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THEME

Well-posedness and stability results in a Timoshenko-type system of thermoelasticity of type III with delay term

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Dedication

To the one whose brow was drenched in sweat and the one who taught me that success only comes with patience and determination, to the light that illuminated my path, and the lamp whose light never goes out in my heart, the one who made precious and precious sacrifices and from whom I derived my strength and self-esteem, my dear father.

To the one who placed paradise under her feet and eased hardships for me with her prayers, to the great person who always wished she could open her eyes to see me on a day like this, my dear mother. To my steadfast side and the safety of my days, to those with whom I strengthened my honor and they were springs from which I drank, to the best and clearest of my days, to the apple of my eye, my dear brothers and sisters.

To everyone who has been a help and support on this path, to loyal friends and companions for years, to those in adversity and crises, to those who have poured out their feelings and sincere advice to me, to you, my family, I dedicate to you this achievement and the fruit of my success, which I have always wished for. Today Ihave completed its first fruits thanks to Him, Glory be to Him, so praise be to God for what He has given me and for making me blessed and for helping me wherever I am. So whoever says, "I am hers," then I am hers. And if she refuses, against her will, I bring her. Praise be to God, thanks, love, and gratitude for the beginning, the end, and the last of their prayers. Praise be to God, Lord of the worlds.

Thanks and gratitude

Praise be to Allah and thanks to Allah who guided my steps and helped Allah,who instilled in our hearts the love of knowledge and faith in truth, honesty,and sincerity.

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I am happy to express my thanks and appreciation to everyone who helped me,near or far,in completing this work to raise the flag of science and knowledge.

ملخص

في هذه المذكرة، ندرس نظَام مرن حرَاري خطي أَحَادي من نوع تيموشنكو مع تأَخير، حيث يتم الحصول علَى التوصيل الحرَاري من خلَال نظرية جرين و نجدي. نثبت أَنّ النظَام معرّف جيدًا و إِستقرَاره لحَالَات السرعَات المتسَاوية و غير المتسَاوية لإِنتشَار الموجَات .

كلمَات مفتَاحية : نظَام تيموشنكو، اِلمرونة الحرَارية، عَامل التأَخير، مبرهنة هيل - يوشيدَا.

Résumé Dans cette mémoire, on considère un système thermoélastique linéaire unidimensionnel de type Timoshenko avec retard, où la conduction thermique est donnée par la théorie de Green et Naghdi. On établit que le système est bien défini et stabilité de ce système pour les cas de vitesses de propagation des ondes égales et inégales.

Mots clés: Système de Timoshenko, Thermoélasticité, Terme de retard, Théorème de Hille-Yoshida.

Abstract In this memory, we consider a one-dimensional linear thermoelastic system of Timoshenko type with delay, where the heat conduction is given by Green and Naghdi's theory. We establish the well-posedness and stability of the system for the cases of equal and unequal speeds of wave propagation.

Key words: Timoshenko system, Thermoelasticity, Delay term, Hille-Yoshida theorem.

Contents

Chapter 1

Introduction

The study of existence and stability of Timoshenko systems has attracted a great deal of attention in the last decades. From a physical or engineering point of view, Timoshenko theory is an improvement of Euler-Bernoulli theory. Indeed, in the Euler-Bernoulli beam theory, it is assumed that plane crosssections that are perpendicular to the axis of the beam remain plane and perpendicular to the axis after deformation, which implies that the transverse shear strain is zero. When the rotational inertia and the transverse shear are significant in the beam model, one has to use rather the Timoshenko theory. The transverse vibrations of the beam depend in general on its geometrical properties (length, size and shape, cross-section, moment of inertia, and shear coefficient) and its mechanical properties (density, Young's modulus, and modulus of rigidity). To be more precise, we have the following model, which was developed by Timoshenko on [1] in 1921,

$$\begin{aligned} &\rho u_{tt}(x,t) = \left(K\left(u_x(x,t) - \varphi(x,t)\right)\right)_x, \quad \text{in } (0,L)\times(0,+\infty,) \\ &I_\rho \varphi_{tt}(x,t) = \left(EI\varphi_x(x,t)\right)_x + K\left(u_t(x,t) - \varphi(x,t)\right), \text{ in } (0,L)\times(0,+\infty). \end{aligned} \tag{1.1}$$

Together with boundary conditions of the form.

$$EI\varphi_x|_{x=0}^{x=L} = 0, \quad (u_x - \varphi)|_{x=0}^{x=L} = 0. \tag{1.2}$$

Where $u(x,t)$ is the transverse displacement, $\varphi(x,t)$ is the rotational angle of the beam, ρ denotes the mass density, I_ρ is the moment of mass inertia, EI is the rigidity coefficient, K is the shear modulus of elasticity, and L is the length of the beam.

Due to a surrounding flow of wind, gas or fluid, the beam is subject to mechanical vibrations. These vibrations are of course undesirable because of their damaging and destructing nature. To reduce these harmful vibrations, several control mechanisms have been designed. This is achieved either by incorporating into the structure a smart material actuator as piezoceramic, by acting inside, or at the free edges of the beam. Several researchers employed different types of damping mechanisms to stabilize these systems and to obtain precise rates of decay. For internal or boundary frictional damping, we quote, among others, the work of Kim and Renardy [2], Raposo et al. [4], Soufyane and Wehbe [5], Rivera and Racke [9],[10], and Mustafa and Messaoudi [11]. Regarding Timoshenko systems for material with "finite" or "infinite" memory, we refer to Ammar-Khodja et al. [12], Guesmia and Messaoudi [13], and Fernández Sare and Rivera [14].

For stabilization via heat dissipation, Rivera and Racke [15] established several exponential decay results for linear Timoshenko systems coupled with the classical heat equation, in which the heat flux is given by Fourier's law. Since this theory predicts an infinite speed of heat propagation, to overcome this physical paradox, many theories have emerged. One of which given by Green and Naghdi [16]-[19], suggests replacing Fourier's law by so- called thermoelasticity of type III for heat conduction modeling thermal disturbances as wave-like pulses traveling at finite speed. See [20] for more details.

Taking into account Green and Naghdi's theory, a Timoshenko system of thermoelasticity of type III of the form.

$$\begin{aligned}
&\rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x = 0 && \text{in } (0,\infty)\times(0,1),\\
&\rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \beta\theta_x = 0 && \text{in } (0,\infty)\times(0,1),\\
&\rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0 && \text{in } (0,\infty)\times(0,1).
\end{aligned} \tag{1.3}$$

Where φ, ψ and θ are functions of (x,t) which model the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature respectively was studied by Messaoudi and SaidHouari [22], and an exponential decay result in the case of equal wave speeds $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$ was proved. The case of nonequal speeds $\left(\frac{K}{\rho_1} \neq \frac{b}{\rho_2}\right)$ was studied later by Messaoudi and Fareh [23], and a polynomial decay result was proved for solutions with smooth initial data. A decay result, where a viscoelastic damping of the form $\int_0^t g(t-s)\theta_{xx}(s)\mathrm{d}s$ is acting in the third equation instead of the strong heat dissipation $-k\theta_{txx}$, it was also established by Kafini [24].

Time delays arise in many applications, because most phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [25] and references therein.

In many cases, it was shown that delay is a source of instability unless additional conditions or control terms are used, see [26]. Therefore, the stability issue of systems with delay is of theoretical and practical great importance.

For the system of wave equation with locally distributed damping of the form.

$$\begin{cases} u_{tt}(x,t) - \triangle u(x,t) + a_0 u_t(x,t) + a u_t(x,t-\tau) = 0, & \text{in } \Omega \times (0,\infty), \\ u(x,t) = 0, & x \in \Gamma_0, t > 0, \\ \frac{\partial u}{\partial v}(x,t) = 0, & x \in \Gamma_1, t > 0. \end{cases} \tag{1.4}$$

It is well-known in the absence of delay $(a = 0, a_0 > 0)$, that the system is exponentially stable, see [27]. In the presence of delay $(a > 0)$, Nicaise and Pignotti [28] examined (1.4) and proved, under the assumption that the weight of the feedback with delay is smaller than the one without delay $(a < a_0)$, that the energy is exponentially stable. For the opposite case, they produced a sequence of delays for which the corresponding solution is instable. The same results were obtained for the case of boundary delay, see also [29] for the treatment to this problem in more general abstract form. When the delay term in (1.4) is replaced by the distributed delay.

$$\int_{\tau_1}^{\tau_2} a(s) u_t(x,t-s) \mathrm{d}s. \tag{1.5}$$

Exponential stability results have been obtained in [33] under the condition $\int_{\tau_1}^{\tau_2} a(s)\mathrm{d} < a_0$. Introducing a delay term in the internal feedback of the thermoelastic system may turn a well-behaved system into a wild one. For instance, contrary to the exponential stability of the classical thermoelastic system without delay, Racke [34] proved that any constant delay makes the system instable.

In this chapter, we are concerned with the following Timoshenko system of thermoelasticity of type III with delay of the form.

$$\begin{cases} \rho_1 \phi_{tt} - K(\phi_x + \psi)_x + \mu_1 \phi_t(x,t) + \mu_2 \phi_t(x,t-\tau) = 0 \text{ in } (0,1) \times (0,\infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + K(\phi_x + \psi) + \beta \theta_{tx} = 0 \quad \text{in } (0,1) \times (0,\infty), \\ \rho_3 \theta_{tt} - \delta \theta_{xx} + \gamma \psi_{tx} - k \theta_{txx} = 0 \text{ in } (0,1) \times (0,\infty), \\ \theta(.,0) = \theta_0, \theta_t(.,0) = \theta_1, \psi(.,0) = \psi_0, \psi_t(.,0) = \psi_1, \\ \phi(.,0) = \phi_0, \phi_t(.,0) = \phi_1, \\ \phi_t(x,t-\tau) = f_0(x,t-\tau), \quad t \in (0,\tau), \\ \phi(0,t) = \phi(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \quad \forall t \geq 0. \end{cases} \tag{1.6}$$

Where $\rho_1, \rho_2, \rho_3, K, b, k, \beta, \gamma, \delta, \mu_1$ are positive constants, μ_2 is a real number, and $\tau > 0$ represents the time delay. We prove, under suitable conditions on the initial data that the energy decays exponentially in the case of equal wave speeds in spite of the existence of the delay. The second part of our result is the case of nonequal speeds which is of much importance because practically or physically the speeds are not necessarily equal. In that case, we prove that the energy decays polynomially.

In [35], the well-posedness and stability of the same system (3.1), without delay and with infinite memory considered in the first or second equation of Timoshenko system was proved, and general decay estimates were obtained depending on the growth of the kernel function at infinity and the wave speeds. In [36], the exponential stability of an abstract hyperbolic system with a discrete time delay and an infinite memory was proved under the assumption that the kernel function converges exponentially to zero and the weight of the delay is small enough. The system considered in [36] is not dissipative due to the fact that the unique considered dissipation is generated by the infinite memory. More results are found in [37], [38] and [39].

Chapter 2

Preliminaries

In this chapter, we recall some basic knowledge in fonctional analysis, moste of wich will be used in the subseqent chapter. The reader can easily find the detailed in the related literature, see, e.g.,[32], [17], [3], [21].

2.1 Functional Spaces

We denote by $\mathbb{R}^n$ the Euclid space, $\Omega \subset \mathbb{R}^n$ is bonded smoth domain, $C^k(\Omega)$ is the k^{th} differentiable continuous function space in Ω, $C^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space in Ω, and $C_c^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space with compact support in Ω

Definition 2.1.1. *Let X be a vector space over the filed $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then a semi-norm on X is a function $\|.\| : X \longrightarrow \mathbb{R}$, such that :*

a) $\|x\| \geqslant 0$ for all $x \in X$,

b) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$,

c) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$.

A norm on X is a semi-norm wich also satisfies :

d) $\|x\| = 0 \Rightarrow x = 0$. A vector space X toghether with a norm $\|.\|$ is called a normed linear space, a normed vector space or simply, a normed space.

Definition 2.1.2. *(Convergent and Cauchy sequences). Let X be a normed space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.*

a) $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

i.e. if

$$\forall \varepsilon > 0; \exists N > 0, \forall n \geq N, \|x_n - x\| < \varepsilon.$$

b) $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence if*

$$\forall\varepsilon > 0; \exists N > 0, \forall m, n \geq N, \|x_m - x_n\| < \varepsilon.$$

Normed spaces in which every Cauchy sequence is convergent are called complet normed spaces. In general a normed space is not complete.

Definition 2.1.3. *(Banach Spaces). A normed spaces is called a Banach space if it is complet i.e. if any Cauchy sequence inside the space converges to a point of the space. Its dual space* X' *is the linear space of all contnuous linear functional* $f : X \to .$

Proposition 2.1. X' *equipped with the norm* $\|.\|_{X'}$ *defined by.*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}$$

is olso a Banach space.

Remark 2.1.1. *From* X' *we construct the bidual or secend dual* $X'' = (X')'$*. Futhermore, with each* $u \in X$ *we can define* $\varphi(u) \in X''$ *by* $\varphi(u)(f) = f(u)$*,* $f \in X'$*, this satisfies clearly* $\|\varphi(x)\| \leq \|u\|$*. Moreover, for each* $u \in X$ *there is an* $f \in X'$ *with* $f(u) = \|u\|$ *and* $\|f\| = 1$*, so it follows that* $\|\varphi(x)\| = \|u\|$*.*

Definition 2.1.4. *Since* φ *is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X *onto a closed subspace of* X''*, we denote this by*

$$X \hookrightarrow X''.$$

Definition 2.1.5. *if* φ *(in the above definition) is onto* X'' *we say* X *is reflexive,* $X \approxeq X''$*.*

2.1.1 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analyse, known as Hilbert spaces. Then, we most give some impotant result on these spaces here.

Definition 2.1.6. *A Hilbert space* H*is a vectorial space supplied with inner product* $\langle u, v\rangle$ *such that* $\|u\| = \sqrt{\langle u, u\rangle}$ *is the norm which let* H *complete.*

Theorem 2.1. *Let $(x_n)_{n\in\mathbb{N}}$ is a bounded squence in the Hilbrt sâce H, then it possess a subsequence which converges in the weak topology of H.*

Theorem 2.2. *In the Hilbrt space, all sequence which converges in the weak topology is bounded.*

Theorem 2.3. *Let $(x_n)_{n\in\mathbb{N}}$ be sequence which converges to x, in the weak topology and $(y_n)_{n\in\mathbb{N}}$ is an other squence which converge weakly to y, then*

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

Proposition 2.2. *Let X and Y be tow Hilbert space, let $(x_n)_{n\in\mathbb{N}} \in X$ be a sequence which conveges weakly to $x \in X$, let $A \in \mathcal{L}(X,Y)$. Then, the sequence $(A(x_n))_{n\in\mathbb{N}}$ converges to $A(x)$ in the weak topology of Y.*

2.1.2 The $L^p(\Omega)$ spaces

Definition 2.1.7. *Let $1 \le p \le \infty$, and let Ω be an open domain in $\mathbb{R}^n$, where $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \,\middle|\, f \text{ is measurable and } \int_\Omega |f(x)|^p\,dx < \infty \right\}$$

.

Notation 1 : for $p \in \mathbb{R}$ and $1 \le p < \infty$, denote by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx\right)^{\frac{1}{p}}.$$

. If $p = \infty$, we have

$$L^p(\Omega) = \left\{ f : \Omega \to : f \text{ is measurable and there exists } C \text{ suche that}, |f(x)| \le C \text{ in } \Omega \right\}.$$

Notation 2 : Let $1 \le p \le \infty$,we denote by q the conjugate of p i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 2.4. *It is well known that $L^p(\Omega)$ supplied with the norm $\|.\|_p$ is a Banach space, for all $1 \le p \le \infty$.*

Remark 2.1.2. *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

is a Hilbert space.

Theorem 2.5. *For $1 < p < \infty$, $L^p(\Omega)$ is reflexive space.*

2.1.3 The Sobolev space $W^{m,p}(\Omega)$

Definition 2.1.8.

i) Let $m \in \mathbb{N}$ *and* $p \in [0, \infty]$*. The space* $W^{m,p}(\Omega)$ *is defined as the set of all functions* $f \in L^p(\Omega)$ *such that.*

$$W^{m,p}(\Omega) = \left\{ f \in L^p(\Omega) \,\middle|\, \partial^\alpha f \in L^p(\Omega) \text{ for all } \alpha \in \mathbb{N}^m, \text{ where } |\alpha| = \sum_{j=1}^{n} \alpha_j \leq m \text{ and } \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} \right\}.$$

ii) If $f \in W^{m,p}(\Omega)$*, we define its norm to be.*

$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|<k} \int_\Omega |D^\alpha f|^p \, dx \right)^{\frac{1}{p}}, & (1 \leq p < \infty), \\ \sum_{|\alpha|<k} \operatorname{ess\,sup} |D^\alpha f|, & (p = \infty), \end{cases}$$

Where $D^\alpha f$ *denotes the derivative of* f *with respect to the multi-index* α*.*

Definition 2.1.9. *We denote by*

$$W_0^{m,p}(\Omega)$$

the closure of $C_0^\infty(\Omega)$ *in* $W^{m,p}(\Omega)$*.*

Remark 2.1.3. *i) if* $p = 2$ *we usully write*

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Supplied with the norm

$$\|f\|_{H^m} = \left(\sum_{|\alpha| \leq m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}}.$$

The letter H *is used, since - as we will see -* $H^m(\Omega)$ *is a Hilber space.*

with usual scalar product

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v dx.$$

Note that $H^0(\Omega) = L^2(\Omega)$*.*

Theorem 2.6. .

1. $H^m(\Omega)$ *supplied with inner product* $\langle ., . \rangle_{H^m(\Omega)}$ *is Hilbert space.*

2. If $m \geq m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$*.*

Theorem 2.7. *Assume that* Ω *is an open domain in* أ$n, n \geq 1$*, with smooth boundary* Γ*. Then,*

i) if $1 \leq p \leq n$*, we have* $W^{1,p} \subset L^q(\Omega)$*, for every* $q \in [p, p^*]$*, where* $p^* = \dfrac{np}{n-p}$*.*

ii) if $p = n$ *we have* $W^{1,p} \subset L^q(\Omega)$*, for every* $q \in [p, \infty)$ *.*

iii) if $p > n$ *we have* $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$*, where* $\alpha = \dfrac{p-n}{p}$*.*

2.1.4 The $L^p(0,T,X)$ space

Definition 2.1.10. *Let X be a Banach space, denote by $L^p(0,T,X)$ the space of measurable functions*

$$f :]0,T[\to X$$
$$t \mapsto f(t).$$

Such that

$$\left(\int_0^T \|f(t)\|_X^p dt\right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \qquad 1 \leq p < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t\in]0,T[} ess\|f(t)\|_X.$$

Theorem 2.8. *$L^p(0,T,X)$ equipped with the norm $\|.\|_{L^p(0,T,X)}$ is a Banach space.*

Proposition 2.3. *Let X be a reflexive Banach space, X' it's dual, and $1 \leq p < \infty$, $1 \leq q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0,T,X)$ is identify algebraically and topologically with $L^q(0,T,X')$.*

2.1.5 The weak and weak star topologies:

Let X be a Banach space and $f \in X'$. Denot by

$$\varphi_f : X \to \mathbb{R}$$
$$x \mapsto \varphi_f.$$

When f cover X', we obtain a family $(\varphi_f)_{f\in X'}$ of appmications to X in .

Definition 2.1.11. *The weak topology on X, denoted by $\sigma(X,X')$, is the weakest topology on X for which every $(\varphi_f)_{f\in X'}$ is continuous.*

We will define the topology on X', the weak star topology, denoted by $\sigma(X',X)$. For all $x \in X$. Denote by

$$\varphi_x : X' \to \mathbb{R}$$
$$f \mapsto \varphi_x(f) = \langle f, x\rangle_{X',X}.$$

Definition 2.1.12. *The weak star topology on X' is the weakest topology on X' for wich every $(\varphi_x)_{x\in X'}$ is continuous.*

Remark 2.1.4. *Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X',X)$ is weakest then the topology $\sigma(X',X'')$, and this later is weakest then the strong topology.*

Definition 2.1.13. *A sequence (x_n) in X is weakly convergent to x if and only if*

$$\lim_{n\to\infty} f(x_n) = f(x),$$

for evry $f \in X'$, and this is denoted by $x_n \rightharpoonup x$.

Remark 2.1.5. *:*

1.If the weak limit exist, it is unique.

2.If $x_n \to x \in X$ (strongly), then $x_n \rightharpoonup x$ (weakly).

3.If $\dim X < \infty$, then the weak convergent implise the strong convergent.

Theorem 2.9. *(The Lax-Milgram Theorem) Let X be a Hilbert space and let $a : X \times X \to \mathbb{R}$ be a bilinear functional. Assume that there exist two constants $C < \infty$ and $\alpha > 0$ such that:*
(i) $|a(u, v)| \leq C\|u\| \cdot \|v\|$ for all $(u, v) \in X \times X$ (continuity);
(ii) $a(u, u) \geq \alpha\|u\|^2$ for all $u \in X$ (coerciveness).
Then, for every $f \in X^*$ (the dual space of X), there exists a unique $u \in X$ such that $a(u, v) = \langle f, v\rangle$ for all $v \in X$.

2.2 Some useful inequalities

In this section, we shall recall some inqualities which will be used in the supsequent chapters.

2.2.1 Young inequalities

Theorem 2.10. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, a, b > 0.$$

Theorem 2.11. *(Young inequality with ε) Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,then*

$$ab \leq \varepsilon\frac{a^p}{p} + \frac{1}{\varepsilon^{\frac{q}{p}}}\frac{b^q}{q}, a, b > 0.$$

The Young inequality has several variants in the following.

Corollary 1. *Let $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. Then*
i) $a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.
ii) $a^{\frac{1}{p}}b^{\frac{1}{q}} \leq \frac{a}{p\varepsilon^{\frac{1}{q}}} + \frac{b\varepsilon^{\frac{1}{p}}}{q} \quad ,\forall\varepsilon > 0$.
iii) $a^{\alpha}b^{1-\alpha} \leq \alpha a + (1-\alpha)b \quad ,0 < \alpha < 1$.

2.2.2 The Holder inequalities

Theorem 2.12. *Let* $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,
then if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, *we have*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)}.\|g\|_{L^q(\Omega)}.$$

Theorem 2.13. *(Generalized Holder inequality) Let* $1 \leq p_1, ..., p_m \leq \infty$,
$\frac{1}{p_1} + ... + \frac{1}{p_m} = 1$, *then if* $f_k \in L^{p_k}(\Omega)$ *for* $k = 1, ..., m$, *we have*

$$\int_\Omega |f_1....f_m| dx \leq \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\Omega)}.$$

Remark 2.2.1. *We have the corresponding weighted Holder inequality of the integral form. Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$,$\omega(x) > 0$ *on* Ω. *Then*

$$\int_\Omega |fg| \omega(x) dx \leq \left(\int_\Omega |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q \omega(x) dx \right)^{\frac{1}{q}}.$$

2.2.3 The Minkowski inequality

Theorem 2.14. *Assume* $1 \leq p \leq \infty$, $f, g \in L^p(\Omega)$, *then*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

If $0 \neq p < 1$, *then*

$$\|f + g\|_{L^p(\Omega)} \geq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

In the applications, the integral form from the Minkowski inequality is used frequentely.

2.2.4 The Poincaré inequality

In this subsection, we shall recall the Poincaré inequality in different forms.

Theorem 2.15. *Let* Ω *be a bounded domain in* أn *and* $f \in H_0^1(\Omega)$. *Then there is a positive constant* C *such that*

$$\|f\|_{L^2(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)}, \qquad \forall f \in H_0^1(\Omega).$$

Theorem 2.16. *Let* Ω *be a bounded domain of* C^1 *in* أn. *There is a positive constant* C, *such that for any* $f \in H^1(\Omega)$.

$$\|f - \tilde{f}\|_{L^2(\Omega)} \leq C \|\nabla f\|_{L^2(\Omega)}.$$

Where $\tilde{f} = \frac{1}{|\Omega|} \int_\Omega f(x) dx$ *is the integral average of* f *over* Ω, *and* $|\Omega|$ *is the volume fo* Ω.

Theorem 2.17. *Under assumption of Theorem* (2.16) *for any* $f \in H^1(\Omega)$, *we have*

$$\|f\|_{L^2(\Omega)} \leq C \left(\|\nabla f\|_{L^2(\Omega)} + |\int_\Omega f dx| \right).$$

2.3 Basic theory of semigroups

In this section, we recall some basic knowledge in semigroups,most of whiche will be used in the subsequent chapters. Ageneral rference to this topic is [17], [3].

2.3.1 C_0-Semigroups of Linear Operators

Definition 2.3.1. *(Semigroups)*

Let X be a Banach space, the one-parametre family $S(t), 0 \leq t < \infty$ from X to X is called a Semigroups if

(i) $S(0) = I$ (I is the identity operateur on X),

(ii) $S(t+s) = S(t) + S(s)$ for every $t, s \geq 0$ (the Semigroup property).

Definition 2.3.2. *The linear operator A defined by*

$$D(A) = \left\{ x \in X : \lim_{t \to 0^+} (S(t)x - x)/t \quad exists \right\},$$

and

$$Ax = \lim_{t \to 0^+} (S(t)x - x)/t = \frac{d(S(t)x)}{dt}|_{t=0} \quad for \quad all \quad x \in D(A),$$

is called the infinitesimal generator of the Semigroup $S(t)$, $D(A)$ is called the domain of A.

Definition 2.3.3. *(C_0-Semigroups).*

A Semigroup $S(t)$,$0 \leq t < \infty$,from X to X is called a strong continuous Semigroup of bounded lineaar operators if

$$\lim_{t \to 0^+} S(t)x = x \quad for \quad all \quad x \in X,$$

or

$$\lim_{t \to 0^+} \|S(t)x - x\| = 0 \quad for \quad all \quad x \in X,$$

i.e $S(t)$ C_0-Semigroup.

Definition 2.3.4. *A semigroup $S(t)$,$0 \leq t < \infty$ is called a semigroup of contraction if there exists a constant $\alpha > 0$ $(0 < \alpha < 1)$ such that for all $t > 0$,*

$$\|S(t)x - S(t)y\| \leq \alpha \|x - y\|, \quad for \quad all \quad x, y \in X.$$

2.3.2 Hille-Yoshida Theorem

Definition 2.3.5. *An unbonded linear operator $A : D(A) \subset H \to H$*[1] *is said to be monotone*[2]*if it satisfies*

$$\langle Av, v\rangle \geq 0 \qquad \forall v \in D(A).$$

It is called maximal monotone if, in addition, $R(I + A) = H$ i.e

$$\forall f \in H \quad \exists u \in D(A) \quad such \quad that \quad u + Au = f.$$

Proposition 2.4. *Let A be a maximal monotone operator. Then.*

1. $D(A)$ is dense in H.

2. A is closed operator.

3. For evry $\lambda > 0, (I + \lambda A)$ is bijective from $D(A)$ onto H, $(I + \lambda A)^{-1}$ is a bounded operator, and $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.

Theorem 2.18. *(Hille-Yosida) Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ there exists a unique function.*

$$u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A))$$

Satisfying:

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad on \qquad [0, +\infty) \\ u(0) = u_0. \end{cases}$$

---

[1] H denotes a Hilbert space

[2] Some authors say that A is accretive or $-A$ is dissipative.

Chapter 3

Well-posedness of the problem

In this chapter, we give the existence and uniqueness result for problem(3.4) using the semigroup theory.

3.1 Preliminaries

In this chapter, we are concerned with the following Timoshenko system of thermoelasticity of type III with delay of the form.

$$\begin{cases} \rho_1\phi_{tt} - K\left(\phi_x + \psi\right)_x + \mu_1\phi_t(x,t) + \mu_2\phi_t(x,t-\tau) = 0 \text{ in } (0,1)\times(0,\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\phi_x + \psi\right) + \beta\theta_{tx} = 0 \quad \text{ in } (0,1)\times(0,\infty) \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{tx} - k\theta_{txx} = 0 \text{ in } (0,1)\times(0,\infty) \\ \theta(.,0) = \theta_0, \theta_t(.,0) = \theta_1, \psi(.,0) = \psi_0, \psi_t(.,0) = \psi_1, \\ \phi(.,0) = \phi_0, \phi_t(.,0) = \phi_1, \\ \phi_t(x,t-\tau) = f_0(x,t-\tau), \quad t \in (0,\tau) \\ \phi(0,t) = \phi(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \quad \forall t \geq 0. \end{cases} \tag{3.1}$$

Where $\rho_1, \rho_2, \rho_3, K, b, k, \beta, \gamma, \delta, \mu_1$ are positive constants, μ_2 is a real number, and $\tau > 0$ represents the time delay. We prove, under suitable conditions on the initial data that the energy decays exponentially in the case of equal wave speeds in spite of the existence of the delay. The second part of our result is the case of nonequal speeds which is of much importance because practically or physically the speeds are not necessarily equal. In that case, we prove that the energy decays polynomially.

As in [33], we introduce the new variable.

$$z(x,\rho,t) = \phi_t(x,t-\tau\rho), \quad x \in (0,1), \rho \in (0,1), t > 0. \tag{3.2}$$

We put.

$$f = t - \tau\rho$$

Then,

$$z(x,p,t) = \phi_t(x, t-\tau\rho) = \phi_t(x, f(x))$$

$$z_\rho(x,p,t) = \frac{d}{dt}(z(x,p,t)) = \frac{d}{dt}(\phi_t(x,f)) = \frac{d}{df}(\phi_i(x,t))\frac{dt}{df}(t) = \frac{d}{df}(\phi_i(x,t))$$

Because.

$$\frac{df}{dt} = \frac{d(t-\tau)}{dt} = 1.$$

And from him.

$$\tau z_t(x,p,t) = -\tau\frac{d}{df}\phi_t(x,t)$$

$$z_\rho(x,p,t) = \frac{d}{dp}\phi_t(x,f) = \frac{d}{df}(\phi_t(x,f)) = \frac{d}{df}\phi_t(x,f)\frac{df}{dp} = -\tau\frac{d}{df}\phi_t(x,f).$$

Because

$$\frac{df}{dt} = \frac{d(t-\tau p)}{dt} = -\tau.$$

And from him.

$$z_\rho(x,p,t) = -\tau\frac{d}{df}\phi_t(x,t).$$

Thus, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x \in (0,1), \rho \in (0,1), t > 0. \tag{3.3}$$

So, problem (3.1) is equivalent to.

$$\begin{cases} \rho_1\phi_{tt} - K\left(\phi_x + \psi\right)_x + \mu_1\phi_t(x,t) + \mu_2 z(x,1,t) = 0 \text{ in } (0,1)\times(0,\infty) \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\phi_x + \psi\right) + \beta\theta_{tx} = 0 \quad \text{ in } (0,1)\times(0,\infty) \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{tx} - k\theta_{txx} = 0 \text{ in } (0,1)\times(0,\infty) \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \text{ in } (0,\infty)\times(0,1)\times(0,1) \\ \phi(.,0) = \phi_0, \phi_t(.,0) = \phi_1, z(x,0,t) = \phi_t(x,t) \\ \theta(.,0) = \theta_0, \theta_t(.,0) = \theta_1, \psi(.,0) = \psi_0, \psi_t(.,0) = \psi_1 \\ \phi(0,t) = \phi(1,t) = \psi(0,t) = \psi(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0 \\ z(x,\rho,0) = f_0(x,-\rho\tau), \quad x \in (0,1), \rho \in (0,1). \end{cases} \tag{3.4}$$

In order to be able to use Poincaré's inequality for θ, we introduce.

$$\bar{\theta}(x,t) = \theta(x,t) - t\int_0^1 \theta_1(x)\mathrm{d}x - \int_0^1 \theta_0(x)\mathrm{d}x. \tag{3.5}$$

Then by (3.4).1–3, we have.

$$\int_0^1 \bar{\theta}(x,t)\mathrm{d}x = 0, \quad \forall t \geq 0. \tag{3.6}$$

In this case, Poincaré's inequality is applicable for $\bar{\theta}$, and furthermore, $(\phi, \psi, \bar{\theta}, z)$ satisfies the same equations and boundary conditions of (2.1). In what follows, we will work with $\bar{\theta}$ but, for convenience, we write θ instead of $\bar{\theta}$.

We will assume that.

$$\mu_1 \geq |\mu_2|\,. \tag{3.7}$$

And show the well-posedness of the problem and that this condition is sufficient to prove the uniform decay of the solution energy.

3.2 Well-posedness of the problem

We will use the following standard $L^2(0,1)$ space with the scalar product and norm denoted by.

$$\langle u, v\rangle_{L^2(0,1)} = \int_0^1 uv\mathrm{d}x, \quad \|u\|_2^2 = \int_0^1 |u|^2\mathrm{d}x. \tag{3.8}$$

Respectively. Introducing the vector function $\mathcal{U}(t) = (\phi, \varphi, \psi, u, \theta, v, z)^T$, where $\varphi = \phi_t, u = \psi_t$, and $v = \theta_t$, system (3.4) can be re-written as.

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{U}(t) + \mathcal{A}\mathcal{U}(t) = 0, \quad t > 0 \\ \mathcal{U}(0) = \mathcal{U}_0 = (\phi_0, \phi_1, \psi_0, \psi_1, \theta_0, \theta_1, f_0)^T\,. \end{cases} \tag{3.9}$$

Where the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$ is defined by.

$$\mathcal{A} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{k}{\rho_1}\frac{d_2}{d_x} & \frac{u_1}{\rho_1} & -\frac{k}{\rho_1}\frac{d}{d_x} & 0 & 0 & 0 & -\frac{u_2}{\rho_1} \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \frac{k}{\rho_2}\frac{d}{d_x} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{b}{\rho_2}\frac{d}{d_x} + \frac{k}{\rho_2}\frac{d}{d_x} & 0 & 0 & 0 & \frac{\beta}{\rho_2}\frac{d}{d_x} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{\lambda}{\rho_3}\frac{d2}{d_{x2}} & -\frac{\sigma}{\rho_3}\frac{d2}{d_{x2}} & -\frac{k}{\rho_3}\frac{d2}{d_{x2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\tau}\frac{d}{d_\rho} \end{pmatrix} \tag{3.10}$$

$$\mathcal{A}\mathcal{U} = \begin{pmatrix} -\varphi \\ -\frac{K}{\rho_1}(\phi_x+\psi)_x + \frac{\mu_1}{\rho_1}\varphi + \frac{\mu_2}{\rho_1}z(.,1) \\ -u \\ -\frac{b}{\rho_2}\psi_{xx} + \frac{K}{\rho_2}(\phi_x+\psi) + \frac{\beta}{\rho_2}v_x \\ -v \\ -\frac{\delta}{\rho_3}\theta_{xx} + \frac{\gamma}{\rho_3}u_x - \frac{k}{\rho_3}v_{xx} \\ \frac{1}{\tau}z_\rho . \end{pmatrix} \tag{3.11}$$

Next, we introduce.

$$\begin{aligned} L^2_\star(0,1) &= \{w \in L^2(0,1) : \int_0^1 w(s)\mathrm{d}s = 0\}, \\ H^1_\star(0,1) &= H^1(0,1) \cap L^2_\star(0,1), \\ H^2_\star(0,1) &= \{w \in H^2(0,1) : w_x(0) = w_x(1) = 0\}. \end{aligned} \tag{3.12}$$

And the energy space

$$\mathcal{H} = H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_\star(0,1) \times L^2_\star(0,1) \times L^2((0,1),L^2(0,1)). \tag{3.13}$$

For a positive constant ξ satisfying.

$$\begin{cases} \gamma\tau|\mu_2| < \xi < \gamma\tau(2\mu_1 - |\mu_2|) & \text{if } \mu_1 > |\mu_2| \\ \xi = \gamma\tau|\mu_2| = \gamma\tau\mu_1 & \text{if } \mu_1 = |\mu_2|. \end{cases} \tag{3.14}$$

We equip $\mathcal{H}$ with the inner product.

$$\begin{aligned} \langle \mathcal{U}, \widetilde{\mathcal{U}} \rangle_{\mathcal{H}} =& \gamma \int_0^1 \{\rho_1\varphi\widetilde{\varphi} + \rho_2 u\widetilde{u} + K(\phi_x+\psi)(\widetilde{\phi}_x + \widetilde{\psi}) + b\psi_x\widetilde{\psi}_x\}\mathrm{d}x \\ &+ \beta\int_0^1 \{\rho_3 v\widetilde{v} + \delta\theta_x\widetilde{\theta}_x\}\mathrm{d}x + \xi\int_0^1\int_0^1 z(x,\rho)\widetilde{z}(x,\rho)\mathrm{d}\rho\mathrm{d}x. \end{aligned} \tag{3.15}$$

The domain of $\mathcal{A}$ is.

$$D(\mathcal{A}) = \left\{ \begin{array}{l} \mathcal{U} \in \mathcal{H} \mid \phi, \psi \in H^2(0,1) \cap H^1_0(0,1), \quad \theta, v \in H^1_\star(0,1), \quad \varphi, u \in H^1_0(0,1), \\ \delta\theta + kv \in H^2_\star(0,1), \quad z, z_\rho \in L^2((0,1),L^2(0,1)), \quad z(x,0) = \varphi(x) \end{array} \right\}. \tag{3.16}$$

And it is dense in $\mathcal{H}$.

We have the following existence and uniqueness result:

Theorem 3.1. *Assume $\mathcal{U}_0 \in \mathcal{H}$ and (3.7) holds. Then, there exists a unique solution $\mathcal{U} \in (\mathbb{R}^+, \mathcal{H})$ of problem (3.4). Moreover, if $\mathcal{U}_0 \in D(\mathcal{A})$ then $\mathcal{U} \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

Proof. We use the semigroup approach. So, we prove that $\mathcal{A}$ is a maximal monotone operator. First, we prove that $\mathcal{A}$ is monotone. For any $\mathcal{U} \in D(\mathcal{A})$, we have.

$$\begin{aligned}(AU,U) =& y\int_0^1 \Bigg\{ p_1\Bigg(\left(\frac{-k}{p_1}(\phi_x+\psi)_x + \frac{\mu_1}{p_1}\varphi + \frac{\mu_2}{p_1}z(.,1)\varphi\right) \\ &+ p_2\left(\left(\frac{-b}{p_2}\psi_{xx} + \frac{k}{p_2}(\phi_x+\psi) + \frac{\beta}{p_2}v_x\right)u\right)\Bigg) \\ &+ k(-\varphi_x - u)(\phi_x+\psi) + b(-u_x)(\psi_x)\Bigg\}dx \\ &- \beta\int_0^1 \left(p_3\left(-\frac{\delta}{p_3}\theta_{xx} + \frac{\gamma}{p_3}u_x - \frac{k}{p_3}v_{xx}\right)v + \delta(-v_x)(\theta_x)\right)dx \\ &+ \varepsilon\int_0^1\int_0^1 \left(\frac{1}{\tau}z_\rho\right) z(x,\rho)d\rho.\end{aligned} \tag{3.17}$$

Using integration by parts.

$$\begin{aligned}(AU,U) \geq & y\Bigg(-k(\phi_x+\psi)\varphi\int_0^1 dx + \int_0^1 k(\phi_x+\psi)\varphi_x dx + \mu_1\varphi^2 - \mu_2\int_0^1 \varphi z(..1)dx\Bigg) \\ &- b\psi_x u\int_0^1 dx + b\int_0^1 \psi_x u_x dx + k(\phi_x+\varphi)u + \beta v_x u - \int_0^1 k(\phi_x+\psi)\varphi_x dx \\ &- \int_0^1 k(\phi_x+\psi)u dx - b\int_0^1 \varphi_x u_x dx - \beta\left((-\delta\theta_x v\int_0^1 dx + \delta\int_0^1 \theta_x v_x dx)\right) \\ &+ (yuv\int_0^1 dx + y\int_0^1 uv_x dx) - kv_x v\int_0^1 dx + k\int_0^1 v_x^2 dx - \delta v_x\theta_x dx + \frac{\varepsilon}{\tau}\int_0^1\int_0^1 zz_\rho d\rho d\varepsilon.\end{aligned} \tag{3.18}$$

$$(\mathcal{AU},\mathcal{U})_{\mathcal{H}} = \gamma\mu_1\int_0^1 \varphi^2 \,\mathrm{d}x + \beta k\int_0^1 v_x^2 \,\mathrm{d}x + \gamma\mu_2\int_0^1 \varphi z(.,1)\mathrm{d}x + \frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho \mathrm{d}\rho\mathrm{d}x. \tag{3.19}$$

By using Young's inequality, the third term in the right-hand side of (3.19) gives.

$$-\mu_2\int_0^1 \varphi z(.,1)\mathrm{d}x \leq \frac{|\mu_2|}{2}\int_0^1 \varphi^2 \,\mathrm{d}x + \frac{|\mu_2|}{2}\int_0^1 z^2(.,1)\mathrm{d}x. \tag{3.20}$$

Also, using integration by parts and the fact that $z(x,0) = \varphi(x)$, the last term in the right-hand side of (3.19) gives.

$$\int_0^1\int_0^1 zz_\rho \mathrm{d}\rho\mathrm{d}x = -\frac{1}{2}\int_0^1 \varphi^2 \,\mathrm{d}x + \frac{1}{2}\int_0^1 z^2(.,1)\mathrm{d}x. \tag{3.21}$$

Consequently, (3.19) yields.

$$(\mathcal{AU},\mathcal{U})\mathcal{H} \geq \frac{1}{2\tau}\left(\gamma\tau\left(2\mu 1 - |\mu_2|\right) - \xi\right)\int_0^1 \varphi^2 \,\mathrm{d}x + \frac{1}{2\tau}\left(\xi - \gamma\tau\,|\mu_2|\right)\int_0^1 z^2(.,1)\mathrm{d}x + \beta k\int_0^1 v_x^2 \,\mathrm{d}x. \tag{3.22}$$

And by using (3.14), we get.

$$(\mathcal{AU},\mathcal{U})\mathcal{H} \geq m0\left(\int_0^1 \varphi^2 \,\mathrm{d}x + \int_0^1 z^2(.,1)\mathrm{d}x\right) + k\beta\int_0^1 v_x^2 \,\mathrm{d}x. \tag{3.23}$$

For some constant $m_0 \geq 0$. Thus, $\mathcal{A}$ is monotone. Next, we prove that the operator $I + \mathcal{A}$ is surjective. Given $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7)^T \in \mathcal{H}$, we prove that there exists a unique $\mathcal{U} \in D(\mathcal{A})$ such that.

$$(I + \mathcal{A})\mathcal{U} = F. \tag{3.24}$$

That is,

$$\begin{cases} -\varphi + \phi = f_1 & \text{in } H_0^1(0,1) \\ -K\left(\phi_x + \psi\right)_x + \left(\mu_1 + \rho_1\right)\varphi + \mu_2 z(.,1) = \rho_1 f_2 & \text{in } L^2(0,1) \\ -u + \psi = f_3 & \text{in } H_0^1(0,1) \\ -b\psi_{xx} + K\left(\phi_x + \psi\right) + \beta v_x + \rho_2 u = \rho_2 f_4 & \text{in } L^2(0,1) \\ -v + \theta = f_5 & \text{in } H_\star^1(0,1) \\ -\delta\theta_{xx} + \gamma u_x - k v_{xx} + \rho_3 v = \rho_3 f_6 & \text{in } L_\star^2(0,1) \\ z_\rho + \tau z = \tau f_7 & \text{in } L^2\left((0,1), L^2(0,1)\right). \end{cases} \tag{3.25}$$

Using $(3.25)_7$ and the fact that $z(x,0) = \varphi(x)$, we get.

$$z(x,\rho) = \varphi(x)e^{-\tau\rho} + \tau e^{-\tau\rho}\int_0^\rho e^{\tau s} f_7(x,s)\mathrm{d}s. \tag{3.26}$$

In order to solve (3.25), we consider the following variational formulation.

$$B\left((\phi,\psi,\theta),(\phi_1,\psi_1,\theta_1)\right) = G\left(\phi_1,\psi_1,\theta_1\right), \tag{3.27}$$

Where $B : \left[H_0^1(0,1) \times H_0^1(0,1) \times H_\star^1(0,1)\right]^2 \longrightarrow \mathbb{R}$ is the bilinear form defined by.

$$\begin{aligned} B\left((\phi,\psi,\theta),(\phi_1,\psi_1,\theta_1)\right) = \gamma K\int_0^1 (\phi_x + \psi)(\phi_{1x} + \psi_1)\mathrm{d}x + \beta(\delta + k)\int_0^1 \theta_x\theta_{1x}\ \mathrm{d}x + b\gamma\int_0^1 \psi_x\psi_{1x}\ \mathrm{d}x \\ + \rho_2\gamma\int_0^1 \psi\psi_1\ \mathrm{d}x + \beta\gamma\int_0^1 \theta_x\psi_1\ \mathrm{d}x + \beta\rho_3\int_0^1 \theta\theta_1\ \mathrm{d}x + \beta\gamma\int_0^1 \theta_1\psi_x\ \mathrm{d}x \\ + \gamma(\mu_1 + \rho_1 + \mu_2 e^{-\tau})\int_0^1 \phi\phi_1\ \mathrm{d}x. \end{aligned} \tag{3.28}$$

And

$$G : \left[H_0^1(0,1) \times H_0^1(0,1) \times H_\star^1(0,1)\right] \longrightarrow \mathbb{R}$$

is the linear functional given by.

$$\begin{aligned} G\left(\phi_1,\psi_1,\theta_1\right) = & \gamma\rho_1\int_0^1 f_2\phi_1\ \mathrm{d}x + \gamma\left(\mu_1 + \rho_1\right)\int_0^1 f_1\phi_1\ \mathrm{d}x + \gamma\rho_2\int_0^1 f_4\psi_1\ \mathrm{d}x \\ & + \gamma\beta\int_0^1 f_{5x}\psi_1\ \mathrm{d}x + \gamma\rho_2\int_0^1 f_3\psi_1\ \mathrm{d}x + \beta\rho_3\int_0^1 f_6\theta_1\ \mathrm{d}x + \beta\rho_3\int_0^1 f_5\theta_1\ \mathrm{d}x \\ & + \gamma\beta\int_0^1 f_{3x}\theta_1\ \mathrm{d}x + \beta k\int_0^1 f_{5x}\theta_{1x}\ \mathrm{d}x - \gamma\tau\mu_2 e^{-\tau}\int_0^1 \phi_1\int_0^1 e^{\tau s} f_7(x,s)\mathrm{d}s. \end{aligned} \tag{3.29}$$

Now, for $V = H_0^1(0,1) \times H_0^1(0,1) \times H_\star^1(0,1)$ equipped with the norm.

$$\|\phi, \psi, \theta\|_V^2 = \|(\phi_x + \psi)\|_2^2 + \|\phi\|_2^2 + \|\psi_x\|_2^2 + \|\theta\|_2^2 + \|\theta_x\|_2^2 . \tag{3.30}$$

Using integration by parts, we have,

$$\begin{aligned} B((\phi,\psi,\theta),(\phi,\psi,\theta)) =& \gamma K \int_0^1 (\phi_x + \psi)^2 \, \mathrm{d}x + \gamma \left(\mu_1 + \rho_1 + \mu_2 e^{-\tau}\right) \int_0^1 \phi^2 \, \mathrm{d}x \\ & + \beta(\delta + k) \int_0^1 \theta_x^2 \, \mathrm{d}x + b\gamma \int_0^1 \psi_x^2 \, \mathrm{d}x + \rho_2 \gamma \int_0^1 \psi^2 \, \mathrm{d}x \\ & + \beta\rho_3 \int_0^1 \theta^2 \, \mathrm{d}x \geq \alpha_0 \|\phi, \psi, \theta\|_V^2 . \end{aligned} \tag{3.31}$$

For some $\alpha_0 > 0$. Thus, B is coercive.

On the other hand, using Cauchy-Schwarz and Poincaré's inequalities, we obtain.

$$\begin{aligned} & \mid B((\phi,\psi,\theta),(\phi_1,\psi_1,\theta_1)) \mid \\ & \leq \gamma K \left\|\phi_x + \psi\right\|_2 \left\|\phi_{1x} + \psi_1\right\|_2 + \gamma b \left\|\psi_x\right\|_2 \left\|\psi_{1x}\right\|_2 + \gamma \left|\mu_1 + \rho_1 + \mu_2 e^{-\tau}\right| \|\phi\|_2 \left\|\phi_1\right\|_2 \\ & + \gamma\rho_2 \|\psi\|_2 \left\|\psi_1\right\|_2 + \gamma\beta \left\|\theta_x\right\|_2 \left\|\psi_1\right\|_2 + \beta(\delta + k) \left\|\theta_x\right\|_2 \left\|\theta_{1x}\right\|_2 + \beta\rho_3 \|\theta\|_2 \left\|\theta_1\right\|_2 \\ & + \gamma\beta \left\|\psi_x\right\|_2 \left\|\theta_1\right\|_2 \\ & \leq c \left(\|\phi_x + \psi\|_2 + \|\phi\|_2 + \|\psi_x\|_2 + \|\theta\|_2 + \|\theta_x\|_2\right) \times \\ & \left(\left\|\phi_{1x} + \psi_1\right\|_2 + \left\|\phi_1\right\|_2 + \left\|\psi_{1x}\right\|_2 + \left\|\theta_1\right\|_2 + \left\|\theta_{1x}\right\|_2\right) \\ & \leq c \|\phi, \psi, \theta\|_V \left\|\phi_1, \psi_1, \theta_1\right\|_V . \end{aligned} \tag{3.32}$$

Similarly,

$$\begin{aligned} \left|G\left(\phi_1, \psi_1, \theta_1\right)\right| \quad & \leq c \Big(\|f_1\|_{H_0^1(0,1)} + \|f_2\|_2 + \|f_3\|_{H_0^1(0,1)} + \|f_4\|_2 + \|f_5\|_{H_\star^1(0,1)} + \|f_6\|_2 + \|f_7\|_{L^2((0,1),L^2(0,1)} \\ & \times \Big(\|\phi_1\|_{H_0^1(0,1)} + \|\psi_1\|_{H_0^1(0,1)} + \|\theta_1\|_{H_*^1(0,1)} \Big) \\ & \leq c \left\|\phi_1, \psi_1, \theta_1\right\|_V . \end{aligned} \tag{3.33}$$

Consequently, Lax-Milgram lemma guarantees the existence of a unique.

$$(\phi, \psi, \theta) \in H_0^1(0,1) \times H_0^1(0,1) \times H_\star^1(0,1).$$

Satisfying

$$((\phi,\psi,\theta),(\phi_1,\psi_1,\theta_1)) = G(\phi_1,\psi_1,\theta_1) \quad \forall(\phi_1,\psi_1,\theta_1) \in V.$$

The substitution of ϕ, ψ, and θ into $(3.4)_1$, $(3.4)_3$, and $(3.4)_5$ yields.

$$(\varphi,u,v) \in H_0^1(0,1) \times H_0^1(0,1) \times H_\star^1(0,1). \tag{3.34}$$

Moreover, if we take $(\phi_1,\theta_1) \equiv (0,0) \in H_0^1(0,1) \times H_\star^1(0,1)$ in (3.2), we get.

$$\begin{aligned} &K\int_0^1 (\phi_x+\psi)\psi_1 \,\mathrm{d}x + b\int_0^1 \psi_x\psi_{1x} \,\mathrm{d}x + \rho_2\int_0^1 \psi\psi_1 \,\mathrm{d}x + \beta\int_0^1 \theta_x\psi_1 \,\mathrm{d}x \\ &= \rho_2\int_0^1 f_4\psi_1 \,\mathrm{d}x + \beta\int_0^1 f_{5x}\psi_1 \,\mathrm{d}x + \rho_2\int_0^1 f_3\psi_1 \,\mathrm{d}x. \end{aligned} \tag{3.35}$$

By recalling $(3.25)_3$ and $(3.25)_5$, we arrive at.

$$\begin{aligned} &K\int_0^1 (\phi_x+\psi)\psi_1 \,\mathrm{d}x + b\int_0^1 \psi_x\psi_{1x} \,\mathrm{d}x + \rho_2\int_0^1 \psi\psi_1 \,\mathrm{d}x + \beta\int_0^1 \theta_x\psi_1 \,\mathrm{d}x \\ &= \rho_2\int_0^1 f_4\psi_1 \,\mathrm{d}x + \beta\int_0^1 (\theta_x - v_x)\psi_1 \,\mathrm{d}x + \rho_2\int_0^1 (\psi-u)\psi_1 \,\mathrm{d}x. \end{aligned} \tag{3.36}$$

Hence, we obtain.

$$b\int_0^1 \psi_x\psi_{1x} \,\mathrm{d}x = \int_0^1 \left[\rho_2 f_4 - K\left(\phi_x+\psi\right) - \beta v_x - \rho_2 u\right]\psi_1 \,\mathrm{d}x, \quad \forall\psi_1 \in H_0^1(0,1). \tag{3.37}$$

By noting that

$$\left[\rho_2 f_4 - K\left(\phi_x+\psi\right) - \beta v_x - \rho_2 u\right] \in L^2(0,1). \tag{3.38}$$

Then,

$$\psi \in H^2(0,1) \cap H_0^1(0,1). \tag{3.39}$$

And, consequently, (3.37) takes the form.

$$b\int_0^1 \left[-\psi_{xx} + K\left(\phi_x+\psi\right) + \beta v_x + \rho_2 u - \rho_2 f_4\right]\psi_1 \,\mathrm{d}x = 0 \quad \forall\psi_1 \in H_0^1(0,1). \tag{3.40}$$

Therefore, we obtain.

$$-\psi_{xx} + K\left(\phi_x+\psi\right) + \beta v_x + \rho_2 u = \rho_2 f_4. \tag{3.41}$$

This gives (3.25). Similarly, if we take $(\psi_1,\theta_1) \equiv (0,0) \in H_0^1(0,1) \times H_\star^1(0,1)$ in (3.2), we can show that.

$$\phi \in H^2(0,1) \cap H_0^1(0,1). \tag{3.42}$$

And $(3.25)_2$ are satisfied. Also, if we take $(\phi_1,\psi_1) \equiv (0,0) \in H_0^1(0,1) \times H_0^1(0,1)$ in (3.2), then using $(3.25)_3$ and $(3.25)_5$, we get.

$$\delta\theta_{xx} + kv_{xx} = \rho_3 f_6 - \gamma u_x - \rho_3 v \text{ in } L_\star^2(0,1). \tag{3.43}$$

And we conclude that.

$$(\delta\theta + kv) \in H^2(0,1). \tag{3.44}$$

Furthermore, it is obvious from

$$\delta\theta_x + kv_x = \rho_3 \int_0^x f_6 \, \mathrm{d}x - \gamma u - \rho_3 \int_0^x v \, \mathrm{d}x. \tag{3.45}$$

That is

$$(\delta\theta_x + kv_x)(0) = (\delta\theta_x + kv_x)(1) = 0. \tag{3.46}$$

Thus, we get.

$$(\delta\theta + kv) \in H^2_\star(0,1). \tag{3.47}$$

Finally, it follows, from (3.26), that.

$$z(x,0) = \varphi(x) \text{ and } z, z_\rho \in L^2\left((0,1), L^2(0,1)\right). \tag{3.48}$$

Hence, there exists a unique $\mathcal{U} \in D(\mathcal{A})$ such that (3.24) is satisfied. Therefore, $\mathcal{A}$ is a maximal monotone operator. Consequently, the well-posedness result follows from the Hille-Yosida theorem. (see [31]) $\square$

The associated solution energy is given by.

$$E(t) = \left(\int_0^1 \frac{\gamma}{2}(\rho_1\phi_t^2 + \rho_2\psi_t^2 + K|\phi_x + \psi|^2 + b\psi_x^2)\, dx\right) + \left(\int_0^1 \frac{\beta}{2}(\rho_3\theta_t^2 + \delta\theta_x^2)\, dx\right) + \frac{\varepsilon}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\, d\rho\, dx. \tag{3.49}$$

Where, as in (3.50),

$$\begin{cases} \gamma\tau|\mu_2| < \xi < \gamma\tau(2\mu_1 - |\mu_2|) & \text{if} \quad \mu_1 > |\mu_2| \\ \xi = \gamma\tau|\mu_2| = \gamma\tau\mu_1 & \text{if} \quad \mu_1 = |\mu_2|. \end{cases} \tag{3.50}$$

The following lemma shows that the associated energy is decreasing in time.

Lemma 3.1. *Let* (ϕ, ψ, θ, z) *be the solution of* (1.3). *Then, for some* $C \geq 0$,

$$E'(t) \leq -\beta k \int_0^1 \theta_{tx}^2 \, dx - C \int_0^1 (\phi_t^2 + z^2(x,1,t))\, dx \leq 0. \tag{3.51}$$

Proof. Multiplying equation $(3.4)_1$ by $\gamma\phi_t$, $(3.4)_2$ by $\gamma\psi_t$, and $(3.4)_3$ by $\beta\theta_t$ and integrating over $(0,1)$ and $(3.4)_4$ by $(\xi/\tau)z$ and integrating over $(0,1)\times(0,1)$ with respect to ρ and x summing up, we get.

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 \gamma(\rho_1\phi_t^2 + \rho_2\psi_t^2 + K|\phi_x + \psi|^2 + b\psi_x^2) + \beta(\rho_3\theta_t^2 + \delta\theta_x^2)\, dx\right] \\ &\quad + \frac{\xi}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_0^1 z^2(x,\rho,t)\, d\rho\, dx. \end{aligned} \tag{3.52}$$

$$p_1\phi_{tt} = k(\phi_x+\psi)_x - \mu_1\phi_t(x,1,t) - \mu_2 Z(x,1,t). \tag{3.53}$$

$$p_2\psi_{tt}\psi_t = b\psi_{xx} - k(\phi_x+\psi) - \beta\theta_{tx}. \tag{3.54}$$

$$p_3\theta_{tt} = \delta\theta_{xx} - y\psi_{tx} + k\theta_{txx}. \tag{3.55}$$

$$Z_t(x,\rho,t) = \frac{-Z\rho(x,\rho,t)}{\tau}. \tag{3.56}$$

Substituting (3.53) and (3.54) and (3.55) and (3.56) into (3.52), we get.

$$\begin{aligned} E'(t) =& y\phi_t \left(\int_0^1 \left((k(\phi_x+\psi)_x - \mu_1\phi_t(x,t) - \mu_2 Z(x,1,t)) \right) dx \right) + y\psi_t \int_0^1 ((b\psi_{xx} - k(\phi_x+\psi) - \beta\theta_{tx})) \\ &+ ky(\phi_x+\psi)(\phi_{xt}+\psi_t) + yb\psi_x\psi_{xt}\,dx + \beta\left((\delta\theta_{xx} - y\psi_{tx} + k\theta_{txx})\theta_t + \delta\theta_x\theta_{xt} \right) dx \\ &- \frac{\xi}{\tau}\int_0^1\int_0^1 zz\rho(x,\rho,t)\,d\rho\,dx. \end{aligned} \tag{3.57}$$

Using integration by parts, we obtain.

$$\begin{aligned} E'(t) =& y\left(k(\phi_x+\psi)\phi_t \int_0^1 \left(-k\int_0^1 (\phi_x+\psi)\phi_{tx} - \mu_1\phi_t^2(x,t) - \mu_2\phi_t Z(x,1,t) \right) dx \right) \\ &+ yb\psi_x\psi_t \int_0^1 \left(-\int_0^1 b\psi_x\psi_{tx} - k(\phi_x+\psi)\psi_t - \beta\int_0^1 \theta_{tx}\psi_t + k(\phi_x+\psi)(\phi_{xt}+\psi_t) \right) dx \\ &+ \beta\left(\delta\theta_x\theta_t \int_0^1 \left(-\int_0^1 \delta\theta_x\theta_{tx} \right) + \beta\left(-y\psi_t\theta_t\int_0^1 + \int_0^1 y\psi_t\theta_{tx} \right) \right) dx \\ &+ \beta\left(k\theta_{tx}\theta_t \int_0^1 \left(-k\int_0^1 \theta_{tx}^2 \right) \right) + \beta\int_0^1 \delta\theta_x\theta_{xt}\,dx - \frac{\xi}{\tau}\int_0^1\int_0^1 zz\rho(x,\rho,t)\,d\rho\,dx. \end{aligned} \tag{3.58}$$

We, now, estimate the last two terms of the right-hand side of (3.52)as follows.

$$\begin{aligned} -\frac{\xi}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)\,d\rho\,dx &= -\frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho} z^2(x,\rho,t)\,d\rho\,dx \\ &= \frac{\xi}{2\tau}\int_0^1 \left(z^2(x,0,t) - z^2(x,1,t) \right) dx = \frac{\xi}{2\tau}\left(\int_0^1 \phi_t^2\,dx - \int_0^1 z^2(x,1,t)\,dx \right) \\ &- \gamma\mu_2\int_0^1 \phi_t z(x,1,t)\,dx \le \frac{\gamma\,|\mu_2|}{2}\left(\int_0^1 \phi_t^2\,dx + \int_0^1 z^2(x,1,t)\,dx \right). \end{aligned} \tag{3.59}$$

We conclude, then,

$$\frac{\mathrm{d}E(t)}{\mathrm{d}} \le -\beta k\int_0^1 \theta_{tx}^2\,dx - \gamma\left(\mu_1 - \frac{\xi}{2\tau\gamma} - \frac{|\mu_2|}{2} \right)\int_0^1 \phi_t^2\,dx - \gamma\left(\frac{\xi}{2\tau\gamma} - \frac{|\mu_2|}{2} \right)\int_0^1 z^2(x,1,t)\,dx. \tag{3.60}$$

Using (3.50), we have, for some $C \ge 0$,

$$E'(t) \le -\beta k\int_0^1 \theta_{tx}^2\,dx - C\int_0^1 \left(\phi_t^2 + z^2(x,1,t) \right) dx \le 0. \tag{3.61}$$

. $\square$

Chapter 4

Decay of solutions

In this chapter, we state and prove our stability result.

Theorem 4.1. *Suppose that* $\mu_1 \geq |\mu_2|$. *Then the energy* $E(t)$ *satisfies* $\forall t > 0$,

$$E(t) \leq CE(0)e^{-\alpha t}, \quad \text{if} \quad \frac{\rho_1}{K} = \frac{\rho_2}{b}, \tag{4.1}$$

In order to prove this result, we introduce various functionals and prove several lemmas.

We note here that the functionals I_1, J and the function $q(x)$, used in Lemma4.33, were first introduced in [12]. Similarly, the function I_3 was first introduced in [28].

Lemma 4.1. *Let* (ϕ, ψ, θ, z) *be the solution of* (3.4). *Then the functional.*

$$I_1(t) := \int_0^1 (\rho_1 \phi_t \omega + \rho_2 \psi_t \psi)\ dx. \tag{4.2}$$

. *Satisfies* $\forall \varepsilon_1 > 0$,

$$\begin{aligned} I_1'(t) \leq & \left(-\frac{b}{2} + \varepsilon_1(\mu_1 + |\mu_2|)\right) \int_0^1 \psi_x^2 \ \mathrm{d}x + \left(\varepsilon_1 \rho_1 + \frac{\mu_1}{4\varepsilon_1}\right) \int_0^1 \phi_t^2 \ \mathrm{d}x \\ & + \left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) \int_0^1 \psi_t^2 \ \mathrm{d}x + \frac{\beta^2}{2b} \int_0^1 \theta_{tx}^2 \ \mathrm{d}x + \frac{|\mu_2|}{4\varepsilon_1} \int_0^1 z^2(x,1,t)\mathrm{d}x, \end{aligned} \tag{4.3}$$

Where

$$\omega(x,t) = -\int_0^x \psi(y,t)\mathrm{d}y + x\int_0^1 \psi(y,t)\mathrm{d}y \tag{4.4}$$

Proof. By differentiating I_1 we get.

$$I_1'(t) = \int_0^1 (p_1 \phi_{tt} \omega + p_1 \phi_t \omega_t + p_2 \psi_{tt} \psi + p_2 \psi_t^2) dx. \tag{4.5}$$

From the first and second equations of problem (3.4) we have.

$$p_1 \phi_{tt} = k(\phi_x + \psi)_x - \mu_1 \phi_t - \mu_2 \mathrm{Z}(x,1,t). \tag{4.6}$$

$$\mathrm{p}_2\psi_{tt} = b\psi_{tt} - k(\phi_x + \psi) - \beta\theta_x. \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.5) we get.

$$\begin{aligned} I_1'(t) = \int\limits_0^1 \Big\{ & k(\phi_x+\psi)_x \omega dx - \int\limits_0^1 \mu_1\phi_t\omega - \int\limits_0^1 \mu_2 \mathrm{Z}(x,1,t)\omega \\ & + \int\limits_0^1 p_1\phi_t\omega_t + \gamma\psi_{xx}\psi - k\int\limits_0^1 (\phi_x+\psi)\psi - \beta\int\limits_0^1 \theta_{tx}\psi + p_2\int\limits_0^1 \psi_t^2 dx \Big\}. \end{aligned} \tag{4.8}$$

$$\begin{aligned} I_1'(t) :=& -b\int_0^1 \psi_x^2 \mathrm{~d}x + \rho_2\int_0^1 \psi_t^2 \mathrm{~d}x - K\int_0^1 \psi^2 \mathrm{~d}x - \beta\int_0^1 \psi\theta_{tx} \mathrm{~d}x \\ & + K\int_0^1 \omega_x^2 \mathrm{~d}x + \rho_1\int_0^1 \phi_t\omega_t \mathrm{~d}x - \mu_1\int_0^1 \phi_t\omega \mathrm{d}x - \mu_2\int_0^1 \omega z(x,1,t)\omega \mathrm{d}x. \end{aligned} \tag{4.9}$$

By exploiting the inequalities

$$\begin{aligned} & \int_0^1 \omega_x^2 \mathrm{~d}x \leq \int_0^1 \psi^2 \mathrm{~d}x \leq \int_0^1 \psi_x^2 \mathrm{~d}x \\ & \int_0^1 \omega_t^2 \mathrm{~d}x \leq \int_0^1 \omega_{tx}^2 \mathrm{~d}x \leq \int_0^1 \psi_t^2 \mathrm{~d}x \end{aligned} \tag{4.10}$$

And Young's inequality, the result follows.

$$-\mu_1\int\limits_0^1 \phi_t\omega dx \leq \frac{\mu_1}{4\varepsilon_1}\int\limits_0^1 \omega_t^2 + \varepsilon_1\mu_1\int\limits_0^1 \omega^2 \leq \frac{\mu_1}{4\varepsilon_1}\int\limits_0^1 \phi_t^2 + \varepsilon_1\mu_1\int\limits_0^1 \psi_x^2 dx. \tag{4.11}$$

$$-\mu_2\int\limits_0^1 \mathrm{Z}(x,1,t)\omega \leq \frac{|\mu_2|}{4\varepsilon_1}\int\limits_0^1 \mathrm{Z}^2(x,1,t) + |\mu_2|\varepsilon_1\int\limits_0^1 \omega^2 \leq \frac{|\mu_2|}{4\varepsilon_1}\int\limits_0^1 \mathrm{Z}^2(x,1,t) + |\mu_2|\varepsilon_1\int\limits_0^1 \psi_x^2. \tag{4.12}$$

$$\int\limits_0^1 p_1\phi_t\omega_t \leq \varepsilon_1 p_1\int\limits_0^1 \phi_t^2 + \frac{p_1}{4\varepsilon_1}\int\limits_0^1 \omega_t^2 \leq \varepsilon_1 p_1\int\limits_0^1 \psi_t^2 + \frac{p_1}{4\varepsilon_1}\int\limits_0^1 \psi_t^2. \tag{4.13}$$

$$-\beta\int\limits_0^1 \theta_{tx}\psi \leq \frac{\beta^2}{2b}\int\limits_0^1 \theta_{tx}^2 + \frac{b}{2}\int\limits_0^1 \psi^2 \leq \frac{\beta^2}{2b}\int\limits_0^1 \theta_{tx}^2 + \frac{b}{2}\int\limits_0^1 \psi_x^2. \tag{4.14}$$

By exploiting estimates (4.11),(4.12),(4.13)and (4.14)in addition to (4.9)we can obtain (4.3). □

Lemma 4.2. *Let (ϕ, ψ, θ, z) be a solution of* (3.4). *Then the functional.*

$$I_2(t) := \rho_2\rho_3\int_0^1 \psi_t(x,t)\int_0^x \theta_t(y,t)\mathrm{d}y \mathrm{~d}x - \delta\rho_2\int_0^1 \theta_x\psi \mathrm{d}x \tag{4.15}$$

. *Satisfies,* $\forall\varepsilon_2 > 0$,

$$I_2'(t) \leq -\frac{\rho_2\gamma}{2}\int_0^1 \psi_t^2 \mathrm{~d}x + \varepsilon_2\int_0^1 \psi_x^2 \mathrm{~d}x + \varepsilon_2\int_0^1 \phi_x^2 \mathrm{~d}x + C\left(\varepsilon_2\right)\int_0^1 \theta_{tx}^2 \mathrm{~d}x. \tag{4.16}$$

.

Proof. By differentiating I_2 we get.

$$I_2'(t) = p_2p_3\int_0^1 \psi_{tt}(x,t)\int_0^x \theta_t(y,t) + p_2p_3\int_0^1 \psi_t(x,t)\int_0^x \theta_{tt}(y,t)dydx - \sigma p_2\int_0^1 \theta_{tx}\psi + \theta_x\psi_t dx \tag{4.17}$$

. From the second and third equations of the problem (3.4) we have.

$$p_2\psi_{tt} = b\psi_{xx} - k(\phi_x+\psi) - \beta\theta_{tx} \tag{4.18}$$

$$\mathrm{p}_3\theta_{tt} = \delta\theta_{xx} - y\psi_{tx} + k\theta_{txx} \tag{4.19}$$

. Substituting(4.18)and (4.19) into(4.17) we get.

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_2\rho_3\int_0^1 \psi_t(x,t)\int_0^x \theta_t(y,t)\mathrm{d}y\ \mathrm{d}x\right)\\
&=\int_0^1 \rho_2\psi_t\int_0^x \left(\delta\theta_{xx} - \gamma\psi_{tx} + k\theta_{txx}\right)\mathrm{d}y\ \mathrm{d}x\\
&\quad+\int_0^1 \left(b\psi_{xx} - K\left(\phi_x+\psi\right) - \beta\theta_{tx}\right)\int_0^x \rho_3\theta_t(y,t)\mathrm{d}y\ \mathrm{d}x\\
&=\int_0^1 \rho_2\psi_t\left(\delta\theta_x - \gamma\psi_t + k\theta_{tx}\right)\mathrm{d}x - \rho_3K\int_0^1 \psi\int_0^x \theta_t(y,t)\mathrm{d}y\ \mathrm{d}x\\
&\quad-\rho_3b\int_0^1 \theta_t\psi_x\ \mathrm{d}x + \rho_3K\int_0^1 \theta_t\phi\mathrm{d}x + \beta\rho_3\int_0^1 \theta_t^2\ \mathrm{d}x\\
&\quad+\left[\rho_3\left(\int_0^x \theta_t(y,t)\mathrm{d}y\right)\left(b\psi_x - K\phi - \beta\theta_t\right)\right]_{x=0}^{x=1}.
\end{aligned} \tag{4.20}$$

By recalling that θ stands for $\bar{\theta}$, we have

$$\int_0^1 \theta_t(y,t)\mathrm{d}y = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta(y,t)\mathrm{d}y = 0 \tag{4.21}$$

Consequently.

$$\left[\rho_3\left(\int_0^x \theta_t(y,t)\mathrm{d}y\right)\left(b\psi_x - K\phi - \beta\theta_t\right)\right]_{x=0}^{x=1} = 0 \tag{4.22}$$

Thus, we obtain.

$$\begin{aligned}
I_2'(t) =& -\gamma\rho_2\int_0^1 \psi_t^2\ \mathrm{d}x - \delta\rho_2\int_0^1 \psi\theta_{tx}\ \mathrm{d}x + k\rho_2\int_0^1 \theta_{tx}\psi_t\ \mathrm{d}x\\
&-K\rho_3\int_0^1\int_0^x \theta_t(y,t)\mathrm{d}y\psi\mathrm{d}x + b\rho_3\int_0^1 \theta_t\psi_x\ \mathrm{d}x\\
&-K\rho_3\int_0^1 \theta_t\phi\mathrm{d}x + \beta\rho_3\int_0^1 \theta_t^2\ \mathrm{d}x.
\end{aligned} \tag{4.23}$$

The assertion of the lemma then follows, using Young's and Poincare's inequalities.

$$kp_2\int_0^1 \theta_{tx}\psi_t \le \frac{p_2k^2}{2\sqrt{\sigma}}\int_0^1 \theta_{tx}^2 + \frac{p_2\sigma}{2}\int_0^1 \psi_t^2. \tag{4.24}$$

$$bp_3\int_0^1 \theta_t\psi_x \le \frac{b^2p_3^2}{4\varepsilon_2}\int_0^1\theta_t^2 + \frac{1}{2}\varepsilon_2\int_0^1\psi_x^2 \le \frac{c_1b^2p_3^2}{2}\int_0^1\theta_{tx}^2 + \frac{\varepsilon_2}{2}\int_0^1\psi_x^2. \tag{4.25}$$

$$\delta p_2\int_0^1\psi\theta_{tx}dx \le \frac{\delta^2p^2}{\varepsilon_2}\int_0^1\theta_{tx}^2 + \frac{\varepsilon_2}{4}\int_0^1\psi^2 \le \frac{\delta^2p_2^2}{\varepsilon_2}\int_0^1\theta_{tx}^2 + c_2\frac{\varepsilon_2}{4}\int_0^1\psi_x^2. \tag{4.26}$$

$$-kp_3\int_0^1\theta_t\phi \le \frac{k^2p_2^2}{\varepsilon_2 4}\theta_{tx}^2 + \varepsilon_2\int_0^1\phi^2 \le \frac{c_1k^2p_2^2}{4\varepsilon_2}\int_0^1\theta_{tx} + \varepsilon_2c_3\int_0^1\phi_x^2. \tag{4.27}$$

$$\begin{aligned}-kp_3\int_0^1\left(\int_0^x\theta dy\right)\psi dx &\le \frac{k^2p_3^2}{\varepsilon_2}\int_0^1\left(\int_0^x\theta_t dx\right)^2 + \frac{\varepsilon^2}{4}\int_0^1\psi_t^2 \le \frac{k^2p_3^2}{\varepsilon_2}c_4\int_0^1\theta_t^2 + \frac{c_2\varepsilon_2}{4}\int_0^1\psi_x \\ &\le \frac{k^2p_3^2}{\varepsilon_2}c_4c_1\int_0^1\theta_{tx}^2 + \frac{c_2\varepsilon_2}{4}\int_0^1\psi_x^2.\end{aligned} \tag{4.28}$$

$$\beta p_3\int_0^1\theta_t^2 \le \beta p_3c_1\int_0^1\theta_{tx}^2. \tag{4.29}$$

$$\begin{aligned}I_2'(t) \le &\left(\frac{c_2\varepsilon_2}{4} + \frac{c_2\varepsilon_2}{4} + \frac{\varepsilon_2}{2}\right)\int_0^1\psi_x^2dx \\ &+ \left(\frac{\delta^2p_2^2}{2\sqrt{\sigma}} + \frac{k^2p_3}{\varepsilon_2}c_4c_1 + \frac{c_1k^2p_3^2}{\varepsilon_2 4} + \frac{cb^2p_3^2}{2} + \beta p_3c_1\right)\int_0^1\theta_{tx}^2dx \\ &+ \varepsilon_2c_3\int_0^1\phi_x^2dx \\ &+ \left(-yp_2 + \frac{p_2\sigma}{2}\right)\int_0^1\psi_t^2dx.\end{aligned} \tag{4.30}$$

We put

$$c(\varepsilon_h) = \frac{p_2k^2}{2\sqrt{\sigma}} + \frac{c_1b^2p_3^2}{2} + \frac{\delta^2p_2^2}{\varepsilon_2} + \frac{c_1k^2p_3^2}{4\varepsilon_2} + \frac{k^2p_3}{\varepsilon_2}c_4c_1 + c_1\beta p_3. \tag{4.31}$$

□

Lemma 4.3. *Let (ϕ,ψ,θ,z) be a solution of (2.1). Then the functional.*

$$J(t) := \rho_2\int_0^1\psi_t(\phi_x+\psi)\mathrm{d}x + \rho_2\int_0^1\psi_x\phi_t\ \mathrm{d}x. \tag{4.32}$$

Satisfies,

$$\begin{aligned}J'(t) \le &[b\phi_x\psi_x]_{x=0}^{x=1} - \frac{K}{2}\int_0^1(\phi_x+\psi)^2\ \mathrm{d}x \\ &+ \rho_2\int_0^1\psi_t^2\ \mathrm{d}x + \frac{\beta^2}{2K}\int_0^1\theta_{tx}^2\ \mathrm{d}x \\ &+ \frac{\rho_2\mu_1^2}{2\rho_1}\int_0^1\phi_t^2\ \mathrm{d}x + \frac{\rho_2}{\rho_1}\int_0^1\psi_x^2\ \mathrm{d}x \\ &+ \frac{\rho_2\mu_2^2}{2\rho_1}\int_0^1 z^2(x,1,t)\mathrm{d}x \\ &+ \left(\frac{\rho_2K}{\rho_1} - b\right)\int_0^1\psi_x\left(\phi_x+\psi\right)_x\ \mathrm{d}x.\end{aligned} \tag{4.33}$$

Proof. By differentiation of $J(t)$, and integration by parts, we get.

$$J'(t) = p_2\int_0^1\psi_{tt}(\phi_x+\psi)dx + \psi_t(\phi_x+\psi)_t + p_2\int_0^1\psi_{xt}\phi_t + \phi_{tx}\psi_xdx. \tag{4.34}$$

From the first and second equations of problem (3.4) we have.

$$p_2\psi_{tt} = b\psi_{xx} - k(\phi_x + \psi) - \beta\theta_{tx}. \tag{4.35}$$

$$\phi_{tt} = \frac{k(\phi_x + \psi)_x - \mu_1\phi_t(x,t) - \mu_2 z(x,1,t)}{p_1}. \tag{4.36}$$

Substituting(4.35)and (4.36) into (4.34) we get.

$$\begin{aligned} J'(t) = &\int_0^1 (b\psi_{xx} - k(\phi_x - \psi) - \beta\theta_{tx})(\phi_x + \psi) \\ &+ p_2 \int_0^1 \psi_t(\phi_x + \psi)_t + p_2 \\ &+ \int_0^1 \psi_{xt}\phi_t + \frac{p_2}{p_1}\int_0^1 k(\phi_x + \psi)_x \\ &- \frac{p_2}{p_1}\int_0^1 \mu_1\phi_t(x,t) - \frac{p_2}{p_1}\int_0^1 \mu_2 z(x,1,t)\psi_x. \end{aligned} \tag{4.37}$$

$$\begin{aligned} J'(t) = &[b\phi_x\psi_x]_{x=0}^{x=1} + \left(\frac{\rho_2 K}{\rho_1} - b\right)\int_0^1 \psi_x\left(\phi_x + \psi\right)_x \,\mathrm{d}x \\ &- K\int_0^1 \left(\phi_x + \psi\right)^2 \,\mathrm{d}x - \beta\int_0^1 \left(\phi_x + \psi\right)\theta_{tx}\,\mathrm{d}x \\ &+ \rho_2\int_0^1 \psi_t^2\,\mathrm{d}x - \frac{\rho_2\mu_1}{\rho_1}\int_0^1 \phi_t\psi_x\,\mathrm{d}x \\ &- \frac{\rho_2\mu_2}{\rho_1}\int_0^1 \psi_x z(x,1,t)\mathrm{d}x. \end{aligned} \tag{4.38}$$

Young's inequality leads to (4.33). ☐

. Next, to handle the boundary terms, appearing in(4.33), we exploit the following function.

$$q(x) = 2 - 4x, \quad x \in (0,1). \tag{4.39}$$

Lemma 4.4. *Let* (ϕ, ψ, θ, z) *be a solution of* (3.4)*. Then we have,* $\forall \varepsilon_3 > 0$,

$$\begin{aligned} &[b\phi_x\psi_x]_{x=0}^{x=1} \\ &\leq -\frac{\varepsilon_3}{K}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_1 q\phi_t\phi_x\,\mathrm{d}x - \frac{b\rho_2}{4\varepsilon_3}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 q\psi_t\psi_x\,\mathrm{d}x \\ &+ 3\varepsilon_3\int_0^1 \phi_x^2\,\mathrm{d}x + \left(\varepsilon_3 + \frac{3b^2}{4\varepsilon_3} + \frac{b^2}{4\varepsilon_3^3}\right)\int_0^1 \psi_x^2\,\mathrm{d}x \\ &+ \frac{\rho_2 b}{2\varepsilon_3}\int_0^1 \psi_t^2\,\mathrm{d}x + \frac{K^2}{4}\varepsilon_3\int_0^1 \left(\phi_x + \psi\right)^2\,\mathrm{d}x + \frac{\beta^2}{4\varepsilon_3}\int_0^1 \theta_{tx}^2\,\mathrm{d}x \\ &+ \left(\frac{2\rho_1\varepsilon_3}{K} + \frac{\mu_1^2}{\varepsilon_3}\right)\int_0^1 \phi_t^2\,\mathrm{d}x + \frac{\mu_2^2}{\varepsilon_3}\int_0^1 z^2(x,1,t)\mathrm{d}x. \end{aligned} \tag{4.40}$$

Proof. By using Young's inequality, we easily see that, $\forall \varepsilon_3 > 0$, ☐

$$[b\phi_x\psi_x]_{x=0}^{x=1} \leq \varepsilon_3\left[\phi_x^2(1)+\phi_x^2(0)\right] + \frac{b^2}{4\varepsilon_3}\left[\psi_x^2(1)+\psi_x^2(0)\right]. \tag{4.41}$$

Also,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 b\rho_2 q\psi_t\psi_x \,\mathrm{d}x = \frac{b^2}{2}\left[q\psi_x^2\right]_{x=0}^{x=1} - \frac{b^2}{2}\int_0^1 q_x\psi_x^2 \,\mathrm{d}x \\ -\frac{\rho_2 b}{2}\int_0^1 q_x\psi_t^2 \,\mathrm{d}x - Kb\int_0^1 q\psi_x\left(\phi_x+\psi\right)\mathrm{d}x \\ -\beta b\int_0^1 q\psi_x\theta_{tx}\,\mathrm{d}x,\end{aligned} \tag{4.42}$$

Which gives

$$\begin{aligned}\frac{\mathrm{d}}{t}\int_0^1 b\rho_2 q\psi_t\psi_x \,\mathrm{d}x \leq -b^2\left[\psi_x^2(1)+\psi_x^2(0)\right] + 3b^2\int_0^1 \psi_x^2\,\mathrm{d}x \\ +2\rho_2 b\int_0^1 \psi_t^2\,\mathrm{d}x + \varepsilon_3^2 K^2\int_0^1 \left(\phi_x+\psi\right)^2\,\mathrm{d}x \\ +\frac{b^2}{\varepsilon_3^2}\int_0^1 \psi_x^2\,\mathrm{d}x + \beta^2\int_0^1 \theta_{tx}^2\,\mathrm{d}x.\end{aligned} \tag{4.43}$$

Similarly, we have.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_1 q\phi_t\phi_x \,\mathrm{d}x \leq -K\left[\phi_x^2(1)+\phi_x^2(0)\right] \\ +(2\varepsilon_3+3K)\int_0^1 \phi_x^2\,\mathrm{d}x + K\int_0^1 \psi_x^2\,\mathrm{d}x \\ +2\rho_1\int_0^1 \phi_t^2\,\mathrm{d}x + \frac{\mu_1^2}{\varepsilon_3}\int_0^1 \phi_t^2\,\mathrm{d}x + \frac{\mu_2^2}{\varepsilon_3}\int_0^1 z^2(x,1,t)\mathrm{d}x.\end{aligned} \tag{4.44}$$

A combination of (4)-(4.57) then yields the desired result.

Lemma 4.5. *Let (ϕ,ψ,θ,z) be a solution of (3.4).*

Then, the functional.

$$\Theta(t) = \int_0^1\left(\rho_3\theta_t^2 + \frac{k}{2}\theta_x^2 + \gamma\psi_x\theta\right)dx. \tag{4.45}$$

. *This satisfies, $\forall\varepsilon_2>0$.*

$$\Theta'(t) \leq -\delta\int_0^1 \theta_x^2 dx + \varepsilon_2\int_0^1 \psi_x^2 dx + \left(\frac{\gamma^2}{4\varepsilon_2}+\rho_3\right)\int_0^1 \theta_t^2 dx. \tag{4.46}$$

.

Proof. A simple differentiation, using equation (3.4) and Young's inequality, gives the result.

$$\begin{aligned}\theta'(t) &= \int_0^1 \left(p_3\theta_t^2 + p_3\theta_{tt}\theta + k\theta_x\theta_{tx} + \delta\psi_{tx}\theta + \delta\psi_x\theta_t\right) dx\\ &= \int_0^1 \left(p_3\theta_t^2 + \theta(\delta\theta_{xx} - \delta\psi_{tx} + k\theta_{txx}) + k\theta_x\theta_{tx} + \delta\psi_{tx}\theta + \delta\psi_x\theta_t\right) dx\\ &= \int_0^1 \left(p_3\theta_t^2 + \delta\theta\theta_x - \delta\int_0^1 \theta_x^2 - \delta\int_0^1 \psi_{tx}\theta dx + k\theta\theta_{tx} - k\int_0^1 \theta_x\theta_{tx} + \delta\int_0^1 \psi_{tx}\theta + \delta\int_0^1 \psi_x\theta_t\right) dx\\ &= \int_0^1 \left(p_3\theta_t^2 - \delta\int_0^1 \theta_x^2 + \delta\int_0^1 \psi_x\theta_t\right) dx.\end{aligned} \tag{4.47}$$

By using Young's inequality, we get.

$$\delta\int_0^1 \psi_x\theta_t \leq \varepsilon_2\int_0^1 \psi_x^2 dx + \frac{\delta^2}{4\varepsilon_2}\int_0^1 \theta_t^2. \tag{4.48}$$

□

Lemma 4.6. *Let* (ϕ, ψ, θ, z) *be a solution of* (3.4).

Then, the functional.

$$I_3(t) := \int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x. \tag{4.49}$$

. *Satisfies, for some* $m_0 > 0$,

$$I_3'(t) \leq -m_0\int_0^1\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x - \frac{c}{\tau}\int_0^1 z^2(x,1,t)\mathrm{d}x + \frac{1}{\tau}\int_0^1 \phi_t^2\ \mathrm{d}x. \tag{4.50}$$

Proof. By differentiation of I_3 we get.

$$I_3'(t) = \int\limits_0^1\int\limits_0^1 2e^{-2\tau\rho} z(x,\rho,t) z_t(x,\rho,t) d\rho dx. \tag{4.51}$$

From the fourth equation of the problem (3.4) we have.

$$z_t(x,\rho,t) = \frac{-z_\rho(x,\rho,t)}{\tau}. \tag{4.52}$$

Substituting(4.52)into (4.51) we get.

$$\begin{aligned}I_3'(t) &= -\frac{2}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho} z z_\rho(x,\rho,t)\mathrm{d}\rho\mathrm{d}x\\ &= -2\int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x - \frac{1}{\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho} z^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x\\ &\leq -m_0\int_0^1\int_0^1 z^2(x,\rho,t)\mathrm{d}\rho\mathrm{d}x - \frac{1}{\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho} z^2(x,\rho,t)\right)\mathrm{d}\rho\mathrm{d}x.\end{aligned} \tag{4.53}$$

Simple integration of the last term gives the result.

$$\begin{aligned}
I_3'(t) &= \frac{-2}{\tau}\left(\frac{1}{2}e^{-2\tau\rho}z^2\right)\int_0^1 dx - \left(\frac{-2}{\tau}\right)(-2\tau)\int_0^1\int_0^1 e^{-2\tau\rho}z^2 dx d\rho \\
&= \frac{-1}{\tau}\int_0^1 e^{-2\tau\rho}z^2(x,1,t) + \frac{1}{\tau}\int_0^1 e^0 z^2(x,0,t)dx - 4\int_0^1\int_0^1 e^{-2\tau\rho}z^2 dx d\rho \\
&= \frac{-e^{2\tau}}{\tau}\int_0^1 z^2(x,1,t) + \frac{1}{\tau}\int_0^1 \phi_t^2 - 4e^{-2\tau\rho}\int_0^1\int_0^1 z^2 dx d\rho.
\end{aligned} \tag{4.54}$$

☐

proof of theorem 4.1. To finalize the proof, we define the Lyapunov functional $\mathcal{L}$ as follows. ☐

$$\begin{aligned}
\mathcal{L}(t) := {} & NE(t) + N_1 I_1(t) + N_2 I_2(t) + J(t) + \Theta(t) + I_3(t) \\
& + \frac{b\rho_2}{4\varepsilon_3}\int_0^1 q\psi_t\psi_x \text{ d}x + \frac{\varepsilon_3}{K}\int_0^1 \rho_1 q\phi_t\phi_x \text{ d}x,
\end{aligned} \tag{4.55}$$

Where N, N_1, N_2 are positive constants to be chosen properly later.

A combination of (3.51),(4.3),(4.33),(4.40),(4.46),(4.50) and we use

$$\int_0^1 \theta_t^2 \text{ d}x \le \int_0^1 \theta_{tx}^2 \text{ d}x, \quad \int_0^1 \phi_x^2 \text{ d}x \le 2\int_0^1 (\phi_x+\psi)^2 \text{ d}x + 2\int_0^1 \psi_x^2 \text{ d}x. \tag{4.56}$$

We arrive

$$\begin{aligned}
\mathcal{L}'(t) \le & \left[-\beta k N + N_1\frac{\beta^2}{2b} + N_2 C(\varepsilon_2)\frac{\beta^2}{4\varepsilon_3} + \frac{\gamma^2}{4\varepsilon_2} + \rho_3\right]\int_0^1 \theta_{tx}^2 \text{ d}x \\
& + \left[N_1\left(-\frac{b}{2} + \varepsilon_1(\mu_1+|\mu_2|)\right) + 3\varepsilon_2 N_2 + \frac{\rho_1}{\rho_2} + \frac{3b^2}{4\varepsilon_3} + \frac{b^2}{4\varepsilon_3^3} + 7\varepsilon_3\right]\int_0^1 \psi_x^2 \text{ d}x \\
& + \left[-NC + N_1\left(\varepsilon_1\rho_1 + \frac{\mu_1}{4\varepsilon_1}\right) + \frac{\rho_2\mu_1^2}{2\rho_1} + \left(\frac{2\rho_1\varepsilon_3}{K} + \frac{\mu_1^2}{\varepsilon_3}\right) + \frac{1}{2\tau}\right]\int_0^1 \phi_t^2 \text{ d}x \\
& + \left[-\frac{N_2\rho_2\gamma}{2} + N_1\left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) + \rho_2 + \frac{\rho_2 b}{2\varepsilon_3}\right]\int_0^1 \psi_t^2 \text{ d}x \\
& + \left[-\frac{K}{2} + \left(\frac{K^2}{4} + 6\right)\varepsilon_3 + 2N_2\varepsilon_2\right]\int_0^1 (\phi_x+\psi)^2 \text{ d}x - \delta\int_0^1 \theta_x^2 \text{ d}x \\
& + \left[-NC - \frac{c}{2\tau} + N_1\frac{|\mu_2|}{4\varepsilon_1} + \frac{\mu_2^2}{\varepsilon_3} + \frac{\rho_2\mu_2^2}{2\rho_1}\right]\int_0^1 z^2(x,1,t)\text{d}x - m_0\int_0^1 z_0^1(x,\rho,t)\text{d}\rho\text{d}x \\
& + \left(\frac{\rho_2 K}{\rho_1} - b\right)\int_0^1 \psi_x(\phi_x+\psi)_x \text{ d}x.
\end{aligned} \tag{4.57}$$

At this point, we choose our constants carefully. First, we take

$$\varepsilon_3 \le \frac{K}{2}\left(\frac{K^2}{4}+6\right)^{-1}, \quad \varepsilon_1 \le \frac{b}{2(\mu_1+|\mu_2|)}. \tag{4.58}$$

Now, select N_1 large enough such that.

$$-N_1\left(\frac{b}{2} - \varepsilon_1(\mu_1+|\mu_2|)\right) + \frac{\rho_1}{\rho_2} + \frac{3b^2}{4\varepsilon_3} + \frac{b^2}{4\varepsilon_3^3} + 7\varepsilon_3 = k_1 < 0. \tag{4.59}$$

And then N_2 large so that.

$$-\frac{N_2\rho_2\gamma}{2} + N_1\left(\rho_2 + \frac{\rho_1}{4\varepsilon_1}\right) + \rho_2 + \frac{\rho_2 b}{2\varepsilon_3} < 0. \tag{4.60}$$

Next, pick ε_2 so small that.

$$k_1 + 3\varepsilon_2 N_2 < 0, \quad -\frac{K}{2} + \left(\frac{K^2}{4} + 6\right)\varepsilon_3 + 2N_2\varepsilon_2 < 0. \tag{4.61}$$

Finally, choose N so large that.

$$\begin{aligned} -\beta k N + N_1\tfrac{\beta^2}{2b} + N_2 C\left(\varepsilon_2\right) + \tfrac{\beta^2}{4\varepsilon_3} + \tfrac{\gamma^2}{4\varepsilon_2} + \rho_3 < 0 \\ -NC + N_1\left(\varepsilon_1\rho_1 + \tfrac{\mu_1}{4\varepsilon_1}\right) + \tfrac{\rho_2\mu_1^2}{2\rho_1} + \left(\tfrac{2\rho_1\varepsilon_3}{K} + \tfrac{\mu_1^2}{\varepsilon_3}\right) + \tfrac{1}{2\tau} < 0 \\ -NC - \tfrac{c}{2\tau} + N_1\tfrac{|\mu_2|}{4\varepsilon_1} + \tfrac{\mu_2^2}{\varepsilon_3} + \tfrac{\rho_2\mu_2^2}{2\rho_1} < 0. \end{aligned} \tag{4.62}$$

And further, for some $\beta_1, \beta_2 > 0$, we have.

$$\beta_1 E(t) \leq \mathcal{L}(t) \leq \beta_2 E(t), \quad t \geq 0. \tag{4.63}$$

Therefore, (4.57) becomes.

$$\begin{aligned} \mathcal{L}'(t) \leq & -\eta_1 \int_0^1 \left(\theta_t^2 + \theta_{xt}^2 + \psi_x^2 + \psi_t^2 + \phi_t^2 + (\phi_x + \psi)^2\right) \mathrm{d}x \\ & - m_0 \int_0^1 \int_0^1 z^2(x,\rho,t) \mathrm{d}\rho \mathrm{d}x + \left(\frac{\rho_2 K}{\rho_1} - b\right) \int_0^1 \psi_x(\phi_x + \psi)_x \, \mathrm{d}x \\ \leq & -C_1 E(t) + \left(\frac{\rho_2 K}{\rho_1} - b\right) \int_0^1 \psi_x(\phi_x + \psi)_x \, \mathrm{d}x. \end{aligned} \tag{4.64}$$

Where η_1 and C_1 are two positive constants.

Now, let's correct the text following equation (4.64):

- Case $\frac{\rho_1}{K} = \frac{\rho_2}{b}$: In this case, equation (4.64) becomes:

$$\mathcal{L}'(t) \leq -C_1 E(t). \tag{4.65}$$

Using equation (4.63), we find $\alpha = \frac{C_1}{\beta_2}$:

$$\mathcal{L}'(t) \leq -\alpha \mathcal{L}(t), \quad \forall t \geq 0. \tag{4.66}$$

Integrating equation (4.66) from 0 to t gives:

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\alpha t}, \quad \forall t \geq 0. \tag{4.67}$$

Hence, recalling equation (4.63), we obtain (4.1)1.

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