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THEME

Compact Operators

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Fatima zahra daoud

CONTENTS

INTRODUCTION

A large area of current research interest is centered around the theory of operators on Hilbert space. and this dissertation will be devoted to this topic and especially the compact operators.

The geometry of Banach space lies in darkness and has attracted the attention of many talented research mathematicians. However, the theory of operators (linear transformations) on a Banach space has very few general results, where As Hilbert space operators have an elegant and well developed general theory.

In addition the theory of operators is fondamental in the functional analysis.

And in this dissertation we discuss an important kind of operators which is " compact operators".

Compact operators are a special class of linear operators that map bounded sets to relatively compact sets. They are useful for studying eigenvalue problems, Fredholm equations, and spectral theory.

This dissertation consists of three chapters. The first chapter was about Hilbert spaces where i discussed a lot of concepts like orthogonality and orthonormal basis and others.

the second chapter was about Baire's theorem and operators and their properties.

And the last one was about compact operators and their properties.

NOTATION

- H the Hilbert space.
- $\mathcal{L}(H)$ the operators set.
- $\mathcal{H}(H)$ the compact operators set.
- $B_H = \{\|x\| \leq 1, x \in H\}$(the the closed unit ball).
- $l^2(\mathbb{C})$ the complex sequences of a summable square set.
- $p_m h$ the orthogonal projection of h on M.
- $RanP = \{P(h), h \in H\}$.
- $KerP = \{h \in H, P(h) = 0, h \in H\}$.
- $\mathbb{L}^2(H, H') = \{f : H \longmapsto H' : \int \| f(h) \|^2 < \infty\}$.
- $H \bigoplus H'$ the direct sum of H and H'.
- $\bigoplus H_i = \sum h_i, h_i \in H_i$.

CHAPTER 1

HILBERT SPACES

Throughout this chapter, we work on a field $\mathbb{K}$ which is $\mathbb{R}$ or $\mathbb{C}$.

1.1 Semi-Inner product

1.1.1.Definition

A semi-inner product on a vector space H is a bilinear map:

$$\langle .,. \rangle : H \times H \to \mathbb{K}$$

$$(g,h) \to \langle g,h \rangle,$$

Satisfying the followings:

1. $\forall f,g,h \in H$ and $\alpha,\beta \in \mathbb{K} : \langle \alpha f + \beta g, h\rangle = \alpha\langle f,h\rangle + \beta\langle g,h\rangle$.
2. $\forall g,h \in H : \langle g,h\rangle = \overline{\langle h,g\rangle}$ (hermitian symmetric).
3. $\forall g \in H : \langle g,g\rangle \geq 0$ (non-negative).
4. $f,g,h \in H$ and $\alpha,\beta \in \mathbb{K} : \langle f, \alpha g + \beta h\rangle = \overline{\alpha}\langle f,g\rangle + \overline{\beta}\langle f,h\rangle$.

1.1.1.Remark

$$\| f \| = \sqrt{\langle f,f\rangle}$$

is a norm on H.

1.2 Inner product

1.2.1.Definition

An inner product is a semi-innerproduct that it also satisfies: $\forall f \in H : \langle f,f\rangle = 0 \Leftrightarrow f = 0$ (positive definite).

1.3 Pri-Hilbert space

1.3.1.Definition

A pre-Hilbert space (inner space) is a vector space H provided with an inner product $\langle .,. \rangle$.

1.4 Hilbert space

1.4.1.Definition

A Hilbert space is a vector space H with an inner product $\langle f, g\rangle$ such that the norm defined by:

$$\parallel f \parallel = \sqrt{\langle f, f\rangle}.$$

turns H into a complete normed space.

1.4.1.Remark

- If the normed space defined by the norm is not complete, then H is instead known as an inner product space or a pre-Hilbert space.
- A pre-Hilbert space with a finit dimension is a complete space, then it is a Hilbert space.

1.4.1 Example of a Hilbert space

The set $l^2(\mathbb{C})$ of complex sequences of a summable square such that:

$$l^2(\mathbb{C}) = \{(x_n)_{n\in\mathbb{N}} : \sum_{n=0}^{\infty} \mid x_n \mid^2 < \infty\}.$$

provided with the inner product:

$$\forall (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C}) : \langle x_n, y_n\rangle = \sum_{n\in\mathbb{N}} x_n\overline{y_n}$$

Is a Hilbert space
Firstly we are going to show that $\langle x_n, y_n\rangle$ is an inner product on $l^2(\mathbb{C})$

- The hermitian symmetric: for all $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C})$, we have:

$$\begin{aligned} \langle (x_n), (y_n)\rangle &= \textstyle\sum_{n\in\mathbb{N}} x_n\overline{y_n} \\ &= \overline{\textstyle\sum_{n\in\mathbb{N}} y_n\overline{x_n}} \\ &= \overline{\langle y_n, x_n\rangle} \end{aligned}$$

- The linearity for the first variable:for all $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C})$ and for all $\lambda, \mu \in \mathbb{C}$, we have:

$$\begin{aligned} \langle (\lambda x_n + \mu y_n), z_n\rangle &= \textstyle\sum_{n\in\mathbb{N}}(\lambda x_n + \mu y_n)\overline{z_n} \\ &= \textstyle\sum_{n\in\mathbb{N}} \lambda x_n\overline{z_n} + \sum_{n\in\mathbb{N}} \mu y_n\overline{z_n} \\ &= \lambda \textstyle\sum_{n\in\mathbb{N}} x_n\overline{z_n} + \mu \sum_{n\in\mathbb{N}} y_n\overline{z_n} \\ &= \lambda\langle (x_n), (z_n)\rangle + \mu\langle (y_n), (z_n)\rangle \end{aligned}$$

- none-negative : for all $(x_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C})$:

$$\begin{aligned}\langle x_n, x_n\rangle &= \textstyle\sum_{n\in\mathbb{N}} x_n\overline{x_n} \\ &= \textstyle\sum_{n\in\mathbb{N}} \mid x_n \mid^2\end{aligned}$$

  It is obviously a positive quantity.

- Positive definite: let $(x_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C})$:

$$\begin{aligned}\langle x_n, x_n\rangle = 0 &\Leftrightarrow \textstyle\sum_{n\in\mathbb{N}} \mid x_n \mid^2 \\ &\Leftrightarrow \forall n \in \mathbb{N} :\mid x_n \mid= 0 \\ &\Leftrightarrow \forall n \in \mathbb{N} : (x_n) = 0\end{aligned}$$

Thus $\langle ., .\rangle$ is an inner product on $l^2(\mathbb{C})$. secondly we prove that $l^2(\mathbb{C}$ is complet for the norm induced by the inner product:

$$\forall (x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C}) : \langle x_n, y_n\rangle = \sum_{n\in\mathbb{N}} x_n\overline{y_n}$$

This norm is:

$$\forall (x_n)_{n\in\mathbb{N}} \in l^2(\mathbb{C}) : \|(x_n)\| = (\sum_{n\in\mathbb{N}} \mid x_n \mid^2)^{\frac{1}{2}}.$$

Let then $x^m = (x_n^m)_{n\in\mathbb{N}}$ be a cauchy sequence of elements in $l^2(\mathbb{C})$. Prisising the notations, we denote the exponent ^{m} as the number of the element x^m in this cauchy sequence (which is a sequence itself in $l^2(\mathbb{C})$. And by the index $_n$ the terms $x_n^m \in \mathbb{C}$ of each element x^m.
Then every element x^m is a sequence (infinite) in $l^2(\mathbb{C})$, which is written as:

$$x^m = (x_0^m, x_1^m, x_2^m, ...).$$

And because (x^m) is cauchy in $l^2(\mathbb{C})$, we have:

$$\forall \varepsilon > 0, \exists M > 0, \forall m, p > M : \|x^m - x^p\|^2 < \varepsilon. \qquad (*)$$

Then:

$$\sum_{n\in\mathbb{N}} \mid x_n^m - x_n^p \mid^2 < \varepsilon.$$

In paticular for all $n \in \mathbb{N}$:

$$\forall \varepsilon > 0, \exists M > 0, \forall m, p > M : \|x_n^m - x_n^p\|^2 < \varepsilon.$$

This exactly means that the sequence of the complex numbers $(x_n^m)_{m\in\mathbb{N}}$ (which is then a sequence of n-th components of the "sequence of sequences "x^m). is cauchy in $\mathbb{C}$.This sequence admets a limit,and let it be $x_n^* \in \mathbb{C}$.
Now we have to verify that the sequence $x^* = (x_0^*, x_1^*, ...)$(which has as a components the limits

of the components of (x^m))is a limit for the sequence x^m in $l^2(\mathbb{C})$. The problem here is that we don't know even if $x^m \in l^2(\mathbb{C})$.
Since x^m is cauchy and bounded(regardless if the considered vector space is complet or no), there exists a $C > 0$ such that:

$$\forall m \in \mathbb{N}, \|x^m\|^2 = \sum_{n=0}^{\infty} | x_n^m |^2 \leq C.$$

In particular for $N \geq 0$:

$$\sum_{n=0}^{N} | x_n^m |^2 \leq C.$$

Passing to the limit in the inequality above(every term x_n^m converges to x_n^*, and because the sum above includs only a finite number of terms), we have:

$$\sum_{n=0}^{N} | x_n^* |^2 \leq C.$$

Since this is true for all $N \in \mathbb{N}$(always with same value of C), we deduce that the series$\sum_{n=0}^{N} | x_n^* |^2$of a positive general term converges, meaning that $x^* = (x_n^*)_{n\in\mathbb{N}}$ is in $l^2(\mathbb{C})$.
Now proving that x^m converges to this element x^* in $l^2(\mathbb{C})$, meaning that :

$$\| x^m - x^* \|^2 = \sum_{n\in\mathbb{N}} | x_n^m - x_n^* |^2 \longrightarrow 0.$$

when $n \longrightarrow \infty$. to do this we pass to the limit in $(*)$, expressing the fact that x^m is cauchy is difficult because of the presence of the infinite sum, which requires the use of the limit-sum interversion theorem

Sum-limit interversion theorem

Let E be a Banach space, and let $l \in \overline{E}$
if $\lim_{x\longrightarrow l} f_n(x)$ exists for all $n \in \mathbb{N}$ and $\sum f_k$ converges uniformaly in the neighborhood of l, then:

$$\lim_{x\longrightarrow l} \sum_{n\geq 0} f_n(x) = \sum_{n\geq 0} \lim_{x\longrightarrow l} f_n(x).$$

To overcome this difficulty we have to do the same thing as before, to get a finite sum, we have:

$$\forall \varepsilon > 0, \exists M > 0, \forall N > 0, \forall m, p > M : \sum_{n=0}^{N} | x^m - x^p |^2 < \varepsilon.$$

Passing to the limit when $p \longrightarrow \infty$, we get :

$$\forall \varepsilon > 0, \exists M > 0, \forall N > 0, \forall m > M : \sum_{n=0}^{N} | x^m - x^* |^2 < \varepsilon,$$

And then:

$$\forall \varepsilon > 0, \exists M > 0, \forall m > M : \sum_{n=0}^{\infty} \mid x^m - x^* \mid^2 < \varepsilon.$$

This shows exactly that (x^m) converges to (x^*) in $l^2(\mathbb{C})$.

1.4.2 Cauchy-Schwarz inequality

Let $f, g \in H$ (H be an inner product space), then:

$$\mid \langle f, g \rangle \mid \leq \parallel f \parallel \parallel g \parallel .$$

Where

$$\parallel f \parallel = \sqrt{\langle f, f \rangle}. \tag{1.1}$$

Proof:

By the nonnegativity of the inner product, we get:

$$\langle \alpha f - \beta g, \alpha f - \beta g \rangle \geq 0.$$

For all $f, g \in \mathbb{H}$ and $\alpha, \beta \in \mathbb{C}$.

The expansion of the inner product gives us:

$$\langle \alpha f - \beta g, \alpha f - \beta g \rangle = \overline{\alpha} \langle f, \alpha f - \beta g \rangle - \overline{\beta} \langle g, \alpha f - \beta g \rangle.$$

We know that:

. $\overline{\alpha} \langle f, \alpha f - \beta g \rangle = \overline{\alpha} \alpha \langle f, f \rangle - \overline{\alpha} \beta \langle f, g \rangle$.

. $-\overline{\beta} \langle g, \alpha f - \beta g \rangle = -\alpha \overline{\beta} \langle g, f \rangle + \overline{\beta} \beta \langle g, g \rangle$.

We get :

$$\langle \alpha f - \beta g, \alpha f - \beta g \rangle = \overline{\alpha} \alpha \langle f, f \rangle - \overline{\alpha} \beta \langle f, g \rangle - \alpha \overline{\beta} \langle g, f \rangle + \overline{\beta} \beta \langle g, g \rangle.$$

We already have :$\parallel f \parallel^2 = \langle f, f \rangle$ and $\alpha \overline{\alpha} = \mid \alpha \mid^2$.

Then:

$$\langle \alpha f - \beta g, \alpha f - \beta g \rangle = \mid \alpha \mid^2 \parallel f \parallel^2 + \mid \beta \mid^2 \parallel g \parallel^2 - \overline{\alpha} \beta \langle f, g \rangle - \alpha \overline{\beta} \langle g, f \rangle.$$

Since:

$$\langle \alpha f - \beta g, \alpha f - \beta g \rangle \geq 0.$$

We get:

$$\mid \alpha \mid^2 \parallel f \parallel^2 + \mid \beta \mid^2 \parallel g \parallel^2 - \overline{\alpha} \beta \langle f, g \rangle - \alpha \overline{\beta} \langle g, f \rangle \geq 0.$$

$$\mid \alpha \mid^2 \parallel f \parallel^2 + \mid \beta \mid^2 \parallel g \parallel^2 \geq \overline{\alpha} \beta \langle f, g \rangle + \alpha \overline{\beta} \langle g, f \rangle.$$

If $\langle f, g\rangle = re^{i\varphi}$, where $r =| \langle f, g\rangle |$ and $\varphi = arg(\langle f, g\rangle)$.
Then we choose : $\alpha =\| g \| e^{i\varphi}$ and $\beta =\| f \|$.

$$\| g \| e^{-i\varphi} \| f \| | \langle f, g\rangle | e^{i\varphi} + \| g \| e^{i\varphi} \| f \| | \langle f, g\rangle | e^{-i\varphi} \leq \| g \|^2 \| f \|^2 + \| f \|^2 \| g \|^2.$$

$$\| g \| \| f \| | \langle f, g\rangle | + \| g \| \| f \| | \langle f, g\rangle | \leq 2 \| g \|^2 \| f \|^2.$$

$$| \langle f, g\rangle | \leq \| f \| \| g \| .$$

We can deduce immediately the following formula :

$$\| f \| = \sup_{g\in H, \|g\|\leq 1} | \langle f, g\rangle | .$$

1.4.3 Theorem

Let H be an inner product space, the inner product is a continuous map from $H \times H \to \mathbb{C}$.
Proof:
$\forall x, x', y, y' \in H$, the Cauchy-Schwarz inequality implies that:

$$\begin{aligned} | \langle x, y\rangle - \langle x', y'\rangle | &= | \langle x - x', y\rangle + \langle x, y - y'\rangle | \\ &\leq | \langle x - x', y\rangle | + | \langle x, y - y'\rangle | \\ &\leq \| x - x' \| \| y \| + \| x \| \| y - y' \| . \end{aligned}$$

This estimate implies the continuity of the inner product.

1.5 Orthogonality

Let H be a Hilbert space, we denote its inner product $\langle ., .\rangle$, the inner product sturcture of a Hilbert space allows us to define the concept of orthogonality, which makes possible to visualize vector subspaces of a Hilbert space geometrically .

1.5.1.Definition

Let $f, g \in H$

- f and g are orthogonal (written $f \perp g$)if $< f, g >= 0$.
- A subsets $\mathbb{A}$ and $\mathbb{B}$ are orthogonal (written $\mathbb{A} \perp \mathbb{B}$) if $x \perp y$, for all $x \in \mathbb{A}$ and $y \in \mathbb{B}$.
- The orthogonal complement $\mathbb{A}^\perp$ of a subset $\mathbb{A}$ of H is the set of vectors orthogonal to $\mathbb{A}$.

$$\mathbb{A}^\perp = \{f \in H, f \perp g | \forall g \in \mathbb{A}\}.$$

1.5.1.Example

Let the space $E = \mathcal{C}([-\pi, \pi], \mathbb{R})$ be provided with the inner product :

$$\int_{-\pi}^{\pi} f(t)g(t)dt.$$

Let f_n be a family of E such that $f_n(t) = cos(nt)$, for all $n, m \in \mathbb{N}, n \neq m$, we have:

$$\begin{aligned} \langle f_n, f_m \rangle &= \int_{-\pi}^{\pi} cos(nt)cos(mt)dt \\ &= \frac{1}{2}\int_{-\pi}^{\pi} cos((n+m)t)cos((n-m)t)dt \\ &= 0 \end{aligned}$$

Thus $\{f_n\}$ is an orthogonal family.

1.5.1 Theorem

The orthogonal complement of a subset of a Hilbert space is a closed linear subspace.

Proof:

Let $\mathbb{A} \subset H$ (H is a Hilbert space), if $h, g \in \mathbb{A}^{\perp}$ and $\alpha, \beta \in \mathbb{C}$, then the linearity of the inner product implies that:

$$< \alpha g + \beta h, f >= \alpha < g, f > +\beta < h, f >, \forall f \in \mathbb{A}.$$

Because $h, g \in \mathbb{A}^{\perp} :< g, f >=< h, f >= 0$,so we get:$< \alpha g+\beta h, f >= 0$. Therefore $\alpha g+\beta h \in \mathbb{A}^{\perp}$, so $\mathbb{A}^{\perp}$ is a linear subspace.

To prove that $\mathbb{A}^{\perp}$ is closed, we have to prove that if $(g_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{A}^{\perp}$. Then its limite g is also in $\mathbb{A}^{\perp}$.

Let $f \in \mathbb{A}$ from theorem(1.4.3), the inner product is continuous and therefore :

$$< f, g >=< f, \lim_{n\longrightarrow\infty} g_n >= \lim_{n\longrightarrow\infty} < f, g_n >= 0.$$

Therefore $g \in \mathbb{A}^{\perp}$.

1.5.1.Remark

$\mathbb{M} \subset H$ is dense if and only if $\mathbb{M}^{\perp} = \{0\}$.

1.5.1.Corollary

Let $\mathbb{M}$ be a closed subset of a Hilbert space H, then H decomposes into a direct sum $H = \mathbb{M} \oplus \mathbb{M}^{\perp}$.

We can state then:

1.5.2 Pythagoras theorem

Let H be a Hilbert space (real or complex), and let $f_1, ..., f_n \in H$ are two by two orthogonal, meaning that $\langle f_i, f_j \rangle = 0$ if $i \neq j$. then :

$$\| \sum_{i=1}^{n} f_i \|_H^2 = \sum_{i=1}^{n} \| f_i \|_H^2. \qquad (*)$$

Proof:

We are going to prove it by recurrence.

for $n = 1$ the equality $(*)$ is satisfied.

supposing that $(*)$ is satisfied for n, and proving it for $n + 1$.

Let $f_1, ..., f_{n+1} \in H$. Putting $y = \| \sum_{i=1}^{n} f_i \|$, we have:

$$\begin{aligned} \| \textstyle\sum_{i=1}^{n+1} f_i \|_H^2 &= \| y + f_{n+1} \|_H^2 \\ &= \langle y + f_{n+1}, y + f_{n+1} \rangle \\ &= \langle y, y \rangle + \langle y, f_{n+1} \rangle + \langle f_{n+1}, y \rangle + \langle f_{n+1}, f_{n+1} \end{aligned}$$

Since $\langle y, f_{n+1} \rangle = \langle f_{n+1}, y \rangle = 0$, we get:

$$\| \sum_{i=1}^{n+1} f_i \|_H^2 = \sum_{i=1}^{n+1} \| f_i \|_H^2.$$

1.5.3 The identity of the parallelogram theorem

If $f, g \in H$, then:

$$\| f + g \|^2 + \| f - g \|^2 = 2\| f \|^2 + 2\| g \|^2. \qquad (1.2)$$

Proof:

Using $\| f \| = \sqrt{< f, f >}$ to write norms in terms of inner product, we get :

$$\| f + g \|^2 + \| f - g \|^2 = 2\| f \|^2 + 2\| g \|^2.$$

$\forall f, g \in H$.

$$\begin{aligned} \langle f + g, f + g \rangle + \langle f - g, f - g \rangle &= \langle f + g, f \rangle + \langle f + g, g \rangle + \langle f - g, f \rangle - \langle f - g, g \rangle. \\ &= \langle f, f \rangle + \langle g, f \rangle + \langle f, g \rangle + \langle g, g \rangle + \langle f, f \rangle - \langle g, f \rangle - \langle f, g \rangle + \langle g, g \rangle. \\ &= 2\langle f, f \rangle + 2\langle g, g \rangle. \\ &= 2\| f \|^2 + 2\| g \|^2. \end{aligned}$$

Conversly if a norm satisfies (1.2), then the equation :

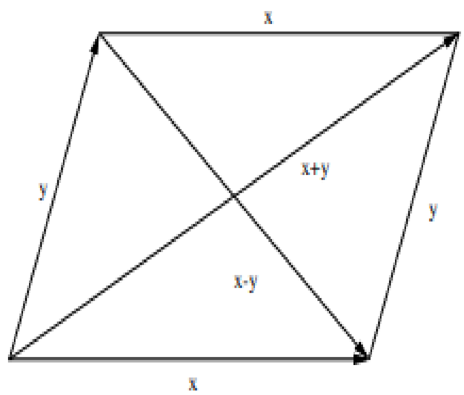
1. If $\mathbb{K} = \mathbb{R}$.

$$< x, y > = \frac{1}{4}(\| x + y \|^2 - \| x - y \|^2).$$

2. If $\mathbb{K} = \mathbb{C}$.

$$< x, y >= \frac{1}{4}(\| x + y \|^2 - \| x - y \|^2 - i\| x + iy \|^2 + i\| x - iy \|^2).$$

The identity of the parallelograms geometrical interpretation is the sum of the squares of the sides of a parallelogram is equal to the sum of the squares of the diagonals .



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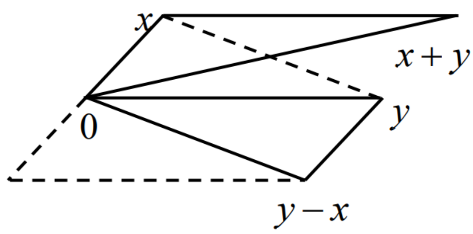


figure(1.2)

1.5.2.Definition

Let H be Banach space, and let $x, y \in H$. H is uniformly convex if :

$$\forall \varepsilon > 0, \exists \delta > 0, \forall (x, y) : (\|x\| \leq 1, \|y\| \leq 1 and \|x - y\| \geq \varepsilon) \Rightarrow \|\frac{x + y}{2}\| \leq 1 - \delta.$$

1.5.1.Proposition

H is uniformly convex and then H is reflexif.

Proof:

Let $\varepsilon > 0$ and $f, g \in H$ such that : $\mid f \mid < 1$ and $\mid g \mid < 1$ and $\mid f - g \mid > \varepsilon$, thanks to the identity of the parallelogram, we have :

$$\begin{aligned} \mid \tfrac{f+g}{2} \mid^2 + \mid \tfrac{f-g}{2} \mid^2 &= 2(\mid \tfrac{f}{2} \mid^2 + \mid \tfrac{g}{2} \mid^2). \\ &\leq 2(\mid \tfrac{1}{2} \mid^2 + \mid \tfrac{1}{2} \mid^2) \\ &\leq 1. \end{aligned}$$

So

$$\begin{aligned} \mid \tfrac{f+g}{2} \mid^2 &\leq 1 - \mid \tfrac{f-g}{2} \mid^2 \\ &\leq 1 - \tfrac{\mid f-g \mid^2}{4} \\ &\leq 1 - \tfrac{\varepsilon^2}{4}. \end{aligned}$$

Thus:

$$\mid \frac{f+g}{2} \mid < 1 - \delta.$$

with $\delta = 1 - (1 - \frac{\varepsilon^2}{4})^{\frac{1}{2}} > 0$.

1.5.4 Orthogonal projection theorem

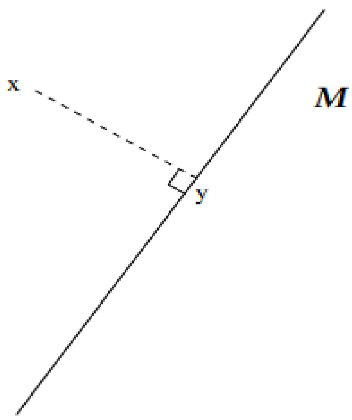
Let $\mathbb{M}$ be a closed linear subspace of a Hilbert space H:

(a) For each $h \in H$ there is a unique closest element $p_m h \in \mathbb{M}$ such that :

$$\| h - p_m h \| = \min_{z \in \mathbb{M}} \| h - z \|.$$

(b) The element $p_m h \in \mathbb{M}$ closest to $h \in H$ is the unique element of $\mathbb{M}$ that satisfies the property:

$$(h - p_m h) \perp \mathbb{M}.$$



figure(1.3)

Poof:

Let d be the distance of h from $\mathbb{M}$.

$$d =| \{\| h - z \|, z \in \mathbb{M}\}. \qquad (1.3)$$

(a) First we prove that there is a closest element $p_m h \in \mathbb{M}$ at which the infimum is reached. Meaning that $\| h - p_m h \|= d$.

From the definition of d there is a sequence of elements $(p_m h)_n$ such that :

$$\lim_{n \longrightarrow \infty} \| h - (p_m h)_n \|= d.$$

Thus, $\forall \varepsilon > 0$, there is an N such that :

$$\| h - (p_m h)_n \|\leq d + \varepsilon.$$

When $n > N$.

(b) We show that the sequence $(p_m h)_n$ is Cauchy. From the parallelogram identity, we have:

$$\| (p_m h)_p - (p_m h)_q \|^2 + \| 2h - (p_m h)_p - (p_m h)_q \|^2 = 2\| h - (p_m h)_p \|^2 + 2\| h - (p_m h)_q \|^2.$$

Since $\frac{(p_m h)_p - (p_m h)_q}{2} \in \mathbb{M}$, equation (1.3) implies that:

$$\| h - \frac{(p_m h)_p - (p_m h)_q}{2} \|\geq d.$$

Combining the equations,we find that fo all $p, q > N$

$$\begin{aligned} \| (p_m h)_p - (p_m h)_q \|^2 &= 2\| h - (p_m h)_p \|^2 + 2\| h - (p_m h)_q \|^2 - \| 2h - (p_m h)_p - (p_m h)_q \|^2. \\ &\leq 4(d + \varepsilon)^2 - 4d^2. \\ &\leq 4\varepsilon(2d + \varepsilon). \end{aligned}$$

Therefore $(p_m h)_n$is Cauchy. Since a Hilbert space is complete there is a $p_m h$ such that : $(p_m h)_n \longrightarrow p_m h$, and since $\mathbb{M}$ is closed, we have $p_m h \in \mathbb{M}$.

1.5.5 Theorem

Let $\mathbb{M}$ be a a closed linear subset of H and $h \in H$, let Ph be the unique point in $\mathbb{M}$ such that $(h - Ph) \perp \mathbb{M}$. Then:

(a) P is a linear transformation on H.

(b) $\| Ph \|\leq\| h \|$, for every $h \in H$.

(c) $P^2 = P$(here P^2 means the composition of P with itself).

(d) $KerP = \mathbb{M}^{\perp}$ and $RanP = \mathbb{M}$.

Proof:

(Keep in mind that:$\forall h \in H, h - Ph \in \mathbb{M}^{\perp}$ and $\| h - Ph \| = dist(h, \mathbb{M})$. With $dist(h, \mathbb{M}) = \min_{z \in \mathbb{M}} \| h - z \|$.)

(a) Let $h_1, h_2 \in H$ and $k_1, k_2 \in \mathbb{K}$, then:

$$\langle (k_1 h_1 + k_2 h_2) - (k_1 P h_1 + k_2 P h_2), f \rangle = k_1 \langle h_1 - P h_1, f \rangle + k_2 \langle h_2 - P h_2 \rangle = 0.$$

And by the uniqueness statement of theorem(1.5.5):

$$P(k_1 h_1 + k_2 h_2) = k_1 P h_1 + k_2 P h_2.$$

(b) If $h \in H$, then $h = (h - Ph) + Ph, Ph \in \mathbb{M}, h - Ph \in \mathbb{M}^{\perp}$. Thus:

$$\| h \|^2 = \| h - Ph \|^2 + \| Ph \|^2 \geq \| Ph \|^2.$$

(c) If $f \in \mathbb{M}$, then $Pf = f$. For any $h \in \mathbb{M}, Ph \in \mathbb{M}$, hence $P^2 h = P(Ph) = Ph.$. Therefore $P^2 = P$.

(d) If $Ph \in KerP$, then $Ph = 0$, then $h = h - Ph \in \mathbb{M}^{\perp}$. Conversely if $h \in \mathbb{M}^{\perp}$, then 0 is the unique vector such that $h - 0 = h \perp \mathbb{M}$, therefore $Ph = 0$, then $Ph \in KerP$, thus $KerP = \mathbb{M}^{\perp}$.
$RanP = \mathbb{M}$ is clear.

1.5.3.Definition

Let $\mathbb{M}$ be a closed linear subspace of H, and P is the linear map defined in the preceding theorem, then P is called the orthogonal projection of H onto $\mathbb{M}$. And let it be denoted as $P_{\mathbb{M}}$.

1.6 Orthonormal bases

1.6.1.Definition

A set of vectors $(u_i)_{i \in I}$ is a orthonormal basis of a Hilbert space H, if :

1. $\| u_i \| = 1$.
2. The generated space $\mathcal{VECT}\{u_i\}$ is dense in H.
3. $(u_i)_{i \in I}$ is orthogonal, meaning that if $i \neq j$, then $u_i \perp u_j$.

In which case the vectors u_i are said to be normalized.

1.6.1.Remark

1. Every Hilbert space has an orthonormal basis, which may be finite, countably infinite, or uncountable.

2. A basis for H is a maximal orthonormal set.

3. Two Hilbert spaces whose orthonormal bases have the same cardinality are isomorphic.

4. Any linear map that identifies basis elements is an isomorphism.

1.6.1.Examples

1. A set of vectors $\{e_1, ..., e_n\}$ is an orthonormal basis of the finite dimentional space $\mathbb{C}^n$ if :

   (a) $\langle e_j, e_k \rangle = \delta_{jk}$ for $1 \leq j, k \leq n$.

   (b) For all $x \in \mathbb{C}^n$ there are a unique coordinates $x_k \in \mathbb{C}$ such that :

   $$x = \sum_{k=1}^{n} x_k e_k.$$

   Where δ_{jk} is the cronecker delta defined by :

   $$\delta_{jk} = \begin{cases} 1 & if \quad i = j. \\ 0 & if \quad i \neq j. \end{cases}$$

   The orthonormality of the basis implies that $x_k = \langle e_k, x \rangle$. For example, the standard orthonormal basis of $\mathbb{C}^n$ consists of the vectors

   $$e_1 = (1, 0, ..., 0), e_2 = (0, 1, ..., 0), ..., e_n = (0, 0, ..., 1).$$

2. The set of functions $\{e_n(x) | n \in \mathbb{Z}\}$, given by:

   $$e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}.$$

   Is an orthonormal basis of the space $\mathbb{L}^2(\mathbb{T})$ of 2π-periodic functions, called the Fourier basis.

1.6.2.Definition

Let $\{h_\alpha \in H \mid \alpha \in I\}$, be an indexed set of a Hilbert space H, where the index set I may be countable or uncountable. For each finite subset J of I, the partial sum S_J is defined by:

$$S_J = \sum_{\alpha \in J} h_\alpha.$$

The unordered sum of the indexed set $\{h_\alpha \mid \alpha \in I\}$ converges to $h \in H$, written

$$h = \sum_{\alpha \in I} h_\alpha.$$

if for every $\varepsilon > 0$ there is a finite subset J^ε of I such that $\| S_J - x \| < \varepsilon$ for all finite subset J of I that contains J^ε. An unordered sum is said to converge unconditionally.

1.6.1.Lemma

Let $U = \{u_\alpha | \alpha \in I\}$ be an indexed orthogonal subset of a Hilbert space H.
The sum $\sum_{\alpha\in\mathbb{I}} u_\alpha$ converges unconditionally if and only if $\sum_{\alpha\in\mathbb{I}} \|u_\alpha\|^2 < \infty$ and in that case :

$$\| \sum_{\alpha\in\mathbb{I}} u_\alpha \|^2 = \sum_{\alpha\in\mathbb{I}} \|u_\alpha\|^2. \qquad (1.4)$$

Proof:

For any finite set $\mathbb{J}$ we have:

$$\| \sum_{\alpha\in\mathbb{J}} u_\alpha \|^2 = \sum_{\alpha,\beta\in\mathbb{J}} \langle u_\alpha, u_\beta\rangle = \sum_{\alpha\in\mathbb{J}} \langle u_\alpha, u_\alpha\rangle = \sum_{\alpha\in\mathbb{J}} \|u_\alpha\|^2.$$

It follows that the Cauchy criterion is satisfied for $\sum_{\alpha\in\mathbb{J}} u_\alpha$, if and only if it is satisfied for $\sum_{\alpha\in\mathbb{J}} \|u_\alpha\|^2$. Thus, one of these sums converges unconditionally if and only if the other does. The equation (1.2) follows because the sum is a limite of a sequence of a finite partial sums and the norm is a continuous function.
When combined with the theorem bellow (Bessel's inequality), this lemma will imply that every element of a Hilbert space can be expanded with respect to an orthonormal basis.

1.6.1 Bessel's inequality

Let $\mathbb{U} = \{u_\alpha | \alpha \in I\}$ be an orthonormal set in a Hilbert space H and $x \in H$, then:

(a) $\sum_{\alpha\in\mathbb{I}} | \langle u_\alpha, x\rangle |^2 \leq \| x \|^2$.

(b) $x_\mathbb{U} = \sum_{\alpha\in\mathbb{I}} \langle u_\alpha, x\rangle u_\alpha$ is a convergent sum.

(c) $x - x_\mathbb{U} \in \mathbb{U}^\perp$.

Proof:

We begin by computing :

$$\| x - \sum_{\alpha\in\mathbb{J}} \langle u_\alpha, x\rangle u_\alpha \| .$$

$$\begin{aligned}
\| x - \textstyle\sum_{\alpha\in\mathbb{J}} \langle u_\alpha, x\rangle u_\alpha \|^2 &= \langle (x - \textstyle\sum_{\alpha\in\mathbb{J}} \langle u_\alpha, x\rangle u_\alpha), (x - \textstyle\sum_{\beta\in\mathbb{J}} \langle u_\beta, x\rangle u_\beta)\rangle. \\
&= \langle x, x\rangle - \textstyle\sum_{\beta\in\mathbb{J}} \langle u_\beta, x\rangle \langle x, u_\beta\rangle - \textstyle\sum_{\alpha\in\mathbb{J}} \overline{\langle u_\alpha, x\rangle} \langle u_\alpha, x\rangle. \\
&+ \textstyle\sum_{\alpha,\beta\in\mathbb{J}} \overline{\langle u_\alpha, x\rangle} \langle u_\beta, x\rangle \langle u_\alpha, u_\beta\rangle. \\
&= \|x\|^2 - \textstyle\sum_{\alpha\in\mathbb{J}} |\langle u_\alpha, x\rangle|^2.
\end{aligned}$$

Hence :

$$\sum_{\alpha\in\mathbb{J}} |\langle u_\alpha, x\rangle|^2 = \|x\|^2 - \| \ x - \sum_{\alpha\in\mathbb{J}}\langle u_\alpha, x\rangle u_\alpha \ \|^2 \leq \|x\|^2.$$

Since $\sum_{\alpha\in\mathbb{J}} |\langle u_\alpha, x\rangle|^2$ is a sum of nonnegative numbers that is bounded from above by $\|x\|^2$,it is Cauchy. Therefore the sum converges and satisfies (a).

To prove the convergence in (b) we apply the previous lemma, then we get :

$$\sum_{\alpha\in\mathbb{J}} |\langle u_\alpha, x\rangle|^2\|u_\alpha\|^2 \leq \|x\|^2\|u_\alpha\|^2.$$

$$\sum_{\alpha\in\mathbb{J}} |\langle u_\alpha, x\rangle|^2\|u_\alpha\|^2 \leq \|xu_\alpha\|^2.$$

Since $\|xu_\alpha\|^2$ is Cauchy, therefore it converges, thus :

$xu = \sum_{\alpha\in\mathbb{I}}\langle u_\alpha, x\rangle u_\alpha$ is a convergent sum.

In order to prove (c), we consider $u_{\alpha_0} \in \mathbb{U}$. Using the orthonormality of $\mathbb{U}$ and the continuity of the inner product, we find that:

$$\begin{aligned} \langle x - \textstyle\sum_{\alpha\in\mathbb{I}}\langle u_\alpha, x\rangle u_\alpha, u_{\alpha_0}\rangle &= \langle x, u_{\alpha_0}\rangle - \textstyle\sum_{\alpha\in\mathbb{I}} \overline{\langle u_\alpha, x\rangle}\langle u_\alpha, u_{\alpha_0}\rangle. \\ &= \langle x, u_{\alpha_0}\rangle - \langle x, u_{\alpha_0}\rangle = 0. \end{aligned}$$

Hence $x - \sum_{\alpha\in\mathbb{I}}\langle u_\alpha, x\rangle u_\alpha \in \mathbb{U}^\perp$.

Given the subset $\mathbb{U}$ of H, we define the closed linear span $[\mathbb{U}]$ of $\mathbb{U}$ by:

$$[\mathbb{U}] = \{\sum_{u\in\mathbb{U}} c_u u | c_u \in \mathbb{C}\}. \tag{1.5}$$

Where $\sum_{u\in\mathbb{U}} c_u u$ converges unconditionally .

Equivalently, $[\mathbb{U}]$ is the smallest closed linear subspace that contains $\mathbb{U}$.

1.6.2 Theorem

If $\mathbb{U} = \{u_\alpha | \alpha \in \mathbb{I}\}$ is an orthonormal subset of a Hilbert space H, then the following conditions are equivalent :

1. $\langle u_\alpha, x\rangle = 0$ for all $\alpha \in \mathbb{I} \Rightarrow x = 0$.
2. $x = \sum_{\alpha\in\mathbb{I}}\langle u_\alpha, x\rangle u_\alpha$ for all $x \in H$.
3. $\|x\|^2 = \sum_{\alpha\in\mathbb{I}} | \ \langle u_\alpha, x\rangle \ |^2$ for all $x \in H$.
4. $[\mathbb{U}] = H$.
5. $\mathbb{U}$ is the maximal orthonormal set.

Proof:

We prove that (a) implies (b),(b) implies (c),(c) implies (d),(d) implies (e),(e) implies (a).
The condition in (a) states that $\mathbb{U}^{\perp} = \{0\}$.
From Bessel's inequality we have $x - x_{\mathbb{U}} \in \mathbb{U}^{\perp}$, so $x - x_{\mathbb{U}} \in \{0\}$, so $x = xu$.Which implies (b).
The implication from (b) to (c) is a direct application of (1.4).
We notice that (c) implies $\mathbb{U}^{\perp} = \{0\}$, which implies $[\mathbb{U}]^{\perp} = \{0\}$, so$[\mathbb{U}] = H$.
Condition (e) means that if $\mathbb{V}$ is a subset of H that contains $\mathbb{U}$ and is strictly larger than $\mathbb{U}$, then $\mathbb{V}$ is not orthonormal .
Since $[\mathbb{U}] = H$, then $\forall v \in H v = \sum_{\alpha \in \mathbb{I}} c_\alpha u_\alpha$, where $c_\alpha = \langle u_\alpha, v \rangle$.therefore , if $v \perp \mathbb{U}$, then $c_\alpha = 0 \forall \alpha$ meaning that $v = 0$, so $\mathbb{U} \cup \{v\}$ is not orthonormal.
Finally , (e) implies (a) ,since (a) is just a reformulation of (e).
In view of this theorem, we can make the following definition.

1.6.2.Definition

An orthonormal subset $\mathbb{U} = \{u_\alpha | \alpha \in \mathbb{I}\}$ of a Hilbert space H is complete if it satisfies any of the equivalent conditions (1)-(5) in the theorem above.
A complete orthonormal subset of H is called an orthonormal basis of H.

1.6.3 Parseval's identity theorem

Supposing that $\mathbb{U} = \{u_\alpha | \alpha \in \mathbb{I}\}$ is an orthonormal basis of a Hilbert space H. If $x = \sum_{\alpha \in \mathbb{I}} a_\alpha u_\alpha$, and $y = \sum_{\alpha \in \mathbb{I}} b_\alpha u_\alpha$.
Where $a_\alpha = \langle u_\alpha, x \rangle$, and $b_\alpha = \langle u_\alpha, y \rangle$, then:

$$\langle x, y \rangle = \sum_{\alpha \in \mathbb{I}} \overline{a_\alpha} b_\alpha.$$

1.6.2.Remark

To show that a Hilbert space has an orthonormal basis, we use zorn's lemma, which states that a nonempty partially ordered set with the property that every totally ordered subset has an upper bound has a maximal element.

1.6.4 Theorem

Every Hilbert space H has an orthonormal basis.
If U is an orthonormal set, then H has an orthonormal basis containing U.

Proof:

If $H = \{0\}$, then the statement is trivially true with $U = \phi$, so we assume that $H \neq \{0\}$. We introduce a partial ordering $\leq$ on an orthonormal subsets of H by inclusion so that:

$$U \leq V \Leftrightarrow U \subset V.$$

If $\{U_\alpha | \alpha \in \mathbb{A}\}$ is a totally ordered family of orthonormal sets, meaning that for any $\alpha, \beta \in \mathbb{A}$, we have either $U_\alpha \leq U_\beta$ or $U_\beta \leq U_\alpha$, then $\cup_{\alpha \in \mathbb{A}} U_\alpha$ is an orthonormal set and is an upper bound in the sense of inclusion, of the family $\{U_\alpha | \alpha \in \mathbb{A}\}$.

Zorn's lemma implies that the family of all orthonormal sets in H has a maximal element.This element satisfies (e) in the previous theorem and hence is a basis.

To prove that any orthonormal set U can be extended to an orthonormal basis of H we apply the same argument to the family of all orthonormal sets containing U .

The existance of othonormal bases would not be useful if we did not have a means of constructing them. The Gram Schmidt orthonormalization procedure is an algorithm for the construction of an orthonormal basis from any countable linearly independent set whose linear span is dense in H.

Let V be a countable set of linearly independent vectors in a Hilbert space H. The Gram Schmidt orthonormalization procedure is a method of constructing an orthonormal set U such that $[U] = [V]$, where $[V]$ is the closed linear span of V. We denote the elements of V by v_n. The orthonormal set $U = \{u_n\}$ is then constructed inductively by setting $u_1 = \frac{v_1}{\|v_1\|}$, and :

$$u_{n+1} = c_{n+1}(v_{n+1} - \sum_{k=1}^{n} \langle u_k, v_{n+1} \rangle u_k).$$

for all $n \geq 1$. Here $c_{n+1} \in \mathbb{C}$ is chosen so that $\|u_{n+1}\| = 1$ it is straightforward to check that $[\{v_1, ..., v_n\}] = [\{u_1, ..., u_n\}]$ forall $n \geq 1$, and hence that:

$$[V] = \overline{\cup_n [\{v_1, ..., v_n\}]} = \overline{\cup_n [\{u_1, ..., u_n\}]} = [U].$$

1.6.2.Examples of an orthonormal basis

1. The space $\mathbb{K}^n$ provided with the scalar product defined by:

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}.$$

   Is a Hilbert space. The canonical basis of $\mathbb{K}^n$ is an orthonormal basis.

2. The space:

$$l^2(\mathbb{N}) = \{(u_n)_{n \in \mathbb{N}} : \sum_{n \in \mathbb{N}} \mid u_n \mid^2 < \infty\}.$$

   Provided with the scalar product:

$$\langle u, v \rangle = \sum_{n \in \mathbb{N}} u_n \overline{v_n}.$$

   For all $k \in \mathbb{N}$, we denote e_k the sequence of which all the terms are nuls, except for the k equals to 1, then $(e_k)_{k \in \mathbb{N}}$ is an orthonormal basis of l_2.

1.7 Dual of a Hilbert space

1.7.1.Definition

Let H be a Hilbert space , and let $\mathbb{K} = \mathbb{R} \vee \mathbb{C}$. The norm $\| x \| = \sqrt{\langle x, x\rangle}$ is always the euclidian norm. For all $y \in H$, the map $f_y : H \longrightarrow K$, defined by:

$$\forall x \in H : f_y(x) = \langle x, y\rangle.$$

Is a linear form such that $f_y \in H^*$, where H^* is the space of all linear forms of H.

1.8 Riesz representation theorem

1. Let H be a Hilbert space, and let f be a linear continuous form over H. Then there exists a unique element a of H such that :

   $$\forall x \in H, f(x) = \langle x, a\rangle.$$

   and $\|f\| = \|a\|$.

2. Conversely, every element a of H defines a linear continuous formf_a over H by the formula :

   $$f_a(x) = \langle x, a\rangle, \forall x \in H.$$

   Thus the map :

   $$g : H \longrightarrow H^*.$$

   $$a \longrightarrow f_a.$$

   Is an isomorphism

Proof:
Let f be a linear continuous form not identically zero, then $\mathbb{F} = ker(f)$ is a closed hyperplane of H (since f is continuous) and $\mathbb{F} \neq H$.
Since $H = \mathbb{F} \bigoplus \mathbb{F}^{\perp}$.
Let y_0 be a non zero element of $\mathbb{F}^{\perp}$, then:

$$\forall x \in H : x = y + y_0, y \in \mathbb{F}.$$

$$\forall x \in H : f(x) = f(y + y_0), y \in \mathbb{F}.$$

$$\forall x \in H : f(x) = 0 + f(y_0).$$

So $f(y_0) \neq 0$.
For all $x \in H$, we put $u = x - \frac{f(x)}{f(y_0)} y_0$.

It's clear that $f(u) = 0$, which means that $u \in \mathbb{F}$, and since $y_0 \in \mathbb{F}^{\perp}$, we get :$\langle u, y_0 \rangle = 0$.
Replacing u by its expression, we obtain:

$$\langle u, y_0 \rangle - \|y_0\|^2 \frac{f(x)}{f(y_0)} = 0.$$

We deduce:

$$f(x) = \langle u, y_0 \rangle \frac{f(y_0)}{\|y_0\|^2}, \forall x \in H.$$

Considering $a = \|y_0\|^{-2}\overline{f(y_0)}y_0$.
It is easy to see that the element a is unique.

1.8.1.Proposition

Let H be a Hilbert space and $L : H \longrightarrow H$ a linear map. Then the following asserions are equivalent:

1. L is continuous.
2. L is continuous at 0.
3. There exists $h \in H$ such that L is continuous at h.
4. There exists $C > 0$ such that $\| L(h) \| \leq C \| h \|$, for all $h \in H$.

Proof:
Since L is continuous then it is continuous at 0, and since it is continuous at 0 it is also continuous at any ponit of H. So it is clear that (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3)
Let's show that (3) $\Longrightarrow$ (1). supposing that L is continuous ath_0, and $h \in H$. If the sequance $h_n \longrightarrow h$, then $h_n - h + h_0 \longrightarrow h_0$. By assumption :

$$\begin{aligned} L(h_0) &= \lim L(h_n - h + h_0) \\ &= \lim L(h_n) - L(h) + L(h_0) \end{aligned}$$

We get $0 = \lim L(h_n) - L(h)$, then $\lim L(h_n) = L(h)$. meaninig that L is continuous.
Now showing that (4) $\Longrightarrow$ (2).
From (4) we have $\| L(h) \| \leq C \| h \|$, for all $h \in H$.
Putting $h = h_1 - h_2, h_1, h_2 \in H$ we get:

$$\| L(h_1 - h_2) \| \leq C \| h_1 - h_2 \|,$$

for all $h_1, h_2 \in H$. We notice that L is C-Lipschitzian, then L is continuous.
Now showing that (2) $\Longrightarrow$ (4).
The definition of continuity at 0 implies that $L^{-1}\big(\{a \in H : |a| < 1\}\big)$ contains an open ball

about 0. So there is $\delta > 0$ such that $B(0,\delta) \subseteq L^{-1}\big(\{a \in H : |a| < 1\}\big)$. That is, $\parallel h \parallel < \delta$ implies $\parallel L(h) \parallel < 1$, If h is arbitrary element of H and $\varepsilon > 0$, then $\frac{\delta h}{\|h\|+\varepsilon} < \delta$. Hence :

$$1 > L\Big[\frac{\delta h}{\parallel h \parallel + \varepsilon}\Big] = \frac{\delta}{\parallel h \parallel + \varepsilon} \parallel h \parallel,$$

Thus

$$L(h) < \frac{1}{\delta}(\parallel h \parallel + \varepsilon).$$

Letting $\varepsilon \longrightarrow 0$, we see that (4) holds with $C = \frac{1}{\delta}$.

CHAPTER 2

BAIRE'S THEOREM AND OPERATORS

2.1 Introduction

Baire's theorem is very fondqmental, when it comes to work on an infinite dimension. The results of this chapter have nothing specific to the Hilbert spaces, and they are correct in any Banach space.

The important property here is the completeness.

2.2 Operators

2.2.1.Definition

Let A be a continuous linear map such that :

$$A : H \longrightarrow H.$$

$$x \longrightarrow Ax.$$

A is called an operator over H.
We denote the set of the operators over H by $\mathcal{L}(H)$.
$(\mathcal{L}(H), +, .)$ is a vector space. Where $(+)$ designates the addition of maps and $(.)$ designates the multiplication of a map by a scalar.
If $A \in \mathcal{L}(H)$, we define the operator norm by:

$$\| A \|_{op} = \sup\{\|Ah\| : h \in H, \|h\| \leq 1\}.$$

2.2.1 Properties of the operator norm

1. If $A \in \mathcal{L}(H)$, then $\| A \|= 0 \Leftrightarrow A = 0$.
2. If $A, B \in \mathcal{L}(H)$, then $A + B \in \mathcal{L}(H)$ and $\| A + B \|\leq\| A \| + \| B \|$.
3. If $\alpha \in \mathbb{K}$, and $A \in \mathcal{L}(H)$,then $\alpha A \in \mathcal{L}(H)$,and $\| \alpha A \|= |\alpha|. \| A \|$.
4. If $A, B \in \mathcal{L}(H)$, then $AB \in \mathcal{L}(H)$ and $\| AB \|\leq\| A \| . \| B \|$ (AB is the conjugacy of maps $A \circ B$).

Proof:

1. Let $A \in \mathcal{L}(H)$, then:

$$\begin{aligned} \| A \|_{op} = 0 &\Leftrightarrow \sup_{\|h\|\leq 1} \| Ah \|= 0 \\ &\Leftrightarrow \| Ah \|= 0, \forall h \in H, \|h\| \leq 1 \\ &\Leftrightarrow A = 0 \end{aligned}$$

2. Let $A, B \in \mathcal{L}(H)$, then:

$$\begin{aligned} \| A + B \|_{op} &= \sup_{\|h\|\leq 1} \| (A + B)h \| \\ &= \sup_{\|h\|\leq 1} \| Ah + Bh \| \\ &\leq \sup_{\|h\|\leq 1} \| Ah \| + \sup_{\|h\|\leq 1} \| Bh \|, \\ &\leq \| A \|_{op} + \| B \|_{op} \end{aligned}$$

3. Let $A \in \mathcal{L}(H)$ and $\alpha \in \mathbb{K}$, then:

$$\begin{aligned} \| \alpha A \|_{op} &= \sup_{\|h\|\leq 1} \| \alpha A h \| \\ &= \sup_{\|h\|\leq 1} |\alpha| \| Ah \| \\ &= |\alpha| \sup_{\|h\|\leq 1} \| Ah \| \\ &= |\alpha| \| A \|_{op} \end{aligned}$$

4. Let $A, B \in \mathcal{L}(H)$ and $k \in H$, then:

$$\| Ak \|_{op} \leq \| A \|_{op} \| k \|$$

Putting $k = Bh$

$$\begin{aligned} \| AB \|_{op} = \sup_{\|h\|\leq 1} \| ABh \| &\leq \| A \|_{op} \| B \|_{op} \| h \| \\ &\leq \| A \|_{op} \| B \|_{op} \end{aligned}$$

2.2.1.Remark

From the first 3 properties we notice that $\| . \|$ defines a norm over $\mathcal{L}(H)$.

2.2.1.Examples:

1. The operator of volterra, defined over $\mathbb{L}^2[0,1]$ by:

$$Vf(x) = \int_0^x f(y)dy.$$

2. Another important example of operators is the shift S, defined over $\mathbb{L}^2(\mathbb{N})$ by:

$$S(\alpha_1, \alpha_1, ...) = (0, \alpha_1, \alpha_2, ...)$$

we have $\| S \| = 1$.

3. Let $(\mathbb{X}, \Omega, \mu)$ a $\sigma-$ finite measure space and $H = \mathbb{L}^2(\mu)$.If $\varphi \in \mathbb{L}^\infty(\mu)$, we can define an operator $M_\varphi \in \mathcal{L}(H)$ such that :

$$M_\varphi f = \varphi f.$$

Then we get :$\| M_\varphi \| = \| \varphi \|_\infty$.

Proof:

Considering :

$$\| \varphi \|_\infty = \inf\{c > 0 : \mu(| \varphi | > c) = 0\}.$$

We can even if it means modifying φ over a zero measure set, suppose that :
$| \varphi | \leq \| \varphi \|_\infty$, for all $x \in \mathbb{X}$, and for $f \in \mathbb{L}^2(\mu)$, we then have :

$$\| M_\varphi f \|^2 = \int | \varphi f |^2 d\mu \leq \| \varphi \|_\infty^2 \int | f |^2 d\mu \leq \| \varphi \|_\infty^2 \| f \|^2.$$

Which implies that $M_\varphi \in \mathcal{L}(H)$, and $\| M_\varphi \| \leq \| \varphi \|_\infty$.
To prove the equality, let $\varepsilon > 0$, then μ being $\sigma-$ finite ,there existe $\Delta \in \Omega$ such that : $0 < \mu(\Delta) < \mu(\Omega)$, and $| \varphi(x) | \geq \| \varphi \|_\infty - \varepsilon$ for all $x \in \Delta$, then putting, $f = (\mu(\Delta))^{\frac{-1}{2}} 1_\Delta$, then $f \in \mathbb{L}^2(\mu)$ and $\| f \|_2 = 1$ thus:

$$\| M_\varphi \|^2 \geq \| \varphi f \|_2^2 = (\mu(\Delta))^{-1} \int_\Delta | \varphi |^2 d\mu \geq (\| \varphi \|_\infty - \varepsilon)^2.$$

When $\varepsilon \longrightarrow 0$, we obtaine $\| M_\varphi \| \geq \| \varphi \|_\infty$.
Thus the operator M_φ defined is called the operator of multiplication, and the function φ is called the symbole of M_φ

4. Let $(\mathbb{X}, \Omega, \mu)$ a measure space and $k : \mathbb{X} \times \mathbb{X} \longrightarrow \mathbb{K}$, a $\Omega \otimes \Omega$ measurable function such there exists a constants c_1, c_2 verifying:

$$\int_{\mathbb{X}} | k(x,t) | d\mu(y) \leq c_1 \quad \mu - almost.everywhere.$$

$$\int_{\mathbb{X}} | k(x,t) | d\mu(y) \leq c_2 \quad \mu - almost.everywhere.$$

Then the map $\mathbb{K} : \mathbb{L}^2(\mu) \longrightarrow \mathbb{L}^2(\mu)$ defiened by:

$$(Kf)(x) = \int_{\mathbb{X}} k(x,t) f(y) d\mu(y).$$

Defines an operator over $H = \mathbb{L}^2(\mu)$, and $\| K \| \leq \sqrt{c_1 c_2}$.

Proof:

If $f \in \mathbb{L}^2(\mu)$, then we write :

$$\begin{array}{rcl} | Kf(x) | & \leq & \int | k(x,t) | . | f(y) | d\mu(y). \\ & \leq & \int | k(x,t) |^{\frac{1}{2}} | k(x,t) |^{\frac{1}{2}} . | f(y) | d\mu(y). \\ & \leq^{c-s} & \Big(\int | k(x,t) | d\mu(y) \Big)^{\frac{1}{2}} . \Big(\int | k(x,t) | .| f(y) |^2 d\mu(y) \Big)^{\frac{1}{2}}. \\ & \leq & \sqrt{c_1} \Big(\int | k(x,t) | .| f(x) |^2 d\mu(y) \Big)^{\frac{1}{2}}. \end{array}$$

We deduce:

$$\begin{array}{rcl} \int | Kf(x) |^2 d\mu(x) & \leq & c_1 \int \int \big(| k(x,t) | | f(y) |^2 d\mu(x) d\mu(y). \\ & = & c_1 \int | f(x) |^2 \Big(\int | k(x,t) | d\mu(x)) d\mu(y) \Big). \\ & \leq & c_1 c_2 \| f \|^2. \end{array}$$

Which shows that Kf is well defined, and that $Kf \in \mathbb{L}^2(\mu)$, and that Kf is an operator over $\mathbb{L}^2(\mu)$, verifying $\| Kf \|^2 \leq c_1 c_2 \| f^2$, then $\| K \| \leq \sqrt{c_1 c_2}$.
The operator K defined, then is called the integral operator, and k is called the kernel of K .

2.2.2.Remark

The kernel here is not Ker$K$ which is a subspace of $\mathbb{L}^2(\mu)$

2.2.2.Definition

A metric space is a pair (M, d) where M is a none-empty set and d is a function $d : M \times M \to \mathbb{R}$ satisfying:

(a) $d(x, y) \geq 0, \forall x, y \in M$.

(b) $d(x, y) = 0 \Leftrightarrow x = y$.

(c) Symmetry: $d(x, y) = d(y, x), \forall x, y \in M$.

(d) The triangle inequality: $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in M$.

The function d is said to be a metric on M.

2.2.3.Definition of the density in a metric space

Let $\mathbb{A}$ be a subset of $\mathbb{X}$, when the topology of $\mathbb{X}$ is given by a metric, we have:

$$\overline{\mathbb{A}} = \mathbb{A} \cup \{\lim_{n\to\infty} a_n : a_n \in \mathbb{A}, \forall n \in \mathbb{N}\},$$

then $\mathbb{A}$ is dense in $\mathbb{X}$ if $\overline{\mathbb{A}} = \mathbb{X}$

2.3 Baire's theorem

Let $\mathbb{X}$ be a complete metric space, and $(U_n)_n \geq 1$ a sequence of dense open sets in $\mathbb{X}$. Then $\cap U_n$ is dense in $\mathbb{X}$.
The same thing if $(F_n)_n \geq 1$ a sequence of closed sets of an empty interior in $\mathbb{X}$. Then $\cup F_n$ is of an empty interior in $\mathbb{X}$.
Proof:
Let $B_0 \subset \mathbb{X}$ be an open set such that $B_0 \neq \phi$. Since U_1 is dense, there exists a ball $B_1(x_1, r_1)$, $(r_1 \leq 1)$, such that $\overline{B_1} \subset U_1 \cap B_0$. Sence U_2 is dense, there exists a ball $B_2(x_2, r_2)$, $(r_2 \leq \frac{1}{2})$, such that $\overline{B_2} \subset U_2 \cap B_1$.
step by step we construct $B_n(x_n, r_n)$, with $(r_n \leq \frac{1}{n})$, such that $\overline{B_n} \subset U_n \cap B_{n-1}$, in particular the sequence $\overline{B_n}$ is decreasing, and the sequence x_n is Cauchy, thus converges.
Let x be its limite, for all$n$ $x \in \overline{B_n} \subset U_n$, therefore $x \in \cap U_n$.
Also since $x \in B_0$, we have already shown that $\cap U_n$ intersectes all open sets, therefore it is dense.

2.4 The open map theorem

If $A \in \mathcal{L}(H)$ is a surjective operator, then for all open set $\mathbb{U} \subset H$, $A(\mathbb{U})$ is open.

Proof:

Since A is linear, we only show that the unit ball containes a neighborhood of 0.

Let $B(0,1)$ be the open unit ball of H, and since A is surjective we can write

$$A = \cup_{n\in\mathbb{N}} \overline{nA(B)}$$

, by the theorem of Baire, there exists$n \in \mathbb{N}$ such that $\overline{nA(B)}$ is of a non empty interior. Let $x \in H$,and $\varepsilon > 0$ such that $B(x,\varepsilon) \subset \overline{nA(B)}$.We also have $B(-x,\varepsilon) \subset \overline{nA(B)}$,therefore $\overline{nA(B)}$ being convex,$B(0,\varepsilon) \subset \overline{nA(B)}$, therefore $B \subset \overline{A(\lambda B)}$, where $\lambda = \frac{n}{\varepsilon}$.

Showing that this implies that $B \subset T(2\lambda B)$.

Let $z \in B$, since $z \in \overline{A(\lambda B)}$ there exists x_1 with $\| x_1 \| < \lambda$ and $\| z - Ax_1 \| < \frac{1}{2}$.

The same thing, since $z - Ax_1 \in \overline{A(\frac{\lambda}{2}B)}$, there exists x_2 with $\| x_2 \| < \frac{\lambda}{2}$ and $\| z - Ax_1 - Ax_2 \| < \frac{1}{4}$, thus we constructe by recurrence a sequence (x_k) verifying $\| x_k \| < \frac{\lambda}{2^{k-1}}$ and $\| z - (Ax_1 + ... + Ax_k \| < \frac{1}{2^k}$. The series converging normally, we can put $x = \sum x_k$,$\| x \| < 2\lambda$, therefore $x \in 2\lambda B$.

since A is continuous, we have $Ax = z$.thus $z \in A(2\lambda B)$ which concludes the proof.

2.5 Inverse operator theorem

If $A \in \mathcal{L}(H)$ is a bijective operator, then A^{-1} is also an operator.

If H is a Hilbert space, we can also provide $H \times H$ with the sturcture of a Hilbert space using the scalar product :

$$\langle (x_1, y_1), (x_2, y_2) \rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle.$$

Proof:

Let $A : H \to H$ be a bijective operator, since A is bijective, then let $x', y' \in H$ such that $x' = Ax, y' = Ay$ with $x, y \in H$, and let $\lambda \in \mathbb{C}$:

$$\begin{aligned} A^{-1}(x' + \lambda y') &= A^{-1}(Ax + \lambda Ay) \\ &= A^{-1}A(x + \lambda y) \\ &= x + \lambda y \\ &= (A^{-1}A)x + \lambda(A^{-1}A)y \\ &= A^{-1}Ax + \lambda A^{-1}Ay \\ &= A^{-1}x' + \lambda A^{-1}y' \end{aligned}$$

Then A^{-1} is linear, and since A is a bijective operator, then A^{-1} is continuous.

Thus A^{-1} is an operator.

2.6 Closed graph theorem

The graph of a linear map $A : H \longrightarrow H$ is the part of $H \times H$ defined as :

$$G(A) = \{(h, Ah) : h \in H\}.$$

If $G(A)$ is closed in $H \times H$, then A is continuous.

Proof:

Providing $G(A)$ with the norm :

$$\|.\|_{G(A)} = \|.\| + \|A\|$$

where $\|.\|$ is the norm of H.

Considering the Cauchy sequence (x_n) such that:

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m, n > N : \|x_n - x_m\| + \|A(x_n - x_m)\| < \varepsilon.$$

Since the sequence is Cauchy for the norm, then $x_n \longrightarrow x$. But the sequence (Ax_n) converges as well to a certain element y. But $G(A)$ being closed, meaning that $y = Ax$, therefore (x_n) converges to x for the norm of the graph.

Thus there exists a certain constant C such that :

$$\|x\| + \|Ax\| \leq C\|x\|,$$

Then :

$$\|Ax\| \leq (C - 1)\|x\|.$$

Therefore A is continuous.

2.7 Adjoint of an operator

2.7.1.Proposition-Definition

Let H be a Hilbert space and $A \in \mathcal{L}(H)$, there exists then a unique operator $A^* \in \mathcal{L}(H)$, called adjoint of A which verifies the following relation:

$$\forall x, y \in H : \langle Ax, y\rangle = \langle x, A^*y\rangle.$$

Moreover, we have: $\|A\|_{op} = \|A^*\|_{op}$.

Proof:

According to Cauchy-Schwarz inequality and the definition of the operator norm, we have the following inequality:

$$|\langle Ax, y\rangle| \leq \|A\|_{op}. \parallel x \parallel . \parallel y \parallel .$$

Thus, the map $l_y : x \longrightarrow \langle Ax, y\rangle$ is a continuous linear map over H, and by the Riesz representation theorem, there exists a unique element in H (note it $A^*(y)$), such that :

$$\langle Ax, y\rangle = \langle x, A^*y\rangle.$$

Easily verifying that for all $y, z \in H$ and $\lambda \in \mathbb{K}$, $A^*(y)+\lambda A^*(z)$ verifies the property which defines $A^*(y+\lambda z)$.
By uniqueness $A^*(y)+\lambda A^*(z) = A^*(y+\lambda z)$. Which proves that A^* is linear.
Finally we calculate the operator norm of A^*.

$$\begin{aligned}
\|A^*\|_{op} &= \sup_{\|x\|\leq 1, \|y\|\leq 1} \mid \langle T^*y, x\rangle|. \\
&= \sup_{\|x\|\leq 1, \|y\|\leq 1} \mid \overline{\langle x, A^*y\rangle} \mid . \\
&= \sup_{\|x\|\leq 1, \|y\|\leq 1} \mid \langle x, A^*y\rangle \mid . \\
&= \sup_{\|x\|\leq 1, \|y\|\leq 1} \mid \langle Ax, y\rangle|. \\
&= \|A\|_{op}.
\end{aligned}$$

2.7.2.Proposition

Let $A, B \in \mathcal{L}(H)$ and $\alpha \in \mathbb{C}$, then:

(a) $(\alpha A+B)^* = \overline{\alpha}A^* + B^*$.($A \longrightarrow A^*$ is antilinear).

(b) $(AB)^* = B^*A^*$.

(c) $(A^*)^* = A$.

(d) If A is invertible of a inverse A^{-1}, then A^* is invertible and $(A^*)^{-1} = (A^{-1})^*$.

Proof:

(a) Using the definition of the inner product and the adjoint. For $x, y \in H, A, B \in \mathcal{L}(H)$, and $\alpha \in \mathbb{C}$, we have:

$$\begin{aligned}
\langle x, (\alpha A+B)^*(y)\rangle &= \langle(\alpha A+B)(x), y\rangle. \\
&= \alpha\langle Ax, y\rangle + \langle Bx, y\rangle. \\
&= \langle x, \overline{\alpha}A^*(y)\rangle + \langle x, B^*(y)\rangle. \\
&= \langle x, (\overline{\alpha}A^* + B^*)(y)\rangle.
\end{aligned}$$

Thus $(\alpha A+B)^* = \overline{\alpha}A^* + B^*$.

(b) By the definition of the adjoint.

$$\begin{aligned}
\langle(AB)^*(x), y\rangle &= \langle x, ABy\rangle. \\
&= \langle A^*x, By\rangle. \\
&= \langle B^*A^*x, By\rangle.
\end{aligned}$$

Thus $(AB)^* = B^*A^*$.

(c)

$$\begin{aligned}\langle Ax, y\rangle &= \langle x, A^*y\rangle.\\ &= \overline{\langle A^*y, x\rangle}.\\ &= \overline{\langle y, (A^*)^*x\rangle}.\\ &= \langle (A^*)^*x, y\rangle.\end{aligned}$$

. Thus $(A^*)^* = A$.

2.7.1.Remark

$A \in \mathcal{L}(H)$ is invertible if there exists $B \in \mathcal{L}(H)$ such that :

$$AB = BA = Id.$$

The operator B then is unique, and we note it A^{-1}.

2.7.3.Proposition

If $A \in \mathcal{L}(H)$, then :

$$\| A \| = \| A^* \| = \| AA^* \|^{\frac{1}{2}}.$$

meaning that A^* is isometric.

Proof:

For $h \in H$ with $\| h \| \leq 1$, we have :

$$\| Ah \|^2 = \langle Ah, Ah\rangle = \langle A^*Ah, h\rangle \leq \| A^*Ah \| \,.\, \| h \| \leq \| A^*A \| \leq \| A^* \| \,.\, \| A \| \,.$$

And then we consider the supremum on h,$\| A \|^2 \leq \| A^*A \| \leq \| A^* \| \,.\, \| A \|$.

Simplifying by $\| A \|$, we obtain:$\| A \| \leq \| A^* \|$, replacing $\| A \|$ by $\| A^* \|$.

We obtain $\| A^* \| \leq \| A^{**} \| = \| A \|$, thus $\| A \| = \| A^* \|$

2.7.4.Proposition

Let $S : l^2 \longrightarrow l^2$ be the shift defined by $S(\alpha_1, \alpha_2, ...) = (0, \alpha_1, \alpha_2, ...)$,then S^* is defined by $S^*(\alpha_1, \alpha_2, ...) = (\alpha_2, \alpha_3, ...)$.

Proof:

Let (α_n) and (β_n) in l^2. Then:

$$\begin{aligned}\langle S^*(\alpha_n), (\beta_n)\rangle &= \langle (\alpha_n), S(\beta_n)\rangle.\\ &= \langle (\alpha_1, \alpha_2, ...), (0, \beta_1, \beta_2, ...).\\ &= \alpha_2\overline{\beta_1} + \alpha_3\overline{\beta_2} + ...\\ &= \langle (\alpha_2, \alpha_3, ...), (\beta_1, \beta_2, ...)\rangle.\end{aligned}$$

Thus S^* is well given by the annonced formula above.

2.7.2.Definition

An operator $A \in \mathcal{L}(H)$ is called:

1)**Hermetian** or self-adjoint if: $A^* = A$.

2) **Positive** if it is hermtian and $\langle Ah, h\rangle \geq 0, \forall h \in H$.

3) **Unitary** if A is invertible and $A^* = A^{-1}$.

4)**Normal** if $A^*A = AA^*$.

2.7.5.Proposition(the norm of a self-adjoint operator)

If $A \in \mathcal{L}(H)$ is self-adjoint, then:

$$\| A \| = \sup_{\|h\|=1} |\langle Ah, h\rangle|.$$

Proof:

Let $M = \sup\{\langle Ah, h\rangle : \|h\| = 1\}$. If $\|h\| = 1$, then $|\langle Ah, h\rangle| \leq \| A \|$ which implies $M \leq \| A \|$. On the other hand and first of all noticing that for all $f \in H$:

$$|\langle Af, f\rangle| \leq M \| A \|^2.$$

If $\| f \| = \| g \|$, then:

$$\langle A(h \pm g), h \pm g\rangle = \langle Ah, h\rangle \pm \langle Ah, g\rangle \pm \langle Ag, h\rangle + \langle Ag, g\rangle.$$

$$= \langle Ah, h\rangle \pm \langle Ah, g\rangle \pm \langle g, A^*h\rangle + \langle Ag, g\rangle.$$

And because $A = A^*$, this implies:

$$\langle A(h \pm g), h \pm g\rangle = \langle Ah, h\rangle \pm 2\mathcal{R}\langle Ah, g\rangle + \langle Ag, g\rangle.$$

Subtracting one from the other of these inequalities, we obtain:

$$4\mathcal{R}\langle Ah, g\rangle = \langle A(h + g), h + g\rangle - \langle A(h - g), h - g\rangle \leq M(\| h + g \|^2 + \| h - g \|^2).$$

We obtain then, by the identity of the parallelogram:

$$4\mathcal{R}\langle Ah, g\rangle \leq 2M(\| h \|^2 + \| g \|^2) = 4M.$$

Now let $\theta \in [0, 2\pi]$, such that $\langle Ah, g\rangle = e^{i\theta} \mid \langle Ah, g\rangle \mid$.Applying the previous inequality with $e^{i\theta}h$ instead of h, we obtain then $\mid \langle Ah, g\rangle \mid \leq M.$, when $\| h \| = \| g \| = 1$.and taking the supremum on g and h, we obtain $\| A \| \leq M$.

2.7.1.Corollary

If $A = A^*$ and $\langle Ah, h\rangle = 0$ for all h, then $A = 0$.

2.7.2.Remark

This corollary is not true for $A \neq A^*$, for example $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over $\mathbb{R}^2$.

Let $h = (h_1, h_2)$, we have:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = (h_2, -h_1)$$

and

$$\langle Ah, h\rangle = (h_2, h_1) \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = h_2h_1 - h_1h_2$$

Then $\langle Ah, h\rangle = 0$ for all $h \in \mathbb{R}^2$.

But

$$A^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{Tr} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq A$$

If H is a complex Hilbert space, and if $A \in \mathcal{L}(H)$,then the operators $B = \frac{A+A^*}{2}, C = \frac{A-A^*}{2i}$are self-adjoint and $A = B + iC$.

The operators B and C are respectively called the real part and the imaginary part of A.

2.7.6.Proposition

Let $A \in \mathcal{L}(H)$, the following assertions are equivalent:

a) A is normal.

b) $\| Ah \| = \| A^*h \|$ for all h.

In the complex case these assertions are equivalent to :

The real and imaginary parts of A commute.

Proof:

If $h \in H$,then: $\| Ah \|^2 - \| A^*h \|^2 = \langle Ah, Ah\rangle - \langle A^*h, A^*h\rangle = \langle (A^*A - AA^*)h, h\rangle$. Because $(A^*A - AA^*)^* = A^*A - AA^*$(hermetian), and since A is normal meaning that $AA^* = A^*A$, then $\langle (A^*A - AA^*)h, h\rangle = 0$, then according to the previous corollary .

$\| Ah \|^2 - \| A^* \|^2 = 0$,which means $\| Ah \| = \| A^* \|$ for all h.

If B and$C$ are the real and imaginary parts of A, then we calculate:

$$\begin{aligned} A^*A &= B^2 - iCB + iBC + C^2. \\ AA^* &= B^2 + iCB - iBC + C^2. \end{aligned}$$

Then$A^*A = AA^*$ if and only if $BC = CB$

2.7.2.Remark

An operator $A \in \mathcal{L}(H)$ is called isometry if it preserves the norm, which means:

$$\| Ah \| = \| h \|, \forall h.$$

2.7.7.Proposition

Let $A \in \mathcal{L}(H)$, the following assertions are equivalent:

(a) A is an isometry.

(b) $A^*A = Id$.

(c) $\langle Ah, Ag\rangle = \langle h, g\rangle$,for all $h, g \in H$.

Proof:
We know that: $\langle A^*Ah, g\rangle = \langle Ah, Ag\rangle$, and putting $h = g$ we obtain $\langle A^*Ah, h\rangle = \langle Ah, Ah\rangle$, and since A is an isometry then: $\langle A^*Ah, h\rangle = \langle Ah, Ah\rangle = \langle h, h\rangle$ therefore $A^*A = Id$
since $\langle A^*Ah, g\rangle = \langle Ah, Ag\rangle$, we can see easily that (2) and(3) are eauivalent.
Putting $g = h$ in (3),we get $\langle Ah, Ah\rangle = \langle h, h\rangle \Leftrightarrow \| Ah \| = \| h \|$,then A is an isometry.

2.7.1 Theorem

Let $A \in \mathcal{L}(H)$, then:

$$KerA = (RanA^*)^{\perp}.$$

Proof:
If $h \in KerA$ and $g \in H$ then:

$$\langle h, A^*g\rangle = \langle Ah, g\rangle = 0.$$

Then:

$$KerA \subset (RanA^*)^{\perp}.$$

For the other way if $h \perp RanA^*$ and $g \in H$, then:$\langle Ah, g\rangle = \langle h, A^*g\rangle = 0$, and therefore :

$$(RanA^*)^{\perp} \subset KerA.$$

2.7.1.Examples of the adjoint:

(a) Let $H = \mathbb{L}^2(\Omega, \mu)$ et $f \in \mathbb{L}^\infty(\Omega, \mu)$.Let M_f defined by:

$$M_f(g) = fg.$$

M_f is linear, continuous, of a norm $\|f\|_\infty$, and $M_f^* = M_{\overline{f}}$.
Because: let $g, h \in \mathbb{L}^\infty(\Omega, \mu)$

$$\begin{aligned}\langle M_f g, h\rangle &= \overline{\langle h, M_f g\rangle}.\\ &= \overline{\langle h, fg\rangle}.\\ &= \overline{f}\overline{\langle h, g\rangle}.\\ &= \overline{f}\langle g, h\rangle.\\ &= \langle g, \overline{f}h\rangle.\\ &= \langle g, M_{\overline{f}}h\rangle.\end{aligned}$$

Thus $M_f^* = M_{\overline{f}}$.

(b) Let $H = \mathbb{L}^2([-\pi, \pi])$, considering the integral operator which for all $x \in H$ associates :

$$(Ax)(t) = \int_{-\pi}^{\pi} cos(t-s)x(s)ds(t \in [-\pi, \pi]).$$

$$\begin{aligned}\langle Ax, y\rangle &= \textstyle\int_{-\pi}^{\pi}(Ax)(t)\overline{y(t)}dt.\\ &= \textstyle\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} cos(t-s)x(s)ds\right)\overline{y(t)}dt.\\ &= \textstyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x(s)\overline{y(t)}cos(t-s)dsdt.\\ &= \textstyle\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x(s)\left(\overline{y(t)}cos(s-t)dt\right)ds.\\ &= \textstyle\int_{-\pi}^{\pi} x(s)\overline{Ay(s)}ds.\\ &= \langle x, Ay\rangle.\end{aligned}$$

By the definition of A^*. $\langle Ax, y\rangle = \langle x, A^*y\rangle$, thus $\langle x, Ay\rangle = \langle x, A^*y\rangle$ for all $x \in H$, then $Ay = A^*y$, therefore T is self adjoint

2.7.8.Proposition

Let $\mathbb{E}, \mathbb{F}$ be two Hilbert spaces, and let $A \in \mathcal{L}(\mathbb{E}, \mathbb{F})$. Then:

$$\mathbb{F} = Ker(A^*)\bigoplus^{\perp}\overline{Ran(A)}, \mathbb{E} = Ker(A)\bigoplus^{\perp}\overline{Ran(A^*)}.$$

Where $\overline{Ran(A}$ and $\overline{Ran(A^*)}$ are the closures (for the norm)of $Ran(A)$ and (A^*) respectevely.

Proof:

It is enough to prove only the first assertion, the second one is obtained by changing the roles of A and A^*. We have the following equivalences:

$$\begin{aligned} y \in KerA^* &\Leftrightarrow \forall x \in \mathbb{E} : \langle A^* y, x \rangle = 0. \\ &\Leftrightarrow \forall x \in \mathbb{E} : \langle y, Ax \rangle = 0. \\ &\Leftrightarrow y \perp Ran(A). \end{aligned}$$

The continuity of the scalar product (a result of Cauchy Schwarz inequality)implies that:

$$y \perp Ran(A) \Leftrightarrow y \perp \overline{Ran(A)}.$$

Thus the orthogonal of $KerA^*$ is the closure of the image of A.

CHAPTER 3

COMPACT OPERATORS

3.1 Introduction

We introduce in this chapter the concept of compact operators, which in a certain way behave the same way as the operators in a finite dimension.

Noting $B_H = \{\|x\| \leq 1, x \in H\}$ the closed unit ball of H.

3.2 Compact operators

3.2.1.Definition

Let $A \in \mathcal{L}(H)$.
A is a compact operator if $\overline{A(B_H)}$ is compact in H.
Noting $\mathcal{H}(H)$ the set of compact operators.

3.2.1 Theorem

Let H be a Hilbert space, and B_H is its unit ball. Then B_H is compact if and only if H is of a finite dimension.

3.2.1.Remark

If H is infinite dimensional then the preceding theorem implies that Id is not compact.

3.2.1.Proposition

$\mathcal{H}(H)$ is a closed ideal of the Algebra $\mathcal{L}(H)$, meaning that :

(a) $\mathcal{H}(H)$ is a closed subspace of $\mathcal{L}(H)$(for the operator norm).

(b) If $A \in \mathcal{L}(H)$ and $B \in \mathcal{H}(H)$, then AB and BA are compacts .

3.2.2.Proposition

Let $A \in \mathcal{L}(H)$. A is compact operator if and only if every bounded sequence $(x_n) \in H$ has a sub-sequence (x_{n_j}) for which (Ax_{n_j}) converges.

3.2.2 Theorem

Let $A, B \in \mathcal{L}(H)$ be a compact operators, and let $\alpha \in \mathbb{C}$. Then $A + B$ and αA are compact.

Proof:

We use the criterion from the preceding proposition. Given a sequence $(x_n) \in H, \| x_n \| \leq C$, Picking a sub-sequence (x'_n) for which (Ax'_n) converges and the a sub-subsequence $x"_n$ for which $(Bx"_n)$ converges, too. Then $(A + B)(x"_n), \alpha(Ax"_n)$ all converge

3.2.3 Schauder's theorem

Let $A \in \mathcal{L}(H)$, we claim that its dual operator $A^* : H \to H$ is also compact.

3.2.2.Definition

An operator $A \in \mathcal{L}(H)$ is said to be of a finit rank, if $Ran(A)$ is of a finite dimension. Its rank is the dimension of $Ran(A)$.

3.2.2.Remark

If an operator $A \in \mathcal{L}(H)$ is of a finit rank, then A is compact.

3.2.4 Theorem

Let $A \in \mathcal{L}(H)$. The following assertions are equivalent:

a. A is compact.

b. A^* is compact.

c. There exists a sequence of operators (A_n) of a finite rank, which converges to A (for the operator norm).

Proof:
$(c) \Rightarrow (a)$. It is a consequence of the fact that $\mathcal{H}(H)$ is closed and that an operator of a finite rank is compact.
$(a) \Rightarrow (c)$. Let $L = \overline{Ran(A)}$. If L is of a finite dimension then the result is evident. Otherwise let $(e_1, e_2, ...)$ be a Hilbertian basis of L. Let P_n be the orthogonal projection over $VECT\{e_1, ..., e_n\}$. Puting $A_n = P_n A$, it is clear that A_n is of a finite rank. We firstly verify that :

3.2.1.Lemma

If $h \in H$, then the sequence $(A_n h)$ converges to Ah.
In fact puting $k = Ah$, and $\alpha_n = \langle k, e_n \rangle$. Then $A_n h = \alpha_1 e_1 + ... + \alpha_n e_n$, and then :

$$\|Ah - A_n h\|^2 = \sum\nolimits_{k=n+1}^{\infty} | \alpha_k |^2.$$

Since this series converges, this quantity tends to 0.
If A is compact, then for all $\varepsilon > 0$, $A(B_h)$ is continuous in the union of a finite balls of the radius $\frac{\varepsilon}{3}$.
Let $(Ah_1, ..., Ah_m)$be the centers of these balls. If $\| h \| \leq 1$, there exists a j such that :

$$\| Ah - Ah_j \| \leq \frac{\varepsilon}{3}$$

And then for all n:

$$\begin{aligned} \| Ah - A_n h \| &\leq \| Ah - Ah_j \| + \| Ah_j - A_n h_j \| + \| Pn(Ah_j - Ah) \| \\ &\leq 2 \| Ah - Ah_j \| + \| Ah_j - A_n h_j \| \\ &\leq \tfrac{2\varepsilon}{3} + \| Ah_j - A_n h_j \| \end{aligned}$$

According to the lemma, there exists an integer n_0 such that if $n \geq n_0$, then:

$$\| Ah_j - A_n h_j \| < \frac{\varepsilon}{3}.$$

For all j.

Thus $\| (A - A_n)h \| \leq \varepsilon$, for all h verifying $\| h \| \leq 1$, and then $\| A - A_n \| \leq \varepsilon$, for all $n \geq n_0$, which shows that the sequence (A_n) converges to A.

$(b) \Leftrightarrow (c)$. This equivalence is a consequence of the equivalence: A is of a finite rank $\Leftrightarrow$ A^* is of a finite rank.

3.2.3.Remark

During the implication $(a) \Longrightarrow (c)$, we have in fact shown the following corollary.

3.2.1.Corollary

If $A \in \mathcal{L}(H)$ is a compact operator, and $e_1, e_2, ...$ is an orthonormal basis of H, and if we denote P_n the orthogonal projection over$VECT\{e_1, ..., e_n\}$, then $\| P_n A - A \| \to 0$.

3.2.3.Definition

Let $A \in \mathcal{L}(H)$.

(a) A complex number λ is an eigen value of A if $Ker(A - \lambda Id) \neq 0$. If h is a non-zero vector in $Ker(A - \lambda Id)$, h is called an eigen vector for A, thus $Ah = \lambda h$.

(b) We call a punctual spectrum of A, $\sigma_p(A)$, the set of the eigen values of A.

(c) The spectrum of A is defined by:

$$\sigma(A) = \{\lambda \in \mathbb{C} : \quad A - \lambda Id \quad \textbf{is not invertible}.\}$$

As an invertible operator is necessarily injective, we have

$$\sigma_p(A) \subset \sigma(A).$$

3.2.3.Proposition

Let A be a compact operator, and $\lambda \in \sigma_p(A)$. If $\lambda \neq 0$, the eigen space $Ker(A - \lambda Id)$ is of a finite dimension.

Proof:

supposing by absurd that $Ker(A - \lambda Id)$ contains an infinite orthonormal sequence e_n. Since A is compact, we can extract a subsequence (e_{n_k}) such that (Ae_{n_k}) converges. But for $n_k \neq n_j$ we have:

$$\| Ae_{n_k} - Ae_{n_j} \| = | \lambda | . \| e_{n_k} - e_{n_j} \| = \sqrt{2}\lambda,$$

which contradicts the fact that (Ae_{n_k}) is Cauchy.

The following proposition is usefull to prove that a compact operator has an eigen values.

3.2.4.Proposition

Let A be a compact operator, and $\lambda \neq 0$. If

$$\inf\{\|(A - \lambda Id)h\| : \|h\| = 1\} = 0,$$

Then $\lambda \in \sigma_p(A)$.

Proof:

Hypothetically thre exists a sequence h_n of vectors of norm equals 1 such that

$$\|(A - \lambda Id)h_n\|h_n \longrightarrow 0.$$

Since A is compact, there exists a subsequence (h_{n_k}) such that (Ah_{n_k}) converges to $f \in H$. Writing $h_{n_k} = \lambda^{-1}\big[Ah_{n_k} - (A - \lambda Id)h_{n_k}\big]$, we can see that (h_{n_k}) converges to $\lambda^{-1}f$. Particularly $\|f\| = |\lambda|$, so $f \neq 0$. In addition, (Ah_{n_k}) converges then to $\lambda^{-1}Af$, which shows that $\lambda^{-1}Af = f$, thus $Af = \lambda f$.

So we found an eigen vector, and $\lambda \in \sigma_p(A)$.

3.3 Diagonalization of self-adjoint and normal compact operators

3.3.1 Theorem

Let $A \in \mathcal{L}(H)$ be a self-adjoint compact operator. Then there exists a real sequence (λ_n) tending to 0, and an orthonormal family $\{e_n\}$ in H such that, if P_n represents the projection over $VECT\{e_n\}$,

$$A = \sum_{n=1}^{\infty} \lambda_n P_n,$$

The convergence is taking place within the meaning of the operator norm.
Before proving the theorem(3.3.1), let's state a few general results.

3.3.1.Definition

A closed subspace E is called invariant for A if $A(E) \subset E$ and $A(E^{\perp}) \subset E^{\perp}$.
In another way saying that E is a reducing subspace for A.
The following lemma is easy

3.3.1.Lemma

Let $E \subset H$ be a closed subspace, and let $A \in \mathcal{L}(H)$. Then E is invariant by A if and only if $E^{\perp}$ is invariant by A^*.

3.3.1.Proposition

If A is a normal operator and $\lambda \in \mathbb{K}$, then :

$$Ker(A - \lambda Id) = Ker(A - \lambda Id)^*$$

In addition the space $Ker(A - \lambda Id)$ is invariant for A and A^*(a reducing space for A and A^*) .

3.3.2.Proposition

If A is a normal operator and λ, μ are 2 distinct scalars, then:

$$Ker(A - \lambda Id) \perp Ker(A - \mu Id).$$

3.3.3.Proposition

If A is a self-adjoint operator, then:

$$\sigma_p(A) \subset \mathbb{R}$$

To prove theorem(3.3.1), we have to show the eigen values of A. And for that, the following lemma is the key.

3.3.2.Lemma

If A is a self-adjoint operator, then one of the two values $\pm \parallel A \parallel$ is an eigen value of A.

Proof:

If $A = 0$ it is evident. If not, then :

$$\parallel A \parallel = \sup\{|\langle Ah, h\rangle|, \|h\| = 1\}$$

This provides a sequence (h_n) of vectors of a norm equals 1 such that:

$$|\langle Ah_n, h_n\rangle| \longrightarrow \parallel A \parallel$$

Even if it means replacing (h_n) by one of its subsequences, we can suppose that :

$$|\langle Ah_n, h_n\rangle| \longrightarrow \lambda$$

With $\lambda = \pm \parallel A \parallel$. We have:

$$0 \leq \parallel (A - \lambda Id)h_n \parallel^2 = \parallel Ah_n \parallel^2 - 2\lambda\langle Ah_n, h_n\rangle + \lambda^2 \longrightarrow 0,$$

and then $\lim \parallel (A - \lambda Id)h_n \parallel = 0$, and we already know that if $\inf\{\|(A - \lambda Id) : \|h\| = 1\} = 0$, then $\lambda = 0$, which implies $\lambda \in \sigma_p(A)$.

Proof of theorem(3.3.1):

From the previous lemma, $\sigma_p(A)$ contains a real number, λ_1, equals $\pm \parallel A \parallel$.

Let $E_1 = Ker(A - \lambda_1 Id)$ and π_1 the projection over E_1. Let $H_2 = E_1^{\perp}$. Thus E_1 and H_2 are invariant of A. We call A_2 the restriction of A to H_2, we verify that $A_2 \in \mathcal{L}(H_2)$ is a self-adjoint compact operator.

Applying the previous lemma to A_2. Then $\sigma_p(A_2)$contains a real number, λ_2, equals $\pm \parallel A_2 \parallel$.

Puting $E_2 = Ker(A_2 - \lambda_2 Id) \subset Ker(A - \lambda_2 Id)$. In addition we necessarily have $\lambda_1 \neq \lambda_2$ because $E_1 \perp E_2$. Denoting π_2 the projection over E_2 and $H_3 = (E_1 \bigoplus E_2)^{\perp}$. Also denoting $\parallel A_2 \parallel \leq \parallel A \parallel$ and $| \lambda_2 | \neq | \lambda_1 |$.

We construct then by recurrence (λ_n) of distinct elements of $\sigma_p(A)$ such that :

1. $\lambda_1 \geq \lambda_2 \geq ...$
2. Si $E_n = Ker(A - \lambda_n)$, then $| \lambda_{n+1} | = \parallel A_{(E_1 \bigoplus ... \bigoplus E_n)^{\perp}} \parallel$.

The first property implies that the sequence (h_n) converges to $\alpha > 0$. Proving that $\alpha = 0$. Othewise we can choose a vector $e_n \in E_n$ with $\parallel e_n \parallel = 1$. We have $Ae_n = \lambda_n e_n$, then $\parallel Ae_n + e_m \parallel^2 = {\lambda_n}^2 + {\lambda_m}^2 \geq 2\alpha^2$. none of the subsequences of (Ae_n) is Cauchy. But since A is compact we can extract from (Ae_n) a convergent subsequence, which is absurd. Noting π_n the projection over E_n, we verify that:

$$\| A - \sum_{j=1}^{n} \lambda_j \pi_j \| = \| A_{(E_1 \bigoplus ... \bigoplus E_n)^{\perp}} \| = | \lambda_{n+1} | \longrightarrow 0.$$

Which proves that the serie $\sum \lambda_n \pi_n$ converges normaly to A. To obtain the wanted formula by the theorem, we choose an orthonormal basis in every eigen space E_n. This induces a writing of π_n as the sum of a finite number of projectors of rank 1. Then we gather the orthonormal bases of E_n in an orthonormal family of H.
Before continuing let's state a theorem about operators which commute with a diagonal operator.

3.3.4.Proposition

Let $\{P_i : i \in I\}$ be a family of pairwise orthogonal projections in $\mathcal{L}(H)$ ($P_iP_j = P_jP_i = 0, i \neq j$), if $h \in H$, then $\sum_{i \in I} P_i h$ converges in H to Ph, where P is the projection of H onto $V\{P_iH, i \in I\}$.

3.3.2.Definition

The partition of the identity on H is a family $\{P_i, i \in \mathbb{Z}\}$ of pairwise orthogonal projections on H such that $V_iP - iH = H$. This might be indicated by $1 = \sum_i P_i$ or $1 = \bigoplus_i P_i$.(Note that 1 is often used to denote the operator$A$ in H such that $Ah = h, \forall h$).

3.3.3.Definition

An operator A on H is diagonalizable if there exists a partition of the identity on H, $\{P_i : i \in I\}$, and a family of scalars $\{\alpha_i : i \in I\}$ such that $sup_i \| \alpha_i \| < \infty$ and
$Ah = \alpha_i h, \forall h \in Ran(P_i)$.
To denote a diagonalizable operator satisfying the conditions of definition(3.3.2), write:

$$A = \sum \alpha_i P_i.$$

Or

$$A = \bigoplus_i \alpha_i P_i.$$

3.3.5.Proposition

An operator A on H is diagonalizable if and only if there is an orthonormal basis for H consisting of eigen vectors.

3.3.2 Theorem

If $A = \bigoplus_i \alpha_i P_i$ is diagonalizable and all the α_i are distinct, then an operator $B \in \mathcal{L}(H)$ satisfies $AB = BA$ if and only if for each i, $RanP_i$ reduces B.

Proof:

If all the α_i are distinct, then $RanP_i = Ker(A - \alpha_i)$. If $AB = BA$ and $Ah = \alpha_i h$, then $ABh = BAh = B(\alpha_i h) = \alpha_i Bh$, hence $Bh \in RanP_i, \forall h \in RanP_i$. Thus $RanP_i$ is left invariant by B. Therefore B leaves $V\{RanP_j, j \neq i\} = \mathcal{N}_i$ invariant but since $\bigoplus_i P_i = 1, \mathcal{N}_i = RanP_i$. Thus P_i reduces B.

Now assume that B is reduced by each $RanP_i$. Thus $BP_i = P_i B$ for all i. If $h \in H$, then $Ah = \sum_i \alpha_i P_i h$. Hence:

$$BAh = \sum_i \alpha_i PBh = ABh.$$

Using the notation of the preceding theorem. If $AB = BA$. Let $B_i = B \mid RanP_i$. then it is appropriate to write $B = \bigoplus_i B_i$ on $H = \bigoplus_i (P_i H)$. One might paraphrase theorem (3.3.2) by saying that B commutes with a diagonalizable operator if and only if B can be diagonalized with operator entries.

3.3.3 Spectral theorem for compact normal operators

Let P_n be a sequence of two by two orthogonal projectors, and λ_n isa non-zero bounded sequence of scalars two by two distinct.

Let $A = \sum_n \lambda_n P_n$ be the diagonal operator. And let $B \in \mathcal{L}(H)$, the following two assertions are equivalent:

1. $AB = BA$.
2. For all n, the subspace $RanP_n$ is invariant by B and B^*.

Proof:

$1 \Rightarrow 2$. Supposing that $AB = BA$ and let $h \in Ran(P_n)$. Since the λ_n are two by two distinct, $RanP_n = Ker(A - \lambda_n Id)$,and so$Ah = \lambda_n h$. Then $\lambda_n Bh = B(\lambda_n h) = BAh = ABh$, and since $\lambda_n \neq 0$ this implies that $B \in RanP_n$ and then $RanP_n$ is invariant by B.

we can apply the same reasoning to $A^* = \sum \overline{\lambda_n} P_n$ and B^*(which commute with A^*). Thus B^* also lets $RanP_n$ invariant.

$2 \Rightarrow 1$. We have $B(RanP_n \subset RanP_n$, which implies that $P_n B P_n = BP_n$. We have then $P_n B = BP_n$. We can easily deduce that $AB = BA$.

In the complex case we can now extend the theorem (3.3.1) to normal compact operators.

3.3.4 Theorem

Let H be a complex Hilbert space and $T \in \mathcal{L}(H)$ a normal compact operator. Then there exists a sequence (λ_n) of a complex numbers tending to 0, and an orthonormal family $\{e_n\}$ in H such that, if P_n represents a projection over $VECT\{e_n\}$,

$$T = \sum_{n=1}^{\infty} \lambda_n P_n,$$

the convergence is in the operator norm sens.

Proof:

Defining the real and imaginary parts of T by :$A = \frac{T+T^*}{2}$ and $B = \frac{T-T^*}{2i}$. Verifying that A and B are two self-adjoint compact operators which commute. We apply theorem(3.3.1) to A, by cutting according to the eigen spaces we obtain:

$$A = \sum \alpha_n P_n$$

where α_n are non-zero real distinct numbers and P_n is a finite rank projectors. Because $AB = BA$, theorem (3.3.2) implies that for all n, $RanP_n$ is invariant by B. Verifying that the restriction $B_{|RanP_n} \in \mathcal{L}(RanP_n)$ is a self-adjoint operator.

Applying theorem(3.3.1) to $B_{|RanP_n}$ (or the version<finite dimention> of the spectral theorem), which allows to decompose

$$B_{|RanP_n} = \sum_k \beta_k^{(n)} Q_k^{(n)},$$

Where $(Q_k^{(n)})$ are the projectors of rank equals 1. Likewise, $KerA$ is invariant by B. Also apply theorem(3.3.1) to$B_{|KerA}$. We gather the obtained orthonormal bases for every application of theorem (3.3.1), which gives a sequence P_N of a two by two orthogonal projectors of a rank equals 1 and two sequences α_n and β_n tending to 0, such that:

$$A = \sum_{n=1}^{\infty} \alpha_n P_n,$$

$$B = \sum_{n=1}^{\infty} \beta_n P_n.$$

Puttin $\lambda_n = \alpha_n + \beta_n$, we get the desired writing for T.

A consequence of theorem (3.3.1) is that we can define $f(A)$ where A is a normal compact operator and f is a bounded function from $\mathbb{C}$ onto$\mathbb{C}$.

3.3.3.Definition

Noting $l^{\infty}(\mathbb{C})$ the set of the bounded functions from $\mathbb{C}$ onto$\mathbb{C}$. If A is a normal compact operator written in the form of theorem (3.3.3) and $\varphi \in l^{\infty}(\mathbb{C})$, we put:

$$\varphi(A) = \sum_{n=1}^{\infty} \varphi(\lambda_n) P_n + \varphi(0) P_0$$

whereP_0 is the orthogonal projection over $KerT$.

3.3.5 Theorem(bounded functional calculus for normal compact operators)

Let A be a normal compact operator over a complex Hilbert space H. The mapping: $\varphi \longmapsto \varphi(A)$, from $l^{\infty}(\mathbb{C})$ onto $\mathcal{L}(H)$, has the following properties:

1. $\varphi \longmapsto \varphi(A)$ is multiplicative linear, in the sense where $(\varphi\psi)(A) = \varphi(A)\psi(A)$.
2. If $\varphi \equiv 1$, then $A = Id$. If $\varphi(z) = z$ for all $z \in \sigma_p(A) \cup \{0\}$, then $\varphi(A) = A$.
3. $\| \varphi(A) \| = \sup\{|\varphi(\lambda)| : \lambda \in \sigma(A)\}$.
4. $\varphi(A)^* = \overline{\varphi}(A)$.
5. If $B \in \mathcal{L}(H)$ verifies $BA = AB$, then $B\varphi(A) = \varphi(A)B$ for all $\varphi \in l^{\infty}(\mathbb{C})$.

3.4 Spectrum of an operator

Starting by an important lemma based on geometric series.

3.4.1.Lemma

Let $A \in \mathcal{L}(H)$ with $\| A - Id \| < 1$. Then A is invertible and we have the following formula(called the series of Neumann)

$$A^{-1} = \sum_{n=0}^{\infty} (Id - A)^n.$$

Proof:

Putting $S = Id - A$, and let $r = \|S\|$, we have $r < 1$. Because $\|S^n\| \leq \|S\|^n = r^n$,the series $\sum_{n\geq 0} \| S^n \|$ converges. And thus the series $\sum_{n=0}^{\infty} S^n$ converges normally. We have:

$$(Id + S + S^2 + ... + S^n)(Id - S) = (Id + S + ... + S^n) - (S + S^2 + ... + S^{n+1}) = Id - S^{n+1}.$$

Since $\| S^{n+1} \| \longrightarrow 0$, we have $R(Id - S) = Id$. Likewise, $(Id - S)R = Id$. Then $A = Id - S$ is invertible and $A^{-1} = R$.

The completness of the space $\mathcal{L}(H)$ is used to justify the convergence of the defined series R.

3.4.1.Corollary

Let A_0 be an invertible operator. If an operator A verifies $\| A - A_0 \| < \| A_0^{-1} \|^{-1}$, then A is invertible.

Proof:

We have $\| A_0^{-1} A - Id \| = \| A_0^{-1}(A - A_0) \| \leq \| A_0^{-1} \| . \| A - A_0 \| < 1$. According to thr previous lemma $A_0^{-1} A$ is invertible.

3.4.2.Lemma

The set G of the invertible elements of $\mathcal{L}(H)$ is open (for the topology induced by the operator norm). Moreover, the mapping $A \longmapsto A^{-1}$ is continuous from G onto G.

Proof:

The fact that G is open is a consequence of the previous corollary.

Firstly proving that $A \longmapsto A^{-1}$ is continuous at Id. Let (A_n) be a converging to Id sequence. Let $0 < \delta < 1$ and supposing that $\| A - Id \| \leq \delta$. By the previous lemma :

$$A_n^{-1} = (Id - (Id - A_n))^{-1} = \sum_{k=0}^{\infty} (Id - A_n)^k = Id + \sum_{k=1}^{\infty} (Id - A_n)^k$$

and then:

$$\|A_n^{-1} - Id\| = \| \sum_{k=1}^{\infty} (Id - A_n)^k \| \leq \sum_{k=1}^{\infty} \| Id - A_n \|^k \leq \frac{\delta}{1 - \delta}.$$

This quantity can be made less than any fixed $\varepsilon > 0$. Chosing a pretty small δ. Then $\| A_n - Id \| < \delta$ implies $\| A_n^{-1} - Id \| < \varepsilon$, which proves the continuity.

Now if a sequence (A_n) of elements of G converges to $A \in G$, the $(A^{-1} A_n)$. By what precedes $(A^{-1n} A) = ((A^{-1} A_n)^{-1})$ converges to Id, multiplying on the right by A^{-1} we obtain that (A_n^{-1}) converges to (A^{-1}).

Reminding of the definition of the spectrum of an operator.

3.4.3.Lemma

The spectrum of A is defined by:

$$\sigma(A) = \{\lambda \in \mathbb{C} : T - \lambda Id \quad \textbf{is not invertible}\}.$$

We call a resolvent set the completness of the spectrum :

$$\rho(A) = \mathbb{C} \backslash \sigma(A).$$

3.4.1 Theorem

Let $T \in \mathcal{L}(H)$. Then $\sigma(T)$ is a non-empty compact set. Moreover $\sigma(T)$ is included in a closed disc of the center 0 and radius $\| T \|$, and the function $\lambda \longmapsto (T - \lambda Id)^{-1}$ is analytic over $\rho(T)$.

Before making the proof, there is a few words are necessary about the analytic functions of a value in a vector space.

If G is open of $\mathbb{C}$ and X is a Banach space, we define the derivative of $f : G \longmapsto X$ at 0 as $\lim_{h \longrightarrow 0} h^{-1}[f(z_0 + h) - f(z_0)]$, since the limit exists. We say that f is analytic if f has a continuous derivative over G.

The whole theory of analytic functions extend to this frame. The statements and proofs of the Cauchy formula of the Liouville theorem are valid word by word in this context. Moreover $f : G \longmapsto X$ is analytic if and only if at every $z_0 \in G$, there exists a sequence (x_n) of elements of X such that:

$$f(z_0) = \sum_{k=0}^{\infty} (z - z_0)^k x_k.$$

For all z such that $| z - z_0 | < r$, where r is the distance of z_0 to ∂G.

Proof of theorem (3.4.1):

The previous corollary implies that $T - \lambda Id$ is invertible since $| \lambda | > \| T \|$. Thus $\sigma(T) \subset D(0, \| T \|)$, then $\sigma(T)$ is bounded.

Let $G \subset \mathcal{L}(H)$ be the open set of invertible operators. The mapping $\lambda \longmapsto \lambda Id - T$ is continuous from $\mathbb{C}$ onto $\mathcal{L}(H)$, then the reciprocal image of G is an open set of $\mathbb{C}$. But this set is none other than $\rho(T)$, and then $\sigma(T)$ is a closed set .

Defining $F : \rho(T) \longmapsto \mathcal{L}(H)$ by $F(\lambda) = (\lambda Id - T)^{-1}$. Using the identity
$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$ with $A = (\lambda + h)Id - T$ and $B = \lambda Id - T$. That gives:

$$\frac{F(\lambda + h) - F(\lambda)}{h} = \frac{((\lambda + h)Id - T)^{-1}(-h)(\lambda Id - T)^{-1}}{h} = -(((\lambda + h)Id - T)^{-1}(\lambda Id - T)^{-1}).$$

According to theorem (3.4.1) $\lim_{h \longrightarrow 0}((\lambda + h)Id - T)^{-1} = (\lambda Id - T)^{-1}$, and then $F'(\lambda)$ exists and $F'(\lambda) = -(\lambda Id - T)^{-2}$. Again according to theorem(3.4.1)

$$F'(\lambda) : \rho(T) \longmapsto \mathcal{L}(H),$$

is continuous, and then F is analytic.

We have seen at the beginig of the proof that $F(z) = z^{-1}(Id - \frac{T}{z})^{-1}$ for $| z | > \| T \|$. Since $Id - \frac{T}{z}$ tends to Id when $| z | \longrightarrow \infty$, its inverse does the same, and then $\lim_{|z| \longrightarrow \infty} F(z) = 0$.

In particular F is bounded.

If $\sigma(T)$ was empty, the the function F would be a bounded integer function, then constant by the theorem of Liouville(vectorial). This constant would be necessarily 0 because of the condition at the infinity, which is absurd because F takes its values in the invertible operators. Thus the spectrum is not empty.

3.4.1.Definition

If A is an operator over H, its spectral radius is defined by:

$$r(A) = \sup_{\lambda \in \sigma(A)} \mid \lambda \mid$$

the supremum is reached(and finite) because $\sigma(A)$ is compact.

We can have $r(A) = 0$ even if $A = 0$, like the matrix $C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ shows in dimension2.

3.4.1.Proposition

The following formula is valid for all operator $A \in \mathcal{L}(H)$

$$r(A) = \lim_{n \longrightarrow \infty} \|A^n\|^{\frac{1}{n}}.$$

Proof :

Putting $G = \{0\} \cup \{z \in \mathbb{C} : z^{-1} \in \rho(A)$. We verify that G is an open set of $\mathbb{C}$. Let $f : G \longmapsto \mathcal{L}(H)$ defined by:

$$\begin{cases} f(0) = 0 \\ f(z) = (z^{-1}Id - A)^{-1} \end{cases}$$

The function f is analytic on $G \setminus \{0\}$ and continuous at 0. It is then analytic on G, and then it is expandable in the neighborhood of 0. The expression

$$A^{-1} = \sum_{n=0}^{\infty} (Id - A)^n$$

explicitly provides this expansion:

$$\begin{aligned} f(z) = (z^{-1}Id - A)^{-1} &= (z^{-1})^{-1}(Id - zA)^{-1} \\ &= z(Id - zA)^{-1} \\ &= z\textstyle\sum_{n=0}^{\infty} (Id - Id + zA)^n \\ &= z\textstyle\sum_{n=0}^{\infty} (zA)^n \\ &= z\textstyle\sum_{n=0}^{\infty} z^n A^n. \end{aligned}$$

The theory of integer series tells us that this series converges when $\mid z \mid < R$ where R is defined as the redius of the biggest open disc that is centred at 0 and contained in G. We verify that:

$$R = \inf\{\mid \lambda \mid : \lambda^{-1} \in \sigma(A)\} = r(A)^{-1}.$$

But the theory of integer series(of a vectorial value) tells us that:

$$R^{-1} = \limsup_n \|A^n\|^{\frac{1}{n}}.$$

We then have shown that:

$$r(A) = \limsup_n \|A^n\|^{\frac{1}{n}}.$$

Moreover if $\lambda \in \mathbb{C}$ and $n \geq 1$, we have the identity:

$$\lambda^n Id - A^n = (\lambda^n Id - A)(\lambda^{n-1} Id + \lambda^{n-2} A + ... + A^{n-1}) = (\lambda^{n-1} Id + \lambda^{n-2} A + ... + A^{n-1})(\lambda^n Id - A).$$

This shows that if $\lambda^n Id - A^n$ is invertible, then $\lambda Id - A$ is also invertible. Thus if $\lambda \in \sigma(A)$, then $\lambda^n \in \sigma(A^n)$, and then by theorem (3.4.1): $|\lambda^n| \leq \|A^n\|$, we deduce then for all $\lambda \in \sigma(A)$, $\mid \lambda \mid \lim_n \inf \|A^n\|^{\frac{1}{n}}$., then taking the supremum on λ we get $r(A) \leq \lim_n \inf \|A^n\|^{\frac{1}{n}}$.
Thus we have:

$$r(A) \leq \liminf_n \|A^n\|^{\frac{1}{n}} \leq \limsup_n \|A^n\|^{\frac{1}{n}} = r(A).$$

Thus $r(A) = \lim_n \|A^n\|^{\frac{1}{n}}$.

3.5 The continuous functional calculus of compact operators

The purpose of this calculus is to define $f(A)$, where $A \in \mathcal{L}(H)$ is a self-adjoint operator and $f : \mathbb{R} \longmapsto \mathbb{R}$ is a continuous function.
In the case where A is diagonalized under the form $A = \sum \lambda_i P_i$, we can put $f(A) = \sum f(\lambda_i) P_i$. This is the case when A is supposed to be compact.
But generally we need another approach.
Starting by studying the spectrum of a self-adjoint operator.

3.5.1.Proposition

Let $A \in \mathcal{L}(H)$. Then the spectrum of A is included in $\mathbb{R}$.

Proof:

Let $A \in \mathcal{L}(H)$, from proposition(3.3.3) we have $\sigma_p(A) \subset \mathbb{R}$.
Now let $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and noting $\alpha =\mid \lambda \mid$. Firstly verifying that $A - \lambda Id$ is of a dense image.
Infact, if $y \perp Im(A - \lambda Id)$, then for all $x \in H$,

$$0 = \langle (A - \lambda Id)x, y \rangle = \langle x, (A - \overline{\lambda} Id)y \rangle$$

Which implies $Ay = \overline{\lambda} y$. But since

$$\overline{\lambda} \in \mathbb{C} \setminus \mathbb{R},$$

it is not an eigen value of A and then $y = 0$
Since $\langle Ax, y\rangle$ is real for all x, we have the inequality:

$$\alpha \| x \|^2 \leq | \langle (A - \lambda Id)x, x\rangle | \leq \| (A - \lambda Id)x \| . \| x \|,$$

hence we deduce $\| (A - \lambda Id)x \| \geq \alpha \| x \|, forall$ x $\in H$. Which implies $A - \lambda Id$ is injective and also it implies that it is of a closed image. In fact, if $((A - \lambda Id)x_n)$ converges to y, this inequality implies that (x_n) is Cauchy, then converges. If we note its limit x, the continuity of $A - \lambda Id$ implies $y = (A - \lambda Id)x$.
Thus the image of $A - \lambda Id$ is closed and dense, then $A - \lambda Id$ is surjective. And we already know that it is injective, so the inverse operator theorem implies that $A - \lambda Id$ is invertible.
The simpliest case is the one with the polynomial $P \in \mathbb{C}[X]$. In this case there is no problem to define $P(A)$. Moreover, the map $\tau : \mathbb{C}[X] \longmapsto \mathcal{L}(H)$ defined by $\tau(P) = P(A)$ is algebras morphism

3.5.2.Proposition

Let $P \in \mathbb{C}[X]$ be a polynomial, and $A \in \mathcal{L}(H)$. Then:

$$P(\sigma(A)) = \sigma(P(A)).$$

In this formula $P(\sigma(A))$ designates the set $\{P(\lambda) : \lambda \in \sigma(A)\}$.
Proof:
$(\subset)$
Let $\lambda \in \sigma(A)$, there exists a polynomial Q such that :

$$P(X) - P(\lambda) = (X - \lambda)Q(X),$$

and then:

$$P(A) - P(\lambda)Id = (A - \lambda Id)Q(A) = Q(A)(A - \lambda Id).$$

This identity shows that if $P(A) - P(\lambda)Id$ was invertible, then $A - \lambda Id$ would be too, and then:

$$P(\sigma(A)) = \sigma(P(A)).$$

$(\supset)$
If $P = 0$, the inclusion is evident. If not, let $\lambda \in \sigma(P(A))$ and writing the polynomial $P(X) - P(\lambda)$ in $\mathbb{C}[X]$:

$$P(X) - P(\lambda) = \gamma(X - \alpha_1)(X - \alpha_2)...(X - \alpha_n).$$

Snce $P \neq 0$, γ is non-zero. The operator $P(A) - P(\lambda)Id$is not invertible, then there exists necessarily an index i such that $A - \alpha_i Id$ is not invertible. In other words $\alpha_i \in \sigma(A)$. Since $\alpha_i = \lambda$, this implies $\lambda \in P(\sigma(A))$.
When we have $\mathbb{K}$ a compact of $\mathbb{R}$, we note $C(\mathbb{K})$ the algebra of the continuous functions from $\mathbb{K}$ onto $\mathbb{R}$. We can provide $C(\mathbb{K})$ with a norm

$$\| f \|_\infty = \max\{|f(x)|, x \in \mathbb{K}\}.$$

The space $(C(\mathbb{K}), \| f \|_\infty)$ is Banach.

3.5.1 Continuous functional calculus theorem

Let H be a Hilbert space (real or complex), and $A \in \mathcal{L}(H)$ is a self-adjoint operator. There exists a unique linear map :

$$\tau : C(\sigma(A)) \longmapsto \mathcal{L}(H),$$

Such that

(a) $\tau(1) = Id$.

(b) $\tau(x \longmapsto x) = A$.

(c) τ is a morphism of algebras, meaning that $\tau(fg) = \tau(f)\tau(g), \forall f, g \in C(\sigma(A))$.

(d) $\forall f \in C(\sigma(A)) \parallel \tau(f) = \| f \|_\infty$.

Generally we denote $f(A)$ rather than $\tau(f)$. Moreover we have the following properties:

(e) $\forall f \in C(\sigma(A)), \sigma(f(A)) = f(\sigma(A))$.

(f) $\forall f \in C(\sigma(A)), \tau(f)$ is self-adjoint.

Starting by some lemmas.

3.5.1.Lemma

Let $A \in \mathcal{L}(H)$ be a normal operator. Then $r(A) = \| A \|$.

Proof:

(Denoting by $\star$ the use of the equality $\| A \|^2 = \| AA^* \|$.)

$$\| A^{2n} \|^2 =^\star \| A^{2n}(A^{2n})^* \| = \| (AA^*)^n((AA^*)^n)^* \| =^\star \| (AA^*)^n \|^2 = \| A^n(A^n)^* \|^2 =^\star \| A^n \|^4.$$

We deduce immediately that $\| A^{2^k} \| = \| A \|^{2^k}$, for all $k \in \mathbb{Z}$. According to proposition(3.4.1), we obtain by letting $k \longrightarrow \infty$ thfollowing:

$$\| A \| = r(A).$$

We will also need the Stone-Weierstrass theorem.

3.5.2 Stone-Weierstrass theorem(lemma)

Let $K \subset \mathbb{R}$ be a compact, and let $\mathcal{P}$ be the set of the restrictions in K of the polynomials in $\mathbb{R}[X]$. Then $\mathcal{P}$ is dense in $C(K)$.

Proof of theorem(3.5.1):

We put $K = \sigma(A)$. Then K is compact, and according to proposition (3.5.1) $K \subset \mathbb{R}$. And let $\mathcal{P} \subset C(K)$ the set of restrictions in K of polynomials of real coefficients. For $P \in \mathcal{P}$, we put $\tau(P) = P(A)$. We have to verify that the definition is well consistent because the different polynomial can coinside at a restriction in K. Let p and q be two polynomials such that $p(x) = q(x)$ for all $x \in K$, then according to proposition(3.5.2), we have

$$\sigma((p-q)A) = (p-q)(\sigma(A)) = \{0\}.$$

But according to lemma(3.5.1), we have $\| \ (p-q)A \ \| = r((p-q)A) = 0$ and then $p(A) = q(A)$. The definition is well consistent, and moreover according to lemma(3.5.1), we have $\| \ \tau(p) \ \| = \| \ p \ \|_\infty$.

According to Stone-Weierstrass theorem, we can prolong by continuity τ in $C(K)$. Verifying that τ has the required properties(properties (a) and (b)) corresponding to $P(X) = 1$ and $P(X) = X$. The property (c) is true when $f, g \in \mathcal{P}$, and by density it extends to $C(K)$. Likewise, (d) is true for $f \in \mathcal{P}$(it is a consequence of proposition (3.5.2), it extends to $C(K)$ by density.

Then the property (e) is the only one left to prove.

Let $\lambda \in \mathbb{C} \setminus f(K)$. Then the function $g = (f-\lambda)^{-1}$ is continuous on K, and the property (c) implies that :

$$g(A)(f(A) - \lambda Id) = (f(A) - \lambda Id)g(A) = Id.$$

Then $f(A) - \lambda Id$ is invertible, and $\lambda \notin \sigma(f(A))$. This gives us the inclusion $\sigma(f(A)) \subset f(\sigma(A))$.

Reciprocally, let λ and $f \in C(K)$. We want to show that $f(\lambda) \in \sigma(f(A))$. By absurd assuming that $f(A) - f(\lambda)Id$ is invertible. Like the set of invertible operators, there exists ε such that the inequality $f(A) - f(\lambda)Id - B \leq \varepsilon$ implies that B is invertible. By the Stone-Weierstrass theorem, we can choose a polynomial $p \in \mathcal{P}$ such that $\| \ f - p \ \|_\infty \leq \frac{\varepsilon}{2}$. Then:

$$\| \ f(A) - f(\lambda)Id - p(A) - p(\lambda)Id \ \| \leq \| \ (f-p)A \ \| + | \ f(\lambda) - p(\lambda) \ | \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2},$$

to majorize the first term we used the property (d). This implies then $p(A) - p(\lambda)Id$ is invertible, and then $p(\lambda) \notin \sigma(p(A))$. But from the proposition (3.5.2), we have $\sigma(p(A)) \subset p(\sigma(A))$, which leads to a contradiction.

Finally, the conditions (a),(b) and (c) ditermines τ uniquely on the algebra generated by the functions $x \longmapsto 1$ and $x \longmapsto x$, which is exactly $\mathcal{P}$. As condition (d) implies that τ is continuous, the extension to $C(K)$ is unique.
Here is a good characterization of positive operators.

3.5.3.Proposition

Let $A \in \mathcal{L}(H)$ be a self-adjoint operator. Then T is positive if and only if its spectrum is contained in $\mathbb{R}^+$.

Proof

Assuming that A is positive. It is clear that $\sigma_p(A) \subset \mathbb{R}^+$. Since $\langle Ax, x\rangle \geq 0$, if $\lambda \leq 0$ then $\langle (A - \lambda Id)x, x\rangle \geq \lambda \| x \|^2$. Then we show that $A - \lambda Id$ is invertible using the same method used to prove proposition (3.5.1). Reciprocally assuming that $\sigma(A) \subset \mathbb{R}^+$. Since the function $x \longmapsto \sqrt{x}$ is continuous on $\sigma(A)$, the functional calculus allows us to define $B = \sqrt{A}$, meaning that an self-adjoint operator such that $\sigma(B) \subset \mathbb{R}^+$ and $B^2 = A$. But then:

$$\langle Ax, x\rangle = \langle B^2x, x\rangle = \langle Bx, Bx\rangle = \|Bx\|^2 \geq 0,$$

and then A is positive.

3.5.1.Corollary(the existance of a square root)

Let $A \in \mathcal{L}(H)$ be a positive self-adjoint operator. Then there exists a positive self-adjoint operator B such that $B^2 = A$.
It is easy to extend the functional calculus of theorem (3.5.1) to functions of a complex value. If f is a continuous function from σA onto $\mathbb{C}$, we can decompose it to $f = \mathcal{R}f + i\mathcal{I}f$, where $\mathcal{R}$ and $\mathcal{I}$ are continuous of a value in $\mathbb{R}$. We can then define :

$$f(A) = (\mathcal{R}f)A + i(\mathcal{I}f)A.$$

Noting then that A is normal, but generaly not self-adjoint.

3.6 Spectrum of compact operators and Fredholm theorem

If A is a normal compact operator, we have seen that it is diagonalizable in an orthonormal basis of an eigen vector. If A is not normal, then it can not be diagonalizable, nevertheless it is possible to specify the structure of its spectrum. It the purpose of the first part of this theorem.

3.6.1 Theorem

Let $K \in \mathcal{L}(H)$ be a compact operator. Then its spectrum is a countable set containing 0. Moreover, if $\lambda \in \sigma(K)$ and $\lambda \neq 0$, then λ is an eigen value of K, and the corresponding eigen space $Ker(K - \lambda Id)$ is of a finite dimension. Finally, if (λ_n) is a sequence of elements of $\sigma(K)$, then $\lim \lambda_n = 0$. We will demonstrate several preliminary results in the form of lemmas

3.6.1.Lemma

Let $K \in \mathcal{L}(H)$ be a compact operator. Putting $A = Id - K$. Let $\mathbb{M} \subset H$ be a closed subspace such that $A_{|\mathbb{M}}$ is injective. Then there exists $c > 0$ such that:

$$\| Ax \| \geq c \| x \|, \forall x \in \mathbb{M},$$

and then $A(\mathbb{M})$ is a closed subspace.

Proof

Firstly, by absurd, assuming that

$$\| Ax \| < c \| x \|, \forall x \in \mathbb{M}.$$

Then there would exists a sequence(x_n) in $\mathbb{M}$ with $\| x_n \| = 1$ and $A(x_n) \longrightarrow 0$. Since K is compact, even if it means passing to a subsequence, we can suppose that (Kx_n) converges. But since $Id = A + K$, the sequence (x_n) converges too. Let x be its limi. We obtain $Ax = 0$ which contradicts the assumption

Secondly to prove that $A(\mathbb{M})$ is closed, let (y_n) be sequence in $\mathbb{M}$ such that $A(y_n)$ converges. The inequality proven previously implies that (y_n) is Cauchy too, thus it converges. Denoting its limit y, then $\lim A(y_n) = Ay \in \mathbb{M}$.

3.6.2.Lemma

Let $K \in \mathcal{L}(H)$ be a compact operator, and $A = Id - K$. Then $KerA$ is of a finite dimension and $RanA$ is closed.

Proof

The first point has already been seen in proposition (3.2.2). Let $\mathbb{M} = (KerA)^{\perp}$. Verifying $A(\mathbb{M}) = A(H) = RanA$ and that $A_{|\mathbb{M}}$ is injective. By the preceding lemma $RanA$ is closed.

3.6.3.Lemma

If $\mathbb{M}$ and $\mathbb{L}$ are two closed spaces of H with $\mathbb{M} \subset \mathbb{L}$ and $\mathbb{M} \neq \mathbb{L}$. Then there exists $y \in \mathbb{L}$ with $\| y \| = 1$ and $d(y, \mathbb{M}) = 1$.

3.6.4.Lemma

Let $K \in \mathcal{L}(H)$ be a compact operator, and $A = Id - K$. It does not exists an infinite sequence $(F_n)_{n\geq 0}$ or $(F_n)_{n\leq 0}$of closed subspaces of H such that :

$$\forall n \in \mathbb{M} : F_n \subset F_{n+1}, F_n \neq F_n, and, A(F_{n+1}) \subset F_n.$$

Proof

Assuming the existance of such a sequence (F_n). By the preceding lemma, we can find for all n a vector $x_{n+1} \in F_n$ such that $\| x_{n+1} \| = 1$ and $d(x_{n+1}, F_n) = 1$. Moreover, the subspace F_n is invariant by K(because it is invariant by Id and by A). If $k < 1$ then the vector $A(x_l) + K(x - k)$ is in F_{l-1}, and then:

$$\| x_l - (A(x_l) + K(x - k)) \| \geq d(x_l, F_{l-1}) = 1.$$

and then $\| Kx_l - Kx_k \| \geq 1$. We will have then in $K(B_H)$ an infinite sequence of points whose mutual distances are superior then1, which contradicts the compactness of K.

3.6.1.Corollary

Let $K \in \mathcal{L}(H)$ be a compact operator, and let $A = Id - K$. Then the increasing sequence $(KerA^n)_{n\geq 0}$ is stationary, and the decreasing sequence $(RanA^n)_{n\geq 0}$ is stationary.

Proof

Assuming that A is surjective and non-injective. Showing by recurrence that $(KerA^n)_{n\geq 0} \neq (KerA^{n+1})_{n\geq 0}$. For the recurrence step, let $x \in (KerA^{n+1})_{n\geq 0} \setminus (KerA^n)_{n\geq 0}$. Since A is surjective, there exists $y \in H$ such that $x = Ay$. Then $y \in (KerA^{n+2})_{n\geq 0} \setminus (KerA^{n+1})_{n\geq 0}$

Likewise, assuming that A is injective and non-surjective. Showing by recurrence that $(RanT^n)_{n\geq 0} \neq (RanA^{n+1})_{n\geq 0}$.

In effect supposing that $x \in (RanA^n)_{n\geq 0} \setminus (RanA^{n+1})_{n\geq 0}$, then $Ax \in (RanA^{n+1})_{n\geq 0}$ and $Ax \notin (RanA^{n+2})_{n\geq 0}$ (using the injectivity of A).

In both cases we conclude by the preceding corollary.

Now we can prove theorem(3.6.1).

Proof of theorem (3.6.1)

Let K be a compact operator. If $\lambda \neq 0$, we see by applying the preceding corollary to $Id - \frac{1}{\lambda}K$ that:

$$\lambda \in \sigma(K) \Leftrightarrow \lambda \in \sigma_p(K).$$

Now let (λ_n) is a sequence of elements in $\sigma(K)$, two by two distinct. We are going to show that $\lim_n \lambda_n = 0$. We can assume for all n, $(\lambda_n) \neq 0$, and then $\lambda \in \sigma_p(K)$. For all n we choose $(x_n) \neq 0$, such that $Kx_n = \lambda_n x_n$. Verifying that the family $\{x_n\}$

is free(considering the minimal length and leading to contradiction by applying to K). Let $\mathbb{M}_n = \mathcal{VECT}\{x_1, ..., x_n\}$. By a preceding lemma we can choose $y_n \in \mathbb{M}_n$ such that $\| y_n \| = 1$ and $d(y_n, \mathbb{M}_{n-1}) = 1$.
Verifying then that $(K - \lambda_n Id)y_n \in \mathbb{M}_{n-1}$. If $n > m$, then:

$$\lambda_n^{-1} K(y_n) - \lambda_m^{-1} K(y_m) = y_n - [\lambda_m^{-1}(K - \lambda_m Id)y_m] - [\lambda_n^{-1}(K - \lambda_n Id)y_n].$$

Since the expression in parantheses is in $\mathbb{M}_{n-1}$.
We have $\| \lambda_n^{-1} K(y_n) - \lambda_m^{-1} K(y_m) \| \geq d(y_n, \mathbb{M}_{n-1}) = 1$. By consequence the sequence $\| \lambda_n^{-1} K(y_n) \|$ does not have a convergent subsequence. But if the sequence $\lambda_n)$ does not tend to 0, then the sequence $(\lambda_n^{-1} y_n)$ have a bounded subsequence. And because K is compact we will have a converging subsequence of $\lambda_n^{-1} K(y_n)$, which is contradiction.
Thus, for all $\varepsilon > 0$, only a finite number of elements of $\sigma(K)$ has a modulus superior than 1. This shows that at most $\sigma(K)$, and the other assertions of the theorem have been already proven.

3.6.2.Definition

Let H be a Hilbert space and $\mathbb{F} \subset H$ a vector subspace (not closed), we call a codimension of $\mathbb{F}$ the dimension of $H \setminus \mathbb{F}$. Noting $codim(\mathbb{F}) = dim(H \setminus \mathbb{F})$. Here the space $H \setminus \mathbb{F}$ is the quotient vector space, which is a purely algebraic notion and does not use the sturcture of a Hilbert space. When $\mathbb{F}$ is closed, we have the direct sum $H = \mathbb{F} + \mathbb{F}^{\perp}$, then $H \setminus \mathbb{F}$ is isomorphic to $\mathbb{F}^{\perp}$ and $codim(\mathbb{F}) = dim(\mathbb{F}^{\perp})$.

3.6.3.Definition

An operator $A \in \mathcal{L}(H)$ is called an operator of Fredholm if $Ker(A)$ is of a finite dimension and $Ran(A)$ is of a finite codimension. Defining the index of A then :

$$ind(A) = dim(KerA) - codim(RanA).$$

3.6.1.Proposition

A Fredholm operator is of a closed image, by consequence we have the formula:

$$ind(A) = dim(KerA) - dim(A^*).$$

Proof
Let A be a Fredholm operator. Even if it means replacing A by $A_{|Ker(A)^{\perp}}$, we can assume that A is injective. Since $RanA$ is of a finite codimension, there exists $y_1, ..., y_n \in H$ such

that $(RanA+y_1,...,Ran(A)+y_n)$ is a basis for the space $H\setminus Ran(A)$. Defining an operator $K : H\bigoplus\mathbb{R}^n \longmapsto H$ by:

$$K(x,v) = Ax + \sum_{j=1}^{n} v_j y_j.$$

Then K is continuous and bijective, and then invertible (by the inverse theorem). Thus there exists a constant $c > 0$ such that $\| K(x,v) \|\geq c \| (x,v) \|$ for all x and v. In particular $\| Tx \|\geq c \| x \|$ for all x, which implies that $RanA$ is closed.

3.6.2 Theorem

Let $A \in \mathcal{L}(H)$ be a compact operator. Then $Id-A$ is of Fredholm and $ind(Id-A) = 0$

Proof

We have already seen that $Ker(Id - A)$ is of a finite dimension when A is compact. Likewise $Ker(Id - A^*)$ is of a finite dimension then $Ran(Id - A)$ (which is closed by a preceding lemma) is of a finite codimension and $Id - A$ is of Fredholm. To show that the index equals 0, we are going to proceed by recurrence. For all $n \in \mathbb{N}$, let P_n be the proposition (for all compact operator A such that $dimKer(Id - A) \leq n$, we have $dimKer(Id - A) = dim(Id - A^*)$).

We have seen that if $Id - A$ is injective, then it is surjective too, and then $Id - A^*$ is injective. Thus P_0 is satisfied. Assuming that A is a compact operator such that $dimKer(Id - A) = n$. Then $Id - A$ is not surjective. And since we know that it is of a closed image we can choose $y_0 \in Ran(Id - A)^{\perp}$, with $y_0 \neq 0$.

Likewise we can choose $x_0 \in Ker(Id - A)$, with $x_0 \neq 0$. Putting then $A'(h) = A(h) + \langle h, x_0\rangle y_0$. Then A' is compact (it is the sum of compact operators and operators of a rank equals 1). Moreover, $(Id-A')(h) = 0$ if and only if $(Id-A')(h) = 0$ and $h \perp x_0$, and then $dimKer(Id - A') = dimKer(Id - A) - 1$. Likewise, we have $A'^*(h) = A'(h) + \langle h, y_0\rangle x_0$ then $dimKer(Id - K'^*) = dimKer(Id - K^*) - 1$.

We know according to the recurrence hypothese that $dimKer(Id - K') = dimKer(Id - K'^*)$, and then $dimKer(Id - K) = dimKer(Id - K^*)$.

This allows us to conclude the recurrence.

3.6.3 Theorem

If $A, B \in \mathcal{L}(H)$ are two Fredholm operators. Then BA is also Fredholm and :

$$ind(BA) = ind(A) + ind(B).$$

3.6.5.Lemma

Assuming that H admets the decompositions $H = \mathcal{M} \bigoplus \mathcal{N} = \mathcal{M}' \bigoplus \mathcal{N}'$, and let A be an operator given by the block matrix :

$$A = \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix} : \mathcal{M} \bigoplus \mathcal{N} \longrightarrow \mathcal{M}' \bigoplus \mathcal{N}'$$

Where $A_1 : \mathcal{M} \longmapsto \mathcal{M}', X : \mathcal{N} \longmapsto \mathcal{M}'$ and $A_2 : \mathcal{N} \longmapsto \mathcal{N}'$. Assuming that A_1 is invertible and that $\mathcal{N}$ and $\mathcal{N}'$ are a finite dimensional. Then A is Fredholm and

$$ind(A) = dim(\mathcal{N}) - dim(\mathcal{N}').$$

Proof

We have $RanA = \mathcal{M}' \bigoplus (\mathcal{N}' \cap RanA)$, where $\mathcal{N}' \cap RanA$ is of a finite dimension then it is closed.

Let $\gamma : Ker(A_2) \longmapsto Ker(A)$ be the map defined by:

$$\gamma(h) = A_1^{-1}X(h) \bigoplus h$$

. It is easy to see that γ is bijective, and then $Ker(A_2) = Ker(A)$. Likewise, it is easy to see that $Ran(A) = \mathcal{M}' \bigoplus Ran(A_2)$, from where we take the orthogonal $Ker(A^*) = Ker(A_2^*)$. Thus A is Fredholm and $ind(A) = dimKerA_2 - dimKerA_2^*$. Let r be the range of the matrix. An elementary calculus of linear algebra gives us :

$$dim(A) = (dim\mathcal{N} - r) - (dim\mathcal{N}' - r) = dim\mathcal{N} - dim\mathcal{N}'.$$

Proof of theorem (3.6.3)

Putting $\mathcal{M}' = (RanA) \cap (KerB)^{\perp}$ and $\mathcal{N}' = \mathcal{M}'^{\perp}$. Now putting $\mathcal{M} = A^{-1}(\mathcal{M}') \cap (KerA)^{\perp}$, $\mathcal{N} = \mathcal{M}^{\perp}, \mathcal{M}" = B(\mathcal{M}')$, and $\mathcal{N}" = M"^{\perp}$. We verify that $\mathcal{N}, \mathcal{M}'$ and $\mathcal{M}"$ are of a finite dimension and that the operators A and B are given by the block matrix :

$$A = \begin{bmatrix} A_1 & X \\ 0 & A_2 \end{bmatrix} : \mathcal{M} \bigoplus \mathcal{N} \longmapsto \mathcal{M}' \bigoplus \mathcal{N}'.$$

$$A = \begin{bmatrix} B_1 & Y \\ 0 & B_2 \end{bmatrix} : \mathcal{M}' \bigoplus \mathcal{N}' \longmapsto \mathcal{M}" \bigoplus \mathcal{N}".$$

Where A_2, B_2 are invertible operators. an elementary calculus shows that BA is written as:

$$BA = \begin{bmatrix} B_1A_1 & Z \\ 0 & B_2A_2 \end{bmatrix} : \mathcal{M} \bigoplus \mathcal{N} \longmapsto \mathcal{M}" \bigoplus \mathcal{N}".$$

According to the preceding lemma, we have $ind(A) = dim\mathcal{N} - dim\mathcal{N}'$
$ind(B) = dim\mathcal{N}' - dim\mathcal{N}"$ and $ind(BA) = dim\mathcal{N} - dim\mathcal{N}"$.which clearly demonstrates the announced formula.

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Abstract

In this dissertation, we have dealt with the compact operators, these kind of operators play a fondamental role in mathematics and they are used to prove a lot of fondamental theorems in different mathematic subfields .

This research consists of three main chapters.

The first chapter was about the Hilbert spaces and theire properties and it was also about orthogonality and orthonormality.

The second one discusses the operators and theire theorems and properties, it also discusses the adjoint of an operator and its properties and definitions.

The third one covers this research object, compact operators, we have dealt in this chapter with the most important theorems and basics of the compact operators.

Eventually i hope that i succeeded in presenting this dissertation throughly and simtly.

Abstrait

Dans ce mémoire, j'ai presentè les opérateurs compact, ce genre des opérateurs joue un role fendamentale dans les mathématiques et ils sont utilisés pour prouver des plusieures théorèmes dans les différent domaines des mathématiques .

Ce mèmoire consiste de trois chapitres principaux.

Le premier chapitre c'était à propos des espaces de Hilbert et leurs propriétés, c'était aussi à propos d'orthogonalité et orthonormalité.

Le deuxiém parle des opérateurs et leurs théorèmes et leurs propriétés, il parle aussi de l'adjoint d'un opérateur et ses propriétés et définitions.

Le troisiéme couvre le but de ce mémoire, les opérateurs compact, j'ai présenté dans ce chapitre les théorèmes trés importants et les concepts de base des opérateurs compact .

Finalement j'éspère que j'ai réussi à présenter ce sujet complétement et simplement.