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THEME

Étude de la stabilité des problèmes d'évolution, Klein-Gordon, Schrödinger

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Abstract

This memory is devoted to the study of stability and instability of solutions for two evolution problems.
In chapter 1, we recall some basic knowldge in functional analysis.
In chapter 2, we study the well-posedness and stability of linear Schrödinger equations with infinite memory.
In chapter 3, we consider a sysytem of nonlinear viscoelastic wave equations with degenerae damping and strong source terms. We prove, with positive initial energy, the global nonexistence of solution by concavity method.

Keywords : Schrödinger equations , infinite memory, well-posedness, stability, global nonexistence, Klein-Gordon.

ملخص

في هذه المذكرة ندرس استقرار و انفجار الحل لبعض الجمل الرياضياتية المتعلقة بالمرونة، وقد قسمنا المذكرة إلى ثلاث فصول.

نذكر في الفصل الأول ببعض المفاهيم الأساسية في التحليل الدالي. نقوم في الفصل الثاني بإثبات الوجود والوحدانية مع الاستقرار الأسي لحل معادلة خطية من نوع شرودينغر وذلك باستعمال طريقة الطاقة .

أما في الفصل الثالث ندرس انفجار الحل لجملة زائدية لمعادلات تفاضلية جزئية غير خطية بإضافة حد يمثل اللزوجة و ذلك باستخدام طريقة الطاقة.

الكلمات المفتاحية: معادلة شرودينغر، الاستقرار، انفجار، كلين-جوردن، وحدانية.

Résumé

Ce mémoire est consacré pour l'étude de la stabilité et de l'instabilité des solutions de deux problèmes d'évolution. Dans le chapitre 1, nous rappelons quelques connaissances de base en analyse fonctionnelle.

Dans le chapitre 2, nous étudions le bien-posé et la stabilité des équations linéaires de Schrödinger à mémoire infinie.

Dans le chapitre 3, nous considérons un système d'équations d'onde viscoélastiques non linéaires avec amortissement dégénéré et termes source. Nous prouvons, avec une énergie initiale positive, l'inexistence globale de solution par méthode de concavité.

Mots clés : équation de Schrödinger , mémoire infinie, le bien posé, stabilité, l'inexistence globale, Klein-Gordon.

Dedication

I dedicate this work

To my parents who have been my source of inspiration and provided me with their encouragement, love, understanding and prayers.

To my beloved family *YAGOUBI* & *DJERADI*.

To all those who have been supportive, caring and patient, I dedicate this simple work.

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A special thanks to my parents, my sisters, my brother and my friends for being
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Fatima Siham Djeradi

Contents

Introduction

The aim of this memory is to investigate the stability and instability of two evolution problems, such as the Schrödinger equation and Klein-Gordon system.
The systems that we treated here are the following.

Schrödinger equation

The Schrödinger equation was invented at a time when electrons, protons and neutrons were considered to be elementery particles. it was extremly successful in what is now called atomic and molecular physics, and it has been applied with great succes to baryons and mesons, especially those made of heavy quark-antiquark pairs. While before World War II approximation methods were developed in a heuristic way, it is only during the post-war period that rigorous results on the energy levels and wave functions have been obtained and these approximation method justified. Impressive global results, such as the proof of the 'stability of matter', were obtained as well as the properties of the two-body Hamiltonians including bounds on the number of bound states. the discovery of quarkonium led to a closer examination of the problem of the order of energy levels from rigorous point of view, and a comparison of that order with what happens in cases of accidental degeneracy such as the Coulom and harmonic oscillator potentials, comparison of these cases also leads to intersting results on purely angular excitations of two-body systems.

Who among us has not written the words 'Schrödinger equation' or ' Schrödinger function' countless times ? the next generation will probably do the same, and keep his name alive.

Klein-Gordon equation

The Klein-Gordon equation is distinguished among other nonlinear hyperbolic equations by its theoretical and practical significance. The nonlinear Klein-Gordon equation appears in the study of some problems of mathematical physics. For example, this equation arises in general relativity, nonlinear optics (e.g., in the study of instability phenomena such as self-focusing), plasma physics, fluid mechanics, radiation theory or in the theory of spin waves [64, 44, 46]. The Cauchy problem for nonlinear Klein-Gordon equation

$$\varphi_{tt} - \Delta\varphi + m\varphi + \varphi_t = f(\varphi), \quad t > 0, \quad x \in R^n \tag{1}$$

$$\varphi(0, x) = \varphi_0(x), \varphi_t(0, x) = \varphi_1(x), x \in R^n \tag{2}$$

has been studied by many authors (see e.g. [37, 58]). Existence and nonexistence of global solutions are the main points of study for the problem (1), (2) in the case $m = 0$, $f(\varphi) \sim |\varphi|^p$ (see e.g. [65]).

In [29, 60], the problem (1), (2) has been investigated in the case $m = 0$, $f(\varphi) \sim |\varphi|^p$, where $1 < p \leq p_c = 1 + \frac{2}{n}$, and the existence of sufficiently small initial data $(\varphi_0; \varphi_1)$ was proved for which the corresponding Cauchy problem has no global solution. In [7, 8] the Klein-Gordon equation has been investigated in the case $m = 0$, $f(\varphi) \sim |\varphi|^p$ when $p > p_c = 1 + \frac{2}{n}$, and the existence of a global solution for the problem (1), (2) has been proved for suffciently small $(\varphi_0; \varphi_1)$. In the case $m > 0$, i.e for the Klein-Gordon equation with mass, the above efects do not occur. In this case, the main objects of study are the corresponding potential well and stability or instability of standing wave. There is a series of works devoted to that problem. The nonexistence of global solutions was studied in [34] for nonlinear wave equations with negative energy and in [35] for a class of abstract equations that, in particular, contains nonlinear wave equations. The nonexistence of global solutions of nonlinear wave equations with positive initial energy was considered in [55]. It was shown in the study of nonlinear wave equations in [33] that there exists initial data with fixed initial energy such that the corresponding Cauchy problem does not have a global solution. This result was improved in [30]. A mixed problem for systems of two semilinear wave equations with viscosity and with memory was studied in [40], where the nonexistence of global solutions with positive initial energy was proved. The nonexistence of global solutions of problem (1), (2) with negative initial energy was studied in [2] for $m = 2$ and in [68] for $m = 2$ and $p_1 = p_2$. The nonexistence of global solutions of a generalized fourth-order Klein–Gordon equation with positive initial energy was analyzed in [52]. A fairly comprehensive picture of the studies in this direction can be gained from the monograph [26].

Our main results in this memory can be summarized as follows.

Chapter 1. In this chapter, we recall some basic knowledge in functional analysis, most of which will be used in the subsequent chapter.

Chapter 2. In this chapter, we study the well-posedness and stability for linear Schrödinger equation in $d-$dimensional open bounded domain under Dirichlet boundry conditions with an infinite memory.

First, we establish the well-posedness in the sens of semigroup theory. Then a decay estimate depending on the smoothness of initial data and the arbitrary growth at infinity of relaxation function is established for each equation with help of multipliers method and some arguments devised in [31] and [32].

Chapter 3. In this chapter we will substantiation that the positive initial-energy solution coupled nonlinear Klein-Gordon equations with degenerate damping and source terms. We prove, with positive initial energy, the global nonexistence of solution by concavity method.

Chapter 1

Preliminaries

All assertions in the first chapter are made without proofs and the scape has been minimised to only material actually needed (see [9], [10] , [57] ,[1] and [69]).

1.1 Functional Spaces

We denote by $\mathbb{R}^n$ the Euclid space, $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $C^k(\Omega)$ is the k^{th} differentiable continuous function space in Ω, $C^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space in Ω, $C_c^\infty(\Omega)$ is the ∞^{th} differentiable continuous function space with compact support in Ω .

Definition 1.1. *Let X be a vector space over the filed $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$). Then a semi-norm on X is a function $\|.\| : X \longrightarrow \mathbb{R}$, such that :*
a) $\|x\| \geqslant 0$ for all $x \in X$,
b) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$,
c) $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in X$.
A norm on X is a semi-norm which also satisfies :
d) $\|x\| = 0 \Rightarrow x = 0$. A vector space X together with a norm $\|.\|$ is called a normed linear space, a normed vector space or simply, a normed space.

Definition 1.2. *(Convergent and Cauchy sequences)*
Let X be a normed space, and let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of elements of X.
a) $\{x_n\}_{n\in\mathbb{N}}$ converges to $x \in X$ if

$$\lim_{n\to\infty} \|x_n - x\| = 0,$$

i.e. if

$$\forall \varepsilon > 0; \exists N \geq 0, \forall n \geq N, \|x_n - x\| < \varepsilon.$$

b) $\{x_n\}_{n\in\mathbb{N}}$ *is a Cauchy sequence if*

$$\forall \varepsilon > 0; \exists N \geq 0, \forall m, n \geq N, \|x_m - x_n\| < \varepsilon.$$

Normed spaces in which every Cauchy sequence is convergent are called complete normed spaces. In general a normed space is not complete.

Definition 1.3. *(Banach Spaces)*
A normed spaces is called a Banach space if it is complete i.e. if any Cauchy sequence inside the space converges to a point of the space. Its dual space X' *is the linear space of all continuous linear functional* $f : X \to \mathbb{R}$.

Proposition 1.1. X' *equipped with the norm* $\|.\|_{X'}$ *defined by*

$$\|f\|_{X'} = \sup\{|f(u)| : \|u\| \leq 1\}$$

is also a Banach space.

Remark 1.1. *From* X' *we construct the bidual or second dual* $X'' = (X')'$. *Furthermore, with each* $u \in X$ *we can define* $\varphi(u) \in X''$ *by* $\varphi(u)(f) = f(u)$, $f \in X'$, *this satisfies clearly* $\|\varphi(x)\| \leq \|u\|$. *Moreover, for each* $u \in X$ *there is an* $f \in X'$ *with* $f(u) = \|u\|$ *and* $\|f\| = 1$, *so it follows that* $\|\varphi(x)\| = \|u\|$.

Definition 1.4. *Since* φ *is linear we see that*

$$\varphi : X \to X'',$$

is a linear isometry of X *onto a closed subspace of* X'', *we denote this by*

$$X \hookrightarrow X''.$$

Definition 1.5. *If* φ *(in the above definition) is onto* X'' *we say* X *is reflexive,* $X \approxeq X''$.

1.1.1 Weak and weak star topologies

Let X be a Banach space and $f \in X'$. Denote by

$$\begin{aligned} \varphi_f : &X \to \mathbb{R} \\ &x \mapsto \varphi_f \end{aligned}$$

When f covers X', we obtain a family $(\varphi_f)_{f\in X'}$ of applications to X in $\mathbb{R}$.

Definition 1.6. *The weak topology on X, denoted by $\sigma(X, X')$, is the weakest topology on X for which every $(\varphi_f)_{f\in X'}$ is continuous.*

We will define the topology on X', the weak star topology, denoted by $\sigma(X', X)$. For all $x \in X$. Denote by

$$\begin{aligned} \varphi_x : &X' \to \mathbb{R} \\ &f \mapsto \varphi_x(f) = \langle f, x\rangle_{X',X} \end{aligned}$$

Definition 1.7. *The weak star topology on X' is the weakest topology on X' for which every $(\varphi_x)_{x\in X'}$ is continuous.*

Remark 1.2. *Since $X \subset X''$, it is clear that, the weak star topology $\sigma(X', X)$ is weakest than the topology $\sigma(X', X'')$, and this later is weaker than the strong topology.*

Definition 1.8. *A sequence (x_n) in X is weakly convergent to x if and only if*

$$\lim_{n\to\infty} f(x_n) = f(x)$$

for evey $f \in X'$, and this is denoted by $x_n \rightharpoonup x$.

Remark 1.3.

1. *If the weak limit exists, it is unique.*
2. *If $x_n \to x \in X$ (strongly), then $x_n \rightharpoonup x$ (weakly).*
3. *If $\dim X < \infty$, then the weak convergent implies the strong convergent.*

1.1.2 Hilbert spaces

The proper setting for the rigorous theory of partial differential equation turns out to be the most important function space in modern physics and modern analyses, known as Hilbert spaces. Then, we must give some important results on these spaces here.

Definition 1.9. *A Hilbert space $\mathcal{H}$ is a vectorial space supplied with inner product $\langle u, v\rangle$ such that $\|u\| = \sqrt{\langle u, u\rangle}$ is the norm which let $\mathcal{H}$ complete.*

Theorem 1.1. *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in the Hilbert space $\mathcal{H}$, then it possess a subsequence which converges in the weak topology of $\mathcal{H}$.*

Theorem 1.2. *In the Hilbert space, all sequence which converges in the weak topology is bounded.*

Theorem 1.3. *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence which converges to x, in the weak topology and $(y_n)_{n\in\mathbb{N}}$ is an other sequence which converges weakly to y, then*

$$\lim_{n\to\infty} \langle x_n, y_n\rangle = \langle x, y\rangle.$$

Proposition 1.2. *Let X and Y be two Hilbert spaces, let $(x_n)_{n\in\mathbb{N}} \in X$ be a sequence which converges weakly to $x \in X$, let $A \in \mathcal{L}(X, Y)$. Then, the sequence $(A(x_n))_{n\in\mathbb{N}}$ converges to $A(x)$ in the weak topology of Y.*

Theorem 1.4. *(The Lax-Milgram Theorem)*
Let X be a Hilbert space and let $a : X \times X \to \mathbb{R}$ be a bilinear functional. Assume that there exist two constants $C < \infty, \alpha > 0$ such that:
i) $|a(u, v)| \leq C\|u\|.\|v\|$ *for all* $(u, v) \in X \times X$ *(continuity),*
ii) $a(u, u) \geq \alpha\|u\|^2$ *for all* $u \in X$ *(coerciveness).*
Then, for every $f \in X^*$ (the dual space of X), there exists a unique $u \in X$ such that $a(u, v) = \langle f, v\rangle$ for all $v \in X$.

1.1.3 $L^p(\Omega)$ spaces

Definition 1.10. *Let $1 \leq p \leq \infty$, and let Ω be an open domain in $\mathbb{R}^n$, $n \in \mathbb{N}$. Define the standard Lebesgue space $L^p(\Omega)$ by*

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} : f \textit{ is measurable and} \int_\Omega |f(x)|^p dx < \infty \right\}$$

Notation 1 : For $p \in \mathbb{R}$ and $1 \leq p < \infty$, denote by

$$\|f\|_p = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}}.$$

If $p = \infty$, we have

$$L^p(\Omega) = \{f : \Omega \to \mathbb{R} : f \ is \ measurable \ and \ there \ exists \ C \ such \ that, |f(x)| \leq C \ in \ \Omega\}$$

. **Notation 2 :** Let $1 \leq p \leq \infty$, we denote by q the conjugate of p i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Theorem 1.5. *It is well known that $L^p(\Omega)$ supplied with the norm $\|.\|_p$ is a Banach space, for all $1 \leq p \leq \infty$.*

Remark 1.4. *In particularly, when $p = 2$, $L^2(\Omega)$ equipped with the inner product*

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f(x)g(x)dx,$$

is a Hilbert space .

Theorem 1.6. *For $1 < p < \infty$, $L^p(\Omega)$ is a reflexive space.*

1.1.4 Sobolev space $W^{m,p}(\Omega)$

Definition 1.11. *i) Let $m \in \mathbb{N}$ and $p \in [0, \infty]$. The $W^{m,p}(\Omega)$ is the space of all $f \in L^p(\Omega)$, defined as*

$$W^{m,p}(\Omega) = \{f \in L^p(\Omega), such \ that \ \partial^\alpha f \in L^p(\Omega) for \ all \ \alpha \in \mathbb{N}^m\}$$

such that $|\alpha| = \sum_{j=1}^n \alpha_j \leq m$ where, $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}....\partial_n^{\alpha_n}$.
ii) if $f \in W^{m,p}(\Omega)$, we define its norm to be

$$\|f\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|<k} \int_\Omega |D^\alpha f|^p dx\right)^{\frac{1}{p}} & ; (1 \leq p < \infty), \\ \sum_{|\alpha|<k} ess \sup |D^\alpha f| & ; (p = \infty). \end{cases}$$

Definition 1.12. *We denote by $W_0^{m,p}(\Omega)$, the closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$.*

Remark 1.5. *i) if $p = 2$ we usually write*

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H_0^m(\Omega) = W_0^{m,2}(\Omega).$$

Supplied with the norm

$$\|f\|_{H^m} = \left(\sum_{|\alpha| \leq m} (\|\partial^\alpha f\|_{L^2})^2 \right)^{\frac{1}{2}}.$$

The letter H is used, since - as we will see - $H^m(\Omega)$ is a Hilbert space.
With usual scalar product

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \int_\Omega \partial^\alpha u \partial^\alpha v \; dx.$$

Note that $H^0(\Omega) = L^2(\Omega)$.

Theorem 1.7. *1. $H^m(\Omega)$ supplied with inner product $\langle ., . \rangle_{H^m(\Omega)}$ is a Hilbert space.*
2. If $m \geq m'$, $H^m(\Omega) \hookrightarrow H^{m'}(\Omega)$.

Theorem 1.8. *Assume that Ω is an open domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary Γ. Then,*
i) if $1 \leq p \leq n$, we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, p^*]$, where $p^* = \dfrac{np}{n-p}$.
ii) if $p = n$, we have $W^{1,p} \subset L^q(\Omega)$, for every $q \in [p, \infty)$.
iii) if $p > n$, we have $W^{1,p} \subset L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$, where $\alpha = \dfrac{p-n}{p}$.

1.1.5 $L^p(0, T, X)$ space

Definition 1.13. *Let X be a Banach space, denote by $L^p(0, T, X)$ the space of measurable functions*

$$\begin{aligned} f :]0, T[&\to X \\ t &\mapsto f(t) \end{aligned}$$

such that

$$\left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}} = \|f\|_{L^p(0,T,X)} < \infty, \qquad 1 \leq p < \infty.$$

If $p = \infty$,

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t \in]0,T[} ess \|f(t)\|_X.$$

Theorem 1.9. *$L^p(0, T, X)$ equipped with the norm $\|.\|_{L^p(0,T,X)}$ is a Banach space.*

Proposition 1.3. *Let X be a reflexive Banach space, X' it's dual, and $1 \leq p < \infty$, $1 \leq q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the dual of $L^p(0, T, X)$ is identify algebraically and topologically with $L^q(0, T, X')$.*

1.2 Some useful inequalities

In this section, we shall recall some inequalities which will be used in the subsequent chapters.

1.2.1 Young inequalities

Theorem 1.10. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad , \ a, \ b > 0.$$

Theorem 1.11. *(Young inequality with ε)*
Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \varepsilon \frac{a^p}{p} + \frac{1}{\varepsilon^{\frac{q}{p}}} \frac{b^q}{q} \quad , a, b > 0.$$

The Young inequality has several variants in the following.

Corollary 1. *Let $a, b > 0$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 < p, q < \infty$. Then*
i) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.
ii) $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p\varepsilon^{\frac{1}{q}}} + \frac{b\varepsilon^{\frac{1}{p}}}{q} \quad , \forall \varepsilon > 0.$

iii) $a^{\alpha} b^{1-\alpha} \leq \alpha a + (1-\alpha) b \quad , 0 < \alpha < 1.$

1.2.2 Hölder inequalities

Theorem 1.12. *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$,*

then, if $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, *we have*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} . \|g\|_{L^q(\Omega)}.$$

Theorem 1.13. *(Generalized Hòlder inequality)*
Let $1 \le p_1, ..., p_m \le \infty$, $\frac{1}{p_1} + ... + \frac{1}{p_m} = 1$, *then, if* $f_k \in L^{p_k}(\Omega)$ *for* $k = 1, ..., m$ *we have*

$$\int_\Omega |f_1 .... f_m| dx \le \prod_{k=1}^{m} \|f_k\|_{L^{p_k}(\Omega)}.$$

Remark 1.6. *We have the corresponding weighted Hölder inequality of the integral form. Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p(\Omega)$, $g \in L^q(\Omega)$, $\omega(x) > 0$ *on* Ω. *Then*

$$\int_\Omega |fg|\omega(x) dx \le \left(\int_\Omega |f(x)|^p \omega(x) dx \right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q \omega(x) dx \right)^{\frac{1}{q}}.$$

1.2.3 Minkowski inequality

Theorem 1.14. *Assume* $1 \le p \le \infty$, $f, g \in L^p(\Omega)$, *then*

$$\|f + g\|_{L^p(\Omega)} \le \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

If $0 \ne p < 1$, *then*

$$\|f + g\|_{L^p(\Omega)} \ge \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

In the applications, the integral from of the Minkowski inequality is used frequently.

1.2.4 Poincaré inequality

In this subsection, we shall recall the Poincaré inequality in different forms.

Theorem 1.15. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *and* $f \in H_0^1(\Omega)$. *Then there is a positive constant* C *such that*

$$\|f\|_{L^2(\Omega)} \le C \|\nabla f\|_{L^2(\Omega)}, \qquad \forall f \in H_0^1(\Omega).$$

Theorem 1.16. *Let* Ω *be a bounded domain of* C^1 *in* $\mathbb{R}^n$. *There is a positive constant* C *, such that for any* $f \in H^1(\Omega)$,

$$\|f - \tilde{f}\|_{L^2(\Omega)} \le C \|\nabla f\|_{L^2(\Omega)}.$$

Where $\tilde{f} = \frac{1}{|\Omega|} \int_\Omega f(x)\, dx$ *is the integral average of* f *over* Ω, *and* $|\Omega|$ *is the volume of* Ω.

Theorem 1.17. *Under assumption of Theorem (1.16) for any $f \in H^1(\Omega)$, we have*

$$\|f\|_{L^2(\Omega)} \leq C \left(\|\nabla f\|_{L^2(\Omega)} + \left| \int_\Omega f dx \right| \right).$$

1.3 Basic theory of semi-groups

In this section, we recall some basic knowledge in semigroups, most of which will be used in the subsequent chapters. A general reference to this topic is [10], [9].

1.3.1 C_0-Semi-groups of Linear Operators

Definition 1.14. *(Semi-groups)*
Let X be a Banach space, the one-parameter family $S(t),\ 0 \leq t < \infty$ from X to X is called a Semigroups if
(i) $S(0) = I$ (I is the identity operator on X).
(ii) $S(t+s) = S(t).S(s)$ for every $t, s \geq 0$ (the Semigroup property).

Definition 1.15. *The linear operator $\mathcal{A}$ defined by*

$$D(\mathcal{A}) = \left\{ x \in X : \lim_{t \to 0^+} \frac{(S(t)x - x)}{t} \quad exists \right\}$$

and

$$\mathcal{A}x = \lim_{t \to 0^+} \frac{(S(t)x - x)}{t} = \frac{d(S(t)x)}{dt}|_{t=0} \quad for \quad all \quad x \in D(\mathcal{A})$$

is called the infinitesimal generator of the Semigroup $S(t)$, $D(\mathcal{A})$ is called the domain of $\mathcal{A}$.

Definition 1.16. *(C_0-Semigroups).*
A Semigroup $S(t),\ 0 \leq t < \infty$, from X to X is called a strong continuous Semigroup of bounded linear operators if

$$\lim_{t \to 0^+} S(t)x = x, \quad for \quad all \quad x \in X,$$

or

$$\lim_{t \to 0^+} \|S(t)x - x\| = 0, \quad for \quad all \quad x \in X.$$

i.e $S(t)$ C_0- Semigroup.

Definition 1.17. *A semigroup* $S(t),\ 0 \leq t < \infty$ *is called a semigroup of contraction if there exists a constant* $\alpha > 0$ $(0 < \alpha < 1)$ *such that for all* $t > 0$,

$$\|S(t)x - S(t)y\| \leq \alpha\|x - y\|, \quad for \;\; all \quad x, y \in X.$$

1.3.2 Hille-Yosida Theorem

Definition 1.18. *An unbounded linear operator* $\mathcal{A} : D(\mathcal{A}) \subset H \to H$[1] *is said to be monotone*[2]*, if it satisfies*

$$\langle Av, v\rangle \geq 0 \qquad \forall v \in D(\mathcal{A}).$$

It is called maximal monotone if, in addition ; $R(I + \mathcal{A}) = H$, *i.e*

$$\forall f \in H, \;\; \exists u \in D(\mathcal{A}) \quad such \;\; that \;\; u + \mathcal{A}u = f.$$

Proposition 1.4. *Let* $\mathcal{A}$ *be a maximal monotone operator. Then*
1. $D(\mathcal{A})$ *is dense in* H.
2. $\mathcal{A}$ *is closed operator.*
3. For every $\lambda > 0, (I + \lambda\mathcal{A})$ *is bijective from* $D(\mathcal{A})$ *onto* H, $(I + \lambda\mathcal{A})^{-1}$ *is a bounded operator, and* $\|(I + \lambda\mathcal{A})^{-1}\|_{\mathcal{L}(H)} \leq 1$.

Theorem 1.18. *(Hille-Yosida)*
Let $\mathcal{A}$ *be a maximal monotone operator. Then, given any* $u_0 \in D(\mathcal{A})$ *there exists a unique function*

$$u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(\mathcal{A})),$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \qquad on \; [0, +\infty) \\ u(0) = u_0. \end{cases}$$

---

[1] H denotes a Hilbert space.

[2] Some authors say that A is accretive or $-\mathcal{A}$ is dissipative.

Chapter 2

Well-posedness and stability for Schrödinger equation with infinite memory

2.1 Introduction

In this chapter we are concerned with the following Schrödinger equation with infinite memory :

$$\begin{cases} iy_t(x,t) + a\Delta y(x,t) - i\int_0^\infty f(s)\Delta y(x,t-s)ds = 0, \quad (x,t) \in \Omega \times \mathbb{R}_+^*, \\ y(x,t) = 0 \qquad x \in \partial\Omega, \quad t > 0, \\ y(x,-t) = y_0(x,t), \qquad (x,t) \in \Omega \times \mathbb{R}_+, \end{cases} \tag{2.1}$$

we denote by the subscript t the derivative with respect to the time variable t, by Δ the Laplacian operator with respect to the space variable x, and by $\Omega \subset \mathbb{R}^d$ an open bounded domain with a smooth boundary Γ, here $d \in \mathbb{N}$, $a \in \mathbb{R}_+^*$, $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a given function, y_0 is a fixed initial data and y in unknown of (2.1).

We would like here to mention some known papers in connection with well-posedness and stability of Schrödinger type equations, which is the subject of our chapter.

When the infinite memory is replaced by a damping, equation (2.1) in the presence or not in $(2.1)_1$ of a semilinear term ; that is

$$iy_t(x,t) + a\Delta y(x,t) + b|y|^p y + icy = 0 \tag{2.2}$$

($a, p, c \in \mathbb{R}_+^*$ and $b \in \mathbb{R}$), has been widely studied in the literature, where it is known that Schrödinger equations are globally well-posed under some smallness conditions on p. In the particular case $a = 1,\ p = 2,\ b \in \{1, -1\}$ and the domain is bounded, the exponential stability of (2.2) was proved in [66], under some smoothness and smallness conditions on the initial data. A generalization to the case of inhomogeneous Dirichlet boundary conditions was given in [54], where the decay rate depends on the regularity of solutions. Some exact controllability results in both Dirichlet and Neumann boundary conditions cases are also known for (2.2). For more general semilinearity: $p = 2$ or not (with $a = 1, b = c$ and the domain is unbounded), some global existence results of solutions as well as the bolw-up phenomena were obtained in [52] for two sets of initial data.

In [18], the authors studied the existence as well as the stability in $\mathbb{R}^d$ of (2.2) with $a = 1,\ b = -1$ and the damping coefficient c is a function on both space and time variables and may vanish when time goes to infinity. Moreover, the uniqueness of solution is proved when $d \in \{1, 2\}$. Similar results were obtained in [6] and [7] in d-dimensional Riemannian manifolds and nonlinear local damping $ic(x)g(y)$ (instead of icy) but with $b = 0$, where g is a given function satisfying some properties.

There exist in the literature several well-posedness and stability (theoretical and numerical) results also for higher order Schrödinger equations. In this direction, see, for example, [22] and the references therein.

For other well-posedness, stability and blow-up results related to Schrödinger types equations cited above, we refer the readers to, for example, [11]-[21], [23] and the refrences therein.

Our goals in the present chapter is studying the existence, uniqueness, regularity and decay of solutions for the linear Schrödinger equations (2.21) , where the unique present dissipation is the one generated by the infinite memory term. This situation is completely different from the ones considered in the literature and cited above, where the dissipation is generated by a (linear or nonlinear) damping.

First, we establish the well-posedness (existence, uniqueness and smoothness of solutions) in the sens of semigroup theory. Then, a decay estimate depending on the smoothness of initial data and the arbitrarily growth at infinity of the relaxation function f is established for the equation (2.1). These two decay estimates imply that any weak solution converges to zero at infinity. In the particular case where $-f'$ converges exponentially to zero at infinity, our decay estimates lead to the decay rate t^{-n}, where $n \in \mathbb{N}^*$ depends on the regularity of initial data. The proofs are based on the semigroup approach, the multipliers method and some arguments devised in [31] and [32]. we denote by the subscript t the derivative with respect to the time variable t, by Δ the Laplacian operator with respect to the space variable x, and by $\Omega \subset \mathbb{R}^d$ an open bounded domain with a smooth boundary Γ, here $d \in \mathbb{N}$, $a \in \mathbb{R}_+^*$, $f : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is a given function, y_0 is a fixed initial data and y in unknown of (2.1).

2.2 Preliminaries and well-posedness results

In this section, we present and proof our well-posedness results for (2.1). For the purpose of simplifying the formulations and to avoid ambiguity, the variables x ,t and s are noted only when it is needed. Let us use $\langle , \rangle$ and $\|.\|$ to denote, respectively, the standard inner product in $L^2(\Omega)$ and its generated norm given by

$$\langle p, q\rangle = \int_\Omega p(x)\bar{q}(x)dx \quad \text{and} \quad \|p\| = \left(\int_\Omega |p(x)|^2 dx\right)^{\frac{1}{2}}.$$

In order to prove the well-posedness of (2.1) by using the semigroup approach, and as in [24], let us consider the variable η^t and its initial data η^0 given by

$$\eta^t(x,s) = \int_{t-s}^t y(x,\tau)d\tau \quad \text{and} \quad \eta^0(x,s) = \int_0^s y_0(x,z)dz \quad (x,s,t) \in \Omega \times \mathbb{R}_+^2. \tag{2.3}$$

Using Leibniz formula

$$\frac{d}{dt}\int_{a(x)}^{b(x)} f(x,t)\ dx = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial t}(x,t)\ dx + f(b(t),t)\ \frac{d}{dt}b(t) - f(a(t),t)\ \frac{d}{dt}a(t),$$

direct calculations show that

$$\eta_t^t(x,s) = y(x,t) - y(x,t-s) \quad \text{and} \quad \eta_s^t(x,s) = y(x,t-s),$$

hence, the functional η^t satisfies

$$\begin{cases} \eta_t^t(x,s) + \eta_s^t(x,s) = y(x,t) & x \in \Omega, \quad s,t \in \mathbb{R}_+^*, \\ \eta^t(x,s) = 0 & x \in \Gamma, \quad s,t \in \mathbb{R}_+^*, \\ \eta^t(x,0) = 0, & x \in \Omega \quad t \in \mathbb{R}_+. \end{cases} \tag{2.4}$$

In order to express in term of η^t the memory integral in (2.1), we assume the

following hypothesis.
(A1) Assume that the function f is non-increasing such that

$$f \in C^2(\mathbb{R}_+), \quad f(0) > 0 \quad \text{and} \quad \lim_{s\to\infty} f(s) = 0. \tag{2.5}$$

Since $f \in C^2(\mathbb{R}_+)$, then $f' \in C^1(\mathbb{R}_+)$, putting $g = -f'$, so $g \in C^1(\mathbb{R}_+)$, and because f is non-increasing $f' \leq 0$, as a result g is non-negative, and

$$\begin{aligned} g_0 :=& \int_0^\infty g(s)ds \\ =& \int_0^\infty -f'(s)ds \\ =& f(0) \in \mathbb{R}_+. \end{aligned}$$

On the other hand, we integrate with respect to s and we use (2.4)$_3$ and the limit in (2.5) we find

$$\begin{aligned} \int_0^\infty g(s)\Delta\eta^t ds &= -\int_0^\infty f'(s)\Delta\eta^t ds \\ &= -\left(f(s)\Delta\eta^t\big|_0^\infty - \int_0^\infty f(s)\Delta\eta_s^t ds \right). \\ &= \int_0^\infty f(s)\Delta\eta_s^t ds. \end{aligned}$$

Using the definition of η^t, we can see that $\eta_s^t = y(t-s)$, consequently

$$\begin{aligned} \int_0^\infty g(s)\Delta\eta^t ds &= \int_0^\infty f(s)\Delta\eta_s^t ds \\ &= \int_0^\infty f(s)\Delta y(t-s)ds. \end{aligned}$$

As a result, we can rewrite the system (2.1) in the form

$$\begin{cases} iy_t + a\Delta y - i\int_0^\infty g(s)\Delta\eta^t ds = 0 \\ \eta_t^t(x,s) + \eta_s^t(x,s) = y(x,t). \end{cases} \tag{2.6}$$

We consider the variable U and its initial data U_0 given by

$$U = (y, \eta^t) \quad \text{and} \quad U_0 = (y_0(.,0), \eta^0). \tag{2.7}$$

Now, we can formulate the system (2.1) in the following initial value problem

$$\begin{cases} U_t(t) = \mathcal{A}U(t), \quad t > 0, \\ U(0) = U_0, \end{cases} \tag{2.8}$$

to define the operator $\mathcal{A}$ we have

$$U_t(t) = (y_t, \eta_t^t), \quad \text{from (2.6) we obtain}$$

$$\begin{cases} iy_t = -a\Delta y + i\int_0^\infty g(s)\Delta\eta^t \, ds \\ \eta_t^t = y - \eta_s^t, \end{cases}$$

that is

$$\begin{cases} y_t = ia\Delta y + \int_0^\infty g(s)\Delta\eta^t \, ds \\ \eta_t^t = y - \eta_s^t, \end{cases}$$

hence, the operator $\mathcal{A}$ is defined by

$$\mathcal{A}U = \begin{pmatrix} ia\Delta y + \int_0^\infty g(s)\Delta\eta^t ds \\ y - \eta_s^t, \end{pmatrix}$$

Let us now consider the space

$$L_1 = \left\{ v : \mathbb{R}_+ \to H_0^1(\Omega), \int_0^\infty g(s)\|\nabla\eta^t\|ds < \infty \right\},$$

equipped with the inner product

$$\langle v, w\rangle_{L_1} = \int_0^\infty g(s)\langle\nabla v(s), \nabla w(s)\rangle ds,$$

let us also consider the energy space

$$\mathcal{H} = L^2(\Omega) \times L_1$$

equipped with the inner product

$$\langle (v_1, v_2)\rangle, (w_1, w_2)\rangle_{\mathcal{H}} = \langle v_1, w_1\rangle + \langle v_2, w_2\rangle_{L_1}.$$

The domain of $D(A)$ is given by

$$D(\mathcal{A}) = \left\{ \begin{matrix} U \in \mathcal{H}, y \in \mathcal{H}_0^1(\Omega), \eta_s^t \in L_1, \eta^t(x,0) = 0 \\ ia\nabla y + \int_0^\infty g(s)\Delta\eta^t ds \in L^2(\Omega) \end{matrix} \right\}.$$

In order to obtain the well-posedness results of (2.8), we need to assume the following additional hypothesis.
(**A2**) : Assuming that g is a non-increasing function such that there exists a positive constant β_0 which satisfies

$$-\beta_0 g \leq g'. \tag{2.9}$$

Theorem 2.1. *Assume that (**A1**) and (**A2**) hold. Then, for any $n \in \mathbb{N}$ and $U_0 \in D(\mathcal{A}_1^n)$, the system (2.8) admits a unique solution U satisfying*

$$U \in \cap_{k=0}^n C^k(\mathbb{R}_+; D(\mathcal{A}^n)).$$

Proof. First, we mention that $\mathcal{H}$ is a Hilbert space and $\mathcal{A}$ is linear. The proof of Theorem (2.1) depends on the Lumer-Philips theorem by proving that the operator $\mathcal{A}$ is dissipative and $I - \mathcal{A}$ is surjective (where I denotes the identity operator) ; in other words $-\mathcal{A}$ is maximal monotone. consequently $\mathcal{A}$ is the infinitesimal generator of a C_0 semigroup of contraction on $\mathcal{H}$ and its domain $D(\mathcal{A}\ )$ is dense in $\mathcal{H}$. The conclusion of Theorem (2.1) follows immediately (see [43] and [57]).
Second, we prove that

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta^t\|^2 ds. \tag{2.10}$$

Where $\Re$ denotes the real part. Hence $\mathcal{A}$ is dissipative, since g is non-increasing and (2.9) guarantees the boundedness of the integrals in (2.10). We use the definition of $\mathcal{A}$ and $\langle , \rangle_{\mathcal{H}}$, we integrate by parts and we use the boundary condition, we have

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \langle ia\nabla y + \int_0^\infty g(s)\Delta\eta^t ds, y\rangle + \langle y - \eta_s^t, \eta^t\rangle_{L_1} \tag{2.11}$$

$$= ia\langle \Delta y, y\rangle + \langle \int_0^\infty g(s)\Delta\eta^t ds, y\rangle + \langle y, \eta^t\rangle_{L_1} - \langle \eta_s^t, \eta^t\rangle_{L_1}$$

$$= ia\int_\Omega \Delta y \times y dx + \int_\Omega\int_0^\infty g(s)\Delta\eta^t y ds dx + \int_0^\infty g(s)\langle\nabla y, \nabla\eta^t\rangle ds - \int_0^\infty g(s)\langle\nabla\eta_s^t, \nabla\eta^t\rangle ds$$

$$= -ia\int_\Omega (\nabla y)^2 dx - \int_0^\infty g(s)\left(\int_\Omega \nabla\eta^t\cdot\nabla y dx\right) ds + \int_0^\infty g(s)\langle\nabla y, \nabla\eta^t\rangle ds - \int_0^\infty g(s)\langle\nabla\eta_s^t, \nabla\eta^t\rangle ds$$

$$= -ia\|\nabla y\|^2 + \int_0^\infty g(s)(\langle\nabla y, \nabla\eta^t\rangle - \langle\nabla\eta^t, \nabla y\rangle - \langle\nabla\eta_s^t, \nabla\eta^t\rangle)ds.$$

That is

$$\langle\mathcal{A}U, U\rangle_{\mathcal{H}} = -ia\|\nabla y\|^2 + \int_0^\infty g(s)(\langle\nabla y, \nabla\eta^t\rangle - \langle\nabla\eta^t, \nabla y\rangle - \langle\nabla\eta_s^t, \nabla\eta^t\rangle)ds. \quad (2.12)$$

Direct computations imply that

$$\begin{aligned}\langle\nabla\eta_s^t, \nabla\eta^t\rangle &= \int_\Omega (\Re\nabla\eta_s^t + i\Im\nabla\eta_s^t)(\Re\nabla\eta^t - i\Im\nabla\eta^t)\ dx\\ &= \int_\Omega (\Re\nabla\eta_s^t\cdot\Re\nabla\eta^t + \Im\nabla\eta_s^t\cdot\Im\nabla\eta^t)\ dx + i\int_\Omega (\Re\nabla\eta^t - \Im\nabla\eta_s^t - \Im\nabla\eta^t - \Re\nabla\eta_s^t)\ dx\\ &= \frac{1}{2}\left(\|\nabla\eta^t\|^2\right)_s + i\int_\Omega (\Re\nabla\eta^t - \Im\nabla\eta_s^t - \Im\nabla\eta^t - \Re\nabla\eta_s^t)\ dx\end{aligned}$$

and

$$\begin{aligned}\langle\nabla y, \nabla\eta^t\rangle - \langle\nabla\eta_s^t, \nabla y\rangle &= \Re\langle\nabla y, \nabla\eta^t\rangle + i\Im\langle\nabla y, \nabla\eta^t\rangle - \Re\langle\nabla y, \nabla\eta^t\rangle + i\Im\langle\nabla y, \nabla\eta^t\rangle\\ &= 2i\Im\langle\nabla y, \nabla\eta^t\rangle,\end{aligned}$$

where $\Im$ denotes the imaginary part. Exploiting these two equalities, we deduce from

(2.11) that

$$\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -ia\|\nabla y\|^2 + \int_0^\infty g(s)(\langle \nabla y, \nabla \eta^t\rangle - \langle \nabla \eta^t, \nabla y\rangle - \langle \nabla \eta^t_s, \nabla \eta^t\rangle)ds \\
&= -ia\|\nabla y\|^2 + \int_0^\infty g(s)\left(2i\Im\langle \nabla y, \nabla \eta^t\rangle - \frac{1}{2}\left(\|\nabla \eta^t\|^2\right)_s\right. \\
&\left. +i\int_\Omega \left(\Re\nabla\eta^t - \Im\nabla\eta^t_s - \Im\nabla\eta^t - \Re\nabla\eta^t_s\right) dx\right) \\
&= -ia\|\nabla y\|^2 + 2i\Im\int_0^\infty g(s)\langle \nabla y, \nabla \eta^t\rangle ds - \int_0^\infty g(s)\frac{1}{2}\left(\|\nabla \eta^t\|\right)^2_s ds \\
&+ i\int_0^\infty g(s)\int_\Omega \left(\Re\nabla\eta^t - \Im\nabla\eta^t_s - \Im\nabla\eta^t - \Re\nabla\eta^t_s\right) dx.
\end{aligned}$$

Hence,

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -ia\|\nabla y\|^2 + 2i\Im\int_0^\infty g(s)\langle \nabla y, \nabla \eta^t\rangle ds \tag{2.13}$$

$$+i\int_0^\infty g(s)\int_\Omega (\Im\nabla\eta^t\cdot\Re\nabla\eta^t_s - \Re\nabla\eta^t\cdot\Im\nabla\eta^t_s)dxds - \frac{1}{2}\int_0^\infty g(s)(\|\nabla\eta^t\|^2)_s.ds$$

Integrating the third integral in (2.13) with respect to s, and by taking the real part of the obtained formula we find (2.10). Using (**A3**), we get

$$\Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^\infty g'(s)(\|\nabla\eta^t\|^2)_s ds \leq 0.$$

Consequently, $\mathcal{A}$ is dissipative.

Third, we prove that $I - \mathcal{A}$ j is surjective. Let $F = (f_1, f_2) \in \mathcal{H}$. We prove that there exists $U \in D(\mathcal{A})$ satisfying

$$U - \mathcal{A}U = F. \tag{2.14}$$

The second equation in (2.14) is reduced to

$$\eta^t - y + \eta^t_s = f_2 \Longrightarrow \eta^t_s + \eta^t = y + f_2. \tag{2.15}$$

Integrating with respect to s and noticing that η^t should satisfy $\eta^t(x,0) = 0$, we obtain

$$\eta^t = (1 - e^{-s})y + \int_0^s e^{\tau - s} f_2(\tau)d\tau. \tag{2.16}$$

The first equation in (2.14) is reduced to

$$y - ia\Delta y - \int_0^\infty g(s)\Delta\eta^t ds = f_1. \tag{2.17}$$

Multiplying (2.17) by $\bar{w}$, with $\bar{w} \in H_0^1(\Omega)$, we find

$$y\ \bar{w} - ia\Delta y\ \bar{w} - \int_0^\infty g(s)\Delta\eta^t ds\ \bar{w} = f_1\bar{w},$$

integrating the obtained formula over Ω and using (2.16), we get

$$\begin{aligned} \int_\Omega y\bar{w}\ dx - ia\int_\Omega \Delta y\bar{w}\ dx - \int_\Omega\left(\int_0^\infty g(s)\Delta\eta^t ds\right)\bar{w}\ dx &= \int_\Omega f_1\bar{w}\ dx \\ \langle y, w\rangle - ia\left(\nabla y\ \nabla\bar{w}|_\Omega - \int_\Omega \nabla y\ \nabla\bar{w}\ dx\right) - \int_0^\infty g(s)\int_\Omega \Delta\eta^t\ \bar{w}\ dx\ ds &= \langle f_1, w\rangle \\ \langle y, w\rangle + ia\langle\nabla y, \nabla w\rangle - \int_0^\infty g(s)\left(\nabla\eta^t\ \nabla\bar{w}|_\Omega - \int_\Omega \nabla\eta^t\ \nabla\bar{w}\ dx\right)ds &= \langle f_1, w\rangle \\ \langle y, w\rangle + ia\langle\nabla y, \nabla w\rangle + \int_0^\infty g(s)\langle\nabla\eta^t, \nabla w\rangle ds &= \langle f_1, w\rangle \\ \langle y, w\rangle + ia\langle\nabla y, \nabla w\rangle + \int_0^\infty g(s)\langle(1-e^{-s})\nabla y + \nabla\int_0^s e^{\tau-s}f_2(\tau)\ d\tau, \nabla w\rangle ds &= \langle f_1, w\rangle \\ \langle y, w\rangle + ia\langle\nabla y, \nabla w\rangle + g_1\langle\nabla y, \nabla w\rangle + \int_0^\infty g(s)\langle\nabla f_3, \nabla w\rangle ds &= \langle f_1, w\rangle, \end{aligned}$$

where

$$g_1 = \int_0^\infty (1-e^{-s})g(s)ds \quad \text{and} \quad f_3(s) = \int_0^s e^{\tau-s}f_2(\tau)d\tau. \tag{2.18}$$

The variational formulation of (2.17) is given by

$$(g_1 + ia)\langle\nabla y, \nabla w\rangle + \langle y, w\rangle = \langle f_1, w\rangle - \langle f_3, w\rangle_{L_1}. \tag{2.19}$$

Using the Fubini theorem and Hölder's inequality, we find

$$\begin{aligned} \int_0^\infty g(s)\|\nabla f_3\|^2 ds &= \int_0^\infty g(s)\left|\left|\int_0^s e^{\tau-s}\nabla f_2(\tau)d\tau\right|\right|^2 ds \\ &\le \int_0^\infty e^{-2s}g(s)\left(\int_0^s e^\tau d\tau\right)\int_0^s e^\tau\|\nabla f_2(\tau)\|^2 d\tau\ ds \end{aligned}$$

$$\begin{aligned}\int_0^\infty g(s)\|\nabla f_3\|^2 ds &\leq \int_0^\infty e^{-s}(1-e^{-s})g(s)\int_0^s e^{\tau}\|\nabla f_2(\tau)\|^2 d\tau\ ds\\ &\leq \int_0^\infty e^{-s}g(s)\int_0^s e^{\tau}\|\nabla f_2(\tau)\|^2 d\tau\ ds\\ &\leq \int_0^\infty e^{\tau}\|\nabla f_2(\tau)\|^2\int_\tau^\infty e^{-s}g(s)ds\ d\tau\\ &\leq \int_0^\infty e^{\tau}g(\tau)\|\nabla f_2(\tau)\|^2\int_\tau^\infty e^{-s}ds\ d\tau\\ &\leq \int_0^\infty g(\tau)\|\nabla f_2(\tau)\|^2\ d\tau = \|f_2\|^2_{L_1} < \infty,\end{aligned}$$

then $f_3 \in L_1$.
Therefore, we can see that, if (2.17) admits a solution y satisfying the required regularity in $D(\mathcal{A})$, then (2.16) implies that η exists and satisfies η^t_s, $\eta^t \in L_1$. To prove the existence of y, we notice that the form

$$F_1(v,w) = (g_1 + ia)\langle\nabla y, \nabla w\rangle + \langle y, w\rangle$$

is bilinear, continuous and coercive, and the form

$$F_2(w) = \langle f_1, w\rangle - \langle f_3, w\rangle_{L_1}$$

is linear and continuous. For the continuity of F_1 and F_2, we have just to apply the classical Poincaré's inequality: there exists $c_* > 0$ such that

$$\|v\|^2 \leq c_*\|\nabla v\|^2, \quad v \in H^1_0(\Omega). \tag{2.20}$$

As for the coercivity of F_1, we have

$$\begin{aligned}F_1(v,v) &= (g_1 + ia)\langle\nabla v, \nabla v\rangle + \langle v, v\rangle\\ &= (g_1 + ia)\|\nabla v\|^2 + \|v\|^2\\ &\geq (g_1 + ia)c'_*\|v\|^2 + \|v\|^2\\ &\geq (g_1 + ia + 1)\|v\|^2,\end{aligned}$$

hence, F_1 is coercive.
So using the Lax-Milgram theorem, we deduce that there exists a unique $y \in H^1_0(\Omega)$ satisfying

$$F_1(y,w) = F_2(w), \quad w \in H^1_0(\Omega).$$

Which implies that (2.19) holds. Hence, classical elliptic regularity arguments imply (2.17) and

$$ia\nabla y + \int_0^\infty g(s)\Delta\eta^t ds \in L^2(\Omega).$$

This proves that (2.14) has a unique solution $U \in D(\mathcal{A})$. $\square$

2.3 Stability results

We prove in this section the stability results of (2.8), where the obtained decay estimate is valid for $U_0 \in D(\mathcal{A}_1^{2n+2})$ in (2.1). We assume the following additional hypothesis on the growth of g at infinity and the size of y_0.

(**A3**) Assume that there exists a positive constant α_0 and an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+^*)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to\infty} G'(t) = \infty \tag{2.21}$$

in such way that

$$g' \leq \alpha_0 g, \tag{2.22}$$

or

$$\int_0^\infty \frac{s^2 g(s)}{G^{-1}(-g'(s))} ds + \sup_{s\in\mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < \infty. \tag{2.23}$$

Furthermore, if (2.22) does not hold, we assume that y_0 satisfies

$$\sup_{t\in\mathbb{R}_+} \max_{k=0}^{n+1} \int_t^\infty \frac{g(s)}{G^{-1}(-g'(s))} \| \int_0^{s-t} \nabla\partial_s^k y_0(.,\tau)d\tau\|^2 ds < \infty \tag{2.24}$$

in (2.1), where ∂_s^k denotes the derivative of order k with respect to s.

Remark 1. 1. *Thanks to (2.23), (2.24) is valid if, for example,* $\|\nabla\partial_s^k y_0\|^2$ *(resp.* $\|\partial_s^k y_0\|^2$*)*, $k = 0, 1, \dots, n+1$, *are bounded with respect to s.*

2. *The class of functions g satisfying (**A1**),(**A2**) and (**A3**) is very wide and contains the ones which converge to zero exponentially like*

$$g_1(s) = d_1 e^{q_1 s}, \tag{2.25}$$

or at a slower rate like

$$g_2(s) = d_2(1+s)^{-q_2} \tag{2.26}$$

with $d_1, q_1, d_2 > 0$ *and* $q_2 > 3$. *Conditions*(2.9) *and* (2.22) *are satisfied by* g_1 *with* $\beta_0 = \alpha_0 = q_1$,*and conditions*(2.9) *and* (2.23) *are satisfied by* g_2 *with*

$$\beta_0 = q_2 \quad \text{and} \quad G(s) = s^p, \quad \text{for any } p > \frac{q_2+1}{q_2-3}. \tag{2.27}$$

In order to announce our stability results, we consider the energy functional E *associated with* (2.1), *and given by*

$$E(t) = \frac{1}{2}\left(\|y\|^2 + \int_0^\infty g(s)\|\nabla\eta^t\|^2 ds\right). \tag{2.28}$$

Theorem 2.1. *Assume that (**A1**),(**A2**) and (**A3**) hold. Let* $n \in \mathbb{N}^*$, $U_0 \in D(\mathcal{A}_1^{2n+2})$. *Then there exists positive constant* α_n *such that*

$$E(t) \leq \alpha_n G_n\left(\frac{\alpha_n}{t}\right) \quad \text{and ,} \quad t \in \mathbb{R}_+^*, \tag{2.29}$$

where

$$G_m(s) = G_1(sG_{m-1}(s)), m = 2, 3, \dots, n, G_1 = G_0^{-1} \text{ and } G_0(s) = \begin{cases} s & \text{if (2.22) holds,} \\ sG'(s) & \text{if (2.23) holds.} \end{cases} \tag{2.30}$$

Remark 2.

1. *We see that* $G_n(0) = 0$ *then* (2.29) *implies that*

$$\lim_{t\to\infty} E_j(t) = 0. \tag{2.31}$$

   By density of $D(\mathcal{A}_1^4)$ *in* $\mathcal{H}$, *we conclude that* (2.31) *is valid, for any* $U_0 \in \mathcal{H}$.

2. *In case* (2.22), $G_m(s) = s^m$, *and so* (2.29) *is reduced to, for some* $\beta_n > 0$,

$$E(t) \leq \beta_n t^{-n}. \tag{2.32}$$

   However, in case(2.23),(2.29) *is weaker that*(2.32). *For the example*(2.26), *where*(2.23) *is satisfied with* G *given in* (2.27), (2.29) *implies that there exist* $\gamma_n > 0$ *such that*

$$E(t) \leq \gamma_n t^{-p_n},$$

   where $p_n = \sum_{m=1}^n p^{-m}$. *Notice that* $p_n \to n$ *when* $p \to 1$; *that is when* $q_2 \to \infty$. *This means that if* g *converges to zero at infinity faster than any polynomial, then* *the decay rate given in(2.29)is arbitrarily close to* t^{-n}.

We start the proof of(2.29) by proving the following identities for the derivatives of E.

Lemma 1. *The energy functional* E *satisfies*

$$E'(t) = \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta^t\|^2 ds \tag{2.33}$$

Proof. We have

$$\begin{aligned} E'(t) =& \Re\frac{1}{2}\left(\|U\|_{\mathcal{H}}^2\right)' \\ &= \Re\frac{1}{2}\left(\langle U, U\rangle_{\mathcal{H}}\right)' \\ &= \Re\langle U_t, U\rangle_{\mathcal{H}}. \end{aligned}$$

Then, using (2.8)$_1$ and (2.10), we get

$$\begin{aligned} E'(t) &= \Re\langle U_t, U\rangle_{\mathcal{H}} \\ &= \Re\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \\ &= \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta^t\|^2 ds. \end{aligned}$$

□

Remark 3.

1. *Thanks to* ($\boldsymbol{A}_2$), E' *is well-defined and non-positive, and so* (2.8) *is dissipative.*

2. *Because* $U_0 \in D(\mathcal{A}_1^{2n+2})$ *in* (2.1) *with* $n \in \mathbb{N}^*$, *we can define the following energy functional of higher order* E_k, *with* $k = 1, 2, 3, 4$

$$E_k(t) = \frac{1}{2}\|\partial_t^k U\|_{\mathcal{H}}^2. \tag{2.34}$$

Because (2.1) *is linear and the coefficient* a *does not depend on* t, *we get (as for* (2.33)*)*

$$E_k'(t) = \frac{1}{2}\int_0^\infty g'(s)\|\nabla\partial_t^k\eta^t\|^2 ds. \tag{2.35}$$

Lemma 2. *There exist positive constants c_1, c_2 and such that*

$$\|\nabla y\|^2 \leq c_1 \int_0^{\infty} g(s)\|\nabla \eta^t\|^2 ds + c_2 \int_{\Omega} (\Im y \Re y_t - \Re y \Im y_t) dx \tag{2.36}$$

in (2.1).

Proof. Multiplying the first equation in (2.6) by $\bar{y}$, integrating over Ω and using the boundary condition, we get

$$i \int_{\Omega} y_t \; \bar{y} \; dx + a \int_{\Omega} \Delta y \; \bar{y} \; dx - i \int_{\Omega} \int_0^{\infty} g(s) \cdot \Delta \eta^t \cdot \bar{y} \; dx = 0$$

$$a \int_{\Omega} \Delta y \; \bar{y} \; dx = -i \int_{\Omega} y_t \; \bar{y} \; dx + i \int_{\Omega} \int_0^{\infty} g(s) \cdot \Delta \eta^t \cdot \bar{y} \; dx,$$

integrating by parts with respect to x and using the boundary conditions, we find

$$-a \int_{\Omega} \nabla y \; \nabla \bar{y} \; dx = -i \langle y_t \; \bar{y} \rangle - i \int_0^{\infty} g(s) \; ds \int_{\Omega} \nabla \eta^t \cdot \nabla \bar{y} \; dx$$

$$-a \langle \nabla y, \nabla \bar{y} \rangle = -i \langle y_t \; \bar{y} \rangle - i \int_0^{\infty} g(s) \langle \nabla \eta^t, \nabla y \rangle \; ds$$

$$a \|\nabla y\|^2 = i \langle y_t, y \rangle + i \int_0^{\infty} g(s) \langle \nabla \eta^t, \nabla y \rangle \; ds. \tag{2.37}$$

Direct computations show that

$$\begin{aligned}
\langle y_t, y \rangle &= \int_{\Omega} y_t \; \bar{y} \; \mathrm{dx} \\
&= \int_{\Omega} (\Re y_t + i \Im y_t)(\Re y - i \Im y) \; dx \\
&= \int_{\Omega} (\Re y_t \cdot \Re y + \Im y_t \cdot \Im y + i \Im y_t \cdot \Re y - i \Re y_t \cdot Imy) \; dx \\
&= \int_{\Omega} (\Re y_t \cdot \Re y + \Im y_t \cdot \Im y) \; dx + i \int_{\Omega} (\Re y \cdot \Im y_t - Imy \cdot \Re y_t) \; dx \\
&= \frac{1}{2} \left(\|y\|^2 \right)_t + i \int_{\Omega} (\Re y \Im y_t - \Im y \Re y_t) \; dx
\end{aligned}$$

hence,

$$\langle y_t, y \rangle = \frac{1}{2} \left(\|y\|^2 \right)_t + i \int_{\Omega} (\Re y \Im y_t - \Im y \Re y_t) dx. \tag{2.38}$$

On the other hand, applying Hölder's and Young's inequalities, we have, for any $\varepsilon > 0$,

$$\left| i \int_0^\infty g(s)\langle \nabla\eta^t, \nabla y\rangle ds \right| \leq \varepsilon\|\nabla y\|^2 + c_\varepsilon \int_0^\infty g(s)\|\nabla\eta^t\|^2 ds, \tag{2.39}$$

where we denote by c_ε a positive constant depending on ε. Combining (2.37) and (2.38), using (2.39) and choosing $\varepsilon = \frac{a}{2}$, we obtain

$$\begin{aligned} a\|\nabla y\|^2 &= i\langle y_t, y\rangle + i\int_0^\infty g(s)\langle \nabla\eta^t, \nabla y\rangle\ ds \\ &= \frac{1}{2}\left(\|y\|^2\right)_t - \int_\Omega (\Re y \Im y_t - \Im y \Re y_t) dx + i\int_0^\infty g(s)\langle \nabla\eta^t, \nabla y\rangle\ ds, \end{aligned}$$

taking the real part, we get

$$a\|\nabla y\|^2 = \int_\Omega (\Im y \Re y_t - \Re y \Im y_t)\ dx + \Re\left(i\int_0^\infty g(s)\langle \nabla\eta^t, \nabla y\rangle\ ds\right),$$

then,

$$\begin{aligned} a\|\nabla y\|^2 &\leq \int_\Omega (\Im y \Re y_t - \Re y \Im y_t)\ dx + \frac{a}{2}\|\nabla y\|^2 + c_\varepsilon \int_0^\infty g(s)\|\nabla\eta^t\|^2 ds, \\ \frac{a}{2}\|\nabla y\|^2 &\leq \frac{2}{a}\int_\Omega (\Im y \Re y_t - \Re y \Im y_t)\ dx + \frac{2}{a}c_\varepsilon \int_0^\infty g(s)\|\nabla\eta^t\|^2 ds, \end{aligned}$$

setting

$$c_1 = \frac{2}{a}c_\varepsilon \quad \text{and} \quad c_2 = \frac{2}{a},$$

we obtain finally

$$a\|\nabla y\|^2 \leq c_1 \int_0^\infty g(s)\|\nabla\eta^t\|^2 ds + c_2 \int_\Omega (\Im y \Re y_t - \Re y \Im y_t)\ dx.$$

□

Now, we prove the following estimation on the last integral in (2.36).

Lemma 3. *For any $\varepsilon > 0$, we have*

$$\int_\Omega (\Im y \Re y_t - \Re y \Im y_t) dx \leq \varepsilon\|\nabla y\|^2 + c_\varepsilon \int_0^\infty g(s)\|\nabla\eta^t_{tt}\|^2 ds - c_\varepsilon E_1'(t) \tag{2.40}$$

in (2.1)

Proof. We proceed as in [27] for Timoshenko systems. Exploiting (2.4) and integrating with respect to s, we have

$$\int_\Omega (\Im y \Re y_t - \Re y \Im y_t)dx = \frac{1}{g_0}\int_\Omega \Im y \int_0^\infty g(s)\Re(\eta^t_{tt}+\eta^t_{st})\ ds\ dx - \frac{1}{g_0}\int_\Omega \Re y \int_0^\infty g(s)\Im(\eta^t_{tt}+\eta^t_{st})\ ds\ dx$$

$$= \frac{1}{g_0}\int_\Omega \Im y \int_0^\infty \Re(g(s)\eta^t_{tt} - g'(s)\eta^t_t)dsdx - \frac{1}{g_0}\int_\Omega \Re y \int_0^\infty \Im(g(s)\eta^t_{tt} - g'(s)\eta^t_t)dsdx. \tag{2.41}$$

Using Hölder's, Young's and Poincaré's inequalities, we obtain, for any $\varepsilon > 0$,

$$\begin{aligned}
\int_\Omega (\Im y \cdot \Re y_t - \Re y \cdot \Im y_t)dx &= \frac{1}{g_0}\int_\Omega \Im y \int_0^\infty g(s)\Re(\eta^t_{tt}+\eta^t_{st})\ ds\ dx - \frac{1}{g_0}\int_\Omega \Re y \int_0^\infty g(s)\Im(\eta^t_{tt}+\eta^t_{st})\ ds\ dx \\
&\le \frac{1}{g_0}\left(\int_\Omega |\Im y|^2\ dx\right)^{\frac12} \cdot \left(\int_0^\infty \int_\Omega \left|\Re\left(g(s)\eta^t_{tt} - g'(s)\eta^t_t\right)\right|^2 ds\ dx\right)^{\frac12} \\
&- \frac{1}{g_0}\left(\int_\Omega |\Re y|^2\ dx\right)^{\frac12} \cdot \left(\int_0^\infty \int_\Omega \left|\Im\left(g(s)\eta^t_{tt} - g'(s)\eta^t_t\right)\right|^2 ds\ dx\right)^{\frac12} \\
&\le \frac{c^2}{g_0}\left(\int_\Omega |\nabla \Im y|^2\ dx\right)^{\frac12} \cdot \left(\int_0^\infty \int_\Omega \left|\Re\left(g(s)\nabla\eta^t_{tt} - g'(s)\nabla\eta^t_t\right)\right|^2 ds\ dx\right)^{\frac12} \\
&- \frac{c^2}{g_0}\left(\int_\Omega |\nabla \Re y|^2\ dx\right)^{\frac12} \cdot \left(\int_0^\infty \int_\Omega \left|\Im\left(g(s)\nabla\eta^t_{tt} - g'(s)\nabla\eta^t_t\right)\right|^2 ds\ dx\right)^{\frac12} \\
&\le \frac{c^2}{2g_0}\int_\Omega |\nabla \Im y|^2\ dx + \frac{c^2}{2g_0}\int_0^\infty \int_\Omega \left|\Re\left(g(s)\nabla\eta^t_{tt} - g'(s)\nabla\eta^t_t\right)\right|^2 ds\ dx \\
&- \frac{c^2}{2g_0}\int_\Omega |\nabla \Re y|^2\ dx - \frac{c^2}{2g_0}\int_0^\infty \int_\Omega \left|\Im\left(g(s)\nabla\eta^t_{tt} - g'(s)\nabla\eta^t_t\right)\right|^2 ds\ dx \\
&\le \frac{c^2}{2g_0}\left(\int_\Omega |\nabla \Im y|^2 - |\nabla \Re y|^2\ dx\right) + \frac{c^2}{2g_0} \\
&\int_0^\infty \int_\Omega \left(\left|\Re\left(g(s)\nabla\eta^t_{tt} - g'(s)\nabla\eta^t_t\right)\right|^2 - \left|\Im\left(g(s)\nabla\eta^t_{tt} - g'(s)\nabla\eta^t_t\right)\right|^2\right) ds\ dx \\
&\le \frac{c^2}{2g_0}\|\nabla y\|^2 + \frac{c^2}{2g_0}\int_0^\infty \left(g(s)\|\nabla\eta^t_{tt}\|^2 - g'(s)\|\nabla\eta^t_t\|^2\right) ds
\end{aligned}$$

$$\int_\Omega (\Im y \Re y_t - \Re y \Im y_t)dx \le \varepsilon\|\nabla y\|^2 + c_\varepsilon \int_0^\infty (g(s)\|\nabla\eta^t_{tt}\|^2 - g'(s)\|\nabla\eta^t_t\|^2)ds. \tag{2.42}$$

Exploiting(2.35)$_1$ for $k = 1$,

$$E_1'(t) = \frac{1}{2}\int_0^\infty g'(s)\|\nabla\eta_t^t\|^2 ds,$$

hence,

$$\int_\Omega (\Im y \Re y_t - \Re y \Im y_t)dx \le \varepsilon\|\nabla y\|^2 + c_\varepsilon \int_0^\infty g(s)\|\nabla\eta_{tt}^t\|^2\, ds - c_\varepsilon \int_0^\infty g'(s)\|\nabla\eta_t^t\|^2 ds,$$

that is

$$\int_\Omega (\Im y \Re y_t - \Re y \Im y_t)dx \le \varepsilon\|\nabla y\|^2 + c_\varepsilon \int_0^\infty g(s)\|\nabla\eta_{tt}^t\|^2\, ds - c_\varepsilon E_1'(t),$$

we notice that (2.42) leads to (2.40). ☐

Now, choosing $\varepsilon = \frac{1}{2c_2}$ in (2.40) and combining with (2.36), we find, for some $c_3 > 0$,

$$\|\nabla y\|^2 \le c_3 \int_0^\infty g(s)\|\nabla\eta^t\|^d s + c_3 \int_0^\infty g(s)\|\nabla\eta_{tt}^t\|^2 ds - c_3 E_1'(t) \tag{2.43}$$

in case (2.1).

Remark 4. *Using (2.9),(2.35)$_1$ (for $k = 1$), we conclude from (2.43) that in case (2.1), for some $c_4 > 0$,*

$$\|\nabla y\|^2 \le c_4(E(t) + E_1(t) + E_2(t)) \le c_4(E(0) + E_1(0) + E_2(0)). \tag{2.44}$$

Therefore, using Hölder's inequality, we get, for $t \ge s \ge 0$,

$$\begin{aligned}\|\nabla\eta^t\|^2 &= \left|\left|\int_{t-s}^t \nabla y(.,\tau)d\tau\right|\right|^2 \\ &\le s\int_{t-s}^t \|\nabla y(.,\tau)\|^2 d\tau \le c_4(E(t) + E_1(t) + E_2(t)) \le c_4(E(0) + E_1(0) + E_2(0))s^2.\end{aligned}$$

For $t \geq s \geq 0$, using the same arguments, we have

$$\begin{aligned}
\|\nabla \eta^t\|^2 &= \left\|\int_0^{s-t} \nabla y_0(.,\tau)d\tau + \int_0^t \nabla y(.,\tau)d\tau\right\|^2 \\
&\leq 2\left\|\int_0^{s-t} \nabla y_0(.,\tau)d\tau\right\|^2 + \left\|\int_0^t \nabla y(.,\tau)d\tau\right\|^2 \\
&\leq 2\left\|\int_0^{s-t} \nabla y_0(.,\tau)d\tau\right\|^2 + 2c_4(E(0)+E_1(0)+E_2(0))s^2.
\end{aligned}$$

Consequently

$$\|\nabla \eta^t\|^2 = \begin{cases} c_4(E(0)+E_1(0)+E_2(0))s^2 & \text{if } 0 \leq s \leq t, \\ 2\left\|\int_0^{s-t} \nabla y_0(.,\tau)d\tau\right\|^2 + 2c_4(E(0)+E_1(0)+E_2(0))s^2 & \text{if } s > t \geq 0 \end{cases} := M(t,s). \tag{2.45}$$

Similarly to(2.45) and since E_2 is non-increasing, we have, for some $c_5 > 0$,

$$\|\nabla \eta^t_{tt}\|^2 = \begin{cases} c_5(E_2(0)+E_3(0)+E_4(0))s^2 & \text{if } 0 \leq s \leq t, \\ 2\left\|\int_0^{s-t} \partial_s^2 y_0(.,\tau)d\tau\right\|^2 + 2c_5(E_2(0)+E_3(0)+E_4(0))s^2 & \text{if } s > t \geq 0 \end{cases} := \tilde{M}(t,s) \tag{2.46}$$

in the case (2.1).
The inequalities (2.45) and (2.46) will be used in the proof of the next lemma in order to estimate the integrals in (2.43). This lemma was introduced in [31] and improved in [32]. Notice that we have used energies of higher order up to E_4 in case (2.1) ; this is why we need initial data $U_0 \in D(\mathcal{A}_1^{2n+2})$ in case (2.1) with $n \in \mathbb{N}^*$.

Lemma 4. *There exist positive constants $d, \tilde{d}$ such that, for any $\varepsilon_0 > 0$, the following inequalities hold :*

$$\frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)} \int_0^\infty g(s)\|\nabla \eta^t\|^2 ds \leq -d\ E'(t) + d\ G_0(\varepsilon_0 E(t)), \tag{2.47}$$

$$\frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)} \int_0^\infty g(s)\|\nabla \eta^t_{tt}\|^2 ds \leq -\tilde{d}E_2'(t) + \tilde{d}G_0(\varepsilon_0 E_1(t)), \tag{2.48}$$

where G_0 is defined in (2.30).

Proof. If (2.22) holds, then (2.33) and (2.35) (for $k = 2$) lead to

$$\int_0^\infty g(s)\|\nabla \eta^t\|^2 ds \leq \frac{-2}{\alpha_0}E'(t), \quad \int_0^\infty g(s)\|\nabla \eta^t_{tt}\|^2 ds \leq \frac{-2}{\alpha_0}E_2'(t). \tag{2.49}$$

So (2.47) and (2.48) hold with $d = \tilde{d} = \frac{2}{\alpha_0}$ and $G_0(s) = s$. When (2.22) does not hold and (2.23) is satisfied, we note first that, without loss of generality, we can assume that $E > 0$ and $g' < 0$ on $\mathbb{R}_+$. Otherwise, if $E(t_1) = 0$, for at least $t_1 \in \mathbb{R}_+$ then $E(t) = 0$ for all $t \geq t_1$, since E is non negative and non-increasing, and consequently, (2.29) is satisfied, since E is bounded. And if $g' < 0$ is not satisfied on $\mathbb{R}_+$, then there exists $s_0 \in \mathbb{R}_+$ such that $g'(s_0) = 0$ and $g' < 0$ on $(0, s_0)$, since $g' \in C(\mathbb{R}_+)$. Therefore(2.23) implies that $g(s_0) = 0$, and so $g(s) = 0$, for all $s \geq s_0$, since g is non-negative and non-increasing. Consequently, the integrals on $\mathbb{R}_+$ in (2.43) are reduced to integrals on $(0, s_0)$ and $g' < 0$ on $(0, s_0)$.

Let $\tau(t,s), \theta(t,s) > 0, \varepsilon_0 > 0$ (which will be fixed later on) and $K(s) = \frac{s}{G^{-1}(s)}$, for $s > 0$. The hypothesis (**A1**) implies that

$$\lim_{s \to 0^+} \frac{s}{G^{-1}(s)} = \lim_{\tau \to 0^+} \frac{G(\tau)}{\tau} = G'(0) = 0,$$

then $K(0) = 0$. The function K is non-decreasing. Indeed, the fact that G^{-1} is concave and $G^{-1}(0) = 0$ implies that, for any $0 \leq s_1 < s_2$,

$$K(s_1) = \frac{s_1}{G^{-1}\left(\frac{s_1}{s_2}s_2 + \left(1 - \frac{s_1}{s_2}\right)G^{-1}(0)\right)} \leq \frac{s_1}{\frac{s_1}{s_2}G^{-1}(s_2) + \left(1 - \frac{s_1}{s_2}\right)G^{-1}(0)} = \frac{s_2}{G^{-1}(s_2)} = K(s_2).$$

Then, using (2.45),

$$K(-\theta(t,s)g'(s)\|\nabla\eta^t\|^2) \leq K(-M(t,s)\theta(t,s)g'(s)). \tag{2.50}$$

Using (2.50), we arrive at

$$\begin{aligned}
\int_0^\infty g(s)\|\nabla\eta^t\|^2 ds = {}& \frac{1}{G'(\varepsilon_0 E(t))}\int_0^\infty \frac{1}{\tau(t,s)} G^{-1}(-\theta(t,s)g'(s)\|\nabla\eta^t\|^2) \\
& \times \frac{\tau(t,s)G'(\varepsilon_0 E(t))g(s)}{-\theta(t,s)g'(s)} K(-\theta(t,s)g'(s)\|\nabla\eta^t\|^2) \\
\leq {}& \frac{1}{G'(\varepsilon_0 E(t))}\int_0^\infty \frac{1}{\tau(t,s)} G^{-1}(-\theta(t,s)g'(s)\|\nabla\eta^t\|^2) \\
& \times \frac{\tau(t,s)G'(\varepsilon_0 E(t))g(s)}{-\theta(t,s)g'(s)} K(-M(t,s)\theta(t,s)g'(s))ds
\end{aligned}$$

$$\int_0^\infty g(s)\|\nabla\eta^t\|^2 ds \le \frac{1}{G'(\varepsilon_0 E(t))}\int_0^\infty \frac{1}{\tau(t,s)} G^{-1}(-\theta(t,s)g'(s)\|\nabla\eta^t\|^2) \times \frac{M(t,s)\tau(t,s)G'(\varepsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\theta(t,s)g'(s))} ds.$$

Let $G^*(s) = \sup\limits_{\tau\in\mathbb{R}_+}\{s\tau - G(\tau)\}$, for $s \in \mathbb{R}_+$, denote the dual function of G. From the hypothesis (**A3**), we see that

$$G^*(s) = s(G')^{-1}(s) - G((G')^{-1}(s)), \quad s \in \mathbb{R}_+.$$

Using Young's inequality : $s_1 s_2 \le G(s_1) + G^*(s_2)$, for

$$s_1 = G^{-1}(-\theta(t,s)g'(s)\|\nabla\eta^t\|^2) \quad \text{and} \quad s_2 = \frac{M(t,s)\tau(t,s)G'(\varepsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\theta(t,s)g'(s))},$$

we obtain

$$\int_0^\infty g(s)\|\nabla\eta^t\|^2 ds \le \frac{1}{G'(\varepsilon_0 E(t))}\int_0^\infty \frac{-\theta(t,s)}{\tau(t,s)} g'(s)\|\nabla\eta^t\|^2 ds$$

$$+\frac{1}{G'(\varepsilon_0 E(t))}\int_0^\infty \frac{1}{\tau(t,s)} G^*\left(\frac{M(t,s)\tau(t,s)G'(\varepsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\theta(t,s)g'(s))}\right) ds.$$

Using the fact that $G^*(s) \le s(G')^{-1}(s)$, we get

$$\int_0^\infty g(s)\|\nabla\eta^t\|^2 ds \le \frac{-1}{G'(\varepsilon_0 E(t))}\int_0^\infty \frac{\theta(t,s)}{\tau(t,s)} g'(s)\|\nabla\eta^t\|^2 ds$$

$$+\int_0^\infty \frac{M(t,s)g(s)}{G^{-1}(-M(t,s)\theta(t,s)g'(s))}(G')^{-1}\left(\frac{M(t,s)\tau(t,s)G'(\varepsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\theta(t,s)g'(s))}\right) ds.$$

Then, using the fact that G^{-1} is non-decreasing and choosing $\theta(t,s) = \frac{1}{M(t,s)}$, we find

$$\int_0^\infty g(s)\|\nabla\eta^t\|^2 ds \le \frac{-1}{G'(\varepsilon_0 E(t))}\int_0^\infty \frac{1}{M(t,s)\tau(t,s)} g'(s)\|\nabla\eta^t\|^2 ds$$

$$+\int_0^\infty \frac{M(t,s)g(s)}{G^{-1}(-g'(s))}(G')^{-1}(m_0 M(t,s)\tau(t,s)G'(\varepsilon_0 E(t)))ds,$$

where $m_0 = \sup\limits_{t\in\mathbb{R}_+}\frac{g(s)}{G^{-1}(-g'(s))}$ (m_0 exists according to (2.23)). Due to (2.23) and the restriction (2.24)) on y_0 (for $k = 0$), we have

$$\sup_{\tau\in\mathbb{R}_+}\int_0^\infty \frac{M_1(t,s)g(s)}{G^{-1}(-g'(s))}ds := m < \infty.$$

Therefore, choosing $\tau(t,s) = \frac{1}{m_0 M(t,s)}$ and using (2.33), we get

$$\begin{aligned}\int_0^\infty g(s)\|\nabla\eta^t\|^2 ds &\le \frac{-m_0}{G'(\varepsilon_0 E(t))}\int_0^\infty g'(s)\|\nabla\eta^t\|^2 ds + \varepsilon_0 E(t)\int_0^\infty \frac{M(t,s)g(s)}{G^{-1}(-g'(s))}ds \\ &\le \frac{-2m_0}{G'(\varepsilon_0 E(t))}E'(t) + m\varepsilon_0 E(t),\end{aligned} \tag{2.51}$$

which, by multiplying (2.51) by $G'(\varepsilon_0 E(t)) = \frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)}$, gives (2.34) with $d = \max\{2m_0, m\}$. As for (2.50),

$$K(-\tilde\theta(t,s)g'(s)\|\eta^t_{tt}\|^2) \le K(-\tilde M(t,s)\tilde\theta(t,s)g'(s))$$

for any positive functions $\tilde\theta$, where $\tilde M$ is defined in (2.46). Consequently, using the above two inequalities and arguing as for (2.51) with $\tilde\tau$ and $\tilde\theta$ instead of τ and θ, respectively, we deduce (2.48) with $\tilde d = \max\{2m_0, \tilde m\}$, where

$$\tilde m = \sup_{t\in\mathbb{R}_+}\int_0^\infty \frac{\tilde M(t,s)g(s)}{G^{-1}(-g'(s))}ds, \quad \tilde\tau = \frac{1}{m_0\tilde M(t,s)} \quad \text{and} \quad \tilde\theta(t,s) = \frac{1}{\tilde M(t,s)}.$$

□

Now, using (2.20) and the definition of E, we see that

$$\frac{2}{c_*}E(t) \le \|\nabla y\|^2 + \frac{1}{c_*}\int_0^\infty g(s)\|\nabla\eta^t\|^2 ds, \tag{2.52}$$

therefore, multiplying (2.52) by $\frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)}$ and using (2.43), we find

$$\begin{aligned}\frac{2}{\varepsilon_0 c_*}G_0(\varepsilon_0 E(t)) \le \left(\frac{1}{c_*} + c_3\right)\frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)}\int_0^\infty g(s)(\|\nabla\eta^t\|^2 + \|\nabla\eta^t_{tt}\|^2)ds \\ -c_3\frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)}E_1'(t),\end{aligned} \tag{2.53}$$

then, combining (2.53) with (2.47) and (2.48), we obtain

$$\left[\frac{2}{c_*} - \varepsilon_0\left(c_3 + \frac{1}{c_*}\right)(d + \tilde d)\right]G_0(\varepsilon_0 E(t)) \le -\varepsilon_0\left(c_3 + \frac{1}{c_*}\right)\left(dE'(t) + \tilde d E_2'(t)\right) \tag{2.54}$$

$$-c_3\frac{G_0(\varepsilon_0 E(t))}{\varepsilon_0 E(t)}E_1'(t).$$

Because E is non-increasing and $H_0(s) := \frac{G_0(s)}{s}$ is non-decreasing, then $H_0(\varepsilon_0 E)$ is non-increasing, and therefore

$$-c_3\frac{G_0(\varepsilon_0 E(t))}{E(t)}E_1'(t) \leq -c_3\frac{G_0(\varepsilon_0 E(0))}{E(0)}E_1'(t). \tag{2.55}$$

Choosing

$$0 < \varepsilon_0 < \frac{2}{(c_* c_3 + 1)(d + \tilde{d})} \quad \text{in case (2.1)},$$

and exploiting (2.54) and (2.55) , we find, for some $c_6 > 0$,

$$G_0(\varepsilon_0 E(t)) \leq -c_6(E'(t) + E_1'(t) + E_2'(t)). \tag{2.56}$$

Finally, integrating (2.56) on $[0, t]$, for $t \in \mathbb{R}_+^*$, and noticing that $G_0(\varepsilon_0 E(t))$ is non-increasing, we arrive at

$$tG_0(\varepsilon_0 E(t)) \leq \int_0^t G_0(\varepsilon_0 E(s))ds \leq c_6(E'(0) + E_1'(0) + E_2'(0)) := c_7.$$

Consequently, because G_0 is irreversible and non-decreasing, we deduce that

$$E(t) \leq \frac{1}{\varepsilon_0}G_0^{-1}\left(\frac{c_7}{t}\right),$$

which gives (2.29), for $n = 1$, with

$$G_1 = G_0^{-1}, \quad \text{and} \quad \alpha_2 = \max\left\{c_7, \frac{1}{\varepsilon_0}\right\}.$$

Because $D(\mathcal{A}_1^5) \subset D(\mathcal{A}_1^4)$ and $D(\mathcal{A}_2^3) \subset D(\mathcal{A}_2^2)$, then (2.29) is still valid for $n = 1$, $U_0 \in D(\mathcal{A}_1^5)$ in case (2.1).
By induction on n, (2.29) holds, for any $n \in \mathbb{N}^*$. Indeed, let $n \in \mathbb{N}^*$ and suppose that (2.29) holds, for any initial data in $D(\mathcal{A}_1^{2n+2})$ in case (2.1). Let $U_0 \in D(\mathcal{A}_1^{2(n+1)+2})$ in case (2.1) and U the corresponding solution of (2.8). We have (thanks to Theorem (2.1))

$$U_0 \in D(\mathcal{A}_1^{2(n+1)+2}) \subset D(\mathcal{A}_1^{2n+2}), \quad U_t(0) \in D(\mathcal{A}_1^{2(n+1)+1}) \subset D(\mathcal{A}_1^{2n+2}) \text{ and } U_{tt}(0) \in D(\mathcal{A}_1^{2n+2})$$

in case (2.1), and then (2.29) holds, for U_0, and implies that, for some $a_{j,n} > 0,\ j = 1,2,$

$$E_{1,j}(t) \leq a_{j,n} G_n\left(\frac{a_{j,n}}{t}\right). \tag{2.57}$$

By integrating (2.56) over $[T, 2T]$, for $T > 0$, noticing that $G_0(\varepsilon_0 E)$ is non-increasing and using (2.57), we get, for some $d_n > 0$,

$$TG_0(\varepsilon_0 E(2T)) \leq \int_T^{2T} G_0(\varepsilon_0 E(t))dt \leq c_6(E(T) + E_1(T) + E_2(T)) \leq d_{1,n} G_n\left(\frac{d_{1,n}}{T}\right). \tag{2.58}$$

Therefore, since G_0 is non-decreasing,

$$E(2T) \leq \frac{1}{\varepsilon_0} G_0^{-1}\left(\frac{2d_{1,n}}{2T} G_n\left(\frac{2d_{1,n}}{2T}\right)\right),$$

that is

$$E(2T) \leq \alpha_{1,n+1} G_{n+1}\left(\frac{\alpha_{1,n+1}}{t}\right), \quad t > 0,$$

where

$$G_{n+1}(s) = G_1(sG_n(s)), \quad \alpha_{1,n+1} = \max\left\{\frac{1}{\varepsilon_0}, 2d_{1,n}\right\}.$$

Which leads to (2.29), for $n+1$ instead of n. This ends the proof of Theorem (2.1).

Chapter 3

Global nonexistence of solutions to system of Klein-Gordon equations with degenerate damping and strong source terms in viscoelasticity

3.1 Introduction

In this chapter, we consider a system of viscoelastic wave equations with degenerate damping and strong nonlinear source terms

$$\begin{cases} u_{tt}-\Delta u+m_1u^2+\displaystyle\int_0^t g\left(t-s\right)\Delta u\left(x,s\right)ds+\left(a\left|u\right|^k+b\left|v\right|^l\right)\left|u_t\right|^{m-1}u_t & =f_1(u,v),\\ v_{tt}-\Delta v+m_2v^2+\displaystyle\int_0^t h\left(t-s\right)\Delta v\left(x,s\right)ds+\left(c\left|v\right|^\theta+d\left|u\right|^\varrho\right)\left|v_t\right|^{r-1}v_t & =f_2(u,v),\end{cases} \tag{3.1}$$

where $m,r>0,k,l,\theta,\varrho\geq 1$ and the functions $f_1\left(u,v\right),f_2\left(u,v\right)$ are defined by

$$\begin{aligned} f_1(u,v)&=a_1|u+v|^{2(\rho+1)}(u+v)+b_1|u|^\rho u|v|^{(\rho+2)}\\ f_2(u,v)&=a_1|u+v|^{2(\rho+1)}(u+v)+b_1|u|^{(\rho+2)}|v|^\rho v,a_1,b_1>0,\end{aligned} \tag{3.2}$$

where $\rho>-1$. In (3.1), $u=u\left(x,t\right),v=v\left(x,t\right)$, where $x\in\Omega$ is a bounded domain of $\mathbb{R}^N\left(N\geq 1\right)$ with a smooth boundary $\partial\Omega$ and $t>0$, $a,b,c,d,m_1,m_2>0$.

To above system (3.1), we add the initial conditions given by

$$(u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1), x \in \Omega \tag{3.3}$$

and boundary conditions given by

$$u(x) = v(x) = 0, x \in \partial\Omega. \tag{3.4}$$

This kind of problems arises in viscoe-lasticity. Dafermos was the first who studied this type in [25], where the general decay was treated. In the last decades, problems related to system (3.1) had a lot of attention and many results appeared on the existence and long time behavior of solutions. See in this directions ([8, 4, 5, 12, 13, 16, 17, 36, 48, 49, 56, 53, 59, 70, 72]) and references therein.

In the absence of viscoelastic term, some special cases of the single wave equations with nonlinear damping and nonlinear source terms in the form

$$u_{tt} - \Delta u + a|u_t|^{m-1}u_t = b|u|^{p-1}u. \tag{3.5}$$

With nonlinear damping and source terms, it arises in the quantum-field and used to describe the movement of charged electromagnetic fields. Equation (3.5) equipped with initial and bounded conditions of Dirichlet type has been extensively studied and many results regarding existence, blow up and asymptotic behavior of solutions have been obtained. Many authors have studied the single wave equations in the presence of various mechanisms of dissipation, damping and non-linear sources. See ([3, 50, 51, 28, 41, 42, 63, 67, 71]) and references therein.

In [47], authors considered the nonlinear viscoelastic system

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + \displaystyle\int_0^t h(t-s)\Delta v(x,s)ds + |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \tag{3.6}$$

where

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{(\rho+2)}|v|^{\rho}v. \end{aligned} \tag{3.7}$$

The global nonexistence theorem for some solutions with positive energy was proved using a method applied in [61].

In [62], the authors studied the nonlinear viscoelastic system in (3.6), where they obtained the decay of solutions for the system. Under some restrictions on the nonlinearities of damping and source terms, they proved that, for some class of relaxation functions and some restrictions on the initial data, the rate of decay of relaxation functions affects the rate of decay of solution for the system.

In this paper, we consider system (3.1)-(3.4) and proved a global nonexistence result of solutions. We extended the result in [47] and [70] to more general cases.

3.2 Preliminaries

In this section, we present some notations and Lemmas.

We assume that the relaxation functions g, $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are of class C^1 and non-increasing differentiable satisfying

$$\begin{cases} 1-\displaystyle\int_0^\infty g(s)ds = l' > 0, \quad g(t) \geq 0, \quad g'(t) \leq 0, \\ 1-\displaystyle\int_0^\infty h(s)ds = k' > 0, \quad h(t) \geq 0, \quad h'(t) \leq 0, \end{cases} \quad t \geq 0. \tag{3.8}$$

We introduce the "*modified*" energy functional E associated to our system

$$2E(t) = \|u_t\|_2^2 + \|v_t\|_2^2 + 2\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) + J(u,v) - 2\int_\Omega F(u,v)\,dx, \tag{3.9}$$

where $F(u,v)$ is defined for all $(u,v) \in \mathbb{R}^2$,

$$\begin{aligned} F(u,v) &= \frac{1}{2(\rho+2)}\left[uf_1(u,v) + vf_2(u,v)\right], \\ &= \frac{1}{2(\rho+2)}\left[|u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}\right] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v), \frac{\partial F}{\partial v} = f_2(u,v),$$

and

$$\begin{aligned} J(u,v) &= \left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \left(1-\int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\ &+ (g\circ\nabla u) + (h\circ\nabla v). \end{aligned} \tag{3.10}$$

Noting by

$$\begin{cases} (g \circ u)(t) = \int_0^t g(t-\tau) \|u(t) - u(\tau)\|_2^2 \, d\tau, \\ (h \circ v)(t) = \int_0^t h(t-\tau) \|v(t) - v(\tau)\|_2^2 \, d\tau. \end{cases} \tag{3.11}$$

We suppose that ρ satisfies

$$\begin{cases} -1 < \rho, & \text{if} \quad N = 1, 2, \\ -1 < \rho \leq \frac{4-N}{N-2} & \text{if} \quad N \geq 3. \end{cases} \tag{3.12}$$

Lemma 1. *[61] There exist two positive constants c_0 and c_1 with the end goal that*

$$\frac{c_0}{2(\rho+2)} \left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)} \right) \leq F(u, v) \leq \frac{c_1}{2(\rho+2)} \left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)} \right).$$

Lemma 2. *Assume that (3.12) holds. There exists $\eta > 0$, such that for any $(u, v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, the inequality*

$$2(\rho+2) \int_\Omega F(u, v) \, dx \leq \eta \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right)^{\rho+2} \tag{3.13}$$

holds.

Lemma 3. *Let $\nu > 0$, be a real positive number and let $L(t)$ be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{3.14}$$

defined in $[0, \infty)$.
If $L(0) > 0$, then the solution does not exist for $t \geq L(0)^{-\nu} \xi^{-\nu} \nu^{-1}$.

Proof. By simple integration of (3.14), we have

$$L^{-\nu}(0) - L^{-\nu}(t) \geq \xi \nu t.$$

Then, we obtain the following estimate

$$L^{\nu}(t) \geq [L^{-\nu}(0) - \xi \nu t]^{-1}. \tag{3.15}$$

Then, the RHS of (3.15) is unbounded for

$$\xi \nu t = L^{-\nu}(0).$$

$\square$

The proof is completed.

3.3 Blow up result

Lemma 4. *Assume that (3.12) holds. Let (u, v) be the solution of the system (3.1)−(3.4) then the energy functional is a non-increasing function, that is, for all $t \geq 0$,*

$$\begin{aligned} E'(t) &= -\int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t(t)|^{m+1}\,dx \\ &- \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t(t)|^{r+1}\,dx \\ &+ \frac{1}{2}(g' \circ \nabla u) + \frac{1}{2}(h' \circ \nabla v) - \frac{1}{2}g(s)\|\nabla u\|_2^2 - \frac{1}{2}h(s)\|\nabla v\|_2^2. \end{aligned} \tag{3.16}$$

Lemma 5. *Suppose that (3.12) holds. Let (u, v) be the solution of the system (3.1)-(3.4), then the energy functional is a non-increasing function, that is, for all $t > 0$,*

$$\begin{aligned} \frac{dE(t)}{dt} &= -\int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t(t)|^{m+1}\,dx \\ &- \int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right)|v_t(t)|^{r+1}\,dx. \end{aligned} \tag{3.17}$$

The proof of Lemma 4 can be done by using a classical calculations.
Our main result reads as follows.

Theorem 1. *Suppose that (3.12) holds. Assume further that*

$$\rho > \max\left(\frac{k+m-3}{2}, \frac{l+m-3}{2}, \frac{\theta+r-3}{2}, \frac{\varrho+r-3}{2}\right), \tag{3.18}$$

and that there exists p such that $2 < p < 2(\rho+2)$, for which

$$\max\left(\int_0^\infty g(s)\,ds, \int_0^\infty h(s)\,ds\right) < \frac{(p/2)-1}{(p/2)-1+1/(2p)}, \tag{3.19}$$

holds. Then any solution of problem (3.1)–(3.4), with initial data satisfying

$$\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2+m_1^2\|u_0\|_2^2+m_2^2\|v_0\|_2^2>\alpha_1^2, \text{ and } E(0)<E_2 \tag{3.20}$$

blows up in finite time, where the constants α_1 *and* E_2 *are defined in (3.21).*

We take $a=b=c=d=1, a_1=b_1=1$ for convenience. We introduce the following constants

$$\begin{aligned} B &= \eta^{\frac{1}{2(\rho+2)}}, \qquad \alpha_1=B^{-\frac{\rho+2}{\rho+1}}, \qquad E_1=\left(\frac{1}{2}-\frac{1}{2(\rho+2)}\right)\alpha_1^2, \\ E_2 &= \left(\frac{1}{p}-\frac{1}{2(\rho+2)}\right)\alpha_1^2, \end{aligned} \tag{3.21}$$

where η is the optimal constant in (3.13).

Lemma 6. *[61] Suppose that (3.12), (3.18) and (3.19) hold. Let* (u,v) *be a solutions of (3.1)–(3.4). Assume further that* $E(0)<E_2$ *and*

$$\|\nabla u_0\|_2^2+\|\nabla v_0\|_2^2+m_1^2\|u_0\|_2^2+m_2^2\|v_0\|_2^2>\alpha_1^2. \tag{3.22}$$

Then, there exists a constant $\alpha_2>\alpha_1$ *such that*

$$J(t)>\alpha_2^2, \tag{3.23}$$

and

$$2(\rho+2)\int_\Omega F(u,v)\,dx \geq (B\alpha_2)^{2(\rho+2)}, \forall t\geq 0. \tag{3.24}$$

Proof of Theorem 1. The proof is similar to one given in [48] with the necessary modification imposed by the nature of our problem. We assume that the solutions exists for all t and we get a contradiction. We set

$$H(t)=E_2-E(t). \tag{3.25}$$

By using the definition of $H(t)$, we obtain

$$\begin{aligned} H'(t) &= -E'(t) \\ &= \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t(t)|^{m+1}\,dx \\ &+ \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t(t)|^{r+1}\,dx \\ &- \frac{1}{2}\left(g^{'}\circ\nabla u\right) - \frac{1}{2}\left(h^{'}\circ\nabla v\right) + \frac{1}{2}g(s)\|\nabla u\|_2^2 + \frac{1}{2}h(s)\|\nabla v\|_2^2 \\ &\geq 0, \forall t \geq 0. \end{aligned} \tag{3.26}$$

Therefore,

$$H(0) = E_2 - E(0) > 0. \tag{3.27}$$

Then,

$$\begin{aligned} 0 &< H(0) \leq H(t) \\ &= E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) - \frac{J(t)}{2} \\ &+ \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right]. \end{aligned} \tag{3.28}$$

Note that from (3.8) and (3.23), we get

$$\begin{aligned} E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{J(t)}{2} &< E_2 - \frac{1}{2}\alpha_2^2 \\ &< E_2 - \frac{1}{2}\alpha_1^2 \\ &< E_1 - \frac{1}{2}\alpha_1^2 \\ &= -\frac{1}{2(\rho+2)}\alpha_1^2 < 0, \forall t \geq 0. \end{aligned} \tag{3.29}$$

Thus, by using (3.29) and Lemma 1, we get

$$\begin{aligned} 0 &< H(0) \leq H(t) \leq \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\ &\leq \frac{c_1}{2(\rho+2)}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \forall t \geq 0. \end{aligned} \tag{3.30}$$

We define the function M as

$$M(t)=\frac{1}{2}\int_{\Omega}\left(u^{2}+v^{2}\right)(x,t)\,dx, \tag{3.31}$$

and let

$$L\left(t\right)=H^{1-\sigma}\left(t\right)+\varepsilon M'(t), \tag{3.32}$$

for ε small to be chosen later and

$$\begin{aligned} 0 \quad < \quad & \sigma\leq\min\left\{\frac{1}{2},\frac{2\rho+3-(k+m)}{2\left(m+1\right)\left(\rho+2\right)},\frac{2\rho+3-(l+m)}{2\left(m+1\right)\left(\rho+2\right)},\right.\\ & \left.\frac{2\rho+3-(\varrho+r)}{2\left(r+1\right)\left(\rho+2\right)},\frac{2\rho+3-(\theta+r)}{2\left(r+1\right)\left(\rho+2\right)},\frac{2\rho+2}{4\left(\rho+2\right)}\right\}. \end{aligned} \tag{3.33}$$

By differentiation of (3.32) with respect to time and using (3.1), we get

$$\begin{aligned} L'\left(t\right) &= \left(1-\sigma\right)H^{-\sigma}\left(t\right)H^{'}\left(t\right)+\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2\right)\\ &- \varepsilon\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)\\ &- \varepsilon\int_{\Omega}u\left(\left|u\left(t\right)\right|^{k}+\left|v\left(t\right)\right|^{l}\right)\left|u_t\right|^{m-1}u_t dx\\ &- \varepsilon\int_{\Omega}v\left(\left|v\left(t\right)\right|^{\theta}+\left|u\left(t\right)\right|^{\varrho}\right)\left|v_t\right|^{r-1}v_t dx\\ &+ \varepsilon\int_{\Omega}\left(uf_1\left(u,v\right)+vf_2\left(u,v\right)\right)dx\\ &+ \varepsilon\int_{\Omega}\nabla u\left(t\right)\int_0^t g\left(t-s\right)\nabla u\left(\tau\right)dxds\\ & +\varepsilon\int_{\Omega}\nabla v\left(t\right)\int_0^t h\left(t-s\right)\nabla v\left(\tau\right)dxds. \end{aligned} \tag{3.34}$$

Then,

$$\begin{aligned}
L'(t) &= (1-\sigma) H^{-\sigma}(t) H^{'}(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\
&- \varepsilon \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\
&- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\
&- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx \\
&+ \varepsilon \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&+ \varepsilon \left(\int_0^t g(s)\,ds\right) \|\nabla u\|_2^2 + \left(\int_0^t h(s)\,ds\right) \|\nabla v\|_2^2 \\
&+ \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) . [\nabla u(\tau) - \nabla u(t)]\, dxds \\
&+\varepsilon \int_0^t h(t-s) \int_\Omega \nabla v(t) . [\nabla v(\tau) - \nabla v(t)]\, dxds. \qquad (3.35)
\end{aligned}$$

By using Cauchy-Schwartz and Young's inequalities, we obtain the following estimate

$$\begin{aligned}
&\int_0^t g(t-s) \int_\Omega \nabla u(t) . [\nabla u(\tau) - \nabla u(t)]\, dxds \\
\leq\ &\int_0^t g(t-s) \|\nabla u\|_2 \|\nabla u(\tau) - \nabla u(t)\|_2\, d\tau \\
\leq\ &\lambda (g \circ \nabla u) + \frac{1}{4\lambda} \left(\int_0^t g(s)\,ds\right) \|\nabla u\|_2^2, \qquad \lambda > 0 \qquad (3.36)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t h(t-s) \int_\Omega \nabla v(t) . [\nabla v(\tau) - \nabla v(t)]\, dxds \\
\leq\ &\lambda (h \circ \nabla v) + \frac{1}{4\lambda} \left(\int_0^t h(s)\,ds\right) \|\nabla v\|_2^2, \qquad \lambda > 0. \qquad (3.37)
\end{aligned}$$

Adding $pE(t)$ and using the definition of $H(t), E_2$ leads to

$$\begin{aligned}
L'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) \\
&+ \varepsilon \left(1+\frac{p}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\
&+ \varepsilon \left(\frac{p}{2} - \lambda\right) \left[(g \circ \nabla u) + (h \circ \nabla v)\right] + p\varepsilon H(t) - p\varepsilon E_2 \\
&- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\
&- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx \\
&+ \varepsilon \left(1 - \frac{p}{2(\rho+2)}\right) \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&+ \varepsilon \left[\left(\frac{p}{2}-1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right) \int_0^\infty g(s)\, ds\right] \|\nabla u\|_2^2 \\
&+ \varepsilon \left[\left(\frac{p}{2}-1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right) \int_0^\infty h(s)\, ds\right] \|\nabla v\|_2^2, \qquad (3.38)
\end{aligned}$$

for some λ such that

$$a_1 = \frac{p}{2} - \lambda > 0,$$

and

$$a_2 = \left[\left(\frac{p}{2}-1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right) \max\left(\int_0^\infty g(s)\, ds,\ \int_0^\infty h(s)\, ds\right)\right] > 0.$$

Then, (3.38) can be estimated as follows

$$\begin{aligned}
L'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(1+\frac{p}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
&+ \varepsilon a_1 \left[(g \circ \nabla u) + (h \circ \nabla v)\right] + p\varepsilon H(t) - p\varepsilon E_2 \\
&- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\
&- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx \\
&+ \varepsilon \left(1 - \frac{p}{2(\rho+2)}\right) \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&+ \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right). \qquad (3.39)
\end{aligned}$$

By taking $c_3 = 1 - \dfrac{p}{\rho+2} - 2E_2 (B\alpha_2)^{-2(\rho+2)} > 0$, since $\alpha_2 > B^{-\frac{2(\rho+2)}{\rho+1}}$. Consequently, (3.39) takes the form

$$\begin{aligned}
L'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) \\
&+ \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
&+ \varepsilon a_1\left[(g\circ\nabla u)+(h\circ\nabla v)\right] \\
&+ \varepsilon a_2\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+p\varepsilon H(t) \\
&+ \varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&- \varepsilon\int_\Omega u\left(|u(t)|^k+|v(t)|^l\right)|u_t|^{m-1}u_t dx \\
&- \varepsilon\int_\Omega v\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v_t|^{r-1}v_t dx. \qquad (3.40)
\end{aligned}$$

By using Young's inequality, we have

$$XY \leq \frac{\delta^\alpha X^\alpha}{\alpha}+\frac{\delta^{-\beta}Y^\beta}{\beta}, \qquad (3.41)$$

where $X, Y \geq 0$, $\delta > 0$ and $\alpha, \beta > 0$ such that $1/\alpha + 1/\beta = 1$, we obtain

$$\left|u|u_t|^{m-1}u_t\right| \leq \frac{\delta_1^{m+1}}{m+1}|u|^{m+1}+\frac{m}{m+1}\delta_1^{-(m+1)/m}|u_t|^{m+1}, \forall delta_1 \geq 0 \qquad (3.42)$$

and

$$\begin{aligned}
&\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)\left|u|u_t|^{m-1}u_t\right|dx \\
\leq\ & \frac{\delta_1^{m+1}}{m+1}\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u|^{m+1}dx \\
&+\frac{m}{m+1}\delta_1^{-(m+1)/m}\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u_t|^{m+1}dx. \qquad (3.43)
\end{aligned}$$

Similarly, for any $\delta_2 > 0$,

$$\left|v|v_t|^{r-1}v_t\right| \leq \frac{\delta_2^{r+1}}{r+1}|v|^{r+1}+\frac{r}{r+1}\delta_2^{-(r+1)/r}|v_t|^{r+1}, \qquad (3.44)$$

which gives

$$\begin{aligned}
&\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)\left|v\,|v_t|^{r-1}v_t\right|dx\\
\leq\;&\frac{\delta_2^{r+1}}{r+1}\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v|^{r+1}dx\\
+\;&\frac{r}{r+1}\delta_2^{-(r+1)/r}\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v_t|^{r+1}dx. \qquad (3.45)
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
L'(t) \;\geq\;& (1-\sigma)\,H^{-\sigma}(t)\,H'(t)\\
+\;&\varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right)\\
+\;&\varepsilon a_1\left[(g\circ\nabla u)+(h\circ\nabla v)\right]\\
+\;&\varepsilon a_2\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+p\varepsilon H(t)\\
+\;&\varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right)\\
-\;&\varepsilon\frac{\delta_1^{m+1}}{m+1}\int_{\Omega}\left(|u(t)|^{k}+|v(t)|^{l}\right)|u|^{m+1}dx\\
-\;&\varepsilon\frac{m}{m+1}\delta_1^{-\frac{(m+1)}{m}}\int_{\Omega}\left(|u(t)|^{k}+|v(t)|^{l}\right)|u_t|^{m+1}dx\\
-\;&\varepsilon\frac{\delta_2^{r+1}}{r+1}\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v|^{r+1}dx\\
-\;&\varepsilon\frac{r}{r+1}\delta_2^{-\frac{(r+1)}{r}}\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v_t|^{r+1}dx. \qquad (3.46)
\end{aligned}$$

Choosing δ_1 and δ_2 such that

$$\delta_1^{-\frac{(m+1)}{m}}=M_1H(t)^{-\sigma},\,\delta_2^{-\frac{(r+1)}{r}}=M_2H(t)^{-\sigma}, \qquad (3.47)$$

for M_1 and M_2 large constants to be fixed later. Thus, by using (3.47), we obtain

$$\begin{aligned}
L'(t) \geq\; & ((1-\sigma)-M\varepsilon)\, H^{-\sigma}(t)\, H'(t) \\
+\; & \varepsilon\left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
+\; & \varepsilon a_1\left[(g\circ\nabla u)+(h\circ\nabla v)\right] \\
+\; & \varepsilon a_2\left(\|\nabla u\|_2^2+\|\nabla v\|_2^2\right)+p\varepsilon H(t) \\
+\; & \varepsilon c_3\left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)}+2\|uv\|_{\rho+2}^{\rho+2}\right) \\
-\; & \varepsilon M_1^{-m} H^{\sigma m}(t)\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u|^{m+1}\,dx \\
-\; & \varepsilon\frac{m}{m+1}\delta_1^{-\frac{(m+1)}{m}}\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u_t|^{m+1}\,dx \\
-\; & \varepsilon M_2^{-r} H^{\sigma r}(t)\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v|^{r+1}\,dx \\
-\; & \varepsilon\frac{r}{r+1}\delta_2^{-\frac{(r+1)}{r}}\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v_t|^{r+1}\,dx,
\end{aligned} \tag{3.48}$$

where $M=m/(m+1)\,M_1+r/(r+1)\,M_2$. Therefore, we have

$$\int_\Omega\left(|u(t)|^k+|v(t)|^l\right)|u|^{m+1}\,dx=\|u\|_{k+m+1}^{k+m+1}+\int_\Omega|v|^l\,|u|^{m+1}\,dx, \tag{3.49}$$

and

$$\int_\Omega\left(|v(t)|^\theta+|u(t)|^\varrho\right)|v|^{r+1}\,dx=\|v\|_{\theta+r+1}^{\theta+r+1}+\int_\Omega|u|^\varrho\,|v|^{r+1}\,dx. \tag{3.50}$$

Also by using Young's inequality, we obtain

$$\begin{aligned}
\int_\Omega|v|^l\,|u|^{m+1} \leq\; & \frac{l}{l+m+1}\delta_1^{(l+m+1)/l}\,\|v\|_{l+m+1}^{l+m+1} \\
+\; & \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}\,\|u\|_{l+m+1}^{l+m+1}, \\
\int_\Omega|u|^\varrho\,|v|^{r+1} \leq\; & \frac{\varrho}{\varrho+r+1}\delta_2^{(\varrho+r+1)/\varrho}\,\|u\|_{\varrho+r+1}^{\varrho+r+1} \\
+\; & +\frac{r+1}{\varrho+r+1}\delta_2^{-(\varrho+r+1)/(r+1)}\,\|v\|_{\varrho+r+1}^{\varrho+r+1}.
\end{aligned}$$

Therefore,

$$\begin{aligned} & H^{\sigma m}(t)\int_\Omega \left(|u(t)|^k+|v(t)|^l\right)|u|^{m+1}\,dx \\ = \; & H^{\sigma m}(t)\,\|u\|_{k+m+1}^{k+m+1}+\frac{l}{l+m+1}\delta_1^{(l+m+1)/l}H^{\sigma m}(t)\,\|v\|_{l+m+1}^{l+m+1} \\ + \; & \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}H^{\sigma m}(t)\,\|u\|_{l+m+1}^{l+m+1}, \end{aligned} \tag{3.51}$$

and

$$\begin{aligned} & H^{\sigma r}(t)\int_\Omega \left(|v(t)|^\theta+|u(t)|^\varrho\right)|v|^{r+1}\,dx \\ = \; & H^{\sigma r}(t)\,\|v\|_{\theta+r+1}^{\theta+r+1}+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}H^{\sigma r}(t)\,\|u\|_{\varrho+r+1}^{\varrho+r+1} \\ + \; & \frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}H^{\sigma r}(t)\,\|v\|_{\varrho+r+1}^{\varrho+r+1}. \end{aligned} \tag{3.52}$$

Since (3.18) holds, we get by using (3.33)

$$\begin{cases} H^{\sigma m}(t)\,\|u\|_{k+m+1}^{k+m+1}\le c_5\left(\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1}+\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\,\|u\|_{k+m+1}^{k+m+1}\right), \\ H^{\sigma r}(t)\,\|v\|_{\theta+r+1}^{\theta+r+1}\le c_6\left(\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1}+\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\,\|v\|_{\theta+r+1}^{\theta+r+1}\right). \end{cases} \tag{3.53}$$

This implies

$$\begin{aligned} & \tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}H^{\sigma m}(t)\,\|v\|_{l+m+1}^{l+m+1} \\ & \le c_7\tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}\left(\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l+m+1}+\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\,\|v\|_{l+m+1}^{l+m+1}\right), \end{aligned} \tag{3.54}$$

and

$$\begin{aligned} & \tfrac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}H^{\sigma r}(t)\,\|u\|_{\varrho+r+1}^{\varrho+r+1} \\ & \le c_8\tfrac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}\left(\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho+r+1}+\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\,\|u\|_{\varrho+r+1}^{\varrho+r+1}\right). \end{aligned} \tag{3.55}$$

Using (3.33) and the algebraic inequality

$$z^\nu\le(z+1)\le\left(1+\frac{1}{a}\right)(z+a),\forall z\ge 0, 0<\nu\le 1, a>0, \tag{3.56}$$

we get, for all $t \geq 0$,

$$\left\{ \begin{array}{l} \|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1} \leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H\left(0\right)\right) \leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H\left(t\right)\right), \\ \|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1} \leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H\left(t\right)\right), \forall t \geq 0, \end{array} \right. \tag{3.57}$$

where $d = 1 + 1/H\left(0\right)$. Similarly

$$\left\{ \begin{array}{l} \|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l(m+1)} \leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H\left(0\right)\right) \leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H\left(t\right)\right), \\ \|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho(r+1)} \leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H\left(t\right)\right), \forall t \geq 0. \end{array} \right. \tag{3.58}$$

Also, since

$$(X+Y)^s \leq C\left(X^s + Y^s\right), \qquad X, Y \geq 0, s > 0, \tag{3.59}$$

by using (3.33) and (3.56) we have

$$\begin{aligned} \|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)} \|u\|_{k+m+1}^{k+m+1} &\leq c_9\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u\|_{k+m+1}^{2(\rho+2)}\right) \\ &\leq c_{10}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u\|_{2(\rho+2)}^{2(\rho+2)}\right), \end{aligned} \tag{3.60}$$

similarly

$$\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)} \|v\|_{\theta+r+1}^{\theta+r+1} \leq c_{11}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \tag{3.61}$$

$$\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)} \|v\|_{l+m+1}^{l+m+1} \leq c_{12}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right) \tag{3.62}$$

and

$$\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)} \|u\|_{\varrho+r+1}^{\varrho+r+1} \leq c_{13}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u\|_{2(\rho+2)}^{2(\rho+2)}\right). \tag{3.63}$$

Taking into account (3.51)-(3.63), then (3.48) written as

$$\begin{aligned}
L^{'}(t) &\geq ((1-\sigma)-M\varepsilon)\, H^{-\sigma}(t)\, H^{'}(t) \\
&+ 2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
&+ \varepsilon\left[2-CM_1^{-m}\left(1+\frac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\frac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right. \\
&- \left. CM_2^{-r}\left(1+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}+\frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}\right)\right]H(t) \\
&+ \varepsilon\left[c_4-CM_1^{-m}\left(1+\frac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\frac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right. \\
&- \left. CM_2^{-r}\left(1+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}+\frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}\right)\right] \\
&\times \left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \qquad (3.64)
\end{aligned}$$

At this point and for large values of M_1 and M_2, we can find positive constants Λ_1 and Λ_2 such that (3.64) becomes

$$\begin{aligned}
L^{'}(t) &\geq ((1-\sigma)-M\varepsilon)\, H^{-\sigma}(t)\, H^{'}(t) \\
&+ 2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
&+ \varepsilon\Lambda_1\left(\|u(t)\|_{2(\rho+2)}^{2(\rho+2)}+\|v(t)\|_{2(\rho+2)}^{2(\rho+2)}\right)+\varepsilon\Lambda_2 H(t). \qquad (3.65)
\end{aligned}$$

Once M_1 and M_2 are fixed (hence Λ_1 and Λ_2), we choose ε small enough so that $((1-\sigma)-M\varepsilon)\geq 0$ and

$$L(0)=H^{1-\sigma}(0)+\varepsilon\int_\Omega\left[u_0.u_1+v_0.v_1\right]dx>0. \qquad (3.66)$$

Therefore, there exists $\Gamma>0$ such that (3.65) can be written as

$$L^{'}(t)\geq\varepsilon\Gamma\left(H(t)+\|u_t\|_2^2+\|v_t\|_2^2+\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \qquad (3.67)$$

Then, we have $L(t)\geq L(0)>0$, for all $t\geq 0$. Next, by using Holder's and Young's inequalities, we have the estimate

$$\begin{aligned}
&\left(\int_\Omega u.u_t(x,t)\,dx+\int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\
\leq\ & C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}}+\|u_t\|_2^{\frac{s}{1-\sigma}}+\|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}}+\|v_t\|_2^{\frac{s}{1-\sigma}}\right), \qquad (3.68)
\end{aligned}$$

for $1/\tau + 1/s = 1$. We takes $s = 2(1-\sigma)$, to get $\dfrac{\tau}{1-\sigma} = \dfrac{2}{1-2\sigma}$. From (3.25) and (3.56), we have

$$\|u\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}} \leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \tag{3.69}$$

and

$$\|v\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}} \leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)} + H(t)\right), \forall t \geq 0. \tag{3.70}$$

Consequently, (3.68) can be written as

$$\begin{aligned} & \left(\int_\Omega uu_t(x,t)\,dx + \int_\Omega vv_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\ \leq\ & c_{14}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2 + \|v_t\|_2^2\right) \\ +\ & c_{14}\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2 + H(t)\right), \forall t \geq 0. \end{aligned}$$

Also, we have

$$\begin{aligned} L^{\frac{1}{1-\sigma}}(t) &= \left(H^{1-\sigma}(t) + \varepsilon\int_\Omega (u.u_t + v.v_t)(x,t)\,dx\right)^{\frac{1}{(1-\sigma)}} \\ &\leq c_{15}\left(H(t) + \left|\int_\Omega (u.u_t(x,t) + v.v_t(x,t))\,dx\right|^{\frac{1}{(1-\sigma)}}\right) \\ &\leq c_{16}\left[H(t) + \|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)} + \|u_t\|_2^2\right] \\ &+ c_{16}\left[\|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right], \forall t \geq 0, \end{aligned} \tag{3.71}$$

from (3.71) and (3.67), we get

$$L^{'}(t) \geq a_0 L^{\frac{1}{1-\sigma}}(t), \forall t \geq 0. \tag{3.72}$$

Finally, a simple integration of (3.72) gives the desired result. □

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