta\beta}{k} \int_0^L q u\, dx \end{aligned} \tag{3.2.51}$$

Using the inequality of young

$$2\varepsilon\eta \int_0^L \theta u_x dx \le 2\varepsilon\eta \frac{1}{2} \int_0^L |\theta|^2\, dx + 2\varepsilon\eta \frac{1}{2} \int_0^L |u_x|^2\, dx$$

$$2\varepsilon\eta\int_0^L \theta u_x dx \leq \varepsilon\eta\,\|\theta\|_2^2 + \varepsilon\eta\,\|u_x\|_2^2$$

$$-\varepsilon\eta\,\|\theta\|_2^2 - \varepsilon\eta\,\|u_x\|_2^2 \leq 2\varepsilon\eta\int_0^L \theta u_x dx \leq \varepsilon\eta\,\|\theta\|_2^2 + \varepsilon\eta\,\|u_x\|_2^2$$

$$2\varepsilon\eta\int_0^L \theta u_x dx \geq -\varepsilon\eta\,\|\theta\|_2^2 - \varepsilon\eta\,\|u_x\|_2^2 \tag{3.2.52}$$

$$\int_0^L p(x)\,|u_x|^2\,dx \leq \|p(x)\|_\infty\,\|u_x\|_2^2$$

$$-\varepsilon\int_0^L p(x)\,|u_x|^2\,dx \geq -\varepsilon\,\|p(x)\|_\infty\,\|u_x\|_2^2 \tag{3.2.53}$$

$$-\int_0^L m(x)\,|u_x|^2\,dx \leq -\|m(x)\|_\infty\,\|u_t\|_2^2$$

$$\varepsilon\int_0^L m(x)\,|u_x|^2\,dx \geq \varepsilon\,\|m(x)\|_\infty\,\|u_t\|_2^2 \tag{3.2.54}$$

$$\frac{\varepsilon\eta\tau}{k}\int_0^L q\,u_t(t)dx \leq \frac{\varepsilon\eta\tau}{k}\frac{1}{2}\int_0^L |q|^2\,dx + \frac{\varepsilon\eta\tau}{k}\frac{1}{2}\int_0^L |u_t|^2\,dx$$

$$\frac{\varepsilon\eta\tau}{k}\int_0^L q\,u_t(t)dx \leq \frac{\varepsilon\eta\tau}{2k}\,\|q\|_2^2 + \frac{\varepsilon\eta\tau}{2k}\,\|u_t\|_2^2$$

$$-\frac{\varepsilon\eta\tau}{2k}\,\|q\|_2^2 - \frac{\varepsilon\eta\tau}{2k}\,\|u_t\|_2^2 \leq \frac{\varepsilon\eta\tau}{k}\int_0^L q\,u_t(t)dx \leq \frac{\varepsilon\eta\tau}{2k}\,\|q\|_2^2 + \frac{\varepsilon\eta\tau}{2k}\,\|u_t\|_2^2$$

$$\frac{\varepsilon\eta\tau}{k}\int_0^L q\,u_t(t)dx \geq -\frac{\varepsilon\eta\tau}{2k}\,\|q\|_2^2 - \frac{\varepsilon\eta\tau}{2k}\,\|u_t\|_2^2 \tag{3.2.55}$$

and

$$\begin{aligned}\int_0^L \mu z(x,1,t)u\,dx &\leq |\mu|\left\{\frac{1}{2}\int_0^L |z(x,1,t)|^2\,dx + \frac{1}{2}\int_0^L |u|^2\,dx\right\}\\ &\leq |\mu|\left\{\frac{\xi_1}{2}\int_0^L |z(x,1,t)|^2\,dx + \frac{1}{2\xi_1}\,\|u\|_2^2\right\}\end{aligned}$$

$$-\varepsilon \int_0^L \mu z(x,1,t) u \, dx \geq -\varepsilon |\mu| \left\{ \frac{\xi_1}{2} \int_0^L |z(x,1,t)|^2 \, dx + \frac{1}{2\xi_1} \|u\|_2^2 \right\} \tag{3.2.56}$$

$$\begin{aligned} \frac{\eta\beta}{k} \int_0^L q \, u \, dx &\leq \frac{\eta\beta}{k} \left\{ \frac{1}{2} \int_0^L |q|^2 \ dx + \frac{1}{2} \int_0^L |u|^2 \ dx \right\} \\ &\leq \frac{\eta\beta}{k} \left\{ \frac{\xi_2}{2} \|q\|_2^2 + \frac{1}{2\xi_2} \|u\|_2^2 \right\} \end{aligned}$$

$$-\varepsilon \frac{\eta\beta}{k} \int_0^L q \, u \, dx \geq -\varepsilon \frac{\eta\beta}{k} \left\{ \frac{\xi_2}{2} \|q\|_2^2 + \frac{1}{2\xi_2} \|u\|_2^2 \right\} \tag{3.2.57}$$

substituting (3.2.52)-(3.2.57)into(3.2.51) we get

$$\begin{aligned} L'(t) \geq& (1-\alpha) H^{-\alpha}(t) H'(t) - \varepsilon\eta \|\theta\|_2^2 - \varepsilon\eta \|u_x\|_2^2 - \varepsilon \|p(x)\|_\infty \|u_x\|_2^2 \\ &+ \varepsilon \|m(x)\|_\infty \|u_t\|_2^2 - \varepsilon \frac{\eta\tau}{2k} \|u_t\|_2^2 - \varepsilon \frac{\eta\tau}{2k} \|q\|_2^2 \\ &- \varepsilon |\mu| \left\{ \frac{\xi_1}{2} \int_0^L |z(x,1,t)|^2 \, dx + \frac{1}{2\xi_1} \|u\|_2^2 \right\} \\ &+ \varepsilon \frac{\eta\beta}{k} \left\{ \frac{\xi_2}{2} \|q\|_2^2 + \frac{1}{2\xi_2} \|u\|_2^2 \right\} \\ &+ \varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx. \end{aligned}$$

$$\begin{aligned} L'(t) \quad \geq \quad & (1-\alpha) H^{-\alpha}(t) H'(t) + \varepsilon \left\{ \|m(x)\|_\infty - \frac{\eta\tau}{2k} \right\} \|u_t\|_2^2 \\ & -\varepsilon \left\{ \|p(x)\|_\infty + \eta \right\} \|u_x\|_2^2 - \varepsilon\eta \|\theta\|_2^2 \\ & -\varepsilon \left\{ \frac{\tau\eta + \beta\eta\xi_2}{2k} \right\} \|q\|_2^2 + \varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx \\ & -\varepsilon |\mu| \frac{\xi_1}{2} \int_0^L |z(x,1,t)|^2 \, dx - \varepsilon \|u\|_2^2 \left\{ \frac{|\mu|}{2\xi_1} + \frac{\eta\beta}{2\xi_2 k} \right\} \end{aligned} \tag{3.2.58}$$

We obtain from(3.2.37)and(3.2.58)the following

$$\begin{aligned} L'(t) \geq& (1-\alpha) H^{-\alpha}(t) H'(t) - \varepsilon \frac{\beta\eta\xi_2}{2k} \|q\|_2^2 - \varepsilon |\mu| \frac{\xi_1 k}{2k} \int_0^L |z(x,1,t)|^2 \, dx \\ &+ \varepsilon \left\{ \|m(x)\|_\infty - \frac{\eta\tau}{2k} \right\} \|u_t\|_2^2 - \varepsilon \left\{ \|p(x)\|_\infty + \eta \right\} \|u_x\|_2^2 - \varepsilon\eta \|\theta\|_2^2 - \varepsilon \frac{\tau\eta}{2k} \|q\|_2^2 \\ &+ \varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx - \varepsilon \left\{ \frac{|\mu| \, k}{2\xi_1 k} + \frac{\eta\beta}{2\xi_2 k} \right\} \|u\|_2^2 \end{aligned} \tag{3.2.59}$$

$$H'(t) = 2 \|\delta(x)\|_\infty \cdot \|\nabla u_t\|_2^2 + \beta \|q\|_2^2 + |\mu| \int_0^L |z(x,1,t)|^2 \, dx$$

We have

$$H'(t) \geq |\mu| \int_0^L |z(x,1,t)|^2 \, dx, \tag{3.2.60}$$

$$H'(t) \geq \beta \|q\|_2^2 . \tag{3.2.61}$$

Multiplying (3.2.60) by $-\varepsilon \frac{\xi_1 k}{2k}$ and (3.2.61) by$-\varepsilon \frac{\eta \xi_2}{2k}$ we find

$$-\varepsilon \frac{\xi_1 k}{2k} H'(t) \leq -\varepsilon \frac{|\mu| \, \xi_1 k}{2k} \int_0^L |z(x,1,t)|^2 \, dx, \tag{3.2.62}$$

$$-\varepsilon \frac{\eta \xi_2}{2k} H'(t) \leq -\varepsilon \frac{\beta \eta \xi_2}{2k} \|q\|_2^2 . \tag{3.2.63}$$

Combining equations (3.2.62) and (3.2.63) side by side, we get:

$$\left(-\varepsilon \frac{\eta \xi_2}{2k} - \varepsilon \frac{\xi_1 k}{2k} \right) H'(t) \leq -\varepsilon \frac{|\mu| \, \xi_1 k}{2k} \int_0^L |z(x,1,t)|^2 \, dx - \varepsilon \frac{\beta \eta \xi_2}{2k} \|q\|_2^2 . \tag{3.2.64}$$

By substituting (3.2.64) in (3.2.59) we find

$$\begin{aligned} L'(t) \geq{}& (1-\alpha) H^{-\alpha}(t) H'(t) + \left(-\varepsilon \frac{\xi_1 k}{2k} - \varepsilon \frac{\eta \xi_2}{2k} \right) H'(t) \\ &+ \varepsilon \left\{ \|m(x)\|_\infty - \frac{\eta \tau}{2k} \right\} \|u_t\|_2^2 - \varepsilon \left\{ \|p(x)\|_\infty + \eta \right\} \|u_x\|_2^2 - \varepsilon \eta \|\theta\|_2^2 \\ &- \varepsilon \frac{\tau \eta}{2k} \|q\|_2^2 + \varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx - \varepsilon \left\{ \frac{|\mu| \, k}{2\xi_1 k} + \frac{\eta \beta}{2\xi_2 k} \right\} \|u\|_2^2 . \end{aligned}$$

$$\begin{aligned} L'(t) \geq{}& \left\{ (1-\alpha) H^{-\alpha}(t) - \varepsilon \left(\frac{k\xi_1 + \eta \xi_2}{2k} \right) \right\} H'(t) \\ &+ \varepsilon \left\{ \|m(x)\|_\infty - \frac{\eta \tau}{2k} \right\} \|u_t\|_2^2 - \varepsilon \left\{ \|p(x)\|_\infty + \eta \right\} \|u_x\|_2^2 - \varepsilon \eta \|\theta\|_2^2 \\ &- \varepsilon \frac{\tau \eta}{2k} \|q\|_2^2 + \varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx - \varepsilon \left\{ \frac{|\mu| \, k}{2\xi_1 k} + \frac{\eta \beta}{2\xi_2 k} \right\} \|u\|_2^2 . \end{aligned} \tag{3.2.65}$$

We also set $\xi_1 = \xi_2 = H^{-\alpha}(t)$, Hence(3.2.65) Where $C = \frac{k+\eta}{2k}$ and $M = |\mu| \, k + \eta \beta$ are strictly positive constants gives

$$\begin{aligned} L'(t) \geq{}& \left\{ (1-\alpha) H^{-\alpha}(t) - \varepsilon \left(\frac{k H^{-\alpha}(t) + \eta H^{-\alpha(t)}}{2k} \right) \right\} H'(t) \\ &+ \varepsilon \left\{ \|m(x)\|_\infty - \frac{\eta \tau}{2k} \right\} \|u_t\|_2^2 - \varepsilon \left\{ \|p(x)\|_\infty + \eta \right\} \|u_x\|_2^2 - \varepsilon \eta \|\theta\|_2^2 \\ &- \varepsilon \frac{\tau \eta}{2k} \|q\|_2^2 + \varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx - \varepsilon \left\{ \frac{|\mu| \, k}{2H^{-\alpha}(t) k} + \frac{\eta \beta}{2H^{-\alpha}(t) k} \right\} \|u\|_2^2 . \end{aligned}$$

$$\begin{aligned} L'(t) \geq{}& \left\{ (1-\alpha) H^{-\alpha}(t) - \varepsilon c \, H^{-\alpha}(t) \right\} H'(t) \\ &+ \varepsilon \left\{ \|m(x)\|_\infty - \frac{\eta \tau}{2k} \right\} \|u_t\|_2^2 - \varepsilon \left\{ \|p(x)\|_\infty + \eta \right\} \|u_x\|_2^2 - \varepsilon \eta \|\theta\|_2^2 \\ &- \varepsilon \frac{\tau \eta}{2k} \|q\|_2^2 + \varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx - \varepsilon \frac{M}{2k} H^\alpha(t) \|u\|_2^2 . \end{aligned}$$

$$
\begin{aligned}
L'(t) \geq & \{(1-\alpha) - \varepsilon c\} H^{-\alpha}(t) H'(t) \\
& + \varepsilon \left\{ \|m(x)\|_{\infty} - \frac{\eta\tau}{2k} \right\} \|u_t\|_2^2 - \varepsilon \left\{ \|p(x)\|_{\infty} + \eta \right\} \|u_x\|_2^2 - \varepsilon\eta \|\theta\|_2^2 \\
& - \varepsilon \frac{\tau\eta}{2k} \|q\|_2^2 + \varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx - \varepsilon \frac{M}{2k} H^\alpha(t) \|u\|_2^2 . \qquad (3.2.66)
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx &= (\varepsilon - \varepsilon a + \varepsilon a) \int_0^L |u|^p \ln |u|^\gamma \, dx \\
&= \varepsilon \int_0^L |u|^p \ln |u|^\gamma \, dx - \varepsilon a \int_0^L |u|^p \ln |u|^\gamma \, dx + \varepsilon a \int_0^L |u|^p \ln |u|^\gamma \, dx. \qquad (3.2.67)
\end{aligned}
$$

Substituting (3.2.67) into (3.2.66) we find that

$$
\begin{aligned}
L'(t) \geq & \{(1-\alpha) - \varepsilon c\} H^{-\alpha}(t) H'(t) \\
& + \varepsilon \left\{ \|m(x)\|_{\infty} - \frac{\eta\tau}{2k} \right\} \|u_t\|_2^2 - \varepsilon \left\{ \|p(x)\|_{\infty} + \eta \right\} \|u_x\|_2^2 - \varepsilon\eta \|\theta\|_2^2 \\
& - \varepsilon \frac{\tau\eta}{2k} \|q\|_2^2 + \varepsilon \int_0^L |u|^p \, Ln \, |u|^\gamma \, dx - \varepsilon a \int_0^L |u|^p \, Ln \, |u|^\gamma \, dx + \varepsilon a \int_0^L |u|^p \, Ln \, |u|^\gamma \, dx - \varepsilon \frac{M}{2k} H^\alpha(t) \|u\|_2^2 . \qquad (3.2.68)
\end{aligned}
$$

From (3.2.34) we get

$$
\begin{aligned}
\frac{1}{p} \int_0^L |u|^p \ln |u|^\gamma dx =& H(t) + \frac{1}{2} \left(\|m(x)\|_\infty \|u_t\|_2^2 \right) + \frac{1}{2} \left(\|p(x)\|_\infty \|u_x\|_2^2 \right) + \frac{\tau}{2} \|q\|_2^2 + \frac{1}{2} \|\theta\|_2^2 \\
& + \frac{\tau_0 |\mu|}{2} \int_0^L \int_0^1 |z(x,\rho,t)|^2 \, d\rho dx + \frac{\gamma}{p^2} \|u\|_p^p.
\end{aligned}
$$

$$
\begin{aligned}
\int_0^L |u|^p \ln |u|^\gamma dx =& p \, H(t) + \frac{p}{2} \left(\|m(x)\|_\infty \|u_t\|_2^2 \right) + \frac{p}{2} \left(\|p(x)\|_\infty \|u_x\|_2^2 \right) + \frac{\tau p}{2} \|q\|_2^2 + \frac{p}{2} \|\theta\|_2^2 \\
& + \frac{\tau_0 p |\mu|}{2} \int_0^L \int_0^1 |z(x,\rho,t)|^2 \, d\rho dx + \frac{\gamma}{p} \|u\|_p^p. \qquad (3.2.69)
\end{aligned}
$$

Substituting (3.2.69) into (3.2.68) we find

$$
\begin{aligned}
L'(t) \geq & \{(1-\alpha) - \varepsilon C\} H^{-\alpha}(t) H'(t) \\
& + \varepsilon \left\{ \|m(x)\|_\infty \left(1 + \frac{p}{2}(1-a) \right) - \frac{\eta\tau}{2k} \right\} \|u_t\|_2^2 \\
& + \varepsilon \left\{ \|p(x)\|_\infty \left(\frac{p}{2}(1-a) - 1 \right) + \eta \right\} \|u_x\|_2^2 \\
& + \varepsilon \left\{ -\eta + \frac{p\varepsilon(1-a)}{2} \right\} \|\theta\|_2^2 + \varepsilon a \int_0^L |u|^p \ln |u|^\gamma \, dx \\
& + \varepsilon \left\{ -\frac{\tau\eta}{2k} + \frac{p\tau(1-a)}{2} \right\} \|q\|_2^2 + \frac{\gamma\varepsilon(1-a)}{2} \|u\|_p^p \\
& + \varepsilon \frac{\tau_0 p (1-a)}{2} |\mu| \int_0^L \int_0^1 |z(x,\rho,t)|^2 \, d\rho dx \\
& - \varepsilon \frac{M}{2k} H^\alpha(t) \|u\|_2^2 + p\varepsilon(1-a) H(t) \qquad (3.2.70)
\end{aligned}
$$

Using (3.2.32) , (3.2.39) and Young's inequality, we find

$$\begin{aligned} H^{\alpha}(t)\|u\|_2^2 &\leq \left(\int_0^L |u|^p \ln|u|^{\gamma} dx\right)^{\alpha} \|u\|_2^2 \\ &\leq C\left[\left(\int_0^L |u|^p \ln|u|^{\gamma} dx\right)^{\alpha+\frac{2}{p}} + \left(\int_0^L |u|^p \ln|u|^{\gamma} dx\right)^{\alpha} \|u_x\|_2^{\frac{4}{p}}\right] \\ &\leq C\left[\left(\int_0^L |u|^p \ln|u|^{\gamma} dx\right)^{\frac{(\alpha p+2)}{p}} + \|u_x\|_2^2 + \left(\int_0^L |u|^p \ln|u|^{\gamma} dx\right)^{\frac{\alpha p}{p-2}}\right] \end{aligned} \tag{3.2.71}$$

Exploiting(3.2.41), we have

$$2 < \alpha p + 2 \leq p \text{ and } 2 < \frac{\alpha p^2}{p-2} \leq p \tag{3.2.72}$$

Thus, *lemma* 1 yields

$$H^{\alpha}(t)\|u\|_2^2 \leq C\left\{\int_0^L |u|^p \ln|u|^{\gamma} dx + \|u_x\|_2^2\right\} \tag{3.2.73}$$

Combining (3.2.70) and (3.2.73), we obtain

$$\begin{aligned} L'(t) \geq\ & \{(1-\alpha) - \varepsilon C\} H^{-\alpha}(t) H'(t) \\ &+\varepsilon\left\{\|m(x)\|_{\infty}\left(1+\frac{p}{2}(1-a)\right) - \frac{\eta\tau}{2k}\right\}\|u_t\|_2^2 \\ &+\varepsilon\left\{\|p(x)\|_{\infty}\left(\frac{p}{2}(1-a)-1\right) + \eta - C\frac{M}{2k}\right\}\|u_x\|_2^2 \\ &+\varepsilon\left\{-\eta + \frac{p\varepsilon(1-a)}{2}\right\}\|\theta\|_2^2 + \varepsilon\left\{a - C\frac{M}{2k}\right\}\int_0^L |u|^p \ln|u|^{\gamma} dx \\ &+\varepsilon\left\{-\frac{\tau\eta}{2k} + \frac{p\tau(1-a)}{2}\right\}\|q\|_2^2 + \frac{\gamma\varepsilon(1-a)}{2}\|u\|_p^p \\ &+\varepsilon\frac{\tau_0 p(1-a)}{2}|\mu|\int_0^L\int_0^1 |z(x,\rho,t)|^2 d\rho dx \\ &+p\varepsilon(1-a)H(t) \end{aligned} \tag{3.2.74}$$

At this point, we choose $a > 0$ so small that

$$\begin{aligned} -\eta + \frac{p\varepsilon(1-a)}{2} &> 0 \\ \left(\frac{p}{2}(1-a)-1\right) &> 0 \\ \frac{\tau_0 p(1-a)}{2} &> 0 \end{aligned}$$

and k so large that

$$\begin{aligned} \|p(x)\|_\infty \left(\frac{p}{2}(1-a)-1\right)+\eta-C\frac{M}{2k} &> 0 \\ a-C\frac{M}{2k} &> 0 \\ \|m(x)\|_\infty \left(1+\frac{p}{2}(1-a)\right)-\frac{\eta\tau}{2k} &> 0 \\ -\frac{\tau\eta}{2k}+\frac{p\tau(1-a)}{2} &> 0 \end{aligned}$$

Once C and a are fixed, we pick ε so small so that

$$(1-\alpha)-\varepsilon C>0$$

And

$$\begin{aligned} A_1 &= \left\{\|m(x)\|_\infty \left(1+\frac{p}{2}(1-a)\right)-\frac{\eta\tau}{2k}\right\}, \\ A_2 &= \left\{\|p(x)\|_\infty \left(\frac{p}{2}(1-a)-1\right)+\eta\right\}, \\ A_3 &= \left\{-\eta+\frac{p\varepsilon(1-a)}{2}\right\}, \\ A_4 &= \left\{-\frac{\tau\eta}{2k}+\frac{p\tau(1-a)}{2}\right\}. \end{aligned}$$

hence (3.2.74) becomes

$$\begin{aligned} L'(t) \geq\ & \{(1-\alpha)-\varepsilon C\} H^{-\alpha}(t) H'(t) \\ & +\varepsilon A_1 \|u_t\|_2^2+\varepsilon A_2 \|u_x\|_2^2+\varepsilon A_3 \|\theta\|_2^2 \\ & +\varepsilon A_4 \|q\|_2^2+\varepsilon\left\{a-C\frac{M}{2k}\right\}\int_0^L |u|^p \ln|u|^\gamma dx \\ & +\varepsilon\frac{\tau_0 p(1-a)}{2}|\mu|\int_0^L\int_0^1 |z(x,\rho,t)|^2 d\rho dx \\ & +\frac{\gamma\varepsilon(1-a)}{2}\|u\|_p^p+p\varepsilon(1-a)H(t) \end{aligned} \tag{3.2.75}$$

where A_1-A_4 are strictly positive constants depending only on p,τ,η,k,a.
Thus, for some $A_0>0$, estimate (3.2.75) becomes

$$L'(t)\geq A_0\left\{\begin{array}{c} H(t)+\|u_t\|_2^2+\|u_x\|_2^2+\|u\|_p^p \\ \|q\|_2^2+\|\theta\|_2^2+\int_0^L |u|^p\ln|u|^\gamma dx \\ +\int_0^L\int_0^1 |z(x,\rho,t)|^2 d\rho dx \end{array}\right\} \tag{3.2.76}$$

and

$$L(t)\geq L(0)>0, \forall t\geq 0. \tag{3.2.77}$$

From (3.2.40) we have

$$L^{\frac{1}{(1-\alpha)}}(t) = H(t) + \left(\varepsilon \int_0^L m(x) u_t u \, dx\right)^{\frac{1}{(1-\alpha)}} \quad + \left(4\varepsilon \int_0^L \delta(x) \left|u_x\right|^2 dx\right)^{\frac{1}{(1-\alpha)}}$$
$$+ \left(\varepsilon \frac{\eta\tau}{k} \int_0^L u \, q \, dx\right)^{\frac{1}{(1-\alpha)}} \tag{3.2.78}$$

Next, using Holder's inequality and the embedding $\left\|u\right\|_2 \leq C \left\|u\right\|_p$, we have

$$\left|\int_0^L m(x) u \, u_t dx\right| \leq \int_0^L \left|m(x) u_t u\right| dx \leq \int_0^L \left|m(x)\right| \left|u_t u\right| dx$$
$$\leq \left\|m(x)\right\|_\infty \int_0^L \left|u_t u\right| dx$$

We put $\left\|m(x)\right\|_\infty = c$

$$\left|\int_0^L m(x) u \, u_t dx\right| \leq c \int_0^L \left|u_t u\right| dx \leq c \int_0^L \left|u_t\right| \left|u\right| dx$$
$$\leq c \left\|u_t\right\|_2 \left\|u\right\|_2 \tag{3.2.79}$$

and exploiting Young's inequality, we obtain

$$\left|\int_0^L m(x) u \, u_t dx\right| \leq \left\|m(x)\right\|_\infty \left\{\left\|u\right\|^r + \left\|u_t\right\|^{r'}\right\}$$

$$\left|\int_0^L m(x) u \, u_t dx\right|^{\frac{1}{1-\alpha}} \leq \left\|m(x)\right\|_\infty^{\frac{1}{1-\alpha}} \left\{\left\|u\right\|^r + \left\|u_t\right\|^{r'}\right\}^{\frac{1}{1-\alpha}}$$

$$\left|\int_0^L m(x) u \, u_t dx\right|^{\frac{1}{1-\alpha}} \leq \left\|m(x)\right\|_\infty^{\frac{1}{1-\alpha}} \left\{\left\|u\right\|^{\frac{r}{1-\alpha}} + \left\|u_t\right\|^{\frac{r'}{1-\alpha}}\right\}$$

We put $c = \left\|m(x)\right\|_\infty^{\frac{1}{1-\alpha}}$

$$\left|\int_0^L m(x) u u_t dx\right|^{\frac{1}{1-\alpha}} \leq C \left\{\left\|u\right\|_p^{\frac{r}{1-\alpha}} + \left\|u_t\right\|_2^{\frac{r'}{1-\alpha}}\right\}, \text{For } \frac{1}{r} + \frac{1}{r'} = 1 \tag{3.2.80}$$

To be able to use Lemma 3, we take $r' = 2\left(1-\alpha\right)$ which gives
$\frac{r}{1-\alpha} = \frac{2}{1-2\alpha} \leq p$

$$\left|\int_0^L m(x) u u_t dx\right|^{\frac{1}{1-\alpha}} \leq C \left(\left\|u\right\|_p^s + \left\|u_t\right\|_2^2\right)$$

Hence, Lemma 5 gives
We put : $\frac{r}{1-\alpha} = \frac{2}{1-2\alpha} = p \quad et \quad \frac{s}{1-\alpha} = 2$

$$\left|\int_0^L m(x)uu_t dx\right|^{\frac{1}{1-\alpha}} \leq C_1 \left(\|u\|_p^p + \|u_t\|_2^2 + \|u_x\|_2^2\right), \forall C_1 > 0 \tag{3.2.81}$$

with the same way, we get

$$\begin{aligned}\left|\frac{\varepsilon\eta\tau}{k}\int_0^L u\,q\,dx\right| &\leq \left|\frac{\varepsilon\eta\tau}{k}\right| \int_0^L |u\,q|\,dx \\ &\leq \left|\frac{\varepsilon\eta\tau}{k}\right| \left\{\|u\|_p^r + \|q\|_2^{s'}\right\}\end{aligned}$$

$$\left|\frac{\varepsilon\eta\tau}{k}\int_0^L u\,q\,dx\right|^{\frac{1}{1-\alpha}} \leq \left|\frac{\varepsilon\eta\tau}{k}\right|^{\frac{1}{1-\alpha}} \left\{\|u\|_p^r + \|q\|_2^{s'}\right\}^{\frac{1}{1-\alpha}}$$

$$\left|\frac{\varepsilon\eta\tau}{k}\int_0^L u\,q\,dx\right|^{\frac{1}{1-\alpha}} \leq c_2 \left\{\|u\|_p^{\frac{r}{1-\alpha}} + \|q\|_2^{\frac{s'}{1-\alpha}}\right\}$$

$$\begin{aligned}\left|4\varepsilon\int_0^L \delta(x)\,|u_x|^2\,dx\right|^{\frac{1}{1-\alpha}} &\leq \left(4\varepsilon\,\|\delta(x)\|_\infty\,\|u_x\|_2^2\right)^{\frac{1}{1-\alpha}} \\ &\leq (4\varepsilon\,\|\delta(x)\|)^{\frac{1}{1-\alpha}} \left(\|u_x\|_2^{\frac{1}{1-\alpha}}\right) \\ &\leq c_3\,\|u_x\|_2^2\end{aligned}$$

We put :$\frac{r}{1-\alpha} = \frac{2}{1-2\alpha} = p \quad et \quad \frac{s}{1-\alpha} = 2$

$$\left|\varepsilon\int_0^L \frac{n\tau}{k} uqdx\right|^{\frac{1}{1-\alpha}} \leq C_2 \left(\|u\|_p^p + \|q\|_2^2\right), \forall C_2 > 0 \tag{3.2.82}$$

$$\left|\varepsilon\int_0^L 4\delta(x)\,|u_x|^2\,dx\right|^{\frac{1}{1-\alpha}} \leq C_3\,\|u_x\|_2^2, \forall C_3 > 0 \tag{3.2.83}$$

From(3.2.81) − (3.2.82) (3.2.83) and we obtain

$$L^{\frac{1}{1-\alpha}}(t) \leq C \left\{\begin{array}{c} H(t) + \|u\|_p^p + \|q\|_2^2 \\ + \|u_x\|_2^2 + \|u_t\|_2^2 \end{array}\right\}; \forall t \geq 0, \forall C > 0 \tag{3.2.84}$$

Combining (3.2.84) and (3.2.76), we arrive at

$$\begin{aligned} L'(t) &\geq a_0 L^{\frac{1}{1-\alpha}}(t) \\ \frac{\mathrm{d}L}{\mathrm{d}t} &\geq a_0 L^{\frac{1}{1-\alpha}}(t) \\ \frac{\mathrm{d}L}{L^{\frac{1}{1-\alpha}}(t)} &\geq a_0 dt \end{aligned} \tag{3.2.85}$$

where $a_0$is a positive constant depending only on A_0 and$C$.
A simple integration of (3.2.85) over $(0,t)$yields

$$\int_0^t \frac{\mathrm{d}L}{L^{\frac{1}{1-\alpha}}(t)} \geq \int_0^t a_0 dt$$

$$-\frac{1}{(\frac{1}{1-\alpha}-1)L^{\frac{1}{1-\alpha}-1}(t)} \geq a_0 t + c$$

$$-\frac{1}{(\frac{\alpha}{1-\alpha})L^{\frac{\alpha}{1-\alpha}}(t)} \geq a_0 t + c$$

$$\left[-\frac{1}{(\frac{\alpha}{1-\alpha})L^{\frac{\alpha}{1-\alpha}}(t)}\right]_0^t \geq [a_0 t + c]_0^t$$

$$-\frac{1}{(\frac{\alpha}{1-\alpha})L^{\frac{\alpha}{1-\alpha}}(t)} + \frac{1}{(\frac{\alpha}{1-\alpha})L^{\frac{\alpha}{1-\alpha}}(0)} \geq a_0 t$$

$$-\frac{1}{(\frac{\alpha}{1-\alpha})L^{\frac{\alpha}{1-\alpha}}(t)} \geq a_0 t - \frac{1}{(\frac{\alpha}{1-\alpha})L^{\frac{\alpha}{1-\alpha}}(0)}$$

$$\frac{1}{(\frac{\alpha}{1-\alpha})L^{\frac{\alpha}{1-\alpha}}(t)} \leq -a_0 t + \frac{1}{(\frac{\alpha}{1-\alpha})L^{\frac{\alpha}{1-\alpha}}(0)}$$

$$\frac{\alpha}{1-\alpha}L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{-a_0 t + \frac{1-\alpha}{\alpha}L^{-\frac{\alpha}{1-\alpha}}(0)}$$

$$\begin{aligned} L^{\frac{\alpha}{1-\alpha}}(t) &\geq \frac{\frac{1-\alpha}{\alpha}}{-a_0 t + \frac{1-\alpha}{\alpha}L^{-\frac{\alpha}{1-\alpha}}(0)} \\ &\geq \frac{1}{\frac{\alpha}{1-\alpha}\left(-a_0 t + \frac{1-\alpha}{\alpha}L^{-\frac{\alpha}{1-\alpha}}(0)\right)} \\ &\geq \frac{1}{L^{\frac{-\alpha}{1-\alpha}}(0) - \frac{\alpha a t}{1-\alpha}} \end{aligned} \tag{3.2.86}$$

Therefore,$L\left(t\right)$blows up in time

$$L^{\frac{-\alpha}{1-\alpha}}(0) - \frac{\alpha a t}{1-\alpha} \geq 0$$

$$L^{\frac{-\alpha}{1-\alpha}}(0) \geq \frac{\alpha a t}{1-\alpha}$$

$$\frac{1-\alpha}{\alpha a_0}L^{\frac{-\alpha}{1-\alpha}}(0) \geq t$$

$$T \leq T^* = \frac{1-\alpha}{\alpha a_0 L^{\frac{\alpha}{1-\alpha}}(t)} \tag{3.2.87}$$

the proof is completed

■

Conclusion

In this work, we are interested with a problem of a logarithmic nonuniform flexible structure with time delay, where the heat flux is given by Cattaneo's law. We show that the energy of any weak solution blows up infinite time if the initial energy is negative. The delay effect that is similar to memory processes is important in the research of applied mathematics such as physics, noninstant transmission phenomena, and biological motivation. In the future work, we will try to study this problem numerically by using the finite element methods.

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