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جامعة عمار ثليجي الأغواط
UNIVERSITE AMAR TELIDJI LAGHOUAT
كلية العلوم
FACULTE DES SCIENCES
قسم الرياضيات
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Abidi Khedidja

THEME

Stability for thermo-elastic Bresse system of second sound with past history and delay term

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Soutenu publiquement devant le jury composé de:

Mr. YAGOUB Ameur	*M.C.A*	*Président*
Mr. OUCHENANE Djamel	*M.C.A*	*Encadreur*
Mr. YAZID Fares	*M.C.B*	*Examinateur*

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Dedication

I dedicate this work

To the man of my life, my eternal example, my moral support and source of joy and happiness, the one who has always sacrificed himself to see me succeed, may God keep you, to you
my father.

In the light of my days, the source of my efforts, the flame of my heart, my life and my happiness; mom whom I adore.

To my beloved family ABIDI & GAOUI.

To all those who have always helped and encouraged me, who were always by my side, and who accompanied me during my path to higher education.

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Khadidja Abidi

ملخص

في هذه المذكرة، ندرس وجود ووحدانية الحل وكذا استقراره للنظام الحراري الخطي أحادي الابعاد من نوع بريس بتأخير مع حد اللزوجة و الصوت الثاني (صدى الصوت). لقد برهنا وجود ووحدانية الحل بإستعمال طريقة الطاقة و برهنا استقرار الحل في حالتين حالة إستقرار أسي ($\tilde{\eta} = 0$)، حالة إستقرار جبري ($\tilde{\eta} \neq 0$) وهذا بإستعمال طريقة **ليبانوف** .

Résumé

Dans ce mémoire, on considère un système thermoélastique linéaire unidimensionnel de type Bresse avec un retard. Nous démontrons que le problème est bien posé, en utilisant la théorie de semi-groupe pour établir l'existence et l'unicité de la solution ainsi la stabilité par la méthode de Lyapunov pour deux cas : stabilité exponentielle et polynomiale.

Mots clés :
Système de Bresse, thermoélasticité, stabilité exponentielle, fonctionnelle de Lyapunov .

Abstract

In this memory, we consider a one-dimensional linear thermo-elastic system of Bresse type with past history and delay term. We prove the well-posedness of problem using the semigroup method. By using the energy method, we discuss stability of the system for two cases. An exponential stability result is obtained in the case where the propagation velocities are equal in equation of vertical displacement and the equation of system rotation angle. Furthermore, a result of algebraic stability is obtained in the case of the different propagation velocities, where the initial data $E_2(0)$ is involved in the decay rate for the case.

Keywords:
Bresse system, thermo-elastic, exponential stability, Lyapunov functional .

Contents

Introduction

The stabilization of one-dimensional coupled systems has attracted the interest of many authors in recent years. C. A. Raposo et *al.* [26], and M. Aassila [1] studied the stabilization of Timoshenko's system and the stabilization results for the Timoshenko system by a single control force acting on the equation of the shear angle were considered. F. Alabau in [4] showed an explicit decay rate of the associate energy for the solutions of the Timoshenko system at infinity in the case of the same speed of propagation in the two equations of the system, and the author obtained an estimate of the polynomial energy decay in the case of the different propagation speed for linear and non-linear Lipschitzian feedbacks. In [20], F. Ammar-Khodja; A. Benabdallah; J. Muoz Rivera; R. Racke, dealt with the exponential and polynomial stabilization of Timoshenko's linear system under the effect of a memory term. Also, an optimality results were obtained.

According to Bresse system, in their study on flexible beam, Lagnese et *al.* [15] derive a general model of three-dimensional nonlinear elastic beams. A special case of this model is a linear model coupling three wave equations. It describes the movement of a planar elastic beam under the effect of small deformations. It is the system of Bresse which is, without feedbacks, given for $t > 0, 0 < x < L, L > 0$, by

$$\begin{cases} \rho_1 \varphi_{tt} = GhQ_x + lEhN, \\ \rho_2 \psi_{tt} = EIM_x - GhQ, \\ \rho_1 w_{tt} = EhN_x - IGhQ. \end{cases} \tag{1}$$

where

$$N = w_x - l\varphi, Q = k\left(\varphi_x + lw + \psi\right), M = \psi_x.$$

The functions φ, ψ and w denote, respectively, the transverse displacement of the beam, the angle of rotation of a filament of the beam and the longitudinal displacement of the beam.

Here $\rho_1, \rho_2, l, G, E, h$ designate a positive constants characterize physical properties of beam.

The propagation velocities in the first equation and second equation are, respectively, given by $v_1 = Gh/\rho_1,\ v_2 = El/\rho_2$. In recent years, with a wide application in elastic systems, ranging from measurement and damping of vibrations in large flexible structures for the control of the dissipative Bresse system, it has become increasingly important to study the stability of an elastic system with different damping mechanism. (See [6], [14], [15], [2], [9], [21]) .
In [27], Alves et *al.* considered the Bresse system in

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + lw + \psi\right)_x - k_0 l\left(w_x - l\varphi\right) = -\gamma_1\varphi_t, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + lw + \psi\right) = -\gamma_2\psi_t, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + lw + \psi\right) = 0. \end{cases} \tag{2}$$

with two frictional dissipations an exponential stability of system has been proved under conditions of equal speeds of wave propagation. Other hand, they showed the lack of the exponential stability. Then, they proved that the solution decays polynomially to zero with optimal decay rate, depending on the regularity of initial data. Concerning thermo-elastic Bresse system describing the motion of a linear planar, shearable thermo-elastic beam, [25] considered

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + lw + \psi\right)_x - k_0 l\left(w_x - l\varphi\right) + l\gamma\theta_1 = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + lw + \psi\right) + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + kl\left(\varphi_x + lw + \psi\right) + \gamma\theta_{1x} = 0, \\ \rho_3\theta_t - \theta_{xx} + \gamma\psi_{tx} = 0, \\ \rho_3\theta_{1x} - \theta_{1xx} + \gamma(w_{tx} - l\varphi_t) = 0. \end{cases} \tag{3}$$

They showed that, when the wave speed of the vertical displacement coincides with the wave speed of longitudinal displacement, the exponentially decay rate is preserved. Thence, the author proved that an algebraic type decay rate can be only obtained. In [19], the authors considered two Cauchy problems related to the Bresse model with two dissipative mechanisms corresponding to the heat conduction coupled to the system. The first of them is the Bresse system with thermo-elasticity of type I

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x - \psi - l\omega)_x - k_0 l(\omega_x - l\varphi) + l\gamma\theta_1 = 0 \qquad \text{in } \mathbb{R}\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} - k(\varphi_x - \psi - l\omega) + \gamma\theta_{2x} = 0 \qquad \text{in } \mathbb{R}\times(0,\infty), \\ \rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x - kl(\varphi_x - \psi - l\omega) + \gamma\theta_{1x} = 0 \qquad \text{in } \mathbb{R}\times(0,\infty), \\ \theta_{1t} - k_1\theta_{1xx} + m_1(\omega_x - l\varphi)_t = 0 \qquad \text{in } \mathbb{R}\times(0,\infty), \\ \theta_{2t} - k_2\theta_{2xx} + m_2\psi_{xt} = 0 \qquad \text{in } \mathbb{R}\times(0,\infty). \end{cases} \tag{4}$$

With the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t, \theta_1, \theta_2)(x,0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_{10}, \theta_{20})(x).$$

The second one, is the Bresse system with thermo-elasticity of Type III

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x - \psi - l\omega)_x - k_0 l(\omega_x - l\varphi) + l\gamma\theta_{1t} = 0 \qquad \text{in } \mathbb{R}\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} - k(\varphi_x - \psi - l\omega) + \gamma\theta_{2xt} = 0 \qquad \text{in } \mathbb{R}\times(0,\infty), \\ \rho_1\omega_{tt} - k_0(\omega_x - l\varphi)_x - kl(\varphi_x - \psi - l\omega) + \gamma\theta_{1xt} = 0 \qquad \text{in } \mathbb{R}\times(0,\infty), \\ \theta_{1tt} - k_1\theta_{1xx} - \alpha_1\theta_{1xxt} + m_1(\omega_x - l\varphi)_t = 0 \qquad \text{in } \mathbb{R}\times(0,\infty), \\ \theta_{2tt} - k_2\theta_{2xx} - \alpha_2\theta_{2xxt} + m_2\psi_{xt} = 0 \qquad \text{in } \mathbb{R}\times(0,\infty). \end{cases} \tag{5}$$

With the initial data

$$(\varphi, \varphi_t, \psi, \psi_t, \omega, \omega_t, \theta_1, \theta_2, \theta_{1t}, \theta_{2t})(x,0) = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_{10}, \theta_{20}, \theta_{11}, \theta_{21})(x),$$

where $\alpha_1,\ \alpha_2,\ \rho_1, \rho_2, \gamma, b, k, k_0, k_1, k_2, l\ m_1$ and m_2 are positive constants. The authors proved that the decay rate of the solutions are very slow in the whole line, where they show that the solutions decay with the rate of $(1+t)^{-1/8}$ in the L^2-norm, whenever the initial data belongs to $L^1(\mathbb{R}) \cap H^s(\mathbb{R})$ for a suitable s.

Here, we review recent works related to the decay rates of the thermoelastic Bresse system under effect of second sound and/or with past history and/or delay term. For stabilization of Timoshenko systems in thermoelasticity with Second sound and delay, Apalara and Messaoudi [5] considered the following one-dimensional problem

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi\right)_x + \mu\varphi_t\left(x, t-\tau_0\right) = 0, \qquad x\in(0,1), \quad t>0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi\right) + \delta\theta_x = 0, \qquad x\in(0,1), \quad t>0, \\ \rho_3\theta_t + q_x + \delta\psi_{tx} = 0, \qquad x\in(0,1), \quad t>0, \\ \tau q_t + \beta q + \theta_x = 0, \qquad x\in(0,1), \quad t>0. \end{cases} \tag{6}$$

where the heat flux is given by Cattaneo's law. Under a smallness condition on the delay and a stability number, the authors proved an exponential decay results. Thermoelastic-Bresse system with second sound is proposed in [13] and the well-posedness of the system was studied. Keddi et *al.* proved that the system is exponentially stable under conditions on the parameters of the system and in other cases, there is a stability in the form of polynomial. In the case of the both effects of second sound and delay, we mention the results obtained in [17], where a one-dimensional thermoelastic-Bresse system with a delay term was considered and by semigroup method, the well posedness was proved. The authors also showed, by the multiplier method, that the dissipation is strong enough to exponentially stabilize the system in the presence of a delay. (See [7], [28]).

Because of their complexity, there are only a few papers devoted to the study of thermoelastic Bresse system by the effect of this kind of dissipation under the case $\tilde{\eta} \neq 0$ given in (2.7). It is very important from the application point of view, where waves are not necessarily of equal speeds. The stability result in this case will be shown in (3.30) which is new and very original. As far we know, there not exist researchers reported this case, especially in thermo-elastic Bresse system.

The memory consists of three chapters as follows .

Chapter1. This chapter, summarizes some concepts, definitions and results which are mostly relevant to the undergraduate curriculum and are thus assumed as basically known, or have specific roots in rather distant areas and have rather auxiliary character with respect to the purpose of this memory.

Chapter2. In this chapter, we study the well-posedness of the problem .

In the last chapter, we use the method of multipliers to demonstrate the exponential and polynomial stability of the system, the principle of this method is to construct a new functional, called the Lyapunov functional, equivalent to classic energy.

Chapter 1

Preliminaries

The main objective of this chapter is to present without proof brief discussion of some concepts and properties related to our problems.Reader should consult [3], [11] and [18] for proofs and more details.

Definition 1.1. *Let* Ω *be a domain in* $\mathbb{R}^n$ *and let* m *be a non-negative integer. We define by* $C^m(\Omega)$ *the linear space of continuous functions on* Ω *whose partial derivatives* $\mathrm{D}^{\alpha}\mathrm{u}, |\alpha| \leq \mathrm{m}$*, exist and continuous, where*

$$D^{\alpha}u(x) = \frac{\partial^{|\alpha|}u(x)}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\ldots.\partial x_n^{\alpha_n}} \ , \tag{1.1}$$

$\alpha = (\alpha_1, \ldots\ldots \alpha_n)$ *is called a multi-index of dimension* n *and*

$$|\alpha| = \sum_{i=1}^{n} \alpha_i. \tag{1.2}$$

Definition 1.2. *The support of a continuous function* f *defined on* $\mathbb{R}^n$ *is the closure of the set of point where* $f(x)$ *is nonzero*

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

The closed and bounded sets in $\mathbb{R}^n$ *are precisely the compact sets, so if* $\sup f$ *is bounded, we say* f *has a compact support and denote the set of such functions by* $C_0(\mathbb{R}^n)$*. Similarly,* $C_0(\Omega)$ *denotes the set of continuous functions on* Ω *whose supports are compact subsets of* Ω*. In addition,* $C_0^{\infty}(\Omega)$ *denotes the class of the functions* u *in* Ω *such that*

a) u *is infinity differentiable, which means that* $D^{\alpha}u$ *is uniformly continuous in* $\bar{\Omega}$*, for any* α

b) u *is compactly supported : supp* u *is a compact subset of* Ω.

Corollary 1.1. *$C^m(\Omega)$ is a Banach space with respect to the norm*

$$\|u\|_{C^m(\Omega)} := \max_{|\alpha|\leq m} \sup_{x\in\Omega} |D^\alpha u(x)| . \tag{1.3}$$

Remark 1.1. *If $m = 0$, we denote $C^0(\Omega) = C(\Omega)$.*

1.1 Lebesgue space

Definition 1.3. *Let Ω be a domain in $\mathbb{R}^n$; for $1 \leq p < \infty$, L^p denote the measurable real-valued functions u on Ω for which*

$$\int_\Omega |u(x)|^p dx < \infty.$$

In addition $L^\infty(\Omega)$ denotes the measurable real valued functions that are essentially bounded (bounded except on a set of measure zero).

For $u \in L^p(\Omega)$, we define the norms

$$||u||_p = \left(\int_\Omega |u(x)|^p dx\right)^{\frac{1}{p}}, \qquad \text{for } 1 \leq p < \infty, \tag{1.4}$$

$$||u||_\infty = ess\ sup|u(x)| = \inf\{M : \mu\{x : u(x) > M\} = 0\}. \tag{1.5}$$

lemma 1.1. *If $1 \leq p < \infty$,and $a, b \geq 0$, then*

$$(a + b)^p \leq 2^{p-1}(a^p + b^p). \tag{1.6}$$

1.1.1 Hölder's inequality

Theorem 1.1. *Let $1 < p < \infty$, and let q denote the conjugate exponent defined by*

$$q = \frac{p}{p-1}, \text{ that is } \frac{1}{p} + \frac{1}{q} = 1,$$

which also satisfies $1 < q < \infty$. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)|dx \leq ||u||_p||v||_q,$$

Equality holds if and only if , for some constants α and β, not both zero,

$$\alpha|u(x)|^p = \beta|v(x)|^q \qquad a.e \text{ in } \Omega.$$

Corollary 1.2. *By taking p=q=2, we obtain the Cauchy-Schwarz inequality*

$$\int\limits_{\Omega} |u(x)v(x)|dx \leq ||u||_2||v||_2. \tag{1.7}$$

1.1.2 Young's inequality

Theorem 1.2. *let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$ and a,b $\geq$ 0. Then for any $\eta > 0$,*

$$ab \leq \eta a^p + C_\eta b^q, \tag{1.8}$$

where $C_\eta = \frac{1}{q(\eta p)^{\frac{q}{p}}}$.
for p=q=2 , the inequality takes the form

$$ab \leq \eta a^2 + \frac{b^2}{4\eta}. \tag{1.9}$$

Theorem 1.3. *$L^p(\Omega)$ equipped with the norm (2.3) is a Banach space if 1$\leq$ p $\leq \infty$.*

Corollary 1.3. *$L^2(\Omega)$ is a Hilbert space with respect to the inner product*

$$\langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}dx.$$

The aassociated norm is then

$$||u||_2^2 = \langle u, u\rangle .$$

1.1.3 Density Theorem

Theorem 1.4. *If f $\in L^p(\Omega)$, $1 \leq p < \infty$, then there exists a sequence $(f_n) \subset C_0^\infty(\Omega)$ which converges to f with respect to the norme $||.||_p$.*
This implies that $C_0^\infty(\Omega)$ is dence in $L^p(\Omega)$

1.2 Sobolev space

Definition 1.4. *(weake derivative)*

If u,v $\in L^p(\Omega)$ *, v is called a weak derivative of order* α *of u, If*

$$\int_\Omega u(x)D^\alpha\phi(x)dx = (-1)^{|\alpha|}\int_\Omega v(x)\phi(x)dx, \forall\phi\in C_0^\infty(\Omega). \tag{1.10}$$

for the definition of α *and* $D^\alpha\phi(x)$*, we refer to (1.1)*

Definition 1.5. *(Sobolev spaces)*

Let Ω *be an open set of* $\mathbb{R}^n$*, then the Sobolov space* $W^{k,p}(\Omega)$*,* $1\le p\le\infty$ *,* $k\in N^*$*(positive integer number), is the set of all function u*$\in L^p(\Omega)$*such that the weak derivatives* $D^\alpha u$ *of order* α*,* $|\alpha|\le k$ *, exist and lie in* $L^p(\Omega)$ *That is*

$$W^{k,p}(\Omega) := \{u\in L^p(\Omega)|D^\alpha u\in L^p(\Omega), |\alpha|\le k\}.$$

$W^{k,p}(\Omega)$ *is equipped with the following norm :*

$$\begin{aligned} \|u\|_{k,p} &= \left(\sum_{|\alpha|\le k}\|D^\alpha u\|_p^p\right)^{\frac{1}{p}}, \qquad 1\le p<\infty, \\ \|u\|_{k,\infty} &= \max_{|\alpha|\le k}||D^\alpha u||_\infty. \end{aligned} \tag{1.11}$$

Remark 1.2. *If* $u\in C^m(\Omega)$*, then all weak derivatives are classical.*

Theorem 1.5. $W^{k,p}(\Omega)$ *is a Banach space with respect to the norm (1.11)*

Remark 1.3. *If* $p=2$*, we denote* $W^{k,2}(\Omega)$ *by* $H^k(\Omega)$ *and it is a Hilbert space with respect to the inner product*

$$\langle u,v\rangle_k = \int_\Omega\sum_{|\alpha|\le k}D^\alpha u(x)D^\alpha v(x)dx, \qquad \forall u,v\in H^k(\Omega). \tag{1.12}$$

Definition 1.6. *(Sobolev spaces of order one in* $\mathbb{R}^n$ *)*

Let Ω *be an open domain of* $\mathbb{R}^n$ *and* $1\le p\le\infty$*. Then*

$$W^{1,p}(\Omega) = \left\{u\in L^p(\Omega)\mid\exists v_i\in L^p(\Omega), \int_\Omega u\frac{\partial\phi}{\partial x_i} = -\int_\Omega v_i\phi, i=1,2,\ldots,n, \forall\phi\in C_0^\infty(\Omega)\right\}, \tag{1.13}$$

is called the Sobole space of order one and it is equipped with the norm

$$\|u\|_{1,p} = \|u\|_p + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_p,$$

or equivalently with

$$\|u\|_{1,p} = \left(\|u\|_p^p + \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|_p^p \right)^{\frac{1}{p}}, \qquad 1 < p < \infty.$$

Remark 1.4. $W^{1,2}(\Omega) = H^1(\Omega)$ *is a Hilbert space with respect to the inner product*

$$\langle u, v\rangle = \int_\Omega uv dx + \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} dx.$$

Definition 1.7. *(The space* $\mathbf{W_0^{1,p}(\Omega)}$*)*
Let Ω *be a domain of* $\mathbb{R}^n$ *and* $1 \leq p < \infty$*, we define the space* $W_0^{1,p}(\Omega)$ *to be the closure of* $C_0^1(\Omega)$ *with respect to the norm of* $W^{1,P}(\Omega)$*.*

Theorem 1.6. *If* $u \in W_0^{1,p}(\Omega) \cap C(\bar{\Omega})$*, then* $u(x) = 0$ *for every* $x \in \partial\Omega$*.*

1.2.1 Poincaré inequality

Theorem 1.7. *Assume that* Ω *is bounded in one direction and* $1 \leq p < \infty$*. Then there is a positive constant* $C = C(\Omega, p)$ *such that*

$$\|u\|_p \leq C_p \|\nabla u\|_p, \qquad \forall u \in W_0^{1,p}(\Omega).$$

Definition 1.8. *Let* V *and* W *be two Banach spaces. We say that* V *is continuously embedded in* W *and we write* $V \hookrightarrow W$*, if we have, for some* $C > 0$*,*

$$\|v\|_W \leq C\|v\|_V, \qquad \forall v \in V.$$

1.2.2 An embedding theorem for L^p spaces

Theorem 1.8. *Suppose that*

$$\text{vol}(\Omega) = \int_\Omega dx < \infty,$$

and $1 \le p \le q \le \infty$. *If* $u \in L^q(\Omega)$, *then* $u \in L^p(\Omega)$, *and*

$$\|u\|_p \le (vol(\Omega))^{\frac{1}{p}-\frac{1}{q}} \|u\|_q \,,$$

hence

$$L^q(\Omega) \hookrightarrow L^p(\Omega).$$

1.2.3 Sobolev embedding theorems

Theorem 1.9. *Let* $\Omega \subset \mathbb{R}^n$ *be a Lipschitz domain,* $m \ge 1$ *and* $1 \le p \le \infty$. *Then, the following mappings represent continuous embeddings*

$$\begin{aligned}
W^{m,p}(\Omega) &\hookrightarrow L^{p^*}(\Omega), \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{m}{n}, \ \textit{if}\ m < \frac{n}{p},\\
W^{m,p}(\Omega) &\hookrightarrow L^{q}(\Omega), \qquad 1 \le q < \infty, \ \textit{if}\ m = \frac{n}{p'},\\
W^{m,p}(\Omega) &\hookrightarrow C^{0,m-\frac{n}{p}}(\bar{\Omega}), \qquad \textit{if}\ \frac{n}{p} < m < \frac{n}{p} + 1,\\
W^{m,p}(\Omega) &\hookrightarrow C^{0,\alpha}(\bar{\Omega}), \quad 0 < \alpha < 1, \qquad \textit{if}\ m = \frac{n}{p} + 1,\\
W^{m,p}(\Omega) &\hookrightarrow C^{0,1}(\bar{\Omega}), \qquad \textit{if}\ m > \frac{n}{p} + 1.
\end{aligned}$$

1.2.4 Sobolev, Gagliardo, Nirenberg

Theorem 1.10. *If* $1 \le p < n$, *then*

$$W^{1,p}\left(\mathbb{R}^n\right) \hookrightarrow L^{p^*}\left(\mathbb{R}^n\right), \qquad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n},$$

and there exists a constant $C = C(n,p)$ *such that*

$$\|u\|_{p^*} \le C \|\nabla u\|_p, \qquad \forall u \in W^{1,p}\left(\mathbb{R}^n\right).$$

Corollary 1.4. *If* $1 \le p < n$, *then*

$$W^{1,p}\left(\mathbb{R}^n\right) \hookrightarrow L^{q}\left(\mathbb{R}^n\right), \qquad 1 \le q \le p^*.$$

Theorem 1.11. *If* $p = n$, *then*

$$W^{1,p}\left(\mathbb{R}^n\right) \hookrightarrow L^q\left(\mathbb{R}^n\right), \quad n \leq q < \infty.$$

1.3 Basic theory of semi-groups

In this section, we recall some basic knowledge in semigroups, most of which will be used in the subsequent chapters.

1.3.1 C_0- Semi-groups of Linear Operators

Definition 1.9. *(Semi-groups)*

Let X *be a Banach space, the one-parameter bounded family* $S(t), 0 \leq t < \infty$ *from* X *to* X *is called a Semigroups if*

(i) $S(0) = I$ *(* I *is the identity operator on* X *).*

(ii) $S(t+s) = S(t)S(s)$ *for every* $t, s \geq 0$ *(the Semigroup property).*

Definition 1.10. *The linear operator* A *defined by*

$$D(A) = \left\{ x \in X : \lim_{t \to 0^+} \frac{(S(t)x - x)}{t} \qquad \textit{exists} \right\}.$$

and

$$Ax = \lim_{t \to 0^+} \frac{(S(t)x - x)}{t} = \left.\frac{d(S(t)x)}{dt}\right|_{t=0} \textit{for all} \quad x \in D(A).$$

is called the infinitesimal generator of the Semigroup $S(t)$, $D(A)$ *is called the domain of* A.

Definition 1.11. *(*C_0- *Semigroups)*

A Semigroup $S(t), \quad 0 \leq t < \infty$, *from* X *to* X *is called a strong continuous Semigroup of bounded linear operators if*

$$\lim_{t \to 0^+} S(t)x = x, \qquad \textit{for all} \quad x \in X,$$

or

$$\lim_{t \to 0^+} \|S(t)x - x\| = 0, \qquad \textit{for all} \quad x \in X.$$

Definition 1.12. *A semigroup* $S(t), \quad 0 \leq t < \infty$ *is called a semigroup of contraction if there exists a constant* $\alpha > 0$ $(0 < \alpha < 1)$ *such that for all* $t > 0$,

$$\|S(t)x - S(t)y\| \leq \alpha \|x - y\|, \quad \textit{for all} \quad x, y \in X.$$

1.3.2 Hille-Yosida Theorem

Definition 1.13. *An unbounded linear operator* $A : D(A) \subset H \to H$[1] *is said to be monotone* [2], *if it satisfies*

$$\langle Av, v\rangle \geq 0 \quad \forall v \in D(A)$$

It is called maximal monotone if, in addition ; $R(I + A) = H$, *i.e*

$$\forall f \in H, \quad \exists u \in D(A) \quad \textit{such that} \quad u + Au = f.$$

Proposition 1.1. *Let A be a maximal monotone operator. Then*

1. $D(A)$ *is dense in* H.

2. *A is closed operator.*

3. *For every* $\lambda > 0, (I + \lambda A)$ *is bijective from* $D(A)$ *onto* $H, (I + \lambda A)^{-1}$ *is a bounded operator, and* $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(H)} \leq 1$.

Theorem 1.12. *(Hille-Yosida)*

Let A be a maximal monotone operator. Then, given any $u_0 \in D(A)$ *there exists a unique function*

$$u \in C^1([0, +\infty); H) \cap C([0, +\infty); D(A)),$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \quad \textit{on } [0, +\infty], \\ u(0) = u_0. \end{cases}$$

---

[1] H denotes a Hilbert space.

[2] Some authors say that A is accretive or $-\mathcal{A}$ is dissipative.

1.4 Problems with a delay

In this section we introduce a large number of problems, both old and new, which are treated using the general theory of differential equations. We attempt to give sufficient description concerning the derivation, solution, and properties of solutions so that the reader will be able to appreciate some of the flavor of the problem. In none of the cases do we give a complete treatment of the problem, but offer references for further study. Economics models The following problem is copied from an elementary text on differential equations by Boyce and DiPrima "A young person with no initial capital invests k dollars per year at an annual interest rate τ. Assume that investments are made continuously and that interest is compounded continuously. If $\tau = 7.5\%$, determine k so that one million dollars will be available at the end of forty years." It is solved by writing

$$S' = 0.075S + k, S(0) = 0.$$

and solving for $S(40)$. Several things are idealized in the problem, but still it is a fair model. It is noted there that in certain contexts continuous investment yields roughly the same as daily investment and it allows the student the opportunity to see the power of differential equations in giving a simple solution to an otherwise tedious problem.

Now the forty years is up and for computational convenience instead of the one million dollars let us say that the person has $\$900,000$ to invest and to live off the proceeds. During times of low interest rates a financial advisor may recommend bank certificates of deposit of 90 -day maturity, automatically renewed at the existing interest rate, but lettered so that $\$10,000$ of the total matures every day and both principal and interest are reinvested. This enables the investor to quickly take advantage of rising rates and to lock in high interest long-term instruments if they become available. We imagine that this is changed to continuous reinvestment, just as the elementary problem imagined continuous investment of k dollars per year. If the total value is again $S(t)$, then from just the investment we would have

$$S'(t) = b(t)S(t - (1/4)).$$

The $b(t)$ represents a product. One factor is the fraction of the total amount of $S(t-1/4)$ which was invested three months earlier and matured today. The other factor is the interest being offered at that time. In addition, the person withdraws a percentage of the total $S(t)$ continuously for living expenses, resulting in an equation

$$S'(t) = -a(t)S(t) + b(t)S(t-1/4), S(t) = \psi(t) \text{ for } -1/4 \leq t \leq t_0.$$

Here, the initial condition is an initial function $\psi : [-1/4, 0] \to \mathbb{R}$ with $\psi(t)$ being exactly that amount $S(t)$ which was invested at time t. We can draw several conclusions of the following type. First, if the solutions are bounded, then times are likely to become difficult since inflation will eat away at the value and medical bills will increase with time; at this time, some studies have shown that those retiring with income sufficient to meet three times their current need approach desperate conditions within fifteen years. Next, we can ask if solutions will tend to zero. If they do, the person will be destined for the poor farm. At a minimum, the retiree must adjust the withdrawals so that the conditions of our theorem are not met.

Clearly, in this example it will make sense for both $a(t)$ and $b(t)$ to vary; $a(t)$ can be negative the day the income tax refund check arrives, and $b(t)$ can be negative when the bank fails and the FDIC assumes control

Controlling a ship Minorsky (1962) designed an automatic steering device for the battleship New Mexico. The following is a sketch of the problem

Let the rudder of the ship have angular position $x(t)$ and suppose there is a friction force proportional to the velocity, say $-cx'(t)$. There is a direction indicating instrument which points in the actual direction of motion and there is an instrument pointing in the desired direction. These two are connected by a device which activates an electric motor producing a certain force to move the rudder so as to bring the ship onto the desired course. There is a time lag of amount $h > 0$ between the time the ship gets off course and the time the electric motor activates the restoring force. The equation for $x(t)$ is

$$x''(t) + cx'(t) + g(x(t-h)) = 0, \tag{1.14}$$

where $xg(x) > 0$ if $x \neq 0$ and c is a positive constant. The object is to give conditions ensuring that $x(t)$ will stay near zero so that the ship closely follows its proper course.

Epidemics (Cooke and Yorke) In the work of Cooke and Yorke (1973) the Lotka assumption is changed so that the number of births per unit time is a function only of the population size, not of the age distribution see [15]. Under this assumption, we let $x(t)$ be the population size and let the number of births be $B(t) = g(x(t))$. Assume each individual has life span L so that the number of deaths per unit time is $g(x(t-L))$. Then the population size is described by

$$x'(t) = g(x(t)) - g(x(t-L)), \tag{1.15}$$

where g is some differentiable function. We note that every constant function is a solution of (1.15). The following model for the spread of gonorrhea is considered by Cooke and Yorke (1973). The population is divided into two classes: (a) $S(t)$ = the number of susceptibles, and (b) $x(t)$ = the number of infectious. The rate of new infection depends only on contacts between susceptible and infectious individuals. Since Set) equals the constant total population minus $x(t)$, the rate is some function $g(x(t))$. Assume that an exposed individual is immediately infectious and stays infectious for a period L (the time for treatment and cure). Then x also satisfies (1.15) holds. Now, at any time $t, x(t)$ equals the sum of capital produced over the period $[t-L, t]$ plus a constant c denoting the value of nondepreciating assets. Thus,

$$x(t) = \int_0^L P(s)g[x(t-s)]ds + c = \int_{t-L}^t P(t-u)g[x(u)]du + c. \tag{1.16}$$

Chapter 2

Position of the problem and well-posedness

In this chapter

2.1 Introduction

We improve these recent results , we are interested in the question of stability for the system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + lw + \psi\right)_x - k_0 l\left(w_x - l\varphi\right) + \mu_1\varphi_t\left(x,t\right) + \mu_2\varphi_t\left(x,t-\tau\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + lw + \psi\right) + \displaystyle\int_0^\infty \Theta\left(p\right)\psi_{xx}\left(x,t-p\right)dp + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + kl\left(\varphi_x + lw + \psi\right) = 0, \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \kappa\theta_x = 0. \end{cases} \tag{2.1}$$

where

$$(x,t) \in (0,1) \times (0,\infty),$$

with initial-boundary conditions

$$\begin{aligned} &\varphi(x,t) = \varphi_x(x,t) = \psi_x(x,t) = \psi(x,t) = 0, \\ &w_x(x,t) = w(x,t) = \theta(x,t) = q(x,t) = 0, \quad x = 0,1. \end{aligned} \tag{2.2}$$

and

$$\begin{cases} \varphi(x,0)=\varphi_0(x), \varphi_t(x,0)=\varphi_1(x) & x\in(0,1),\\ \psi(x,0)=\psi_0(x), \psi_t(x,0)=\psi_1(x) & x\in(0,1),\\ w(x,0)=w_0(x), w_t(x,0)=w_1(x) & x\in(0,1),\\ \theta(x,0)=\theta_0(x), q(x,0)=q_0(x), & \\ \varphi_t(x,t-\tau)=f_0(x,t-\tau). & \end{cases} \tag{2.3}$$

with $\tau>0$ is a time delay, $\mu_1>0$ and $\mu_2\in\mathbb{R}$. The function θ is the temperature difference, q is the heat flux, $\rho_1,\rho_2,\rho_3,k,l,k_0,b,\gamma,\kappa,\alpha,\beta$ are positive constants. The relaxation function Θ satisfies the folowing

(G1) The function $\Theta\in C^1(\mathbb{R}_+,\mathbb{R}_+)$ satisfying

$$\Theta\left(0\right)>0,\ \ b-\Theta_0=L>0,\ \ \Theta_0=\int_0^\infty\Theta\left(p\right)dp>0. \tag{2.4}$$

(G2) For some positive constant ζ, we have

$$\Theta'\left(t\right)\le-\zeta\Theta\left(t\right),\forall t\ge0. \tag{2.5}$$

(G3) The following hypotheses

$$|\mu_2|<\mu_1. \tag{2.6}$$

hold. We prove the well-posedness and establish a stabilities results related with the following

$$\tilde{\eta}=\left(\kappa^2-\frac{\alpha k\rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)-\frac{\gamma^2\alpha}{L}\text{ and }k=k_0. \tag{2.7}$$

2.2 Preliminaries and well-posedness results

In this section, we present and proof our well-posedness results for (2.1)-(2.3) Firstly, we assume the following hypothese:

$$|\mu_2|<\mu_1. \tag{2.8}$$

Using semigroup theory, we will prove that systems (2.1)-(2.3) are well posed by introduce the following new variable [17]

$$z(x,\rho,t)=\varphi_t\left(x,t-\tau\rho\right)\qquad x\in\left(0,1\right),\rho\in\left(0,1\right),t>0. \tag{2.9}$$

such as

$$z_t(x,\rho,t) = \frac{\partial \varphi_t(x,t-\tau\rho)}{\partial(t-\tau\rho)} \times \frac{\partial(t-\tau\rho)}{\partial t} = \frac{\partial \varphi_t(x,t-\tau\rho)}{\partial(t-\tau\rho)},$$

and

$$z_\rho(x,\rho,t) = \frac{\partial \varphi_t(x,t-\tau\rho)}{\partial(t-\tau\rho)} \times \frac{\partial(t-\tau\rho)}{\partial \rho} = -\tau\frac{\partial \varphi_t(x,t-\tau\rho)}{\partial(t-\tau\rho)},$$

Then, we have

$$\tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0 \quad \text{in } (0,1)\times(0,1)\times(0,\infty). \tag{2.10}$$

Further, let

$$\eta^t(x,p) = \psi(x,t) - \psi(x,t-p), \qquad p \geq 0. \tag{2.11}$$

such as

$$\eta^t_t(x,p) = \frac{\partial \psi(x,t)}{\partial t} - \frac{\partial \psi_t(x,t-p)}{\partial(t-p)} \times \frac{\partial(t-p)}{\partial t} = \psi_t(x,t) - \frac{\partial \psi_t(x,t-p)}{\partial(t-p)},$$

and

$$\eta^t_p(x,p) = \frac{\partial \psi(x,t)}{\partial p} - \frac{\partial \psi_t(x,t-p)}{\partial(t-p)} \times \frac{\partial(t-p)}{\partial p} = \frac{\partial \psi_t(x,t-p)}{\partial(t-p)},$$

For this reason, we observe that

$$\eta^t_t(x,p) + \eta^t_p(x,p) = \psi_t(x,t), \tag{2.12}$$

In addition to that

$$\eta^t_{xx}(x,p) = \psi_{xx}(x,t) - \psi_{xx}(x,t-p), \qquad p \geq 0. \tag{2.13}$$

So, problem (2.1) equivalent to

$$\begin{cases} \rho_1\varphi_{tt} - k(\varphi_x + lw + \psi)_x - lk_0(w_x - l\varphi) + \mu_1\varphi_t(x,t) + \mu_2 z(x,1,t) = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \\ \rho_2\psi_{tt} + \psi_{xx}(-b + \int_0^\infty \Theta(p)\,dp) + k(\varphi_x + lw + \psi) - \int_0^\infty \Theta(p)\,\eta^t_{xx}(x,p)\,dp + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + lw + \psi) = 0, \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \kappa\theta_x = 0, \\ \eta^t_t(x,p) + \eta^t_p(x,p) = \psi_t(x,t). \end{cases} \tag{2.14}$$

Therefore, problem (2.1) takes the form

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + lw + \psi\right)_x - lk_0\left(w_x - l\varphi\right) + \mu_1\varphi_t\left(x,t\right) + \mu_2 z\left(x,1,t\right) = 0, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \\ \rho_2\psi_{tt} - L\psi_{xx} + k\left(\varphi_x + lw + \psi\right) - \displaystyle\int_0^\infty \Theta\left(p\right)\eta^t_{xx}\left(x,p\right)dp + \gamma\theta_x = 0, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + lw + \psi\right) = 0, \\ \rho_3\theta_t + \kappa q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \kappa\theta_x = 0, \\ \eta^t_t\left(x,p\right) + \eta^t_p\left(x,p\right) = \psi_t\left(x,t\right). \end{cases} \tag{2.15}$$

with the boundary and initial conditions

$$\begin{cases} \varphi(x,t) = \varphi_x(x,t) = \psi_x(x,t) = \psi(x,t) = w_x(x,t) = w(x,t) = 0, \\ \theta(x,t) = q(x,t) = 0, \;\; x = 0,1, \;\; t \geq 0, \eta^t\left(0,p\right) = \eta^t\left(1,p\right) = 0 \qquad \forall p \geq 0, \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \psi(x,0) = \psi_0(x) \qquad x \in (0,1), \\ \psi_t(x,0) = \psi_1(x), w(x,0) = w_0(x), w_t(x,0) = w_1(x) \qquad x \in (0,1), \\ \theta(x,0) = \theta_0(x), q(x,0) = q_0(x) \qquad x \in (0,1), \\ \varphi_t(x,-t) = f_0(x,t) \qquad \text{in } \left(0,1\right) \times \left(0,\tau\right), \\ z\left(x,1,t\right) = f_0\left(x,t-\tau\right) \qquad \text{in } \left(0,1\right) \times \left(0,\tau\right), \\ \eta^t\left(x,0\right) = 0 \qquad \forall t \geq 0, \\ \eta^0\left(x,p\right) = \eta_0\left(p\right) = 0 \qquad \forall p \geq 0. \end{cases} \tag{2.16}$$

Let $\xi > 0$ such that

$$\tau|\mu_2| < \xi < \tau\left(2\mu_1 - |\mu_2|\right). \tag{2.17}$$

where, τ is a real number with $0 < \tau$ and $\mu_1 > 0$ and μ_2 is a real constant and $(\varphi_0, \varphi_1, f_0, \psi_0, \psi_1, w_0, w_1, \theta_0, q_0, \eta_0)$ belong to a suitable space.

Let us set

$$U = \left(\varphi, \varphi_t, z, \psi, \psi_t, w, w_t, \theta, q, \eta^t\right)^T,$$

then

$$U' = \left(\varphi_t, \varphi_{tt}, z_t, \psi_t, \psi_{tt}, w_t, w_{tt}, \theta_t, q_t, \eta^t_t\right)^T.$$

Then (2.15)-(2.16) can be take the form

$$\begin{cases} U'\left(t\right)-AU\left(t\right)=0, \\ U\left(0\right)=\left(\varphi_0,\varphi_1,f_0(.,-\tau),\psi_0,\psi_1,w_0,w_1,\theta_0,q_0,\eta_0\right), \end{cases} \tag{2.18}$$

and the new dependent variables $\varphi_t=u,\ \ \psi_t=v,\ \ \eta^t=\phi,\ \ \omega_t=\varpi,$

the operator A is given by

$$A\begin{pmatrix}\varphi\\ u\\ z\\ \psi\\ v\\ w\\ \varpi\\ \theta\\ q\\ \phi\end{pmatrix}=\begin{pmatrix} u\\ \frac{k}{\rho_1}\left(\varphi_x+lw+\psi\right)_x+\frac{k_0 l}{\rho_1}\left(w_x-l\varphi\right)-\frac{\mu_1}{\rho_1}u-\frac{\mu_2}{\rho_1}z\left(.,1,.\right)\\ -\left(\frac{1}{\tau}\right)z_\rho\\ v\\ \frac{L}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}\left(\varphi_x+lw+\psi\right)+\frac{1}{\rho_2}\int_0^\infty\Theta\left(p\right)\phi_{xx}\left(p\right)dp-\frac{\gamma}{\rho_2}\theta_x\\ \varpi\\ \frac{k_0}{\rho_1}\left(w_x-l\varphi\right)_x-\frac{kl}{\rho_1}\left(\varphi_x+lw+\psi\right)\\ -\frac{\kappa}{\rho_3}q_x-\frac{\gamma}{\rho_3}v_x\\ -\frac{\beta}{\alpha}q-\frac{\kappa}{\alpha}\theta_x\\ -\phi_p+v \end{pmatrix}. \tag{2.19}$$

The energy space $\mathcal{H}$ is defined as

$$\begin{aligned}\mathcal{H} \ &= \ H_0^1\left(0,1\right)\times L^2\left(0,1\right)\times L^2\left(\left(0,1\right),H_0^1(0,1)\right)\times H_0^1\left(0,1\right)\times L^2\left(0,1\right)\times H_0^1\left(0,1\right)\\ &\quad\times L^2\left(0,1\right)\times L^2\left(0,1\right)\times L^2\left(0,1\right)\times L^2_\Theta\left(\mathbb{R}^+,H_0^1(0,1)\right),\end{aligned}$$

Where $L^2_\Theta\left(\mathbb{R}^+,H_0^1(0,1)\right)$ denotes the Hilbert space of H_0^1-valued functions on $\mathbb{R}^+$, endowed with the inner product

$$\left(V_1,V_2\right)_{L^2_\Theta\left(\mathbb{R}^+,H_0^1(\Omega)\right)}=\int_0^1\int_0^1\Theta\left(p\right)V_{1x}\left(p\right)V_{2x}\left(p\right)dpdx.$$

We are now going to show that A generates a C_0 semigroup on $\mathcal{H}$ under (2.17). For this end, we define on $\mathcal{H}$ for

$$U=\left(\varphi,u,z,\psi,v,w,\varpi,\theta,q,\phi\right)^T,\overline{U}=\left(\overline{\varphi},\overline{u},\overline{z},\overline{\psi},\overline{v},\overline{w},\overline{\varpi},\overline{\theta},\overline{q},\bar{\phi}\right)^T,$$

The inner product

$$\begin{aligned}\left\langle U,\overline{U}\right\rangle_{\mathcal{H}} &= k\int_0^1 \left(\varphi_x+\psi+lw\right)\left(\overline{\varphi}_x+\overline{\psi}+l\overline{w}\right)dx+k_0\int_0^1 \left(w_x-l\varphi\right)\left(\overline{w}_x-l\overline{\varphi}\right)dx+\rho_1\int_0^1 u\overline{u}dx \\ &\quad +\rho_2\int_0^1 v\overline{v}dx+\rho_1\int_0^1 \varpi\overline{\varpi}dx+L\int_0^1 \psi_x\overline{\psi_x}dx+\xi\int_0^1\int_0^1 z\overline{z}d\rho dx \\ &\quad +\rho_3\int_0^1 \theta\overline{\theta}dx+\alpha\int_0^1 q\overline{q}dx+\int_0^1\int_0^\infty \Theta\left(p\right)\phi_x\left(p\right)\bar{\phi}_x\left(p\right)dxdp. \qquad (2.20)\end{aligned}$$

For l small enough since, it is easy to see that $\mathcal{H}$ is a Hilbert space. We define the domain of A as

$$D\left(A\right)=\left\{\begin{array}{c} U\in\mathcal{H}/\varphi\in H_0^2\left(0,1\right);\psi,w\in H_0^2\left(0,1\right),u,\theta\in H_0^1\left(0,1\right);v,\varpi,q\in H_0^1\left(0,1\right), \\ u=z\left(.,0\right),z_\rho\in L^2\left(\left(0,1\right);L^2\left(0,1\right)\right),\varphi_x\left(x\right)=0,w_x\left(x\right)=\psi_x\left(x\right)=0,\ \ x=0,1 \\ \phi_s\in L_\Theta^2\left(\mathbb{R}^+,H_0^1(0,1)\right),\ \ \phi\left(x,0\right)=0.\end{array}\right\}. \qquad (2.21)$$

The following two Lemmas will be useful to prove that A is a maximal monotone operator.

lemma 2.1. *The operator A is dissipative and satisfies, for any $U\in D\left(A\right)$,*

$$\begin{aligned}\left\langle AU,U\right\rangle_{\mathcal{H}} &\leq -\beta\int_0^1 q^2dx+\Big(-\mu_1+\frac{|\mu_2|}{2}+\frac{\xi}{2\tau}\Big)\int_0^1 u^2dx+\Big(\frac{|\mu_2|}{2}-\frac{\xi}{2\tau}\Big)\int_0^1 z^2\left(x,1\right)dx \\ &\quad +\int_0^1\int_0^\infty \Theta'\left(p\right)\left|\phi_x\left(x,p\right)\right|^2dpdx \\ &\leq 0. \qquad (2.22)\end{aligned}$$

Proof. Using the inner product for any $U\in D\left(A\right)$

$$\left\langle AU,U\right\rangle_{\mathcal{H}}=\left\langle\begin{pmatrix} u \\ \frac{k}{\rho_1}\left(\varphi_x+lw+\psi\right)_x+\frac{k_0l}{\rho_1}\left(w_x-l\varphi\right)-\frac{\mu_1}{\rho_1}u-\frac{\mu_2}{\rho_1}z\left(.,1\right) \\ -\left(\frac{1}{\tau}\right)z_\rho \\ v \\ \frac{L}{\rho_2}\psi_{xx}-\frac{k}{\rho_2}\left(\varphi_x+lw+\psi\right)+\frac{1}{\rho_2}\int_0^\infty\Theta\left(p\right)\phi_{xx}\left(p\right)dp-\frac{\gamma}{\rho_2}\theta_x \\ -\varpi \\ \frac{k_0}{\rho_1}\left(w_x-l\varphi\right)_x-\frac{kl}{\rho_1}\left(\varphi_x+lw+\psi\right) \\ -\frac{\kappa}{\rho_3}q_x-\frac{\gamma}{\rho_3}v_x \\ -\frac{\beta}{\alpha}q-\frac{\kappa}{\alpha}\theta_x \\ -\phi_p+v\end{pmatrix},\begin{pmatrix}\varphi \\ u \\ z \\ \psi \\ v \\ w \\ \varpi \\ \theta \\ q \\ \phi\end{pmatrix}\right\rangle_{\mathcal{H}}$$

Then

$$
\begin{aligned}
\langle AU,U\rangle_H &= k\int_0^1 \varphi_{tx}\left(\varphi_x+\psi+lw\right)dx+k\int_0^1 \psi_t\left(\varphi_x+\psi+lw\right)dx\\
&\quad +kl\int_0^1 w_t\left(\varphi_x+lw+\psi\right)dx+k_0\int_0^1 w_{tx}\left(w_x-l\varphi\right)dx-k_0l\int_0^1 \varphi_t\left(w_x-l\varphi\right)dx\\
&\quad +\int_0^1 \varphi_t\left[k\left(\varphi_x+lw+\psi\right)_x+k_0l\left(w_x-l\varphi\right)-\mu_1\varphi_t-\mu_2 z\left(x,1\right)\right]dx\\
&\quad +\int_0^1 w_t\left[k_0\left(w_x-l\varphi\right)_x-lk\left(\varphi_x+lw+\psi\right)\right]dx\\
&\quad +\int_0^1 \psi_t\left[L\psi_{xx}-k\left(\varphi_x+lw+\psi\right)+\int_0^\infty \Theta\left(p\right)\phi_{xx}\left(p\right)dp-\gamma\theta_x\right]dx\\
&\quad +L\int_0^1 \psi_x\psi_{tx}dx-\frac{\xi}{\tau}\int_0^1\int_0^1 z\left(x,\rho\right)z_\rho\left(x,\rho\right)dxd\rho\\
&\quad +\int_0^1\int_0^\infty \Theta\left(p\right)\phi_x\left(p\right)\left(-\phi_p+v\right)_x dxdp\\
&\quad -\int_0^1 \theta q_x dx-\gamma\int_0^1 \theta\psi_{tx}-\beta\int_0^1 q^2dx-\int_0^1 \theta_x q dx\\
&= -\beta\int_0^1 q^2dx-\mu_1\int_0^1 u^2dx-\mu_2\int_0^1 z\left(x,1\right)udx\\
&\quad -\frac{\xi}{\tau}\int_0^1\int_0^1 z\left(x,\rho\right)z_\rho\left(x,\rho\right)d\rho dx+\int_0^1\int_0^\infty \Theta'\left(p\right)\left|\phi_x\left(x,p\right)\right|^2dpdx\\
&= -\beta\int_0^1 q^2dx-\mu_1\int_0^1 u^2dx-\mu_2\int_0^1 z\left(x,1\right)udx\\
&\quad -\frac{\xi}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}z^2\left(x,\rho\right)d\rho dx+\int_0^1\int_0^\infty \Theta'\left(p\right)\left|\phi_x\left(x,p\right)\right|^2dpdx\\
&= -\beta\int_0^1 q^2dx-\mu_1\int_0^1 u^2dx-\mu_2\int_0^1 z\left(x,1\right)udx\\
&\quad -\frac{\xi}{2\tau}\int_0^1 \left\{z^2\left(x,1\right)-z^2\left(x,0\right)\right\}dx+\int_0^1\int_0^\infty \Theta'\left(p\right)\left|\phi_x\left(x,p\right)\right|^2dpdx\\
&= -\beta\int_0^1 q^2dx-\mu_1\int_0^1 u^2dx-\mu_2\int_0^1 z\left(x,1\right)udx-\frac{\xi}{2\tau}\int_0^1 z^2\left(x,1\right)dx\\
&\quad +\frac{\xi}{2\tau}\int_0^1 u^2dx+\int_0^1\int_0^\infty \Theta'\left(p\right)\left|\phi_x\left(x,p\right)\right|^2dpdx. \qquad (2.23)
\end{aligned}
$$

Using Young's inequality , we get

$$
\int_0^1 z\left(x,1\right)u\left(x\right)dx\le\frac{1}{2}\int_0^1 [z^2\left(x,1\right)+u^2\left(x\right)]dx,
$$

We obtient

$$
\langle AU,U\rangle_H \;\le\; -\beta\int_0^1 q^2dx-\mu_3\int_0^1 u^2dx-\mu_4\int_0^1 z^2\left(x,1\right)dx+\int_0^1\int_0^\infty \Theta'\left(p\right)\left|\phi_x\left(x,p\right)\right|^2dpdx,
$$

where

$$\mu_3 = \left(\mu_1 - \frac{|\mu_2|}{2} - \frac{\xi}{2\tau}\right), \quad \mu_4 = \left(\frac{\xi}{2\tau} - \frac{|\mu_2|}{2}\right).$$

By condition (2.17), the desired result yields.

lemma 2.2. *The operator $I - A$ is surjective.*

Proof. For all

$$\mathcal{F} = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9, f_{10})^T \in \mathcal{H}.$$

We show that there exists $U \in D(A)$ such that

$$U - AU = \mathcal{F}, \tag{2.24}$$

that is

$$\begin{cases} -u + \varphi = f_1 \in H_0^1(0,1), \\ -k(\varphi_x + lw + \psi)_x - k_0 l(w_x - l\varphi) + \rho_1 u + \mu_1 u + \mu_2 z(.,1) = \rho_1 f_2 \in L^2(0,1), \\ z + \tau^{-1} z_\rho = f_3 \in L^2((0,1), H^1(0,1)), \\ -v + \psi = f_4 \in H_0^1(0,1), \\ -L\psi_{xx} + k(\varphi_x + lw + \psi) + \rho_2 v - \int_0^\infty \Theta(p)\,\phi_{xx}(p)\,dp + \gamma\theta_x = \rho_2 f_5 \in L^2(0,1), \\ -\varpi + w = f_6 \in H_0^1(0,1) \\ -k_0(w_x - l\varphi)_x + kl(\varphi_x + lw + \psi) + \rho_1 \varpi = \rho_1 f_7 \in L^2(0,1), \\ q_x + \gamma v_x + \rho_3 \theta = \rho_3 f_8 \in L^2(0,1), \\ (\beta + \alpha) q + \theta_x = \alpha f_9 \in L^2(0,1), \\ \phi + \phi_p - v = f_{10} \in L^2(0,1). \end{cases} \tag{2.25}$$

From (2.25), we define

$$\theta = \alpha \int_0^x f_9(y)\,dy - (\beta + \alpha) \int_0^x q(y)\,dy, \tag{2.26}$$

then $\theta(0,t) = 0$. Inserting

$$u = \varphi - f_1, v = \psi - f_4, \varpi = w - f_6, \tag{2.27}$$

and (2.25) into, (2.26) we get

$$\begin{cases} -k\left(\varphi_x+lw+\psi\right)_x-k_0 l\left(w_x-l\varphi\right)+(\rho_1+\mu_1+\mu_2 e^{-\tau})\varphi=h_1\in L^2\left(0.1\right), \\ -L\psi_{xx}+k\left(\varphi_x+lw+\psi\right)+\rho_2\psi-\int_0^\infty \Theta\left(p\right)\phi_{xx}\left(p\right)dp-\gamma\left(\beta+\alpha\right)q=h_2\in L^2, \\ -k_0\left(w_x-l\varphi\right)_x+kl\left(\varphi_x+lw+\psi\right)+\rho_1 w=h_3\in L^2\left(0.1\right), \\ q_x+\left(\beta+\alpha\right)\int_0^x q\left(y\right)dy-\gamma\psi_x=h_4\in L^2\left(0.1\right), \\ z+\tau^{-1}z_\rho=h_5\in L^2\left(0.1\right), \\ \phi+\phi_p-v=h_6\in L^2\left(0.1\right). \end{cases} \tag{2.28}$$

where

$$\begin{cases} h_1=\rho_1\left(f_1+f_2\right)+\left(\mu_1 f_1+\mu_2 z_0\right), \\ h_2=\rho_2\left(f_4+f_5\right)-\alpha\gamma f_9, \\ h_3=\rho_1\left(f_6+f_7\right), \\ h_4=-\gamma f_{4x}-\rho_3\left(f_8-\alpha\int_0^x f_9\left(y\right)dy\right), \\ h_5=f_3, \\ h_6=f_{10}. \end{cases} \tag{2.29}$$

Furthermore, by (2.25) we can find as

$$z\left(x,0\right)=u\left(x\right)\qquad for\ x\in\left(0,1\right), \tag{2.30}$$

Corollary 2.1. *Let f be a continuous function on an interval $I\in\mathbb{R}$, α constant and $t_0\in I$ the general solution of the scalar equation*

$$y'(t)=\alpha y(t)+f(t),$$

Is given by

$$y(t)=Ce^{\alpha t}+\int_{t_0}^t e^{\alpha(t-s)}f(s)ds.$$

where C is a constant.

So , the equation

$$z_\rho=\tau z+f_3.$$

Given by:

$$\begin{aligned} z(x,\rho)&=Ce^{-\tau\rho}+\int_0^\rho \tau e^{-\tau(\rho-p)}f_3(x,p)dp \\ &=Ce^{-\tau\rho}+\tau e^{-\tau\rho}\int_0^\rho f_3(x,p)e^{\tau p}dp, \end{aligned}$$

From (3.5), we get

$$z(x,\rho)=u(x)e^{-\tau\rho}+\tau e^{-\tau\rho}\int_0^{\rho} f_3(x,p)e^{\tau p}dp,$$

From (2.27), we have

$$z(x,\rho)=\varphi(x)e^{-\tau\rho}-f_1e^{-\tau\rho}+\tau e^{-\tau\rho}\int_0^{\rho} f_3(x,p)e^{\tau p}dp,$$

such as

$$z(x,1)=\varphi(x)e^{-\tau}+z_0(x),$$

Or $x\in(0,1)$ and

$$z_0\left(x\right)=-f_1e^{-\tau}+\tau e^{-\tau}\int_0^1 f_3\left(x,p\right)e^{\tau p}dp.$$

Same thing for the equation $(2.28)_6$ with $\phi\left(x,0\right)=0$ has a unique solution given as

$$\begin{aligned}\phi\left(x,p\right) &= \left(\int_0^x e^y\left(f_{10}\left(x,y\right)+v\left(x\right)dy\right)e^{-p}\right)\\ &= \left(\int_0^x e^y\left(f_{10}\left(x,y\right)+\psi\left(x\right)-f_4\left(x\right)dy\right)e^{-p}\right). \end{aligned}\tag{2.31}$$

To solve (2.29) let us consider

$$a\left(\left(\varphi,\psi,w,q\right),\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right)\right)=L\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right),\tag{2.32}$$

where

$$a:\left[H_0^1\left(0,1\right)\times H_0^1\left(0,1\right)\times H_0^1\left(0,1\right)\times L^2\left(0,1\right)\right]^2\longrightarrow\mathbb{R},$$

Is the bilinear form defined by

$$\begin{aligned}a\left(\left(\varphi,\psi,w,q\right),\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right)\right) &= k\int_0^1\left(\varphi_x+lw+\psi\right)\left(\widetilde{\varphi}_x+l\widetilde{w}+\widetilde{\psi}\right)dx+b\int_0^1\psi_x\widetilde{\psi}_xdx\\ &+\left(\beta+\alpha\right)\int_0^1 q\widetilde{q}dx+k_0\int_0^1\left(w_x-l\varphi\right)\left(\widetilde{w}_x-l\widetilde{\varphi}\right)dx+\rho_1\int_0^1 w\widetilde{w}dx\\ &+\rho_2\int_0^1\psi\widetilde{\psi}dx-\gamma\left(\beta+\alpha\right)\int_0^1 q\widetilde{\psi}dx+\rho_1\int_0^1\psi\widetilde{\psi}dx+\gamma\left(\beta+\alpha\right)\int_0^1\psi\widetilde{q}dx\\ &+\int_0^1\varphi\widetilde{\varphi}\left(\rho_1+\mu_1+\mu_2e^{-\tau}\right)dx+\left(\beta+\alpha\right)\int_0^1\left(\int_0^x q\left(y\right)dy\int_0^x\widetilde{q}\left(y\right)dy\right)dx.\end{aligned}$$

and

$$L:\left[H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times L^2(0,1)\right]\longrightarrow \mathbb{R},$$

Is the linear form given by

$$\begin{aligned} L\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right) &= \int_0^1 h_1\widetilde{\varphi}dx+\int_0^1 h_2\widetilde{\psi}dx+\int_0^1 h_3\widetilde{w}dx \\ &+(\alpha+\beta)\int_0^1 h_4\int_0^x \widetilde{q}(y)\,dydx. \end{aligned} \tag{2.33}$$

It is not hard to see that a is continuous and coercive and L is continuous. So by using the Lax-Milgram theorem, we find that

$$\forall\left(\widetilde{\varphi},\widetilde{\psi},\widetilde{w},\widetilde{q}\right)\in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times L^2(0,1),$$

the problem (2.32) admits a unique solution

$$(\varphi,\psi,w,q)\in H_0^1(0,1)\times H_0^1(0,1)\times H_0^1(0,1)\times L^2(0,1).$$

The existence of unique $U\in D(A)$ such that (2.18) is satisfied comes from the regularity theory for the linear elliptic equations. Then, by Lemmas 2.1 and Lemma 2.2, we conclude that A is a maximal monotone operator. Hence, by Hille-Yosida theorem we can state well-posedness result (see [22] and [29]).

Theorem 2.1. *Let* $U_0\in\mathcal{H}$, *then there exists a unique weak solution* $U\in C(\mathbb{R}^+,\mathcal{H})$ *of problem (2.15)-(2.16). Moreover, if* $U_0\in D(A)$, *then* $U\in C(\mathbb{R}^+,D(A))\cap C^1(\mathbb{R}^+,\mathcal{H})$.

Chapter 3

Exponential Stability

Here, we introduce our stabilities results for solution of (2.15)-(2.16). The energy associated with solution is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left[\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+L\psi_x^2+\rho_3\theta^2+\alpha q^2\right. \\ &\quad \left.+k\left(\varphi_x+\psi+lw\right)^2+k_0\left(w_x-l\varphi\right)^2\right]dx+\frac{\xi}{2}\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx \\ &\quad +\frac{1}{2}\int_0^1\int_0^\infty \Theta(p)\left|\eta_x^t(x,p)\right|^2 dpdx. \end{aligned} \tag{3.1}$$

To achieve our goal, we need the following Lemmas.

lemma 3.1. *Let $(\varphi,\psi,w,\theta,q,z,\eta^t)$ be the solution of (2.15)-(2.16). Then the energy functional, defined by (3.1) satisfies*

$$\begin{aligned} E'(t) &\leq -\beta\int_0^1 q^2dx-\mu_3\int_0^1\varphi_t^2dx-\mu_4\int_0^1 z^2(x,1,t)\,dx \\ &\quad +\frac{1}{2}\int_0^1\int_0^\infty \Theta'(p)\left|\eta_x^t(x,p)\right|^2 dpdx. \end{aligned} \tag{3.2}$$

Where $\mu_3=\left(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2}\right)>0$, $\mu_4=\left(\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)>0$.

Proof. Multiplying $(2.15)_1$, $(2.15)_2$, $(2.15)_3$, $(2.15)_4$, and $(2.15)_5$ by φ_t, ψ_t, w_t, θ, and q, respectively, we get

$$\begin{cases} \rho_1\varphi_{tt}(x,t)\varphi_t(x,t) - K\left(\varphi_x + lw + \psi\right)_x \varphi_t(x,t) - lk_0\left(w_x - l\varphi\right)\varphi_t(x,t) + \mu_1\varphi_t^2(x,t) \\ +\mu_2 z(x,1,t)\varphi_t(x,t) = 0, \\ \rho_2\psi_{tt}(x,t)\psi_t(x,t) - L\psi_{xx}(x,t)\psi_t(x,t) + K\left(\varphi_x + lw + \psi\right)(x,t)\psi_t(x,t) + \gamma\theta_x(x,t)\psi_t(x,t) \\ -\displaystyle\int_0^\infty \Theta\left(p\right)\eta_{xx}^t\left(x,p\right)\psi_t(x,t)dp = 0, \\ \rho_1 w_{tt}(x,t)w_t(x,t) - k_0\left(w_x - l\varphi\right)_x(x,t)w_t(x,t) + lk\left(\varphi_x + lw + \psi\right)(x,t)w_t(x,t) = 0, \\ \rho_3\theta_t(x,t)\theta(x,t) + \kappa q_x(x,t)\theta(x,t) + \gamma\psi_{tx}(x,t)\theta(x,t) = 0, \\ \alpha q_t(x,t)q(x,t) + \beta q^2(x,t) + \kappa\theta_x(x,t)q(x,t) = 0. \end{cases} \tag{3.3}$$

The integral by parts gives

$$\begin{cases} \int_0^1 \left\{\rho_1\varphi_{tt}\varphi_t + K\left(\varphi_x + lw + \psi\right)\varphi_{tx} - lk_0\left(w_x - l\varphi\right)\varphi_t + \mu_1\varphi_t^2 + \mu_2 z(x,1,t)\varphi_t\right\}dx = 0, \\ \int_0^1 \left\{\rho_2\psi_{tt}\psi_t + b\psi_x\psi_{tx} + K\left(\varphi_x + \psi\right)\psi_t - \displaystyle\int_0^\infty \Theta\left(p\right)\psi_x\psi_{xt}dp + \gamma\theta_x\psi_t\right\}dx = 0, \\ \int_0^1 \left\{\rho_1 w_{tt}w_t + K_0\left(w_x - l\varphi\right)w_{tx} + lk\left(\varphi_x + lw + \psi\right)w_t\right\}dx = 0, \\ \int_0^1 \left\{\rho_3\theta_t\theta + \kappa q_x\theta + \gamma\psi_{tx}\theta\right\}dx = 0, \\ \int_0^1 \left\{\alpha_0 q_t q + \beta q^2 + \kappa\theta_x q\right\}dx = 0. \end{cases} \tag{3.4}$$

Then we get

$$\int_0^1 \left\{\rho_1\varphi_{tt}\varphi_t + K\left(\varphi_x + lw + \psi\right)\left(\varphi_{tx} + \psi_t + lw_t\right) - k_0\left(w_x - l\varphi\right)\left(l\varphi_t - w_{xt}\right) + \mu_1\varphi_t^2 + \mu_2 z\varphi_t + \rho_2\psi_{tt}\psi_t \right.$$
$$\left. + L\psi_x\psi_{tx}\rho_3\theta_t\theta + \alpha q_t q + \beta q^2\right\}dx = 0,$$

On the other hand we have

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + \rho_3\theta^2 + \alpha q^2\right\}dx = \int_0^1 \left\{\rho_1\varphi_{tt}\varphi_t + \rho_2\psi_{tt}\psi_t + \rho_1 w_{tt}w_t + \rho_3\theta_t\theta + \alpha q_t q\right\}dx,$$

and

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{K\left(\varphi_x + lw + \psi\right)^2 + k_0\left(w_x - l\varphi\right) + L\psi_x^2\right\}dx$$
$$= \int_0^1 \left\{K\left(\varphi_x + lw + \psi\right)\left(\varphi_{tx} + lw_t + \psi_t\right) + k_0\left(w_x - l\varphi\right)\left(w_{xt} - l\varphi_t\right) + L\psi_x\psi_{tx} + q\right\}dx,$$

Then, we find the following legality

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1 w_t^2+\rho_3\theta^2+\alpha q^2\right\}dx+\frac{1}{2}\frac{d}{dt}\int_0^1\left\{K\left(\varphi_x+lw+\psi\right)^2+L\psi_x^2+k_0\left(w_x-l\varphi\right)\right\}dx$$
$$=-\beta\int_0^1 q^2dx-\mu_1\int_0^1\varphi_t^2(x,t)dx-\mu_2\int_0^1\varphi_t(x,t)z(x,1,t)dx. \quad (3.5)$$

Multiply the last equation in (2.15) par$(\xi/\tau)z$ and intégrate the result $(0,1)\times(0,1)$ respectively of ρ and x, we obtain

$$\begin{aligned}\frac{\xi}{2}\frac{d}{dt}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx &= \frac{-\xi}{\tau}\int_0^1\int_0^1 zz_\rho(x,\rho,t)d\rho dx\\ &=\frac{-\xi}{2\tau}\int_0^1\int_0^1\frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho dx\\ &=\frac{\xi}{2\tau}\int_0^1\left\{z^2(x,0,t)-z^2(x,1,t)\right\}dx, \qquad (3.6)\end{aligned}$$

of (3.1), (3.5) and (3.6) we have

$$\frac{dE(t)}{dt}=-\beta\int_0^1 q^2dx-\left(\mu_1-\frac{\xi}{2\tau}\right)\int_0^1\varphi_t^2(x,t)dx-\frac{\xi}{2\tau}\int_0^1 z^2(x,1,t)dx-\mu_2\int_0^1\varphi_t(x,t)z(x,1,t)dx, \quad (3.7)$$

We use Young's inequality, (3.7) rewritten as

$$\frac{dE(t)}{dt}\le-\beta\int_0^1 q^2dx-\left(\mu_1-\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)\int_0^1\varphi_t^2(x,t)dx-\left(\frac{\xi}{2\tau}-\frac{\mu_2}{2}\right)\int_0^1 z^2(x,1,t)dx,$$

Then, using (3.1), we deduce that there is $C>0$ such as

$$\frac{dE(t)}{dt}\le-\delta\int_0^1 q^2dx-C\left(\int_0^1\varphi_t^2(x,t)dx+\int_0^1 z^2(x,1,t)dx\right).$$

The last inequality implies that the energy E is a non-increasing function with respect to t.

lemma 3.2. *Let $(\varphi,\psi,w,\theta,q,z,\eta^t)$ be the solution of (2.15)-(2.16). Then the functional*

$$F_1(t):=\alpha\rho_3\int_0^1\theta\int_0^x q(y)\,dydx, \quad (3.8)$$

satisfies, for any $\varepsilon_1>0$, the estimate

$$F_1'(t)\le-\frac{\rho_3\kappa}{2}\int_0^1\theta^2dx+\varepsilon_1\int_0^1\psi_t^2dx+c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^1 q^2dx. \quad (3.9)$$

Proof. Taking the derivative of F_1, using (2.15)$_5$, (2.15)$_6$, we get

$$F_1'(t) = -\rho_3\kappa\int_0^1 \theta^2 dx + \alpha\kappa\int_0^1 q^2 dx + \alpha\gamma\int_0^1 q\psi_t dx - \beta\rho_3\int_0^1 \theta\int_0^x q(y)\,dydx. \tag{3.10}$$

Thanks to Cauchy–Schwartz and Young's inequalities with $\varepsilon_1 > 0$ to get (3.9).

lemma 3.3. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (2.15)-(2.16). Then the functional*

$$F_2(t) := \frac{\rho_2\rho_3}{\gamma}\int_0^1 \psi_t\int_0^x \theta(y)\,dydx, \tag{3.11}$$

satisfies, for any $\varepsilon_1, \varepsilon_2 > 0$, the estimate

$$\begin{aligned} F_2'(t) \leq\ & -\frac{\rho_2}{2}\int_0^1 \psi_t^2 dx + \varepsilon_2\int_0^1 (\varphi_x+\psi+lw)^2 dx + \varepsilon_3\int_0^1 \psi_x^2 dx \\ & +c\left(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right)\int_0^1 \theta^2 dx + c\int_0^1 q^2 dx \\ & +c\int_0^1\int_0^\infty \Theta(p)\left|\eta_x^t(x,p)\right|^2 dpdx \\ & +\frac{\rho_2 k}{\gamma}q(0,t) + \rho_2\psi_t(0,t) + \frac{L\rho_3}{\gamma}\psi_x(1,t)\int_0^1 \theta(y,t)dy \\ & -\rho_3\theta(1,t)\int_0^1 \theta(y,t)dy + \frac{\rho_3}{\gamma}\int_0^\infty \Theta(p)\eta_x^t(1,p)dp\int_0^1 \theta(y,t)dy. \end{aligned} \tag{3.12}$$

Proof. By differentiating F_2, then exploiting the (2.15)$_2$, (2.15)$_4$ we get

$$\begin{aligned} F_2'(t) =\ & -\rho_2\int_0^1 \psi_t^2 dx - \frac{\rho_2\kappa}{\gamma}\int_0^1 q\psi_t dx + \rho_3\int_0^1 \theta^2 dx - \frac{L\rho_3}{\gamma}\int_0^1 \theta\psi_x dx \\ & -\frac{k\rho_3}{\gamma}\int_0^1 (\varphi_x+\psi+lw)\int_0^x \theta(y)\,dydx \\ & +\frac{\rho_3}{\gamma}\int_0^1\int_0^\infty \Theta(p)\,\theta\eta_x^t(x,p)\,dpdx. \end{aligned} \tag{3.13}$$

We obtain (3.12) by applying the Cauchy–Schwarz and Young's inequalities.

lemma 3.4. *Let $(\varphi, \psi, w, \theta, q, z, \eta^t)$ be the solution of (2.15)-(2.16). Then the functional*

$$F_3(t) := \rho_2\int_0^1 \psi\psi_t dx, \tag{3.14}$$

satisfies

$$
\begin{aligned}
F_3'(t) \leq\; & -\frac{L}{2}\int_0^1 \psi_x^2 dx + \rho_2 \int_0^1 \psi_t^2 dx \\
& + \frac{3k^2}{2cL}\int_0^1 (\varphi_x + \psi + lw)^2 dx + c\int_0^1 \theta^2 dx \\
& + c\int_0^1 \int_0^\infty \Theta(p)\left|\eta_x^t(x,p)\right|^2 dp dx.
\end{aligned} \tag{3.15}
$$

Proof. Taking the derivative of F_3 and using $(2.15)_3$, it follows that

$$
\begin{aligned}
F_3'(t) =\; & -L\int_0^1 \psi_x^2 dx + \rho_2\int_0^1 \psi_t^2 dx + \gamma\int_0^1 \psi_x \theta dx - k\int_0^1 \psi(\varphi_x + \psi + lw) dx \\
& + \int_0^1 \psi_x(x)\int_0^\infty \Theta(p)\eta_x^t(x,p) dp dx.
\end{aligned} \tag{3.16}
$$

Thanks to Young and Poincaré's inequalities, to get (3.15).

lemma 3.5. *Let* $(\varphi, \psi, w, \theta, q, z, \eta^t)$ *be the solution of* (2.15)-(2.16)*. Then the functional*

$$
F_4(t) := -\rho_1\int_0^1 \varphi_t(w_x - l\varphi) dx - \rho_1\int_0^1 w_t(\varphi_x + \psi + lw) dx, \tag{3.17}
$$

satisfies the estimate

$$
\begin{aligned}
F_4'(t) \leq\; & -\frac{lk_0}{2}\int_0^1 (w_x - l\varphi)^2 dx - \frac{l\rho_1}{2}\int_0^1 w_t^2 dx + c\int_0^1 \varphi_t^2 dx \\
& + c\int_0^1 \psi_t^2 dx + lk\int_0^1 (\varphi_x + \psi + lw)^2 dx + c\int_0^1 z^2(x,1,t) dx.
\end{aligned} \tag{3.18}
$$

Proof. By differentiating F_4 and using $(2.15)_1$, $(2.15)_4$, we obtain

$$
\begin{aligned}
F_4'(t) =\; & -lk_0\int_0^1 (w_x - l\varphi)^2 dx - l\rho_1\int_0^1 w_t^2 dx + l\rho_1\int_0^1 \varphi_t^2 dx \\
& + lk\int_0^1 (\varphi_x + \psi + lw)^2 dx - \rho_1\int_0^1 \psi_t w_t dx \\
& + \mu_1\int_0^1 \varphi_t(w_x - l\varphi) dx + \mu_2\int_0^1 z(x,1,t)(w_x - l\varphi) dx.
\end{aligned} \tag{3.19}
$$

We obtain (3.18) by applying Young's inequality.

lemma 3.6. *Let* $(\varphi, \psi, w, \theta, q, z, \eta^t)$ *be the solution of* (2.15)-(2.16) *and let* (2.7) *holds.*

Then the functional

$$\begin{aligned}F_5(t) \quad : \quad &= \rho_2 \int_0^1 \psi_t(\varphi_x+\psi+lw)\,dx + \frac{L\rho_1}{k}\int_0^1 \varphi_t\psi_x dx \\ &+\frac{L\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \theta\varphi_t dx - \frac{L\kappa}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q(\varphi_x+\psi+lw)\,dx \\ &-\frac{Ll^2\rho_2}{k_0}\int_0^1 \psi\psi_t dx + \frac{Ll\rho_1}{k_0}\int_0^1 \psi w_t dx - \frac{\rho_1}{k}\int_0^1 \varphi_t \int_0^\infty \Theta(p)\eta_x^t(p)dpdx,\end{aligned} \tag{3.20}$$

satisfies, for any $\varepsilon_4, \varepsilon_5, \varepsilon_6 > 0$, *the estimate*

$$\begin{aligned}F_5'(t) \quad \leq \quad & -\frac{k}{2}\int_0^1 (\varphi_x+\psi+lw)^2 dx + 2\varepsilon_4 \int_0^1 w_t^2 dx + \left(\frac{L^2l^2}{k}+4\varepsilon_6\right)\int_0^1 \psi_x^2 dx \\ &+2\varepsilon_5\int_0^1 (w_x-l\varphi)^2 dx + c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1 \psi_t^2 dx + c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^1 q^2 dx \\ &+c\left(1+\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6}\right)\int_0^1 \theta^2 dx + c\left(1+\frac{1}{\varepsilon_6}\right)\int_0^1 z^2(x,1,t)dx \\ &+c\left(1+\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6}\right)\int_0^1\int_0^\infty \Theta(p)\left|\eta_x^t(x,p)\right|^2 dpdx + c\left(1+\frac{1}{\varepsilon_6}\right)\int_0^1 \varphi_t^2 dx \\ &-c\int_0^1\int_0^\infty \Theta'(p)\left|\eta_x^t(x,p)\right|^2 dpdx + \frac{L\tilde{\eta}}{\gamma\alpha}\int_0^1 \theta_x(\varphi_x+\psi+lw)\,dx.\end{aligned} \tag{3.21}$$

where $\tilde{\eta} = \left(\kappa^2 - \frac{\alpha k \rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right) - \frac{\gamma^2\alpha}{L}$.

Proof. A simple differentiation of F_5 gives

$$\begin{aligned}F_5'(t) \quad = \quad & \rho_2\int_0^1 \psi_{tt}(\varphi_x+\psi+lw)\,dx + \rho_2\int_0^1 \psi_t(\varphi_x+\psi+lw)_t\,dx \\ &+\frac{L\rho_1}{k}\int_0^1 \varphi_{tt}\psi_x dx - \frac{L\rho_1}{k}\int_0^1 \varphi_t\psi_{xt}dx + \frac{L\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \theta_t\varphi_t dx \\ &+\frac{L\rho_3}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \theta\varphi_{tt}dx - \frac{L\kappa}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q_t(\varphi_x+\psi+lw)\,dx \\ &-\frac{L\kappa}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q(\varphi_x+\psi+lw)_t\,dx - \frac{Ll^2\rho_2}{k_0}\int_0^1 \psi_t^2 dx \\ &-\frac{Ll^2\rho_2}{k_0}\int_0^1 \psi_{tt}\psi dx + \frac{Ll\rho_1}{k_0}\int_0^1 w_{tt}\psi dx + \frac{Ll\rho_1}{k_0}\int_0^1 w_t\psi dx \\ &+\frac{Ll^2\rho_2}{k_0}\int_0^1 \psi_x(x)\int_0^\infty \Theta(p)\,\eta_x^t(x,p)\,dpdx \\ &-\frac{\rho_1}{k}\int_0^1 \varphi_{tt}\int_0^\infty \Theta(p)\eta_x^t(p)dpdx - \frac{\rho_1}{k}\int_0^1 \varphi_t\frac{d}{dt}\left\{\int_0^\infty \Theta(p)\eta_x^t(p)dp\right\}dx.\end{aligned} \tag{3.22}$$

Using (2.15)-(2.16), the terms in (3.22) take the forme

$$\rho_2 \int_0^1 \psi_{tt} \left(\varphi_x + \psi + lw\right) dx = -k \int_0^1 \left(\varphi_x + \psi + lw\right)^2 dx - \gamma \int_0^1 \theta_x \left(\varphi_x + \psi + lw\right) dx \\ - L \int_0^1 \psi_x \left(\varphi_x + \psi + lw\right)_x dx + \int_0^1 \left(\varphi_x + \psi + lw\right) \int_0^\infty \Theta(p) \eta^t_{xx}(p) dp dx, \quad (3.23)$$

and

$$\rho_1 \int_0^1 \varphi_{tt} \psi_x dx = k \int_0^1 \psi_x \left(\varphi_x + \psi + lw\right)_x dx + k_0 l \int_0^1 \left(w_x - l\varphi\right) \psi_x dx - \mu_1 \int_0^1 \psi_x \varphi_t dx - \mu_2 \int_0^1 z(x,1,t) \psi_x dx,$$

and

$$\rho_3 \int_0^1 \theta_t \varphi_t dx = \kappa \int_0^1 q \varphi_{xt} dx + \gamma \int_0^1 \psi_t \varphi_{xt} dx,$$

and

$$\int_0^1 \theta \varphi_{tt} dx = -\frac{k}{\rho_1} \int_0^1 \theta_x \left(\varphi_x + \psi + lw\right) dx + \frac{lk_0}{\rho_1} \int_0^1 \theta \left(w_x - l\varphi\right) dx - \frac{\mu_1}{\rho_1} \int_0^1 \theta \varphi_t dx - \frac{\mu_2}{\rho_1} \int_0^1 \theta z(x,1,t) dx,$$

and

$$-\int_0^1 q_t \left(\varphi_x + \psi + lw\right) dx = \frac{\beta}{\alpha} \int_0^1 q \left(\varphi_x + \psi + lw\right) dx + \frac{\kappa}{\alpha} \int_0^1 \theta_x \left(\varphi_x + \psi + lw\right) dx,$$

and

$$-\rho_2 \int_0^1 \psi_{tt} \psi dx = L \int_0^1 \psi_x^2 dx + k \int_0^1 \psi \left(\varphi_x + \psi + lw\right) dx - \gamma \int_0^1 \theta \psi_x dx - \int_0^1 \psi \int_0^\infty \Theta(p) \eta^t_{xx}(p) dp dx,$$

and

$$-\frac{\rho_1}{k} \int_0^1 \varphi_{tt} \int_0^\infty \Theta(p) \eta^t_x(p) dp dx = -\int_0^1 \left(\varphi_x + \psi + lw\right)_x \int_0^\infty \Theta(p) \eta^t_x(p) dp dx + \frac{\mu_1}{k} \int_0^1 \varphi_t \int_0^\infty \Theta(p) \eta^t_x(p) dp dx \\ -\frac{k_0 l}{k} \int_0^1 \left(\omega_x - l\varphi\right) \int_0^\infty \Theta(p) \eta^t_x(p) dp dx + \frac{\mu_2}{k} \int_0^1 z(x,1,t) \int_0^\infty \Theta(p) \eta^t_x(p) dp dx,$$

and

$$-\frac{\rho_1}{k} \int_0^1 \varphi_t \frac{d}{dt} \left\{ \int_0^\infty \Theta(p) \eta^t_x(p) dp \right\} dx = -\frac{\rho_1}{k} \int_0^1 \varphi_t \left\{ \Theta_0 \psi_t + \int_0^\infty \Theta'(s) \eta^t_x(p) dp \right\} dx,$$

and

$$\rho_1 \int_0^1 w_{tt} \psi dx = -k_0 \int_0^1 \psi_x \left(w_x - l\varphi\right) dx - kl \int_0^1 \psi \left(\varphi_x + \psi + lw\right) dx. \quad (3.24)$$

Substituting (3.23)-(3.24) into (3.22), by (2.7), to get

$$\begin{aligned} F_5'(t) &= -k\int_0^1 (\varphi_x+\psi+lw)^2\,dx + \left(\rho_2 - \frac{Ll^2\rho_2}{k_0}\right)\int_0^1 \psi_t^2 dx \\ &+ \left(l\rho_2+\frac{Ll\rho_1}{k_0}\right)\int_0^1 \psi_t w_t dx + \frac{L\tilde{\eta}}{\alpha\gamma}\int_0^1 \theta_x(\varphi_x+\psi+lw)\,dx \\ &-\frac{L}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q\psi_t dx - \frac{bl}{\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q w_t dx \\ &-\frac{L\mu_1}{k}\int_0^1 \varphi_t\psi_x dx - \frac{L\mu_2}{k}\int_0^1 \psi_x z(x,1,t)dx \\ &+\frac{Llk_0\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \theta\,(w_x-l\varphi)\,dx - \frac{\gamma L l^2}{k_0}\int_0^1 \theta\psi_x dx \\ &-\frac{L\mu_1\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \varphi_t\theta dx - \frac{L\mu_2\rho_3}{\gamma\rho_1}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 \theta z(x,1,t)dx \\ &+\frac{L\beta\kappa}{\alpha\gamma}\left(\frac{\rho_1}{k}-\frac{\rho_2}{L}\right)\int_0^1 q\,(\varphi_x+\psi+lw)\,dx + \frac{L^2l^2}{k_0}\int_0^1 \psi_x^2 dx \\ &+Ll\left(\frac{k_0}{k}-1\right)\int_0^1 \psi_x\,(w_x-l\varphi)\,dx \\ &+\frac{Ll^2\rho_2}{k_0}\int_0^1 \psi_x\int_0^\infty \Theta(p)\,\eta_x^t(p)dpdx + \frac{\mu_1}{k}\int_0^1 \varphi_t\int_0^\infty \Theta(p)\,\eta_x^t(p)dpdx \\ &-\frac{lk_0}{k}\int_0^1 (\omega_x-l\varphi)\int_0^\infty \Theta(p)\,\eta_x^t(p)dpdx \\ &+\frac{\mu_2}{k}\int_0^1 z(x,1,t)\int_0^\infty \Theta(p)\,\eta_x^t(p)dpdx \\ &+\frac{\rho_1\Theta_0}{k}\int_0^1 \varphi_t\psi_t dx + \frac{\rho_1}{k}\int_0^1 \varphi_t\int_0^\infty \Theta'(p)\eta_x^t(p)dpdx. \end{aligned} \tag{3.25}$$

We obtain (3.21) by Cauchy–Schwarz and Young's inequalities since $k = k_0$.

lemma 3.7. *Let $(\varphi,\psi,w,\theta,q,z,\eta^t)$ be the solution of (2.15)-(2.16). Then, we define the functional*

$$F_6(t) := \int_0^1\int_0^1 e^{-2\tau\rho} z^2(x,\rho,t)\,d\rho dx, \tag{3.26}$$

Then the following result holds.

$$F_6'(t) \le -F_6(t) - \frac{c_1}{2\tau}\int_0^1 z^2(x,1,t)\,dx + \frac{1}{2\tau}\int_0^1 \varphi_t^2(x,t)\,dx. \tag{3.27}$$

where $c > 0$.

Proof. Differentiating (3.26) with respect to t and using the equation (2.10), we have

$$\begin{aligned}\frac{d}{dt}\left(\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx\right) &= -\frac{1}{\tau}\int_0^1\int_0^1 e^{-2\tau\rho}zz_\rho(x,\rho,t)\,d\rho dx \\ &= -\int_0^1\int_0^1 e^{-2\tau\rho}z^2(x,\rho,t)\,d\rho dx \\ &= -\frac{1}{2\tau}\int_0^1\int_0^1 \frac{\partial}{\partial\rho}\left(e^{-2\tau\rho}z^2(x,\rho,t)\right)d\rho dx.\end{aligned} \tag{3.28}$$

This implies that there exists a positive constant c_1 such that (3.27) holds.

Let

$$E(t) = E(\varphi,\psi,w,\theta,q,z,\eta^t) = E_1(t),$$

and

$$E_2(t) = E(\varphi_t,\psi_t,w_t,\theta_t,q_t,z_t,\eta_t^t).$$

The main result is given in the next Theorem.

Theorem 3.1. *Assume that (2.4)–(2.6) hold, and (2.7). Then the energy functional (3.1) satisfies,* $\forall t\geq 0$

$$E(t) \leq \lambda_0 e^{-\lambda_1 t} \qquad ,if \quad \tilde{\eta}=0 \tag{3.29}$$

$$E(t) \leq C(E_1(0)+E_2(0))t^{-1} \quad ,if \quad \tilde{\eta}\neq 0 \tag{3.30}$$

where the positive constant λ_0 is directly depending on initial data and the uniform constant λ_1 is depending only on the coefficients of the system.

Proof. We define a Lyapunov functional

$$\mathcal{L}(t) := NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + F_4(t) + N_5F_5(t) + F_6(t), \tag{3.31}$$

where $N, N_1, N_2, N_3, N_5 > 0$.

By differentiating of (3.31), and using (3.2), (3.9), (3.12), (3.15), (3.18), (3.21), and (3.27), we have

$$\begin{aligned}\mathcal{L}'(t) \leq & -\left[\beta N - c_1N_1\left(1+\frac{1}{\varepsilon_1}\right) - cN_2 - c\left(1+\frac{1}{\varepsilon_4}\right)N_5\right]\int_0^1 q^2dx \\ & -\left[N\mu_3 - c\left(1+\frac{1}{\varepsilon_6}\right)N_5 - c\right]\int_0^1 \varphi_t^2dx\end{aligned}$$

$$\begin{aligned}
&-\left[N\mu_4-c\left(1+\frac{1}{\varepsilon_6}\right)N_5-c\right]\int_0^1 z^2\left(x,1,t\right)dx\\
&-\left[\frac{N_1\rho_3\kappa}{2}-cN_2\left(1+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}\right)-cN_3-c\left(1+\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6}\right)N_5\right]\int_0^1\theta^2dx\\
&-\left[N_2\frac{\rho_2}{2}-\varepsilon_1N_1-\rho_2N_3-c\left(1+\frac{1}{\varepsilon_4}\right)N_5-c\right]\int_0^1\psi_t^2dx\\
&-\left[\frac{k}{2}N_5-\varepsilon_2N_2-\frac{3k^2}{2cL}N_3-c\right]\int_0^1\left(\varphi_x+lw+\psi\right)^2dx\\
&-\left[\frac{L}{2}N_3-\varepsilon_3N_2-4\varepsilon_6N_5-\frac{L^2l^2}{k}N_5\right]\int_0^1\psi_x^2dx\\
&-\left[\frac{l\rho_1}{2}-2\varepsilon_4N_5\right]\int_0^1 w_t^2dx\\
&-\left[\frac{lk_0}{2}-2\varepsilon_5N_5\right]\int_0^1\left(w_x-l\varphi\right)^2dx\\
&+\left[cN_2+cN_3+c\left(1+\frac{1}{\varepsilon_5}+\frac{1}{\varepsilon_6}\right)N_5\right]\int_0^1\int_0^\infty\Theta\left(p\right)\left|\eta_x^t\left(x,p\right)\right|^2dpdx\\
&+\left[\frac{N}{2}-cN_5\right]\int_0^1\int_0^\infty\Theta'(p)\left|\eta_x^t\left(x,p\right)\right|^2dpdx\\
&+\left[\frac{L\tilde{\eta}}{\alpha\gamma}N_5\right]\int_0^1\theta_x(\varphi_x+lw+\psi)dx-F_6\left(t\right).
\end{aligned}\tag{3.32}$$

By setting

$$\varepsilon_1=\frac{\rho_2N_2}{4N_1},\varepsilon_2=\frac{kN_5}{8N_2},\varepsilon_3=\frac{LN_3}{8N_2},\varepsilon_6=\frac{LN_3}{32N_5},\varepsilon_4=\frac{l\rho_1}{8N_5},\varepsilon_5=\frac{lk_0}{8N_5},N_3=\frac{cL}{6k}N_5,$$

we obtain

$$\begin{aligned}
\mathcal{L}'\left(t\right)\ \le\ &-\left[\beta N-c_1N_1\left(1+\frac{N_1}{N_2}\right)-cN_2-c\left(1+N_5\right)N_5\right]\int_0^1 q^2dx\\
&-\left[N\mu_3-c\left(1+\frac{N_5}{N_3}\right)N_5-c\right]\int_0^1\varphi_t^2dx-\left[N\mu_4-c\left(1+\frac{N_5}{N_3}\right)N_5-c\right]\int_0^1 z^2\left(x,1,t\right)dx\\
&-\Big[\frac{N_1\rho_3\kappa}{2}-cN_2\left(1+\frac{N_2}{N_5}+\frac{N_2}{N_3}\right)-cN_3-c\left(1+N_5+\frac{N_5}{N_3}\right)N_5\Big]\int_0^1\theta^2dx\\
&-\left[N_2\frac{\rho_2}{4}-\rho_2N_3-c\left(1+N_5\right)N_5-c\right]\int_0^1\psi_t^2dx-\left[\frac{k}{8}N_5-c\right]\int_0^1\left(\varphi_x+lw+\psi\right)^2dx\\
&-\left[\frac{L}{4}(1-l^2\frac{24}{c})N_3\right]\int_0^1\psi_x^2dx-\left[\frac{l\rho_1}{4}\right]\int_0^1 w_t^2dx+\left[\frac{L\tilde{\eta}}{\alpha\gamma}N_5\right]\int_0^1\theta_x(\varphi_x+lw+\psi)dx\\
&+\left[cN_2+cN_3+c\left(1+N_5+\frac{N_5}{N_3}\right)N_5\right]\int_0^1\int_0^\infty\Theta\left(p\right)\left|\eta_x^t\left(x,p\right)\right|^2dpdx\\
&+\left[\frac{N}{2}-cN_5\right]\int_0^1\int_0^\infty\Theta'(s)\left|\eta_x^t\left(x,p\right)\right|^2dpdx-\left[\frac{lk_0}{4}\right]\int_0^1\left(w_x-l\varphi\right)^2dx-F_6\left(t\right).
\end{aligned}\tag{3.33}$$

We need now to choose carefully the constants. We start by choosing N_5 large enough such that

$$\frac{k}{8}N_5 - c > 0,$$

we fixed N_5, and choosing l small enough such that

$$1 - l^2\frac{24}{c} > 0,$$

Moreover, we pick N_2 large enough so that

$$N_2\frac{\rho_2}{4} - \rho_2 N_3 - c\left(1 + N_5\right)N_5 - c > 0,$$

we take N_1 large enough such that

$$N_1\frac{\rho_3\kappa}{2} - cN_2\left(1 + \frac{N_2}{N_5} + \frac{N_2}{N_3}\right) - cN_3 - c\left(1 + N_5 + \frac{N_5}{N_3}\right)N_5 > 0.$$

On the other hand, if we let

$$\mathfrak{L}\left(t\right) = N_1 F_1\left(t\right) + N_2 F_2\left(t\right) + N_3 F_3\left(t\right) + F_4\left(t\right) + N_5 F_5(t) + F_6(t),$$

then

$$\begin{aligned} \left|\mathfrak{L}\left(t\right)\right| \leq\ & \rho_3\alpha N_1\int_0^1\left|\theta\int_0^x q(y)dy\right|dx + \frac{\rho_2\rho_3}{\gamma}N_2\int_0^1\left|\psi_t\int_0^x\theta(y)dy\right|dx \\ & +\rho_2 N_3\int_0^1\left|\psi_t\psi\right|dx + \rho_1\int_0^1\left|\varphi_t(w_x - l\varphi)\right|dx \\ & +\rho_2\int_0^1\left|w_t(\varphi_x + lw + \psi)\right|dx + N_5\rho_2\int_0^1\left|\psi_t\left(\varphi_x + \psi + lw\right)\right|dx \\ & +\frac{L\rho_1}{k}N_5\int_0^1|\varphi_t\psi_x|dx + \frac{L\rho_3}{\gamma}\left|\frac{\rho_1}{k} - \frac{\rho_2}{L}\right|N_5\int_0^1|\theta\varphi_t|dx \\ & +\frac{L\kappa}{\gamma}\left|\frac{\rho_1}{k} - \frac{\rho_2}{L}\right|N_5\int_0^1\left|q\left(\varphi_x + \psi + lw\right)\right|dx \\ & +\frac{\rho_1}{k}N_5\int_0^1\left|\varphi_t\int_0^\infty\Theta(p)\eta_x^t(p)dp\right|dx + \int_0^1\int_0^1 e^{-2\tau\rho}z^2\left(x,\rho,t\right)d\rho dx \\ & +\frac{Ll^2\rho_2}{k_0}N_5\int_0^1|\psi\psi_t|dx + \frac{Ll\rho_1}{k_0}N_5\int_0^1|\psi w_t|dx. \end{aligned} \tag{3.34}$$

Thanks to Young's, Cauchy-Schwartz and Poincaré's inequalities, to get

$$\begin{aligned}|\mathfrak{L}(t)| &\leq c\int_0^1\left(\psi_t^2+\psi_x^2+\varphi_t^2+(\varphi_x+lw+\psi)^2+(w_x-l\varphi)^2+\theta^2+q^2\right)dx\\ &\quad +c\int_0^1\int_0^\infty \Theta(p)|\eta_x^t(x,p)|^2dp+c\int_0^1\int_0^1 z^2(x,\rho,t)\,d\rho dx\leq cE(t).\end{aligned}$$

Then

$$|\mathfrak{L}(t)|=|\mathcal{L}(t)-NE(t)|\leq cE(t),$$

that is

$$(N-c)E(t)\leq\mathcal{L}(t)\leq(N+c)E(t). \tag{3.35}$$

Choosing N large enough such that

$$N-c>0,\beta N-c>0,N\mu_3-c>0,N\mu_4-c>0,\frac{N}{2}-c>0,$$

we get

$$c_2E(t)\leq\mathcal{L}(t)\leq c_3E(t),\forall t\geq0, \tag{3.36}$$

we obtain

$$\begin{aligned}\mathcal{L}'(t) &\leq -k_1E(t)+\alpha_2\int_0^1\int_0^\infty\Theta(p)|\eta_x^t(x,p)|^2dp\\ &\quad +\alpha_1\int_0^1\theta_x(\varphi_x+lw+\psi)dx,\quad\forall t\geq0,\end{aligned} \tag{3.37}$$

for some $k_1,c_2,c_3,\alpha_2>0$, and $\alpha_1=N_5\dfrac{L\tilde{\eta}}{\gamma\alpha}$.

Case 1: If $\tilde{\eta}=0$, in this case, (3.37) takes the from

$$\mathcal{L}'(t)\leq-k_1E(t)+\alpha_2\int_0^1\int_0^\infty\Theta(p)|\eta_x^t(x,p)|^2dp,\quad\forall t\geq0, \tag{3.38}$$

The last term in (3.38) is estimated as following, using (2.5), we have

$$\begin{aligned}\alpha_2\int_0^1\int_0^\infty\Theta(p)|\eta_x^t(x,p)|^2dp &= \frac{\alpha_2}{\zeta}\int_0^1\int_0^\infty\zeta\Theta(p)|\eta_x^t(x,p)|^2dp\\ &\leq -\frac{\alpha_2}{\zeta}\int_0^1\int_0^t\Theta'(p)|\eta_x^t(x,p)|^2dp\\ &\leq -\frac{2\alpha_2}{\zeta}E(t).\end{aligned}$$

Thus, (3.38) becomes

$$\mathcal{L}'(t) \leq -k_1 E(t) - \frac{2\alpha_2}{\zeta} E'(t), \forall t \geq 0,$$

which can be rewritten as

$$\left(\mathcal{L}(t) + \frac{2\alpha_2}{\zeta} E(t)\right)' \leq -k_1 E(t), \forall t \geq 0.$$

By exploiting (3.36), we notice that

$$\mathcal{R}(t) = \mathcal{L}(t) + \frac{2\alpha_2}{\zeta} E(t) \sim E(t), \tag{3.39}$$

Consequently, for some positive constant λ_1, we obtain

$$\mathcal{R}'(t) \leq -\lambda_1 \mathcal{R}(t), \forall t \geq 0. \tag{3.40}$$

where $\lambda_1 = \frac{k_1}{c_2}$.

Finally, the integration of (3.40) and by (3.36) give (3.29).

Case 2: if, $\tilde{\eta} \neq 0$, and

$$|\tilde{\eta}| < \frac{2kk_1\gamma\alpha}{N_5\eta_2 L\rho_1}, \tag{3.41}$$

Then

$$\begin{aligned} E_2'(t) \quad \leq \quad & -\beta \int_0^1 q_t^2 dx - \mu_3 \int_0^1 \varphi_{tt}^2 dx - \mu_4 \int_0^1 z_t^2(x,1,t)\, dx \\ & + \frac{1}{2} \int_0^1 \int_0^\infty \Theta'(p) \left|\eta_{tx}^t(x,p)\right|^2 dp dx. \end{aligned} \tag{3.42}$$

The last term in (3.37), by using $(2.15)_1$, and Young's inequality, and by setting $K = \dfrac{\rho_1|\alpha_1|}{k}$ as follows

$$\begin{aligned} \alpha_1 \int_0^1 \theta_x(\varphi_x + lw + \psi) dx \quad = \quad & -\frac{\alpha_1\rho_1}{k} \int_0^1 \theta\varphi_{tt} dx + \frac{k_0 l\alpha_1}{k} \int_0^1 (w_x - l\varphi)\theta dx \\ & -\frac{\alpha_1\mu_1}{k} \int_0^1 \theta\varphi_t dx - \frac{\mu_2\alpha_1}{k} \int_0^1 z(x,1,t)\theta dx \\ = \quad & \frac{K}{2} \int_0^1 \varphi_{tt}^2 dx + \frac{K}{2} \int_0^1 \{\theta^2 d + \delta_1\theta^2 + \delta_5\theta^2 + \delta_3\theta^2\} dx \\ & + \frac{K}{2}\delta_2 \int_0^1 (w_x - l\varphi)^2 dx + \frac{K}{2}\delta_4 \int_0^1 \varphi_t^2 dx + \frac{K}{2}\delta_6 \int_0^1 z^2(x,1,t) dx. \end{aligned} \tag{3.43}$$

then (3.37)

$$\begin{aligned}\mathcal{L}'(t) &\le -k_2E_1(t)+\alpha_2\int_0^1\int_0^\infty \Theta(p)|\eta_x^t(x,p)|^2dp\\ &\quad+\frac{K}{2}\int_0^1 \varphi_{tt}^2dx+\frac{K}{2}\delta_6\int_0^1 z^2(x,1,t)dx,\end{aligned}\tag{3.44}$$

where

$$\begin{aligned}k_2 &= k_1-\frac{K}{2}\left(\frac{(1+\delta_1+\delta_3+\delta_5)}{\rho_3}+\frac{\delta_2}{k_0}+\frac{\delta_4}{\rho_1}\right)\\ &= k_1-\frac{K}{2}\eta_2,\end{aligned}$$

and by (3.41) we have $k_2=k_1-\dfrac{K}{2}\eta_2>0$,

Let

$$G(t)=\mathcal{L}(t)+N_7(E_1(t)+E_2(t)),\tag{3.45}$$

With (3.36), we obtain

$$G(t) \le c_1E_1(t)+N_7(E_1(t)+E_2(t)),$$

It is not hard to see that

$$m_1(E_1(t)+E_2(t))\le G(t)\le m_2(E_1(t)+E_2(t)),\tag{3.46}$$

where $m_1,m_2>0$. By using (3.44) and (3.45), we obtain

$$\begin{aligned}G'(t) &= \mathcal{L}'(t)+N_7(E_1'(t)+E_2'(t))\\ &\le -k_2E_1(t)+\alpha_2\int_0^1\int_0^\infty \Theta(p)|\eta_x^t(x,p)|^2dp-\left(\mu_3N_7-\frac{K}{2}\right)\int_0^1\varphi_{tt}^2dx-\left(\mu_4N_7-\frac{K}{2}\right)\int_0^1 z^2(x,1,t)dx\end{aligned}$$

we choose N_7 large enough, such that

$$\begin{cases}\mu_3N_7-\frac{K}{2}>0,\\ \\ \mu_4N_7-\frac{K}{2}\delta_6>0,\end{cases}$$

we have

$$G'(t) \leq -k_2 E_1(t) + \alpha_2 \int_0^1 \int_0^\infty \Theta(p)|\eta_x^t(x,p)|^2 dp, \tag{3.47}$$

by using (3.39), we get

$$\mathcal{K}'(t) \leq -k_2 E_1(t), \tag{3.48}$$

where

$$\mathcal{K}(t) = (G(t) + \frac{\alpha_2}{\zeta} E(t)) \sim E_1(t) + E_2(t).$$

Integrating (3.48), we get

$$\int_0^t E_1(y)dy \leq \frac{1}{k_2}(\mathcal{K}(0) - \mathcal{K}(t)) \leq \frac{1}{k_2}\mathcal{K}(0) \leq \frac{m_2}{k_2}(E_1(0) + E_2(0)), \tag{3.49}$$

and by

$$(tE_1(t))' = tE_1'(t) + E_1(t) \leq E_1(t), \tag{3.50}$$

we have

$$tE_1(t) \leq \frac{m_2}{k_2}(E_1(0) + E_2(0)). \tag{3.51}$$

yields (3.30). The proof is now completed.

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